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ELLIPTIC APPROXIMATION OF FORWARD-BACKWARD PARABOLIC EQUATIONS

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Abstract. In this note we give existence and uniqueness result for some elliptic problems depending on a small parameter and show that their solutions converge, when this parameter goes to zero, to the solution of a mixed type equation, elliptic-parabolic, parabolic both forward and backward. The aim is to give an approximation result via elliptic equations of a changing type equation.

1. Introduction.

In [8] and [7] existence results for mixed type equations, in particular forward-backward parabolic equations, are given. The simplest examples are the two following: given $T > 0$, Ω open subset of $\mathbf{R}^n$, $r \in L^\infty(\Omega \times (0,T))$ consider (Δ_p denotes the p-Laplacian for $p \geqslant 2$ and ν the outside normal to $\partial\Omega$)

$$\begin{cases} \dfrac{\partial}{\partial t}\big(r(x,t)u\big) - \Delta_p u = f \qquad \text{or} \qquad r(x,t)\dfrac{\partial u}{\partial t} - \Delta_p u = f \qquad \text{in } \Omega \times (0,T) \\ u = 0 \quad \text{in } \partial\Omega \times (0,T) \qquad \text{or} \qquad \dfrac{\partial u}{\partial \nu} = 0 \quad \text{in } \partial\Omega \times (0,T) \\ u = \varphi \quad \text{in } \{x \in \Omega | r(x,0) > 0\} \times \{0\} \\ u = \psi \quad \text{in } \{x \in \Omega | r(x,T) < 0\} \times \{T\} \end{cases} \tag{1}$$

with f, φ, ψ suitable data and, at least for r assuming both positive and negative sign, suitable assumptions on the two sets $\{x \in \Omega | r(x,0) > 0\} \times \{0\}$ and $\{x \in \Omega | r(x,T) < 0\} \times \{T\}$. Equations of such type arise in the study of some stochastic differential equation, in the kinetic theory, in some physical models like electron scattering or neutron transport. For some references one can see [3] or the much less recent papers [6, 1, 2] (or the references contained therein and in [8] and [7]) where simple equations like

$$\operatorname{sgn}(x)|x|^m u_t - u_{xx} = f$$

are considered, being $m \in \mathbf{N}$.

The aim of the present note is to give an approximation result for abstract forward-backward parabolic equations via elliptic problems (see Theorem 4.1). For

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the problems (1) the result may be stated as follows. In accordance with the equation and with the boundary condition considered in (1) consider one of the two equations

$$-\varepsilon\frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^{p-2}\frac{\partial u}{\partial t}\right)+\frac{\partial}{\partial t}\big(r(x,t)u\big)-\Delta_p u=f$$
$$-\varepsilon\frac{\partial}{\partial t}\left(\left|\frac{\partial u}{\partial t}\right|^{p-2}\frac{\partial u}{\partial t}\right)+r(x,t)\frac{\partial u}{\partial t}-\Delta_p u=f$$

in $\Omega\times(0,T)$ (ε is a positive parameter) with the boundary conditions

$$\begin{aligned}
&u=0\quad\text{in }\partial\Omega\times(0,T)\qquad\text{or}\qquad\frac{\partial u}{\partial\nu}=0\quad\text{in }\partial\Omega\times(0,T)\\
&u=\varphi\quad\text{in }\{x\in\Omega|r(x,0)>0\}\times\{0\}\\
&\frac{\partial u}{\partial\nu}=\frac{\partial u}{\partial t}=0\quad\text{in }\{x\in\Omega|r(x,0)\leqslant 0\}\times\{0\}\\
&u=\psi\quad\text{in }\{x\in\Omega|r(x,T)<0\}\times\{T\}\\
&\frac{\partial u}{\partial\nu}=\frac{\partial u}{\partial t}=0\quad\text{in }\{x\in\Omega|r(x,T)\geqslant 0\}\times\{T\}
\end{aligned}$$

and show that the solutions converge, when ε converge to zero, to the corresponding solution of (1) in the following sense: if u denotes the solution of one of the problems (1) and u_ε the solution of the approximating problem one has

$$\begin{aligned}
&u_\varepsilon\to u\qquad\text{in }L^p(0,T;W^{1,p}(\Omega))\text{-weak},\\
&ru_\varepsilon\to ru\qquad\text{in }L^2(\Omega\times(0,T))\text{-strong},\\
&\varepsilon\left|\frac{\partial u_\varepsilon}{\partial t}\right|^{p-2}\frac{\partial u_\varepsilon}{\partial t}\to 0\qquad\text{in }W^{-1,p'}\big(\Omega\times(0,T)\big)\text{-strong}\cap L^{p'}\big(\Omega\times(0,T)\big)\text{-weak},\\
&r\frac{\partial u_\varepsilon}{\partial t}\to r\frac{\partial u}{\partial t}\qquad\text{in }L^{p'}(0,T;W^{-1,p'}(\Omega))\text{-weak},\\
&\frac{\partial}{\partial t}(ru_\varepsilon)\to\frac{\partial}{\partial t}(ru)\qquad\text{in }L^{p'}(0,T;W^{-1,p'}(\Omega))\text{-weak}.
\end{aligned}$$

This result is similar to that contained in [4], even if our purpose is different: the result of Lions aims to give an existence result for linear parabolic equations with boundary conditions depending on time, we only want to give an approximation result, via more standard equations, of a mixed type equation, even if the technique can be used also in other different environments.

2. **Notations, hypotheses and preliminary results.**

Consider the following family of evolution triplets

$$V(t)\subset H(t)\subset V'(t)\quad t\in[0,T]\tag{2}$$

where $H(t)$ is a separable Hilbert space, $V(t)$ a reflexive Banach space which continuously and densely embeds in $H(t)$ and $V'(t)$ the dual space of $V(t)$, and we suppose there is a constant C_0 which satisfies

$$\|w\|_{V'(t)}\leqslant C_0\|w\|_{H(t)},\qquad\text{and}\qquad\|v\|_{H(t)}\leqslant C_0\|v\|_{V(t)}\tag{3}$$

for every $w\in H(t)$, $v\in V(t)$ and every $t\in[0,T]$.

The framework seems to be similar to the one considered in [4], but we have in mind something different (see the example in the last section). We will suppose the existence of a Banach space U such that

$$U \subset V(t) \qquad \text{and} \qquad U \text{ dense in } V(t) \qquad \text{for a.e. } t \in [0,T] \tag{4}$$

and define, for some $p \geqslant 2$, the set

$$\mathcal{U} := W^{1,p}(0,T;U).$$

Moreover we will suppose that the functions

$$t \mapsto \|u(t)\|_{V(t)}, \quad t \mapsto \|u(t)\|_{H(t)}, \quad t \mapsto \|u(t)\|_{V'(t)}, \quad t \in [0,T],$$

are measurable for every $u \in \mathcal{U}$ and we define the spaces

$$\mathcal{V} \qquad \text{and} \qquad \mathcal{H} \tag{5}$$

as the completion of $\mathcal{U}$ with respect to the natural norms

$$\|v\|_{\mathcal{V}} := \left(\int_0^T \|v(t)\|^p_{V(t)} dt \right)^{1/p}, \qquad \|v\|_{\mathcal{H}} := \left(\int_0^T \|v(t)\|^2_{H(t)} dt \right)^{1/2}.$$

Finally by $\mathcal{V}'$ we denote the dual space of $\mathcal{V}$ endowed with the norm

$$\|f\|_{\mathcal{V}'} := \left(\int_0^T \|f(t)\|^{p'}_{V'(t)} dt \right)^{1/p'}.$$

Definition 2.1. Given a family of linear operators $R(t)$ such that

$$R \quad \text{depends on a parameter } t \in [0,T] \quad \text{ and } R(t) \in \mathcal{L}(H(t)), \tag{6}$$

being $\mathcal{L}(H(t))$ the set of linear and bounded operators from $H(t)$ in itself, instead of (6) we sometimes will write improperly

$$R : [0,T] \longrightarrow \mathcal{L}(H(t)), \qquad t \in [0,T]. \tag{7}$$

Now consider an abstract function $R : [0,T] \longrightarrow \mathcal{L}(H(t))$. We say that R belongs to the class $\mathcal{E}(C_1, C_2)$, $C_1, C_2 > 0$, if it satisfies what follows for every $u, v \in U$:

- ◇ $R(t)$ is self-adjoint and $\|R(t)\|_{\mathcal{L}(H(t))} \leqslant C_1$ for every $t \in [0,T]$,
- ◇ $t \mapsto \big(R(t)u, v\big)_{H(t)}$ is absolutely continuous on $[0,T]$,
- ◇ $\left|\frac{d}{dt}\big(R(t)u, v\big)_{H(t)}\right| \leqslant C_2 \|u\|_{V(t)} \|v\|_{V(t)}$ for a.e. $t \in [0,T]$.

Now, given two positive constants C_1 and C_2, consider $R \in \mathcal{E}(C_1, C_2)$. For every $t \in [0,T]$ we consider the spectral decomposition of $R(t)$ (see, e.g., Section 8.4 in [5]) and define $R_+(t)$, and respectively $R_-(t)$, the operator connected to the positive, respectively negative, part of the spectrum, so that $R(t) = R_+(t) - R_-(t)$ and $R_+(t) \circ R_-(t) = R_-(t) \circ R_+(t) = 0$ and $R_+(t)$ and $R_-(t)$ turn out to be invertible. Equivalently one can define $R_+(t)$ and $R_-(t)$ as follows: since $R(t)$ is self-adjoint we get that $R(t)^2 = R^*(t) \circ R(t)$ is a positive operator; then we can define the square root of $R(t)^2$ (see, e.g., Chapter 3 in [5]), which is a positive operator,

$$|R(t)| = \big(R(t)^2\big)^{1/2}$$

and then define the two positive operators

$$R_+(t) := \frac{1}{2}\big(|R(t)| + R(t)\big), \qquad R_-(t) := |R(t)| - R_+(t).$$

By this decomposition we can also write $H(t) = H_+(t) \oplus H_0(t) \oplus H_-(t)$ where $H_+(t) = (\mathrm{Ker} R_+(t))^\perp$ and $H_-(t) = (\mathrm{Ker} R_-(t))^\perp$ and $H_0(t)$ is the kernel of $R(t)$. Finally we denote $\tilde{H}_0(t) = H_0(t) = \mathrm{Ker} R(t)$ and

$$\tilde{H}(t), \tilde{H}_+(t), \tilde{H}_-(t) = \text{the completion respectively of } H(t), H_+(t), H_-(t) \tag{8}$$

with respect to the norm

$$\|w\|_{\tilde{H}(t)} = \| |R(t)|^{1/2} w \|_{H(t)}.$$

Clearly the operation $\tilde{}$ depends on R. Moreover we consider $P_+(t)$ and $P_-(t)$ the orthogonal projections from $\tilde{H}(t)$ onto $H_+(t)$ and $H_-(t)$ respectively, $P_0(t)$ the projection defined in $H(t)$ onto $H_0(t)$.

Given an operator $R \in \mathcal{E}(C_1, C_2)$ it is possible to define two other linear operators. First we can define the derivative of R which, unlike R, is valued in $\mathcal{L}(V(t), V'(t))$, i.e. the set of linear and bounded operators from $V(t)$ to $V'(t)$: since $R \in \mathcal{E}(C_1, C_2)$ we can define a family of equibounded operators

$$R'(t), \quad t \in [0,T], \qquad R'(t) : V(t) \to V'(t) \qquad \text{by}$$

$$\langle R'(t)u, v \rangle_{V'(t) \times V(t)} := \frac{d}{dt} \big(R(t)u, v \big)_{H(t)}, \qquad u, v \in U.$$

By the density of U in $V(t)$ we can extend $R'(t)$ to $V(t)$. Then we can also define

$$\mathcal{R} : \mathcal{H} \to \mathcal{H}, \qquad (\mathcal{R}u)(t) := R(t)u(t), \tag{9}$$

$$\mathcal{R}_+ : \mathcal{H} \to \mathcal{H}, \qquad (\mathcal{R}_+ u)(t) := R_+(t)u(t), \tag{10}$$

$$\mathcal{R}_- : \mathcal{H} \to \mathcal{H}, \qquad (\mathcal{R}_- u)(t) := R_-(t)u(t), \tag{11}$$

which turn out to be linear and bounded by the constant C_1 and, by density of $\mathcal{U}$ in $\mathcal{V}$, an operator

$$\mathcal{R}' : \mathcal{V} \to \mathcal{V}' \qquad \text{by} \qquad \langle \mathcal{R}'u, v \rangle_{\mathcal{V}' \times \mathcal{V}} := \int_0^T \langle R'(t)u(t), v(t) \rangle_{V'(t) \times V(t)} dt \tag{12}$$

which turns out to be linear, self-adjoint and bounded by C_2.

For a function

$$u : [0,T] \to U$$

we denote by u' the distributional derivative, i.e. the function such that

$$\int_0^T u \varphi' dt = - \int_0^T u' \varphi dt$$

for every $\varphi \in C^1_0([0,T]; \mathbf{R})$. We maintain the same notation for functions belonging to $\mathcal{V}$.

We now could consider for $R \in \mathcal{E}(C_1, C_2)$ the two operators

$$u \mapsto (\mathcal{R}u)' \quad \text{and} \quad \mathcal{R}u'$$

defined respectively in the two spaces

$$\mathcal{W}_1 := \big\{ u \in \mathcal{V} \big| \mathcal{R}u' \in \mathcal{V}' \big\} \qquad \text{and} \qquad \mathcal{W}_2 := \big\{ u \in \mathcal{V} \big| (\mathcal{R}u)' \in \mathcal{V}' \big\}.$$

Since $\mathcal{R}$ admits a derivative one has (see [7]) that $(\mathcal{R}u)' = \mathcal{R}'u + \mathcal{R}u'$ and that $\mathcal{W}_1 = \mathcal{W}_2$ even if we will endow the two spaces respectively with the norms

$$\|u\|_{\mathcal{W}_1} = \|u\|_{\mathcal{V}} + \|\mathcal{R}u'\|_{\mathcal{V}'} \qquad \text{and} \qquad \|u\|_{\mathcal{W}_2} = \|u\|_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'}.$$

Because of that, it will not always be necessary to specify which of the two spaces we are talking about and in those cases we will simply refer to them as

$$\mathcal{W}_{\mathcal{R}}.$$

As done before we can define, in a way analogous to that done for the spaces (8),

$$\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_+, \tilde{\mathcal{H}}_- = \text{the completion respectively of } \mathcal{H}, \mathcal{H}_+, \mathcal{H}_- \tag{13}$$

with respect to the norm $\|w\|_{\tilde{\mathcal{H}}} = \||\mathcal{R}|^{1/2} w\|_{\mathcal{H}}$, where $|\mathcal{R}| = \mathcal{R}_+ + \mathcal{R}_-$.

Analogously, we define $\mathcal{H}_+$ and $\mathcal{H}_-$ and $\mathcal{P}_+$ and $\mathcal{P}_-$ the orthogonal projections from $\tilde{\mathcal{H}}$ onto $\mathcal{H}_+$ and $\mathcal{H}_-$ respectively. $\mathcal{H}_0$ is the kernel of $\mathcal{R}$ and $\mathcal{P}_0$ the projection defined in $\mathcal{H}$ onto $\mathcal{H}_0$.

Now we recall a result which can be found in [7] (see also [8]).

Proposition 2.2. *Suppose $R \in \mathcal{E}(C_1, C_2)$. Then we have that for every $u, v \in \mathcal{W}_{\mathcal{R}}$ the following holds:*

$$\begin{aligned}
&\frac{d}{dt}(\mathcal{R}u(t), v(t))_{H(t)} \\
&= \langle \mathcal{R}u'(t), v(t)\rangle_{V'(t)\times V(t)} + \langle \mathcal{R}v'(t), u(t)\rangle_{V'(t)\times V(t)} + \langle \mathcal{R}'u(t), v(t)\rangle_{V'(t)\times V(t)} \\
&= \langle (\mathcal{R}u)'(t), v(t)\rangle_{V'(t)\times V(t)} + \langle (\mathcal{R}v)'(t), u(t)\rangle_{V'(t)\times V(t)} - \langle \mathcal{R}'u(t), v(t)\rangle_{V'(t)\times V(t)}.
\end{aligned}$$

Moreover the function $t \mapsto (R(t)u(t), v(t))_{H(t)}$ is continuous and there exists a constant c, which depends only on T, such that

$$\begin{aligned}
&\max_{[0,T]} |(R(t)u(t), v(t))_{H(t)}| \\
&\leqslant c\Big[\|\mathcal{R}u'\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} + \|\mathcal{R}v'\|_{\mathcal{V}'}\|u\|_{\mathcal{V}} + \|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} + \|\mathcal{R}\|_{\mathcal{L}(\mathcal{H})}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\Big].
\end{aligned}$$

and

$$\begin{aligned}
&\max_{[0,T]} |(R(t)u(t), v(t))_{H(t)}| \\
&\leqslant c\Big[\|(\mathcal{R}u)'\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} + \|(\mathcal{R}v)'\|_{\mathcal{V}'}\|u\|_{\mathcal{V}} + \|\mathcal{R}'\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}} + \|\mathcal{R}\|_{\mathcal{L}(\mathcal{H})}\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}\Big].
\end{aligned}$$

Finally we recall a classical result (see, e.g., Section 32.4 in [11], in particular Corollary 32.26) for which we need some definitions, which we remind.

We say that an operator $\mathcal{Q} : \mathcal{X} \to \mathcal{X}'$, $\mathcal{X}$ being a reflexive Banach space, is coercive if

$$\lim_{\|x\|\to+\infty} \frac{\langle \mathcal{Q}x, x\rangle}{\|x\|} \to +\infty,$$

The same operator $\mathcal{Q}$ is hemicontinuous if the map

$$t \mapsto \big\langle \mathcal{Q}(u + tv), w\big\rangle_{\mathcal{X}'\times\mathcal{X}} \qquad \text{is continuous in } [0,1] \qquad \text{for every } u, v, w \in \mathcal{X}.$$

A monotone and hemicontinuous operator $\mathcal{Q}$ is of type M if (see, for instance, *Basic Ideas of the Theory of Monotone Operators* in volume B of [11] or Lemma 2.1 in [10]), i.e. it satisfies what follows: for every sequence $(u_j)_{j\in\mathbf{N}} \subset \mathcal{X}$ such that

$$\left.\begin{array}{ll}
u_j \to u & \text{in } \mathcal{X}\text{-weak} \\
\mathcal{Q}u_j \to b & \text{in } \mathcal{X}'\text{-weak} \\
\limsup\limits_{j\to+\infty} \big\langle \mathcal{Q}u_j, u_j\big\rangle_{\mathcal{X}'\times\mathcal{X}} \leqslant \big\langle b, u\big\rangle_{\mathcal{X}'\times\mathcal{X}} &
\end{array}\right| \quad \Longrightarrow \quad \mathcal{Q}u = b. \tag{M}$$

Theorem 2.3. *Let $M : \mathcal{X} \to \mathcal{X}'$ be monotone, bounded, coercive and hemicontinuous. Suppose $L : \mathcal{X} \to 2^{\mathcal{X}'}$ to be maximal monotone. Then for every $f \in \mathcal{X}'$ the following equation has a solution*

$$Lu + Mu \ni f$$

and in particular if L, M are single-valued the equation $Lu+Mu = f$ has a solution. If, moreover, M is strictly monotone the solution is unique.

3. **The approximating problems.** In this section we want to give an existence and uniqueness result for a family of elliptic problems defined below (see (28)). Before we introduce another functional space, denoted by $\mathcal{V}_*$ below. To do that first consider another family of reflexive Banach spaces $K(t)$ such that

$$V(t) \subset K(t) \subset H(t) \quad t \in [0,T] \tag{14}$$

where $V(t)$ continuously embeds in $K(t)$ and $K(t)$ continuously embeds in $H(t)$ and there is a positive constant, which for simplicity we suppose to be C_0, such that

$$\|w\|_{H(t)} \leqslant C_0 \|w\|_{K(t)}, \qquad \text{and} \qquad \|v\|_{K(t)} \leqslant C_0 \|v\|_{V(t)} \tag{15}$$

Then we suppose that the functions

$$t \mapsto \|u(t)\|_{K(t)}, \quad t \in [0,T],$$

are measurable for every $u \in \mathcal{U}$ and we define the space $\mathcal{K}$ as the completion of $\mathcal{U}$ with respect to the natural norm

$$\|v\|_{\mathcal{K}} := \left(\int_0^T \|v(t)\|^p_{K(t)} dt \right)^{1/p} .$$

Notice that if v belongs to the space $\{u \in \mathcal{V} \big| u' \in \mathcal{K}\}$, which is contained in $v \in \{u \in \mathcal{V} \big| u' \in \mathcal{V}'\}$, then

$$t \mapsto \|v(t)\|_{H(t)} \qquad \text{is continuous.}$$

To see that it is sufficient to adapt Proposition 3.4 in [7]. Then we consider the space (the orthogonal projection operators P_+, P_0, P_- are defined in Section 2)

$$\mathcal{V}_* := \Big\{ u \in \mathcal{V} \Big| u' \in \mathcal{K},\, P_+(0)u(0) + \big(P_0(0) + P_-(0)\big)u'(0) = 0 \text{ in } H(0), \\ \big(P_0(T) + P_+(T)\big)u'(T) + P_-(T)u(T) = 0 \text{ in } H(T) \Big\}$$

endowed with the norm

$$\|u\|_{\mathcal{V}_*} := \|u\|_{\mathcal{V}} + \|u'\|_{\mathcal{K}}.$$

We will suppose that

$$\begin{aligned} &\text{if } p = 2 \quad \text{then} \quad K(t) = H(t) \ \text{ and } \ \mathcal{K} = \mathcal{K}' = \mathcal{H}, \\ &\text{if } p > 2 \quad \text{then} \quad K(t) \subsetneq H(t) \ \text{ and } \ \mathcal{K} \subsetneq \mathcal{H}. \end{aligned}$$

We now consider, besides the operator $\mathcal{R}$, two operators $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}', \qquad \mathcal{B} : \mathcal{V}_* \to \mathcal{V}_*' \tag{16}$$

the two following family of problems ($\varepsilon > 0$ is a parameter which, in the following, we will let go to zero)

$$\text{(I)} \quad \varepsilon\mathcal{B}u + \mathcal{R}u' + \mathcal{A}u = f, \qquad \text{(II)} \quad \varepsilon\mathcal{B}u + (\mathcal{R}u)' + \mathcal{A}u = f. \tag{17}$$

(with suitable boundary conditions we will specify below) where $f \in \mathcal{V}'$. These equalities are to be intended in $\mathcal{V}'_*$ as follows:

$$\varepsilon\big\langle \mathcal{B}u, v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} + \big\langle \mathcal{R}u', v\big\rangle_{\mathcal{V}'\times\mathcal{V}} + \big\langle \mathcal{A}u, v\big\rangle_{\mathcal{V}'\times\mathcal{V}} = \big\langle f, v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*},$$
$$\varepsilon\big\langle \mathcal{B}u, v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} + \big\langle (\mathcal{R}u)', v\big\rangle_{\mathcal{V}'\times\mathcal{V}} + \big\langle \mathcal{A}u, v\big\rangle_{\mathcal{V}'\times\mathcal{V}} = \big\langle f, v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*}$$

for every $v \in \mathcal{V}_*$. Notice that, since $f \in \mathcal{V}'$, one has that $\big\langle f, v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*}$ is in fact $\big\langle f, v\big\rangle_{\mathcal{V}'\times\mathcal{V}}$ (since $v \in \mathcal{V}_* \subset \mathcal{V}$).

We suppose there are four positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$ and a function b such that:

$$\begin{aligned}
&\mathsf{b} : [0,T]\times\mathbf{R} \to \mathbf{R} \qquad\qquad \text{satisfying} \\
&\big(\mathsf{b}(t,\xi) - \mathsf{b}(t,\eta)\big)(\xi-\eta) \geqslant \beta_1|\xi-\eta|^p \quad \text{for every } \xi,\eta\in\mathbf{R}, \\
&\big|\mathsf{b}(t,\xi)\big| \leqslant \beta_2|\xi|^{p-1} \quad \text{for every } \xi\in\mathbf{R},
\end{aligned}$$

and suppose that the operator $\mathcal{B}$ is defined as

$$\big\langle \mathcal{B}u, v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} = \int_0^T \big\langle \mathsf{b}(t, u'(t)), v'(t)\big\rangle_{K'(t)\times K(t)} dt$$

in such a way that

$$\big\langle \mathcal{B}u - \mathcal{B}v, u-v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} \geq \beta_1\|u'-v'\|^p_{\mathcal{K}}, \qquad \|\mathcal{B}u\|_{\mathcal{V}'_*} \leq \beta_2\|u'\|^{p-1}_{\mathcal{K}'} \tag{18}$$

and, if we consider problems (17)-(I), we require that

$$\text{for } p=2 \qquad \begin{aligned}&\big\langle \mathcal{A}u - \mathcal{A}v - \frac{1}{2}(\mathcal{R}'u - \mathcal{R}'v), u-v\big\rangle_{\mathcal{V}'\times\mathcal{V}} \geq \alpha_1\|u-v\|^2_{\mathcal{V}}, \\ &\|\mathcal{A}u - \tfrac{1}{2}\mathcal{R}'u\|_{\mathcal{V}'} \leq \alpha_2\|u\|_{\mathcal{V}}\end{aligned} \tag{19}$$

$$\text{for } p>2 \qquad \begin{aligned}&\langle \mathcal{A}u - \mathcal{A}v, u-v\rangle_{\mathcal{V}'\times\mathcal{V}} \geq \alpha_1\|u-v\|^p_{\mathcal{V}}, \quad \|\mathcal{A}u\|_{\mathcal{V}'} \leq \alpha_2\|u\|^{p-1}_{\mathcal{V}} \\ &\langle \mathcal{R}'u, u\rangle_{\mathcal{V}'\times\mathcal{V}} \leq 0\end{aligned} \tag{20}$$

for every $u, v \in \mathcal{V}$; if we consider problems (17)-(II) we require

$$\text{for } p=2 \qquad \begin{aligned}&\big\langle \mathcal{A}u - \mathcal{A}v + \frac{1}{2}(\mathcal{R}'u - \mathcal{R}'v), u-v\big\rangle_{\mathcal{V}'\times\mathcal{V}} \geq \alpha_1\|u-v\|^2_{\mathcal{V}}, \\ &\|\mathcal{A}u + \frac{1}{2}\mathcal{R}'u\|_{\mathcal{V}'} \leq \alpha_2\|u\|_{\mathcal{V}}\end{aligned} \tag{21}$$

$$\text{for } p>2 \qquad \begin{aligned}&\langle \mathcal{A}u - \mathcal{A}v, u-v\rangle_{\mathcal{V}'\times\mathcal{V}} \geq \alpha_1\|u-v\|^p_{\mathcal{V}}, \quad \|\mathcal{A}u\|_{\mathcal{V}'} \leq \alpha_2\|u\|^{p-1}_{\mathcal{V}} \\ &\langle \mathcal{R}'u, u\rangle_{\mathcal{V}'\times\mathcal{V}} \geq 0\end{aligned} \tag{22}$$

for every $u, v \in \mathcal{V}$. If we denote by

$$\mathcal{A}_\varepsilon : \mathcal{V}_* \to \mathcal{V}'_*, \qquad \mathcal{A}_\varepsilon u := \varepsilon\mathcal{B}u + \mathcal{A}u$$

for $p=2$ we have that by (19) one derives

$$\begin{aligned}
&\big\langle \mathcal{A}_\varepsilon u - \mathcal{A}_\varepsilon v - \frac{1}{2}(\mathcal{R}'u - \mathcal{R}'v), u-v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} \\
=&\varepsilon\big\langle \mathcal{B}u - \mathcal{B}v, u-v\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} + \big\langle \mathcal{A}u - \mathcal{A}v - \frac{1}{2}(\mathcal{R}'u - \mathcal{R}'v), u-v\big\rangle_{\mathcal{V}'\times\mathcal{V}} \\
\geqslant&\varepsilon\beta_1\|(u-v)'\|^2_{\mathcal{H}} + \alpha_1\|u-v\|^2_{\mathcal{V}} \\
\geqslant&\frac{1}{2}\min\{\varepsilon\beta_1, \alpha_1\}\|u-v\|^2_{\mathcal{V}_*},
\end{aligned} \tag{23}$$

$$
\begin{aligned}
\|\mathcal{A}_\varepsilon u - \frac{1}{2}\mathcal{R}'u\|_{\mathcal{V}_*'} &\leqslant \varepsilon\|\mathcal{B}u\|_{\mathcal{V}_*'} + \|\mathcal{A}u - \frac{1}{2}\mathcal{R}'u\|_{\mathcal{V}'} \leqslant \varepsilon\beta_2\|u'\|_{\mathcal{H}} + \alpha_2\|u\|_{\mathcal{V}} \\
&\leqslant \max\{\varepsilon\beta_2, \alpha_2\}\|u\|_{\mathcal{V}_*}; \qquad (24)
\end{aligned}
$$

while, similarly, by (21) one gets

$$
\big\langle \mathcal{A}_\varepsilon u - \mathcal{A}_\varepsilon v + \frac{1}{2}(\mathcal{R}'u - \mathcal{R}'v), u - v\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} \geqslant \frac{1}{2}\min\{\varepsilon\beta_1, \alpha_1\}\|u - v\|^2_{\mathcal{V}_*}, \qquad (25)
$$

$$
\|\mathcal{A}_\varepsilon u + \frac{1}{2}\mathcal{R}'u\|_{\mathcal{V}_*'} \leqslant \max\{\varepsilon\beta_2, \alpha_2\}\|u\|_{\mathcal{V}_*}, \qquad (26)
$$

For $p > 2$ by (20) and (22) (c_p being a constant depending only on p) one gets

$$
\begin{aligned}
&\big\langle \mathcal{A}_\varepsilon u - \mathcal{A}_\varepsilon v, u - v\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} \geqslant \varepsilon\beta_1\|u' - v'\|^p_{\mathcal{K}} + \alpha_1\|u - v\|^p_{\mathcal{V}} \\
&\geqslant c_p\min\{\varepsilon\beta_1, \alpha_1\}\|u - v\|^p_{\mathcal{V}_*}, \qquad (27) \\
&\|\mathcal{A}_\varepsilon u\|_{\mathcal{V}_*'} \leqslant \varepsilon\beta_2\|u'\|^{p-1}_{\mathcal{K}} + \alpha_2\|u\|^{p-1}_{\mathcal{V}} \leqslant \max\{\varepsilon\beta_2, \alpha_2\}\|u\|^{p-1}_{\mathcal{V}_*}.
\end{aligned}
$$

Notice that the operators $\mathcal{P}_\varepsilon u := \mathcal{A}_\varepsilon u + \mathcal{R}u'$ and $\mathcal{Q}_\varepsilon u := \mathcal{A}_\varepsilon u + (\mathcal{R}u)'$ defined in $\mathcal{V}_*$ with above assumptions are strictly monotone in $\mathcal{V}_*$. Indeed if (19) in the case $p = 2$ or (20) in the case $p > 2$ holds then

$$
\big\langle \mathcal{A}_\varepsilon u + \mathcal{R}u' - \mathcal{A}_\varepsilon v - \mathcal{R}v', u - v\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} \geqslant \varepsilon\beta_1\|u' - v'\|^p_{\mathcal{K}} + \alpha_1\|u - v\|^p_{\mathcal{V}}.
$$

Similarly if (21) in the case $p = 2$ or (22) in the case $p > 2$ holds then for every $u, v \in \mathcal{V}_*$

$$
\big\langle \mathcal{A}_\varepsilon u + (\mathcal{R}u)' - \mathcal{A}_\varepsilon v - (\mathcal{R}v)', u - v\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} \geqslant \varepsilon\beta_1\|u' - v'\|^p_{\mathcal{K}} + \alpha_1\|u - v\|^p_{\mathcal{V}}.
$$

We now want to apply Theorem 2.3. First we state the following result. Consider the space

$$
\mathcal{V}^0_* := \big\{u \in \mathcal{V}\big| u' \in \mathcal{K}, P_+(0)u(0) = 0 \text{ in } H(0), P_-(T)u(T) = 0 \text{ in } H(T)\big\} \supset \mathcal{V}_*
$$

and the operators

$$
\mathcal{L}_1 u = \mathcal{R}u' + \frac{1}{2}\mathcal{R}'u, \quad \mathcal{L}_2 u = \mathcal{R}u', \quad \mathcal{L}_3 u = (\mathcal{R}u)', \quad D(\mathcal{L}_i) = \mathcal{V}^0_* \quad i = 1, 2, 3.
$$

Lemma 3.1.
i) The operator $\mathcal{L}_1 : \mathcal{V}^0_* \to (\mathcal{V}^0_*)'$ is maximal monotone;
ii) the operator $\mathcal{L}_2 : \mathcal{V}^0_* \to (\mathcal{V}^0_*)'$ is maximal monotone if $\big\langle \mathcal{R}'u, u\big\rangle_{\mathcal{V}'\times\mathcal{V}} \leqslant 0$ for every $u \in \mathcal{V}_*$; iii) the operator $\mathcal{L}_3 : \mathcal{V}^0_* \to (\mathcal{V}^0_*)'$ is maximal monotone if $\big\langle \mathcal{R}'u, u\big\rangle_{\mathcal{V}'\times\mathcal{V}} \geqslant 0$ for every $u \in \mathcal{V}_*$.

Remark 3.2. - Clearly the lemma is true even if the domain of $\mathcal{L}_j$ is $\mathcal{V}_*$.

Proof. We prove the lemma for $\mathcal{L}_1$, being the other proofs similar and, indeed, simpler.

From Proposition 2.2 we have that

$$
\big\langle \mathcal{L}_1 u, u\big\rangle_{\mathcal{V}'\times\mathcal{V}} = \frac{1}{2}\Big[(R_+(T)u(T), u(T))_{H(T)} + (R_-(0)u(0), u(0))_{H(0)}\Big] \geqslant 0
$$

for every $u \in \mathcal{V}^0_*$, and then $\mathcal{L}_1$ is monotone. To see that it is maximal monotone fix $w \in (\mathcal{V}^0_*)'$ and $v \in \mathcal{V}^0_*$ and suppose

$$
\big\langle w - \mathcal{L}_1 u, v - u\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} \geq 0
$$

for every $u \in \mathcal{V}^0_*$. We want to show that $v \in \mathcal{V}^0_*$ and $w = \mathcal{L}_1 v$. Choose $u = \varphi z$ with $\varphi \in C^1_0([0,T])$ and $z \in U$ and get

$$\langle w, v\rangle \geq \langle \mathcal{L}_1 u, v-u\rangle + \langle w, u\rangle$$

that is, since $\langle \mathcal{L}_1 u, u\rangle = 0$, $\mathcal{R}$ and $\mathcal{R}'$ are linear and self adjoint, the following equivalent inequalities:

$$\langle w, v\rangle \geq \langle \varphi'\mathcal{R}z + \varphi\frac{1}{2}\mathcal{R}'z, v\rangle + \langle w, \varphi z\rangle$$

$$\langle w, v\rangle \geq \langle \mathcal{R}z, \varphi' v\rangle + \frac{1}{2}\langle \mathcal{R}'z, \varphi v\rangle + \langle \varphi w, z\rangle$$

$$\langle w, v\rangle \geq \langle \mathcal{R}\varphi' v, z\rangle + \frac{1}{2}\langle \mathcal{R}'\varphi v, z\rangle + \langle \varphi w, z\rangle.$$

Since this holds for each $z \in U$ we can consider λz with $\lambda \in \mathbf{R}$ and get

$$\langle w, v\rangle \geq \lambda\Big[\langle \mathcal{R}\varphi' v, z\rangle + \frac{1}{2}\langle \mathcal{R}'\varphi v, z\rangle + \langle \varphi w, z\rangle\Big].$$

Since this holds both for $\lambda > 0$ and $\lambda < 0$ we derive that

$$\langle \mathcal{R}\varphi' v, z\rangle + \frac{1}{2}\langle \mathcal{R}'\varphi v, z\rangle + \langle \varphi w, z\rangle = 0$$

and since this holds for every $z \in U$ we get that

$$\langle \mathcal{R}\varphi' v, p\rangle + \frac{1}{2}\langle \mathcal{R}'\varphi v, p\rangle + \langle \varphi w, p\rangle = 0$$

where p is a polynomial with coefficients in U, i.e.

$$p(t) = \sum_{k=0}^{N} z_k t^k \qquad \text{for some } N \in \mathbf{N} \text{ and } z_k \in U.$$

Since the space of such polynomials is dense in $\mathcal{U}$ and then in $\mathcal{V}_*$ we finally get that

$$\varphi'\mathcal{R}v + \varphi\frac{1}{2}\mathcal{R}'v + \varphi w = 0 \qquad \text{in } \mathcal{V}'_*,$$

that is

$$(\mathcal{R}v)' = \frac{1}{2}\mathcal{R}'v + w \qquad \Longleftrightarrow \qquad w = \mathcal{R}v' + \frac{1}{2}\mathcal{R}'v = \mathcal{L}_1 v.$$

□

REMARK 3.3. - Notice that in Theorem 3.6 we consider $f \in \mathcal{V}'$, even if, a priori, in (28) one could consider a datum $F \in \mathcal{V}'_*$. If one consider $F \in \mathcal{V}'_*$ there are $f \in \mathcal{V}'$ and $g \in \mathcal{K}'$ such that

$$\langle F, v\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} = \langle f, v\rangle_{\mathcal{V}'\times\mathcal{V}} + \langle g, v'\rangle_{\mathcal{K}'\times\mathcal{K}}.$$

If one confines to consider $F \in \mathcal{V}'_*$ such that $g = 0$ (or, more generally, $g' \in \mathcal{V}'$) F does not act directly on v'. In the following theorem we will confine to consider $F = f \in \mathcal{V}'$ so that

$$\langle F, v\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} = \langle f, v\rangle_{\mathcal{V}'\times\mathcal{V}}.$$

This is needed to have the estimates in Theorem 3.6 with a constant c independent of ε.

We now consider the two following problems: find $u \in \mathcal{V}_*^N$ such that

$$\text{(I)} \begin{cases} \varepsilon\mathcal{B}u + \mathcal{R}u' + \mathcal{A}u = f & \text{in } \mathcal{V}_*' \\ P_+(0)u(0) = \varphi \\ P_-(T)u(T) = \psi, \end{cases} \qquad \text{(II)} \begin{cases} \varepsilon\mathcal{B}u + (\mathcal{R}u)' + \mathcal{A}u = f & \text{in } \mathcal{V}_*' \\ P_+(0)u(0) = \varphi \\ P_-(T)u(T) = \psi, \end{cases} \tag{28}$$

where $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_+(T)$ and

$$\begin{aligned} \mathcal{V}_*^N := \Big\{ u \in \mathcal{V} \Big| u' \in \mathcal{K},\ \big(P_0(0) + P_-(0)\big)\mathsf{b}(0, u'(0)) = 0 \text{ in } H(0), \\ \big(P_0(T) + P_+(T)\big)\mathsf{b}(T, u'(T)) = 0 \text{ in } H(T) \Big\} = \\ \mathcal{V}_*^N := \Big\{ u \in \mathcal{V} \Big| u' \in \mathcal{K},\ \big(P_0(0) + P_-(0)\big)u'(0) = 0 \text{ in } H(0), \\ \big(P_0(T) + P_+(T)\big)u'(T) = 0 \text{ in } H(T) \Big\}. \end{aligned}$$

Before stating the result we will suppose an additional assumption. Consider the following spaces:

$$U_+(0) = \big\{ w \in U \ \big|\ P_+(0)w \in U \big\} = U \cap (\tilde{H}_+(0) \oplus \tilde{H}_0(0)),$$
$$U_-(T) = \big\{ w \in U \ \big|\ P_-(T)w \in U \big\} = U \cap (\tilde{H}_-(T) \oplus \tilde{H}_0(T)).$$

(see (8) for the definition of $\tilde{H}_-, \tilde{H}_0, \tilde{H}_+$). The first assumption is to suppose that

$$U_+(0) \text{ dense in } \tilde{H}_+(0), \qquad U_-(T) \text{ dense in } \tilde{H}_-(T). \tag{29}$$

Remark 3.4. - Assumption (29) is in fact an assumption about $R(0)$ and $R(T)$, which in fact results in an assumption about the sets $\Omega_+(0)$ and $\Omega_-(T)$. Indeed, for example, in the simple situation where

$$R(t) \equiv R \quad \text{for every } t \qquad \text{and} \qquad Ru := r(x)u$$

with

$$r \equiv 1 \text{ in } \Omega_+, \quad r \equiv 0 \text{ in } \Omega_0, \quad r \equiv -1 \text{ in } \Omega_-,$$

$\tilde{H}(t) = H(t) = L^2(\Omega)$ for every t, $U = V(t) = H_0^1(\Omega)$ for every t, then requiring that

$$U_+(0) = \big\{ u \in H_0^1(\Omega) \big| u|_{\Omega_+} \in H_0^1(\Omega_+) \big\}$$

is dense in $L^2(\Omega_+)$ means requiring some regularity on the set Ω_+. For example, this were surely true if Ω_+ is an open subset of Ω with Lipschitz boundary (the analogous clearly holds for Ω_-). For other details we refer to [8] and [7].

Remark 3.5. - Assumption (29) is in fact an assumption about $R(0)$ and $R(T)$, which results in an assumption about the sets $\Omega_+(0)$ and $\Omega_-(T)$. Indeed, for example, in the simple situation where

$$R(t) \equiv R \quad \text{for every } t \qquad \text{and} \qquad Ru := r(x)u$$

with

$$r \equiv 1 \text{ in } \Omega_+, \quad r \equiv 0 \text{ in } \Omega_0, \quad r \equiv -1 \text{ in } \Omega_-,$$

$\tilde{H}(t) = H(t) = L^2(\Omega)$ for every t, $U = V(t) = H_0^1(\Omega)$ for every t and Ω_+ is an open subset of Ω with Lipschitz boundary then

$$U_+(0) = \big\{ u \in H_0^1(\Omega) \big| u|_{\Omega_+} \in H_0^1(\Omega_+) \big\}$$

is dense in $L^2(\Omega_+)$. For other details we refer to [8] and [7].

Theorem 3.6. *Consider $R \in \mathcal{E}(C_1, C_2)$ and the operator $\mathcal{R}$ defined via R as in (9) and $\mathcal{B}$ satisfying (18). Consider $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$ and suppose that (29) holds.*

I) Consider $\mathcal{A}$ and suppose $\mathcal{A}$ and $\mathcal{R}$ satisfy (19) for $p = 2$ and (20) for $p > 2$. Moreover suppose that $\mathcal{A}$ and $\mathcal{B}$ are hemicontinuous. Then there exists a unique $u \in \mathcal{V}_*^N$ satisfying (28)-(I) and there is $c > 0$, depending only on $\alpha_1, \beta_1, \alpha_2, \beta_2, p$, such that (for $\varepsilon \in (0,1]$)

$$\varepsilon\|u'\|^p_{\mathcal{K}} + \|u\|^p_{\mathcal{V}} + \|\mathcal{R}u'\|_{\mathcal{V}'} \leqslant c\,\Bigg[\|f\|_{\mathcal{V}'} + \|f\|_{\mathcal{V}'}^{\frac{p}{p-1}} + \\ + \|R_-^{1/2}(T)\psi\|^{\frac{2}{p}}_{H_-(T)} + \|R_+^{1/2}(0)\varphi\|^{\frac{2}{p}}_{H_+(0)} + \|R_-^{1/2}(T)\psi\|^{2\frac{p-1}{p}}_{H_-(T)} + \|R_+^{1/2}(0)\varphi\|^{2\frac{p-1}{p}}_{H_+(0)}\Bigg].$$

II) Consider $\mathcal{A}$ and suppose $\mathcal{A}$ and $\mathcal{R}$ satisfy (21) for $p = 2$ and (22) for $p > 2$. Moreover suppose that $\mathcal{A}$ and $\mathcal{B}$ are hemicontinuous. Then for every $f \in \mathcal{V}'$ and $\phi \in \{u \in \mathcal{V} \big| u' \in \mathcal{K}\}$ there exists a unique $u \in \mathcal{V}_*^N$ satisfying (28)-(II) and there is $c > 0$, depending only on $\alpha_1, \beta_1, \alpha_2, \beta_2, p$, such that (for $\varepsilon \in (0,1]$)

$$\varepsilon\|u'\|^p_{\mathcal{K}} + \|u\|^p_{\mathcal{V}} + \|(\mathcal{R}u)'\|_{\mathcal{V}'} \leqslant c\,\Bigg[\|f\|_{\mathcal{V}'} + \|f\|_{\mathcal{V}'}^{\frac{p}{p-1}} + \\ + \|R_-^{1/2}(T)\psi\|^{\frac{2}{p}}_{H_-(T)} + \|R_+^{1/2}(0)\varphi\|^{\frac{2}{p}}_{H_+(0)} + \|R_-^{1/2}(T)\psi\|^{2\frac{p-1}{p}}_{H_-(T)} + \|R_+^{1/2}(0)\varphi\|^{2\frac{p-1}{p}}_{H_+(0)}\Bigg].$$

Proof. Estimates - Consider point I) in the case $p > 2$, being the proof in the other cases very similar. Then, as observe in Remark 3.3, we stress that the estimate we are going to show would not be true uniformly in ε for a general $f \in \mathcal{V}_*'$. Precisely, consider $u \in \mathcal{V}_*^N$ and suppose that

$$\mathcal{P}_\varepsilon u := \varepsilon\mathcal{B}u + \mathcal{R}u' + \mathcal{A}u \in \mathcal{V}'. \tag{30}$$

By Proposition 2.2 and since $\mathcal{R}'$ satisfies (20) we have that

$$\begin{aligned} 2\big\langle\mathcal{R}u', u\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} &= 2\big\langle\mathcal{R}u', u\big\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &= -\big\langle\mathcal{R}'u, u\big\rangle_{\mathcal{V}'\times\mathcal{V}} + \big(R(T)u(T), u(T)\big)_{H(T)} - \big(R(0)u(0), u(0)\big)_{H(0)} \\ &\geqslant -\big(R_-(T)u(T), u(T)\big)_{H(T)} - \big(R_+(0)u(0), u(0)\big)_{H(0)}. \end{aligned} \tag{31}$$

Then by (27), (30) and (31) we get that

$$\begin{aligned} \varepsilon\|u'\|^p_{\mathcal{K}} + \|u\|^p_{\mathcal{V}} &\leqslant c_1\big\langle\mathcal{A}_\varepsilon u, u\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} = c_1\Big[\big\langle\mathcal{P}_\varepsilon u, u\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*} - \big\langle\mathcal{R}u', u\big\rangle_{\mathcal{V}_*'\times\mathcal{V}_*}\Big] \\ &\leqslant c_1\Big[\langle\mathcal{P}_\varepsilon u, u\rangle_{\mathcal{V}'\times\mathcal{V}} + \frac{1}{2}\big(R_-(T)u(T), u(T)\big)_{H(T)} + \frac{1}{2}\big(R_+(0)u(0), u(0)\big)_{H(0)}\Big] \\ &\leqslant c_1\Bigg[\frac{1}{q}\left(\frac{1}{p\delta}\right)^{q/p}\|\mathcal{P}_\varepsilon u\|^q_{\mathcal{V}'} + \delta\|u\|^p_{\mathcal{V}} + \big(R_-(T)u(T), u(T)\big)_{H(T)} + \\ &\qquad\qquad + \big(R_+(0)u(0), u(0)\big)_{H(0)}\Bigg] \end{aligned}$$

with $c_1 = c_1(\alpha_1, \beta_1)$ and $q = p/(p-1)$. By that, choosing δ in such a way that $c_1\delta = 1/2$, we get

$$\varepsilon\|u'\|^p_{\mathcal{K}} + \|u\|^p_{\mathcal{V}} \leqslant c_2\Big[\|\mathcal{P}_\varepsilon u\|^{\frac{p}{p-1}}_{\mathcal{V}'} + \big(R_-(T)u(T), u(T)\big)_{H(T)} + \big(R_+(0)u(0), u(0)\big)_{H(0)}\Big] \tag{32}$$

with $c_2 = c_2(\alpha_1, \beta_1, p)$. Now, since $\mathcal{R}u' = \mathcal{P}_\varepsilon u - \mathcal{A}_\varepsilon u$ and $\mathcal{B}u \in \mathcal{V}'$, $\|\mathcal{R}u'\|_{\mathcal{V}}$ can be estimated (see (27)) as follows:

$$\begin{aligned}\|\mathcal{R}u'\|_{\mathcal{V}'} &\leqslant \|\mathcal{P}_\varepsilon u\|_{\mathcal{V}'} + \|\mathcal{A}_\varepsilon u\|_{\mathcal{V}'} \leqslant \|\mathcal{P}_\varepsilon u\|_{\mathcal{V}'} + \varepsilon\beta_2\|u'\|_{\mathcal{K}}^{p-1} + \alpha_2\|u\|_{\mathcal{V}}^{p-1} \\ &\leqslant \|\mathcal{P}_\varepsilon u\|_{\mathcal{V}'} + \varepsilon^{\frac{1}{p}}\beta_2\big(\varepsilon\|u'\|_{\mathcal{K}}^p\big)^{\frac{p-1}{p}} + \alpha_2\big(\|u\|_{\mathcal{V}}^p\big)^{\frac{p-1}{p}} \\ &\leqslant \|\mathcal{P}_\varepsilon u\|_{\mathcal{V}'} + \varepsilon^{\frac{1}{p}}\beta_2\big(c_2\|\mathcal{P}_\varepsilon u\|_{\mathcal{V}'}^{\frac{p}{p-1}}\big)^{\frac{p-1}{p}} + \alpha_2\big(c_2\|\mathcal{P}_\varepsilon u\|_{\mathcal{V}'}^{\frac{p}{p-1}}\big)^{\frac{p-1}{p}} \\ &\leqslant c_3\Big[\|\mathcal{P}_\varepsilon u\|_{\mathcal{V}'} + \big(R_-(T)u(T), u(T)\big)_{H(T)}^{\frac{p-1}{p}} + \big(R_+(0)u(0), u(0)\big)_{H(0)}^{\frac{p-1}{p}}\Big]\end{aligned}$$

where $c_3 = c_3(p, \alpha_1, \beta_1, \alpha_2, \beta_2, \varepsilon^{\frac{1}{p}})$ or simply $c_3 = c_3(p, \alpha_1, \beta_1, \alpha_2, \beta_2)$ if we confine to consider $\varepsilon \in (0, 1]$. Summing this last inequality to (32) we get the thesis.

Existence and uniqueness - Consider first $\varphi = 0$ and $\psi = 0$. By assumptions we have that, both in case I) and in case II), and for every $p \geqslant 2$, the operator $\mathcal{A}_\varepsilon$ is strictly monotone, coercive, bounded and hemicontinuous. By Lemma 3.1 the operator $u \mapsto \mathcal{R}u'$ in case (I) and the operator $u \mapsto (\mathcal{R}u)'$ in case (II) are maximal monotone in $\mathcal{V}_*^0$, and then in $\{v \in \mathcal{V}_*^N \,|\, P_+(0)v(0) = 0, P_-(T)v(T) = 0\}$.

$$\mathcal{V}_*^{0,N} := \Big\{u \in \mathcal{V}_*^N \Big| P_+(0)u(0) = 0 \text{ in } H(0), P_-(T)u(T) = 0 \text{ in } H(T)\Big\} = \mathcal{V}_*^N \cap \mathcal{V}_*^0.$$

Applying Theorem 2.3 we conclude. Now consider $\varphi, \psi \in U$, any $\delta \in (0, T/2)$ and ϕ defined as

$$\phi := \begin{cases} \varphi & t \in [0, \delta] \\ \dfrac{(T - 2\delta) - (t - \delta)}{T - 2\delta}\varphi + \dfrac{t - \delta}{T - 2\delta}\psi & t \in [\delta, T - \delta] \\ \psi & t \in [T - \delta, T]. \end{cases} \tag{33}$$

In this way $\phi \in \mathcal{V}_*^N$. Then a function u satisfies (28)-(I) if and only if the function $v = u - \phi$ satisfies

$$\begin{cases} \varepsilon\mathcal{B}(v + \phi) + \mathcal{R}v' + \mathcal{A}(v + \phi) = f - \mathcal{R}\phi' \\ v \in \mathcal{V}_*^{0,N}. \end{cases}$$

If we define

$$\tilde{\mathcal{B}}v := \mathcal{B}(v + \phi) \qquad \text{and} \qquad \tilde{\mathcal{A}}v := \mathcal{A}(v + \phi) \tag{34}$$

we have the following problem

$$\begin{cases} \varepsilon\tilde{\mathcal{B}}v + \mathcal{R}v' + \tilde{\mathcal{A}}v = f - \mathcal{R}\phi' \\ v \in \mathcal{V}_*^{0,N}. \end{cases} \tag{35}$$

It is not difficult to verify that $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{A}}$ are bounded, coercive, strongly monotone and hemicontinuous, so arguing as before we get a unique solution $v \in \mathcal{V}_*^{0,N}$ satisfying (35), and then a unique $u \in \mathcal{V}_*^N$ satisfying (28)-(I).

Now we use the a priori estimates previously obtained to get the thesis for every admissible datum. Consider now $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$ and two sequences $(\varphi_n)_n, (\psi_n)_n \subset U$ such that (this is possible thanks to assumption (29))

$$\varphi_n \to \varphi \quad \text{in } \tilde{H}_+(0), \qquad \psi_n \to \psi \quad \text{in } \tilde{H}_-(T).$$

In this way the function ϕ_n defined in a way analogous to (33) belong to $\mathcal{V}_*^N$. Similarly as done above to get the a priori estimate one gets (for instance, in case

I))

$$\begin{aligned}\varepsilon\|u_n - u'_m\|^p_{\mathcal{K}} + \|u_n - u_m\|^p_{\mathcal{V}} + \|\mathcal{R}(u'_n - u'_m)\|_{\mathcal{V}'} \\ \leqslant c \Big[\|R_-^{1/2}(T)(\psi_n - \psi_m)\|^{\frac{2}{p}}_{H_-(T)} + \|R_+^{1/2}(0)(\varphi_n - \varphi_m)\|^{\frac{2}{p}}_{H_+(0)} + \\ + \|R_-^{1/2}(T)(\psi_n - \psi_m)\|^{2\frac{p-1}{p}}_{H_-(T)} + \|R_+^{1/2}(0)(\varphi_n - \varphi_m)\|^{2\frac{p-1}{p}}_{H_+(0)}\Big]\end{aligned}$$

for every $n, m \in \mathbf{N}$, and then there is a function $u \in \mathcal{V}^N_*$ such that

$$\begin{aligned} u_n \to u \quad & \text{in } \mathcal{V}, \\ u'_n \to u' \quad & \text{in } \mathcal{K}, \\ \mathcal{R}u'_n \to \mathcal{R}u' \quad & \text{in } \mathcal{V}'. \end{aligned}$$

We also get that

$$\|\mathcal{B}u_n\|_{\mathcal{V}'_*} \leqslant c, \qquad \|\mathcal{A}u_n\|_{\mathcal{V}'} \leqslant c$$

for some positive constant c. Up to select a subsequence we get that $\mathcal{A}u_n$ weakly converge to some $b \in \mathcal{V}'$ and then $\langle \mathcal{A}u_n, u_n\rangle_{\mathcal{V}'\times\mathcal{V}} \to \langle b, u\rangle_{\mathcal{V}'\times\mathcal{V}}$. Since $\mathcal{A}$ is type M we conclude that $b = \mathcal{A}u$. In the same way one has that $\mathcal{B}u_n \to \mathcal{B}u$. Since for every subsequence $(u_{n_j})_{j\in\mathbf{N}}$ we can extract a further subsequence $(u_{n_{j_k}})_{k\in\mathbf{N}}$ such that $\mathcal{A}u_{n_{j_k}} \to \mathcal{A}u$ and $\mathcal{B}u_{n_{j_k}} \to \mathcal{B}u$ we conclude that all the sequence satisfies $\mathcal{A}u_n \to \mathcal{A}u$ and $\mathcal{B}u_n \to \mathcal{B}u$ and u is the solution looked for. □

4. **Taking the limit for $\varepsilon \to 0$.** In this section we want to prove the result which is the goal of the paper: to show that the solutions of problems (28)-(I) (respectively of problems (28)-(II)) converge, in a suitable way, to the solution of (36)-(I) (respectively of (36)-(II)). We recall that the existence of a solution of the following problems has already been proved in [8] and [7]:

$$\text{(I)}\begin{cases} \mathcal{R}u' + \mathcal{A}u = f & \text{in } \mathcal{V}' \\ P_+(0)u(0) = \varphi & \text{in } \tilde{H}_+(0) \\ P_-(T)u(T) = \psi & \text{in } \tilde{H}_-(T), \end{cases} \qquad \text{(II)}\begin{cases} (\mathcal{R}v)' + \mathcal{A}v = f & \text{in } \mathcal{V}' \\ P_+(0)v(0) = \varphi & \text{in } \tilde{H}_+(0) \\ P_-(T)v(T) = \psi & \text{in } \tilde{H}_-(T), \end{cases} \tag{36}$$

In the following three steps we will consider the problem (28)-(I) for $p > 2$. The proofs in other cases, problem (28)-(I) for $p = 2$ and problem (28)-(II) both for $p > 2$ and $p = 2$, are very similar.

Limit in the equation - Consider some $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$ and $\psi \in \tilde{H}_-(T)$ and denote by $u_\varepsilon \in \mathcal{V}_*$ the solution of (28)-(I), $p > 2$. By Theorem 3.6 and boundedness of $\mathcal{A}$ we get that (up to select a sequence $\varepsilon_j \to 0$ which we will still denote by ε for sake of simplicity) letting ε go to 0

$$\begin{aligned} u_\varepsilon \to u \qquad & \text{in } \mathcal{V}\text{-weak}, \\ \varepsilon^{1/p}u'_\varepsilon \to w \qquad & \text{in } \mathcal{K}\text{-weak}, \\ \mathcal{A}u_\varepsilon \to g \qquad & \text{in } \mathcal{V}'\text{-weak}, \\ \mathcal{R}u'_\varepsilon \to z \qquad & \text{in } \mathcal{V}'\text{-weak}. \end{aligned} \tag{37}$$

Notice that, by (18) and since $\varepsilon^{1/p}u'_\varepsilon$ is bounded in $\mathcal{K}$, we also get that

$$\|\varepsilon\mathcal{B}u_\varepsilon\|_{\mathcal{V}'_*} \leqslant \varepsilon\beta_2\|u'_\varepsilon\|^{p-1}_{\mathcal{K}'} = \varepsilon^{1/p}\beta_2\|\varepsilon^{1/p}u'_\varepsilon\|^{p-1}_{\mathcal{K}'} \to_\varepsilon 0. \tag{38}$$

Moreover for every $\eta \in C^1_c([0,T]; U)$

$$\big\langle \mathcal{R}u'_\varepsilon, \eta\big\rangle_{\mathcal{V}'\times\mathcal{V}} = -\big(\mathcal{R}u_\varepsilon, \eta'\big)_{\mathcal{H}} - \big\langle \mathcal{R}'u_\varepsilon, \eta\big\rangle_{\mathcal{V}'\times\mathcal{V}}$$

and taking the limit for $\varepsilon \to 0$ one gets

$$\big\langle z, \eta \big\rangle_{\mathcal{V}' \times \mathcal{V}} = -\big(\mathcal{R}u, \eta'\big)_{\mathcal{H}} - \big\langle \mathcal{R}'\eta, u \big\rangle_{\mathcal{V}' \times \mathcal{V}}$$

by which we derive

$$z = \mathcal{R}u'.$$

With these informations we consider the limit in the equation of problem (28)-(I) and get

$$\mathcal{R}u' + g = f.$$

The goal now is to show that

$$g = \mathcal{A}u. \tag{39}$$

Now we consider problems (28) and multiply by u_ε the equations of problem (28). First observe that $\mathcal{A}u_\varepsilon \to g$ and then

$$\mathcal{A}u_\varepsilon \to_{\varepsilon \to 0} g = f - \mathcal{R}u' \qquad \text{in } \mathcal{V}'\text{-weak.}$$

We get

$$\begin{aligned} &\big\langle \mathcal{R}u'_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{A}u_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &\qquad \leqslant \varepsilon \big\langle \mathcal{B}u_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}'_* \times \mathcal{V}_*} + \big\langle \mathcal{R}u'_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{A}u_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} = \big\langle f, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} \end{aligned}$$

by which

$$\limsup_{\varepsilon \to 0} \Big[\big\langle \mathcal{R}u'_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{A}u_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} \Big] \leqslant \big\langle f, u \big\rangle_{\mathcal{V}' \times \mathcal{V}}. \tag{40}$$

Observe that, since $u \mapsto \mathcal{R}u'$ is monotone in $\mathcal{V}^0_*$ and $u_\varepsilon - u \in \mathcal{V}^0_*$,

$$\begin{aligned} &\big\langle \mathcal{A}u_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} = \big\langle \mathcal{A}u_\varepsilon, u \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{A}u_\varepsilon, u_\varepsilon - u \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &\leqslant \big\langle \mathcal{A}u_\varepsilon, u \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{A}u_\varepsilon, u_\varepsilon - u \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{R}u'_\varepsilon - \mathcal{R}u', u_\varepsilon - u \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &= \big\langle \mathcal{A}u_\varepsilon, u \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{A}u_\varepsilon + \mathcal{R}u'_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} - \big\langle \mathcal{R}u', u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &\quad - \big\langle \mathcal{A}u_\varepsilon, u \big\rangle_{\mathcal{V}' \times \mathcal{V}} - \big\langle \mathcal{R}u'_\varepsilon, u \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{R}u', u \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &= \big\langle \mathcal{A}u_\varepsilon + \mathcal{R}u'_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} - \big\langle \mathcal{R}u', u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} - \big\langle \mathcal{R}u'_\varepsilon, u \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{R}u', u \big\rangle_{\mathcal{V}' \times \mathcal{V}} \end{aligned}$$

and taking the limit and using (40) and since $g = f - \mathcal{R}u'$ we get

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \big\langle \mathcal{A}u_\varepsilon, u_\varepsilon \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &\qquad \leqslant \big\langle f, u \big\rangle_{\mathcal{V}' \times \mathcal{V}} - \big\langle \mathcal{R}u', u \big\rangle_{\mathcal{V}' \times \mathcal{V}} - \big\langle \mathcal{R}u', u \big\rangle_{\mathcal{V}' \times \mathcal{V}} + \big\langle \mathcal{R}u', u \big\rangle_{\mathcal{V}' \times \mathcal{V}} \\ &\qquad = \big\langle f - \mathcal{R}u', u \big\rangle_{\mathcal{V}' \times \mathcal{V}} = \big\langle g, u \big\rangle_{\mathcal{V}' \times \mathcal{V}}. \end{aligned}$$

Since we suppose $\mathcal{A}$ to be hemicontinuous and, as already observed, $\mathcal{A}$ is of type M we get that

$$\mathcal{A}u = f - \mathcal{R}u'$$

that is

$$\mathcal{R}u' + \mathcal{A}u = f.$$

Limit in the boundary conditions - By Lemma 3.19 in [7] given $u \in \mathcal{W}_{\mathcal{R}}$ we have that

$$R(\sigma)u(\sigma) \in \bigcap_{t\in[0,T]} H(t) \text{ for every } \sigma \in [0,T] \quad \text{and}$$

$$\left\| \int_s^t (\mathcal{R}u)'(\tau)d\tau \right\|_{H(\sigma)} \leqslant \int_s^t \|(\mathcal{R}u)'(\tau)\|_{V'(\tau)} d\tau \text{ for every } \sigma \in [0,T] \text{ and } [s,t] \subset [0,T].$$

In particular for the family of solutions u_ε of problems (28)-(I) since $(u_\varepsilon)_\varepsilon$ are bounded in $\mathcal{V}$ and $\mathcal{R}u'_\varepsilon$ are bounded in $\mathcal{V}'$ we have (C_0 is defined in (3))

$$\begin{aligned}
\big\|R(t_2)u_\varepsilon(t_2) - R(t_1)u_\varepsilon(t_1)\big\|_{H(0)} &= \left\| \int_{t_1}^{t_2} \mathcal{R}'u_\varepsilon(s)ds + \int_{t_1}^{t_2} \mathcal{R}u'_\varepsilon(s)ds \right\|_{H(0)} \\
\leqslant & C_0 \left[\int_{t_1}^{t_2} \big\|\mathcal{R}'u_\varepsilon(s)\big\|_{V'(s)} ds + \int_{t_1}^{t_2} \big\|\mathcal{R}u'_\varepsilon(s)\big\|_{V'(s)} ds \right] \\
\leqslant & C_0 |t_2 - t_1|^{1/2} \left[\left[\int_{t_1}^{t_2} \big\|\mathcal{R}'u_\varepsilon(s)\big\|^2_{V'(s)} ds \right]^{1/2} + \left[\int_{t_1}^{t_2} \big\|\mathcal{R}u'_\varepsilon(s)\big\|^2_{V'(s)} ds \right]^{1/2} \right] \\
\leqslant & C_0 |t_2 - t_1|^{1/2} \big(\|\mathcal{R}'u_\varepsilon\|_{\mathcal{V}'} + \|\mathcal{R}u'_\varepsilon\|_{\mathcal{V}'} \big).
\end{aligned}$$

Notice that, since $\mathcal{R}'$ is linear and continuous and $(u_\varepsilon)_\varepsilon$ converge to u in $\mathcal{V}$, we have that

$$\mathcal{R}'u_\varepsilon \to \mathcal{R}'u.$$

Since we also have that $\mathcal{R}u'_\varepsilon \to \mathcal{R}u'$ we derive that the quantity $\|\mathcal{R}'u_\varepsilon\|_{\mathcal{V}'} + \|\mathcal{R}u'_\varepsilon\|_{\mathcal{V}'}$ is bounded with respect to ε and then we got that the family

$$\big(R(t)u_\varepsilon(t)\big)_{\varepsilon>0} \qquad \begin{array}{l}\text{is equibounded and equicontinuous in } [0,T] \\ \text{with respect to the topology of } H(0)\end{array}$$

and then $\big(R(t)u_\varepsilon(t)\big)_{\epsilon>0}$ is weakly relatively compact in $H(0)$ uniformly in time. Precisely, since $\mathcal{R}u_\varepsilon \to \mathcal{R}u$ in $\mathcal{H}$ we get that for every $\eta \in H(0)$

$$\big(R(t)u_\varepsilon(t), \eta\big)_{H(0)} \to \big(R(t)u(t), \eta\big)_{H(0)} \qquad \text{uniformly in } [0,T].$$

The same argument can be used to get that for every $\eta \in H(T)$

$$\big(R(t)u_\varepsilon(t), \eta\big)_{H(T)} \to \big(R(t)u(t), \eta\big)_{H(T)} \qquad \text{uniformly in } [0,T].$$

In particular we get that

$$R_+(0)u(0) = R_+(0)\varphi \quad \text{in } H(0), \qquad R_-(T)u(T) = R_-(T)\psi \quad \text{in } H(T) \tag{41}$$

and also

$$R_-(0)u_\varepsilon(0) \to R_-(0)u(0) \quad \text{in } H(0), \qquad R_+(T)u_\varepsilon(T) \to R_+(T)u(T) \quad \text{in } H(T),$$

but for these we loose the propery to belong to $\mathcal{V}^N$.

Summing up, we have that there exists a sequence of the family of the solutions $(u_\varepsilon)_\varepsilon > 0$ of problems (28)-(I) with $p > 2$ which converge to a function u which satisfies (36)-(I).

The proofs of the other cases are completely similar.

Convergence of the whole family - Since for every subfamily of $(u_\varepsilon)_{\varepsilon>0}$ satisfying (28)-(I) one can repeat the same argument as above and get a limit function u satisfying (36)-(I) by the uniqueness of the solution just of (36)-(I) we get that from every subfamily one can select a sequence converging to the same u. Then we conclude that we do not need to select a sequence, but all the family of solutions is converging.

Summing up, we have proved the following result.

Theorem 4.1. *Consider an operator* $\mathcal{R} : \mathcal{W}_{\mathcal{R}} \to \mathcal{V}'$ *defined via* $R \in \mathcal{E}(C_1, C_2)$, $\mathcal{B} : \mathcal{V}_* \to \mathcal{V}'_*$ *operator satisfying* (18) *and hemicontinuous,* $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ *hemicontinuous. Suppose* (29) *holds and, given* $f \in \mathcal{V}'$, $\varphi \in \tilde{H}_+(0)$, $\psi \in \tilde{H}_-(T)$, *denote by* u_ε *the solution of problem* (28)-(I) *with* $\mathcal{A}$ *and* $\mathcal{R}'$ *satisfying* (19) *or* (20), *denote by* v_ε *the solution of problem* (28)–(II) *with* $\mathcal{A}$ *and* $\mathcal{R}'$ *satisfying* (21) *or* (22). *Then we have that*

$$\lim_{\varepsilon\to 0} \varepsilon\mathcal{B}u_\varepsilon = \lim_{\varepsilon\to 0} \varepsilon\mathcal{B}v_\varepsilon = 0 \qquad \textit{in } \mathcal{V}'_*\textit{-strong,}$$
$$\lim_{\varepsilon\to 0} u_\varepsilon = u, \quad \lim_{\varepsilon\to 0} v_\varepsilon = v \qquad \textit{in } \mathcal{V}\textit{-weak}$$
$$\lim_{\varepsilon\to 0} \mathcal{R}u'_\varepsilon = \mathcal{R}u', \quad \lim_{\varepsilon\to 0} (\mathcal{R}v_\varepsilon)' = (\mathcal{R}v)' \qquad \textit{in } \mathcal{V}'\textit{-weak,}$$

where u *and* v *are the unique solutions respectively of the problems* (36)-(I) *and* (36)-(II) *and satisfy*

$$\|u\|_{\mathcal{V}} + \|\mathcal{R}u'\|_{\mathcal{V}'} \leqslant C\Big[\|f\|_{\mathcal{V}'} + \|f\|_{\mathcal{V}'}^{\frac{1}{p-1}} + \|R_-^{1/2}(T)\psi(T)\|_{H(T)}^{\frac{2}{p}} + \|R_+^{1/2}(0)\varphi(0)\|_{H(0)}^{\frac{2}{p}} + \|R_-^{1/2}(T)\psi(T)\|_{H(T)}^{2\frac{p-1}{p}} + \|R_+^{1/2}(0)\varphi(0)\|_{H(0)}^{2\frac{p-1}{p}}\Big].$$

$$\|v\|_{\mathcal{V}} + \|(\mathcal{R}v)'\|_{\mathcal{V}'} \leqslant C\Big[\|f\|_{\mathcal{V}'} + \|f\|_{\mathcal{V}'}^{\frac{1}{p-1}} + \|R_-^{1/2}(T)\psi(T)\|_{H(T)}^{\frac{2}{p}} + \|R_+^{1/2}(0)\varphi(0)\|_{H(0)}^{\frac{2}{p}} + \|R_-^{1/2}(T)\psi(T)\|_{H(T)}^{2\frac{p-1}{p}} + \|R_+^{1/2}(0)\varphi(0)\|_{H(0)}^{2\frac{p-1}{p}}\Big].$$

with $C = C(p, \alpha_1, \alpha_2)$.

As an immediate consequence we have the following corollaries.

Corollary 4.2. *As a consequence of the previous result,* u_ε, u, v_ε, v *as above, we also get that*

$$\lim_{\varepsilon\to 0} \mathcal{R}u_\varepsilon = \mathcal{R}u, \qquad \lim_{\varepsilon\to 0} \mathcal{R}v_\varepsilon = \mathcal{R}v \qquad \textit{in } \mathcal{H}\textit{-strong.}$$

Proof. The proof follows immediately from Theorem 3.6 and Proposition 3.4 in [7] (see also Theorem 2.14 and Proposition 2.6 in [8]). □

Corollary 4.3. *As a consequence of Theorem* 4.1 *we also get that*

$$\lim_{\varepsilon\to 0} \langle \varepsilon\mathcal{B}u_\varepsilon, u_\varepsilon\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} = \lim_{\varepsilon\to 0} \langle \varepsilon\mathcal{B}v_\varepsilon, v_\varepsilon\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} = 0.$$

Proof. Consider u_ε, the solution of (28)-(I), and u its limit in $\mathcal{W}_{\mathcal{R}}$ satisfying (36)-(I). Since u_ε weakly converge in $\mathcal{V}$ and $\varepsilon\mathcal{B}u_\varepsilon$ strongly converge to zero (see (38)) we immediatly conclude. Similarly one proves the convergence for $\langle \varepsilon\mathcal{B}v_\varepsilon, v_\varepsilon\rangle_{\mathcal{V}'_*\times\mathcal{V}_*}$. □

Corollary 4.4. *As a consequence of the previous corollary,* u_ε *and* v_ε *as above, we also get that*

$$\lim_{\varepsilon\to 0} u_\varepsilon = u, \quad \lim_{\varepsilon\to 0} v_\varepsilon = v \qquad \textit{in } \mathcal{V}\textit{-strong}$$

and, if $\mathcal{A}$ is continuous,

$$\lim_{\varepsilon\to 0}\mathcal{R}u'_\varepsilon = \mathcal{R}u', \quad \lim_{\varepsilon\to 0}(\mathcal{R}v_\varepsilon)' = (\mathcal{R}v)' \qquad \text{in } \mathcal{V}'\text{-strong},$$
$$\lim_{\varepsilon\to 0}\mathcal{A}u_\varepsilon = \mathcal{A}u, \quad \lim_{\varepsilon\to 0}\mathcal{A}v_\varepsilon = \mathcal{A}v \qquad \text{in } \mathcal{V}'\text{-strong}.$$

Proof. As usual we prove the result for u_ε, being the proof for v_ε similar. Subtracting $\delta\mathcal{B}u_\delta + \mathcal{R}u'_\delta + \mathcal{A}u_\delta = f$ to $\varepsilon\mathcal{B}u_\varepsilon + \mathcal{R}u'_\varepsilon + \mathcal{A}u_\varepsilon = f$ we get

$$\varepsilon\mathcal{B}u_\varepsilon - \delta\mathcal{B}u_\delta + \mathcal{R}u'_\varepsilon - \mathcal{R}u'_\delta + \mathcal{A}u_\varepsilon - \mathcal{A}u_\delta = 0.$$

Multiplying by $u_\varepsilon - u_\delta$ we get

$$\begin{aligned} 0 \leqslant & \varepsilon\big\langle \mathcal{B}u_\varepsilon, u_\varepsilon\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} + \delta\big\langle \mathcal{B}u_\delta, u_\delta\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} \\ & + \big\langle \mathcal{R}u'_\varepsilon - \mathcal{R}u'_\delta, u_\varepsilon - u_\delta\big\rangle_{\mathcal{V}'\times\mathcal{V}} + \big\langle \mathcal{A}u_\varepsilon - \mathcal{A}u_\delta, u_\varepsilon - u_\delta\big\rangle_{\mathcal{V}'\times\mathcal{V}} \\ = & \, \delta\big\langle \mathcal{B}u_\delta, u_\varepsilon\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} + \varepsilon\big\langle \mathcal{B}u_\varepsilon, u_\delta\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} \end{aligned}$$

and since $u_\varepsilon - u_\delta \in \mathcal{V}^0$ one gets

$$\alpha_1\|u_\varepsilon - u_\delta\|^p_{\mathcal{V}} \leqslant \delta\big\langle \mathcal{B}u_\delta, u_\varepsilon\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} + \varepsilon\big\langle \mathcal{B}u_\varepsilon, u_\delta\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*}.$$

By Theorem 4.1 one derives that

$$\lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}} \varepsilon\big\langle \mathcal{B}u_\varepsilon, u_\delta\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} = \lim_{\substack{\varepsilon\to 0^+\\ \delta\to 0^+}} \varepsilon\big\langle \mathcal{B}u_\varepsilon, u_\delta\big\rangle_{\mathcal{V}'_*\times\mathcal{V}_*} = 0$$

and then $(u_\varepsilon)_{\varepsilon>0}$ is a Cauchy family in $\mathcal{V}$ and then

$$\lim_{\varepsilon\to 0^+} u_\varepsilon = u \qquad \text{strongly in } \mathcal{V}.$$

Since $\varepsilon\mathcal{B}u_\varepsilon + \mathcal{R}u'_\varepsilon + \mathcal{A}u_\varepsilon = f$ and $\varepsilon\mathcal{B}u_\varepsilon$ strongly converge to zero (see (38)) in $\mathcal{V}'_*$ (and in $\mathcal{V}'$) we also get that

$$\mathcal{R}u'_\varepsilon + \mathcal{A}u_\varepsilon \to f \qquad \text{strongly in } \mathcal{V}'.$$

By the continuity of $\mathcal{A}$ we get that $\mathcal{A}u_\varepsilon \to \mathcal{A}u$ and consequently

$$\lim_{\varepsilon\to 0^+}\mathcal{R}u'_\varepsilon = \mathcal{R}u' \qquad \text{strongly in } \mathcal{V}'.$$

□

5. **Examples.** In this section we present just two examples, since many examples of forward-backward parabolic equations are already given in the two papers [8] and [7].

Before exposing these examples we stress that, obviously, the two simple cases

$$\mathcal{R} \equiv 0 \qquad \text{and} \qquad \mathcal{R} = \mathrm{Id}$$

are admitted. In the first case we approximate an elliptic problem in dimension n with an analogous elliptic problem in dimension $n+1$, while in the second case the limit problem is a parabolic equation (completely forward).

In both the two following examples we consider

$$\mathcal{R} : \mathcal{H} \to \mathcal{H}, \qquad (\mathcal{R}u, v) = \int_0^T\int_\Omega u(x,t)v(x,t)r(x,t)dxdt,$$

but

in the first	r	is bounded,
in the second	r	may be unbounded.

First example: we consider the following situation: $T > 0$, $\Omega \subset \mathbf{R}^n$ an open set with Lipschitz boundary, λ_o, Λ_o positive constants and, for $p \leqslant 2$, we consider the spaces

$$U \equiv V(t) = W_0^{1,p}(\Omega), \quad K(t) = L^p(\Omega), \quad H(t) = L^2(\Omega) \qquad \text{for every } t \in [0,T],$$
$$\mathcal{H} = L^2(\Omega \times (0,T)), \qquad \mathcal{V} = L^2(0,T;W_0^{1,p}(\Omega)), \qquad \mathcal{V}_* = W_0^{1,p}(\Omega \times (0,T))$$

and the operators

$$A(t) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$$
$$\big(A(t)u\big)(x) := -\mathrm{div} a\big(x,t,u(x),Du(x)\big),$$
$$\text{with } a : \Omega \times (0,T) \times \mathbf{R}^n \to \mathbf{R}^n \qquad \text{verifying}$$
$$\lambda_o |\xi|^p \leqslant a(x,t,u,\xi) \cdot \xi \leqslant \Lambda_o |\xi|^p \qquad \text{for every } \xi \in \mathbf{R}^n,$$
$$\mathcal{A} : \mathcal{V} \to \mathcal{V}', \quad \langle \mathcal{A}u, v\rangle_{\mathcal{V}' \times \mathcal{V}} = \int_0^T \int_\Omega \big(a(x,t,u(x,t),Du(x,t)), Dv(x,t)\big) dxdt,$$
$$\mathcal{B} : \mathcal{V}_* \to \mathcal{V}_*', \quad \langle \mathcal{B}u, v\rangle_{\mathcal{V}_*' \times \mathcal{V}_*} = \int_0^T \int_\Omega |u_t|^{p-2}(x,t) u_t(x,t) v_t(x,t) dxdt,$$
$$\mathcal{R} : \mathcal{H} \to \mathcal{H}, \quad (\mathcal{R}u, v) = \int_0^T \int_\Omega u(x,t)v(x,t)r(x,t)dxdt$$

where

$$r : \Omega \times (0,T) \to \mathbf{R}, \qquad r \in L^\infty(\Omega \times (0,T))$$

is such that

$$t \mapsto \int_\Omega u(x)v(x)r(x,t)dx \qquad \text{is absolutely continuous and}$$
$$\left| \frac{d}{dt} \int_\Omega u(x)v(x)r(x,t)dx \right| \leqslant C_2 \left(\int_\Omega |Du|^p(x)dx \int_\Omega |Dv|^p(x)dx \right)^{1/p}.$$

The operator R' is defined as follows: $R'(t) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ and

$$\langle R'(t)u, v\rangle_{W^{-1,p'} \times W_0^{1,p}} := \frac{d}{dt} \int_\Omega u(x)v(x)r(x,t)dx \tag{42}$$

which has to satisfy assumptions (19)-(20). Notice that assumption (29) is in fact an assumption about R, in particular an assumption regarding the regularity of the two sets

$$\{x \in \Omega | r(x,0) \geqslant 0\} \qquad \text{and} \qquad \{x \in \Omega | r(x,T) \leqslant 0\}.$$

About that we refer to example (3) in the last section in [8]. About the sets

$$\Omega_+(0) := \{x \in \Omega | r(x,0) > 0\} \qquad \text{and} \qquad \Omega_-(T) := \{x \in \Omega | r(x,T) < 0\}$$

we supposte that they are measurable sets. Define the spaces

$$\tilde{H}_+(0) := \text{the completion of } C_c^1(\Omega_+(0)) \text{ w.r.t. the topology induced by } \left(\int_\Omega u^2(x) r_+(x,0)dx \right)^{1/2}$$

$$\tilde{H}_-(T) := \text{the completion of } C_c^1(\Omega_-(T)) \text{ w.r.t. the topology induced by } \left(\int_\Omega u^2(x) r_-(x,T)dx \right)^{1/2}.$$

Then, the solutions u_ε of

$$\begin{cases} -\varepsilon\dfrac{\partial}{\partial t}\left(\left|\dfrac{\partial u}{\partial t}\right|^{p-2}\dfrac{\partial u}{\partial t}\right)+r\dfrac{\partial u}{\partial t}-\mathrm{div}a\big(x,t,u(x),Du(x)\big)=f & \text{in } \Omega\times(0,T)\\ u=0 & \text{in } \partial\Omega\times(0,T)\\ u(x,0)=\varphi(x) & \text{in } \Omega_+(0)\times\{0\}\\ \dfrac{\partial u}{\partial t}(x,0)=0 & \text{in } \big(\Omega_0(0)\cup\Omega_-(0)\big)\times\{0\}\\ u(x,T)=\psi(x) & \text{in } \Omega_-(T)\times\{T\}\\ \dfrac{\partial u}{\partial t}(x,T)=0 & \text{in } \big(\Omega_0(T)\cup\Omega_+(T)\big)\times\{0\} \end{cases}$$

under the assumtpions (19)-(20) converge to the solution of

$$\begin{cases} r\dfrac{\partial u}{\partial t}-\mathrm{div}a\big(x,t,u(x),Du(x)\big)=f & \text{in } \Omega\times(0,T)\\ u=0 & \text{in } \partial\Omega\times(0,T)\\ u=\varphi & \text{in } \partial\Omega_+(0)\\ u=\psi & \text{in } \partial\Omega_-(T). \end{cases}$$

Similarly, mutatis mutandis, i.e. under the assumtpions (21)-(22), one has the same result substituting in the two previous problems $r\frac{\partial u}{\partial t}$ by $\frac{\partial}{\partial t}(ru)$.
Second example: to consider an example where the spaces are depending on t one can consider λ_o and Λ_o depending on time. In this way one must introduce some weighted spaces. First suppose that

$$\lambda_o=\lambda(x,t)\quad\text{and}\quad\Lambda_o=L\lambda(x,t)$$

for some $L\geqslant 1$ and for $\lambda>0$ almost everywhere (and possibly unbounded). We will denote by $\lambda(t)$ the function $x\mapsto\lambda(x,t)$ and by $r(t)$ the function $x\mapsto r(x,t)$ (also r could be unbounded). One can suitably define the spaces (see [9])

$$H(t):=L^2(\Omega,\mu(t)),\quad K(t):=L^p(\Omega,\mu(t)),\quad V(t):=\mathcal{W}_0^{1,p}(\Omega,|r(t)|,\lambda(t))$$

which may be defined as the completion of $C^1_c(\Omega)$ with respect to the topologies induce by the norms

$$\left(\int_\Omega|u(x,t)|^q\mu(x,t)\right)^{1/q},\ q=2\text{ or }q=p\quad\text{and}\quad\left(\int_\Omega|Du(x,t)|^p\lambda(x,t)\right)^{1/p},$$

provided that a suitable Poincaré inequality holds

$$\left(\int_\Omega|u(x,t)|^p\mu(x,t)\right)^{1/p}\leqslant c\left(\int_\Omega|Du(x,t)|^p\lambda(x,t)\right)^{1/p}\qquad\text{for } u\in C^1_c(\Omega)$$

and where μ is a suitable extension of $|r|$, i.e.

$$\mu(x,t)=|r(x,t)|\quad\text{where } |r|>0\qquad\text{and}\qquad\mu>0\text{ almost everywhere.}$$

Under suitable assumptions about r and λ there is $s\in\mathbf{R}$ such that $W^{1,s}_0(\Omega)$ is a dense subset of $\mathcal{W}_0^{1,p}(\Omega,|r(t)|,\lambda(t))$ for every $t\in[0,T]$ and then one can consider

$$U:=W^{1,s}_0(\Omega)\qquad\text{for such } s.$$

If r is unbounded we consider

$$R_+(t)=P_+(t)$$

i.e. $R_+(t)$ is the orthogonal projection from $H(t)$ onto $L^2(\Omega_+(t), r_+(t))$ (analogous is the definition of $R_-(t)$). With the other simple and obvious adaptations one can conclude as in the previous example.

In this case the operator $\mathcal{B}$ could be $\mathcal{B}u := \big(|u_t|^{p-2}u_t\mu\big)_t$, i.e.

$$\big\langle \mathcal{B}u, v\big\rangle = \iint |u_t|^{p-2}u_t v_t \mu(x,t)\,dxdt.$$

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