ASYMPTOTIC BEHAVIOUR FOR OPERATORS OF GRUSHIN TYPE: INVARIANT MEASURE AND SINGULAR PERTURBATIONS

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Abstract. This paper concerns singular perturbation problems where the dynamics of the fast variable evolve in the whole space according to an operator whose infinitesimal generator is formed by a Grushin type second order part and a Ornstein-Uhlenbeck first order part.

We prove that the dynamics of the fast variables admits an invariant measure and that the associated ergodic problem has a viscosity solution which is also regular and with logarithmic growth at infinity. These properties play a crucial role in the main theorem which establishes that the value functions of the starting perturbation problems converge to the solution of an effective problem whose operator and initial datum are given in terms of the associated invariant measure.

1. **Introduction.** This paper is devoted to study with PDE's methods some asymptotic features of processes described by the dynamics

$$dZ_t = b(Z_t)dt + \sqrt{2}\sigma(Z_t)dW_t \quad \text{for } t \in (0, +\infty), \qquad Z_0 = x^0 \in \mathbb{R}^2, \qquad (1)$$

where W_t is a 2-dimensional Brownian motion while the matrix σ is degenerate and of Grushin type and the drift b is of Ornstein-Uhlenbeck type, namely

$$\sigma(x) = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix}, \ b(x) = -\alpha x, \ \alpha > 0. \qquad (2)$$

The columns of σ in (2), X_1 and X_2, satisfy Hörmander condition: actually $X_1 = (1, 0)$ and $[X_1, X_2] = (0, 1)$ span all $\mathbb{R}^2$. Hence, we have that $[X_1, X_2] = \partial_{x_2}$.

In particular, we shall investigate:

1) existence and uniqueness of the invariant measure of this process;

2) existence, uniqueness and regularity of the solution for the ergodic problem of the infinitesimal generator $\mathcal{L}$ (see (4));

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3) the asymptotic behaviour as $\epsilon \to 0$ of the value function of optimal control problems driven by

$$\begin{cases} dX_t = \tilde{\phi}(X_t, Y_t, u_t)dt + \sqrt{2}\tilde{\sigma}(X_t, Y_t, u_t)dW_t, & X_0 = x \in \mathbb{R}^n \\ dY_t = \frac{1}{\epsilon}b(Y_t)dt + \frac{\sqrt{2}}{\sqrt{\epsilon}}\sigma(Y_t)dW_t, & Y_0 = y \in \mathbb{R}^2. \end{cases} \tag{3}$$

The paper is organized as follows: in Section 2 we prove existence and uniqueness of the invariant measure. In Section 3 we establish our main result on perturbation problem; to this end, we introduce the approximated ergodic problems and investigate the regularity of their solutions.

2. **Existence of the invariant measure.** We consider the stochastic dynamics (1) with coefficients as in (2). The main aim of this section is to prove existence and uniqueness of the invariant measure m associated to the process (1). To this goal, we use a Liouville property for the infinitesimal generator of (1) (see [4] for other Liouville properties for Grushin operator in a semilinear framework with a superlinear growth for the zeroth order term).

Let us recall from [3] that a probability measure m on $\mathbb{R}^2$ is an *invariant measure* for process (1) if, for each $u_0 \in \mathbb{L}^\infty(\mathbb{R}^2)$, it satisfies

$$\int_{\mathbb{R}^2} u(x,t)m(x)\,dx = \int_{\mathbb{R}^2} u_0(x)m(x)\,dx$$

where $u(x,t) = \mathbb{E}_x(u_0(X_t))$ is the solution to the parabolic Cauchy problem

$$\partial_t u + \mathcal{L}u = 0 \quad \text{in } (0,+\infty) \times \mathbb{R}^2, \qquad u(x,0) = u_0(x) \quad \text{on } \mathbb{R}^2$$

where

$$\mathcal{L}(y,q,Y) := -tr(\sigma\sigma^T Y) - b \cdot q \equiv -Y_{11} - y_1^2 Y_{22} - b(y)q \tag{4}$$

is the *infinitesimal generator* of process (1). For the sake of completeness, let us recall the result in [12, Example 5.1].

Theorem 2.1. *The diffusion process* (1) *admits exactly one invariant probability measure* m.

Proof. Under our assumptions it is easy to check that the matrix $A_\rho = \sigma_\rho \sigma_\rho^T$ where

$$\sigma_\rho(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 & \rho \end{pmatrix}$$

is such that $A_\rho \to A$ in L^∞ and A_ρ is locally definite positive. Moreover taking

$$W(x) = \frac{1}{12}x_1^4 + \frac{1}{2}x_2^2,$$

we have

$$-tr(A_\rho(x)D^2 W(x))) = -2x_1^2 - \rho^2$$
$$-b(x)DW = \frac{1}{3}\alpha x_1^4 + \alpha x_2^2.$$

Then $\mathcal{L}_\rho W = -tr(\sigma_\rho \sigma_\rho^T D^2 W) + b \cdot DW \geq 1$ is equivalent to the following condition

$$\frac{1}{3}\alpha x_1^4 + \alpha x_2^2 \geq 2x_1^2 + 1 + \rho^2,$$

hence W satisfies

$$\mathcal{L}_\rho W \geq 1 \quad \text{in } \overline{B(0,R_0)}^C, \quad W \geq 0 \quad \text{in } \overline{B(0,R_0)}^C, \quad \lim_{|x| \to \infty} W = \infty$$

for ρ sufficiently small.

Then following the procedure used in [12, Proposition 2.1], using the function W, there exists an unique invariant measure m_ρ for the process with diffusion σ_ρ, and arguing as in [12, Theorem 2.1 (proof)] and using again the function W we obtain the existence of the invariant measure associated to the process (1). For the proof of uniqueness, we refer the reader to [12, Theorem 2.1]. $\square$

Remark 1. Lions-Musiela in [11] have considered a similar degenerate case but in their paper the elements of the matrix are bounded in $\mathbb{R}^2$ in this way

$$\sigma(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{x_1}{\sqrt{1+x_1^2}} \end{pmatrix}.$$

Remark 2. Following [12, Example 5.1] one can obtain results similar to Theorem 2.1 with a more general drift term b:

$$b_i(x) = b_i(x_i), \qquad \begin{cases} b_i(x_i) \leq -\frac{C_i}{|x_i|^{1-\beta}} & \text{for } x_i \geq R \\ b_i(x_i) \geq \frac{C_i}{|x_i|^{1-\beta}} & \text{for } x_i \leq -R \end{cases}$$

for $\beta \geq 0$, $R > 0$ and suitably chosen $C_i > 0$ $(i = 1, 2)$.

Remark 3. As applications of the existence of an invariant measure we obtain, arguing as in [12] the following results:

$$\lim_{\delta \to 0^+} \delta u_\delta(x) = \lim_{t \to +\infty} u(t,x) = \lim_{t \to +\infty} \frac{v(t,x)}{t} = \int_{\mathbb{R}^2} f \, dm,$$

where m is the invariant measure and u_δ, u and v are the solutions respectively of

$$\delta u_\delta + \mathcal{L} u_\delta = f(\cdot), \qquad \text{in } \mathbb{R}^2,$$

$$u_t + \mathcal{L} u = 0 \qquad \text{in } (0, +\infty) \times \mathbb{R}^2, \qquad u(0, \cdot) = f(\cdot) \qquad \text{on } \mathbb{R}^2,$$

$$v_t + \mathcal{L} v = f(\cdot) \qquad \text{in } (0, +\infty) \times \mathbb{R}^2, \qquad v(0, \cdot) = 0 \qquad \text{on } \mathbb{R}^2,$$

and $\mathcal{L}$ is defined in (4).

3. **Asymptotic behaviour for a singular perturbation problem.** In this section, we investigate the limit of the value function

$$V^\epsilon(t, x, y) := \sup_{u \in \mathcal{U}} \mathbb{E}\left[\int_t^T f(X_s, Y_s, u_s) ds + e^{a(t-T)} g(X_T) \right]$$

where $\mathbb{E}$ denotes the expectation, $\mathcal{U}$ is the set of progressively measurable processes with values in a compact metric set U and a is a fixed positive parameter and (X_t, Y_t) are driven by (3) (note that V^ϵ depends on ϵ through the coefficients of the dynamics).

Throughout this section, we shall assume

  i) the function $f = f(x, y, u)$ is Lipschitz continuous in (x, y) uniformly in u and, for some $C_f > 0$, it satisfies

$$|f(x, y, u)| \leq C_f(1 + |x|) \qquad \forall (x, y, u) \in \mathbb{R}^n \times \mathbb{R}^2 \times U;$$

  ii) the function g is continuous in (x, y) and there exits C_g such that

$$|g(x, y)| \leq C_g(1 + |x|) \qquad \forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^2;$$

  iii) $\tilde{\phi}(x, y, u)$ and $\tilde{\sigma}(x, y, u)$ are Lipschitz continuous and bounded in (x, y) uniformly on u: $|\tilde{\phi}(x, y, u)| \leq C_{\tilde{\phi}}$, $|\tilde{\sigma}(x, y, u)| \leq C_{\tilde{\sigma}}$.

Problems of this type arise from models where the variables Y evolve much faster than the variables X. We refer to [2] and [13] for the financial models which inspired this research.

By standard theory (see [7]), the value function V^ϵ is the unique (viscosity) solution to the following Cauchy problem

$$\begin{cases} -\partial_t V^\epsilon + H\left(x, y, D_x V^\epsilon, D_{xx}^2 V^\epsilon, \frac{D_{xy}^2 V^\epsilon}{\sqrt{\epsilon}}\right) \\ \quad + \frac{1}{\epsilon}\mathcal{L}(y, D_y V^\epsilon, D_{yy} V^\epsilon) + a V^\epsilon = 0 & \text{in } (0,T) \times \mathbb{R}^n \times \mathbb{R}^2 \\ V^\epsilon(T, x, y) = g(x, y) & \text{on } \mathbb{R}^n \times \mathbb{R}^2 \end{cases} \qquad (5)$$

where $\mathcal{L}$ is the operator defined in (4) and

$$H(x, y, p, X, Z) \quad := \quad \min_{u \in U}\left\{-tr(\tilde{\sigma}\tilde{\sigma}^T X) - \tilde{\phi} \cdot p - 2tr(\tilde{\sigma}\sigma^T Z) - f(x, y, u)\right\}.$$

Our aim is to establish that, as $\epsilon \to 0^+$, the function V^ϵ converges locally uniformly to a function $V = V(t, x)$ (which will be independent of y) which can be characterized as the unique (viscosity) solution to the *effective* Cauchy problem

$$\begin{cases} -\partial_t V + \overline{H}\left(x, D_x V, D_{xx}^2 V\right) + aV = 0 & \text{in } (0,T) \times \mathbb{R}^n \\ V(T, x) = \overline{g}(x) & \text{on } \mathbb{R}^n. \end{cases} \qquad (6)$$

The effective Hamiltonian and the effective terminal datum are given by

$$\overline{H}(x, p, X) \quad := \quad \int_{\mathbb{R}^2} H(x, y, p, X, 0) dm(y) \qquad (7)$$

$$\overline{g}(x) \quad := \quad \int_{\mathbb{R}^2} g(x, y) dm(y) \qquad (8)$$

and m is the invariant measure established in Theorem 2.1. As a matter of facts, $\overline{H}(x, p, X)$ is the ergodic constant λ of the cell problem

$$- tr(\sigma(y)\sigma^T(y)D^2 w(y)) - b(y)Dw(y) + H(x, y, p, X, 0) = \lambda \quad y \in \mathbb{R}^2, \qquad (9)$$

(the solution w to this equation is called *corrector*) while $\overline{g}(x)$ is the constant obtained in the long time behaviour of the parabolic Cauchy problem

$$\partial_t w^* - \mathcal{L}w^* = 0 \quad \text{in } (0, \infty) \times \mathbb{R}^2, \qquad w^*(0, y) = g(x, y) \quad \text{on } \mathbb{R}^2,$$

(namely $\overline{g} = \lim_{t \to +\infty} w^*(t, y)$).

The main issues of this setting are: 1) the fast variables evolve in the whole space, 2) the infinitesimal generator of their operator is degenerate with unbounded coefficients, 3) the variables y lack a group structure. In order to overcome these issues, we shall use the following tools: 1) there exists a superlinear Lyapunov function, 2) a Liouville type result applies to operator $\mathcal{L}$, 3) there exists an invariant measure, 4) the cell problem admits a regular solution (we shall first prove that it is globally Lipschitz continuous and then we make a bootstrap argument) with an at most logarithmic growth.

In order to prove the existence and the properties of (λ, w) satisfying (9), we introduce the approximated problems

$$\delta u_\delta - tr(\sigma(y)\sigma^T(y)D^2 u_\delta) - b(y)Du_\delta = F(y) \qquad \text{in } \mathbb{R}^2, \qquad (10)$$

where $\delta > 0$ and $F(y) := -H(x, y, p, X, 0)$ with (x, p, X) fixed. In the next subsection we investigate the properties of the approximated correctors u_δ; in the last subsection these properties will be inherited by the corrector w.

3.1. **Regularity of the approximated correctors.** In this section we shall establish two results on the regularity of u_δ in two different setting for F: a global Lipschitz continuity and a local Hölder continuity. In our opinion, both these results have their own interest because we apply two different techniques: the former follows the ones of [5, 10] while the latter one follows the ones of [8]. However, in the rest of the paper we shall only need the former one.

Throughout this section we assume

$$|F(y)| \leq C_F(|y| + 1) \quad \forall y \in \mathbb{R}^2. \tag{11}$$

Let us recall from [13, Lemma 3.3] the following result on the growth of u_δ; for the proof, we refer the reader to [13].

Lemma 3.1. *Under assumptions* (11), *there exists a constant C such that*

$$|u_\delta(y)| \leq C\left(|y| + \frac{1}{\delta}\right), \quad y \in \mathbb{R}^2. \tag{12}$$

3.1.1. *Global Lipschitz continuity of the approximated corrector.*

Proposition 1. *Assume b as in* (2) *with $\alpha > 1$ and that F is Lipschitz continuous in $\mathbb{R}^2$ with Lipschitz constant L. Let u_δ be the unique continuous solution of* (10) *which satisfies* (12). *Then, for $\bar{L} > L/(\alpha - 1)$, there holds*

$$|u_\delta(x) - u_\delta(y)| \leq \bar{L}|x - y| \qquad \forall x, y \in \mathbb{R}^2, \delta > 0.$$

Proof. The proof follows the same arguments of the proof of [13, Theorem 3.2]. For completeness, we briefly sketch the main steps. For each $\eta > 0$, we introduce the function

$$\Psi(x, y) = u_\delta(x) - u_\delta(y) - \bar{L}|x - y| - \eta|x|^2 - \eta|y|^2.$$

Our statement is equivalent to the following inequality

$$\Psi(x, y) \leq \frac{4\eta}{\delta} \qquad \forall x, y \in \mathbb{R}^2, \eta \in (0, 1). \tag{13}$$

In order to prove (13), we argue by contradiction. Using the Lemma [5, Lemma 3.2], we follow the same calculation up to equation [13, eq.(3.24)]. By our choice of the matrix σ, we obtain the desired contradiction. $\square$

Remark 4. As in [13], for $b(x) = (-\alpha_1 x_1, -\alpha_2 x_2)$, we obtain the same result when $\alpha_1 > 1$, $\alpha_2 > 0$ and $\bar{L} > L/l$ where $l = \min\{\alpha_1 - 1, \alpha_2\}$.

3.1.2. *Local Hölder continuity of the approximated corrector.*

Proposition 2. *Assume b as in* (2) *with $\alpha > 1$,* (11) *and*

$$|F(x) - F(y)| \leq C_F|x - y|^\gamma(\Phi(x) + \Phi(y)), \ x, y \in \mathbb{R}^2, \gamma \in (0, 1], C_F > 0$$

where $\Phi(x) = x_1^4 + x_2^2 + M$, $M \geq 1$. Let u_δ be the unique continuous solution of (10) *which satisfies* (12). *Then there is a constant $C > 0$, independent on δ such that*

$$|u_\delta(x) - u_\delta(y)| \leq C|x - y|^\gamma(\Phi(x) + \Phi(y)), \quad \forall \ x, y \in \mathbb{R}^2. \tag{14}$$

Proof. We follow the procedure of [8, Theorem 4.3]. We define the functions $w_\delta(x, y) = u_\delta(x) - u_\delta(y)$ and $\tilde{g}(x, y) = C_F|x - y|^\gamma(\Phi(x) + \Phi(y) + A)$ where A will be chosen suitably large. If we prove that $w_\delta \leq \tilde{g}$ in $\mathbb{R}^2 \times \mathbb{R}^2$ then we obtain (14) with a suitable C, since $\Phi > 1$.

We argue by contradiction, we suppose that $\sup_{\mathbb{R}^2 \times \mathbb{R}^2}(w_\delta - \tilde{g}) > 0$. From the linear growth of u_δ (see (12)) we know that $\lim_{|x|+|y| \to +\infty}(w_\delta(x, y) - \tilde{g}(x, y)) = -\infty$

hence we have that $w_\delta \leq \tilde{g}$ in $(\mathbb{R}^2 \times \mathbb{R}^2) \setminus B_R$ for a suitable ball $B_R \subset \mathbb{R}^2 \times \mathbb{R}^2$. Let $(\hat{x}, \hat{y}) \in \overline{B_R}$ be maximum point of $w_\delta(x, y) - \tilde{g}(x, y)$: $w_\delta(\hat{x}, \hat{y}) - \tilde{g}(\hat{x}, \hat{y}) > 0$, $\hat{x} \neq \hat{y}$. At this point we introduce the operator Ξ defined as:

$$\Xi g(x, y) = tr(\Sigma(x, y)D^2 g(x, y)) + 2 \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} g(x, y)$$

where

$$\Sigma(x, y) = \begin{pmatrix} \sigma(x)\sigma^T(x) & \sigma(x)\sigma^T(y) \\ \sigma(y)\sigma^T(x) & \sigma(y)\sigma^T(y) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & x_1^2 & 0 & x_1 y_1 \\ 1 & 0 & 1 & 0 \\ 0 & x_1 y_1 & 0 & y_1^2 \end{pmatrix}$$

(the matrix σ is defined in (2)). This operator is elliptic.

We observe that $w_\delta(x, y)$ satisfies for any $x, y \in \mathbb{R}^2$

$$\delta w_\delta - \Xi w_\delta + \alpha x D_x w_\delta + \alpha y D_y w_\delta = F(x) - F(y)$$
$$\leq C_F |x - y|^\gamma (\Phi(x) + \Phi(y)).$$

Hence from the maximum principle we have that

$$\delta \tilde{g}(\hat{x}, \hat{y}) - \Xi \tilde{g}(\hat{x}, \hat{y}) + \alpha \hat{x} D_x \tilde{g}(\hat{x}, \hat{y}) + \alpha y D_y \tilde{g}(\hat{x}, \hat{y}) \leq C_F |\hat{x} - \hat{y}|^\gamma (\Phi(\hat{x}) + \Phi(\hat{y})). \quad (15)$$

At this point to find a contradiction we compute

$$\delta \tilde{g}(\hat{x}, \hat{y}) - \Xi \tilde{g}(\hat{x}, \hat{y}) + \alpha \hat{x} D_x \tilde{g}(\hat{x}, \hat{y}) + \alpha y D_y \tilde{g}(\hat{x}, \hat{y})$$

directly by the definition of $\tilde{g}$.

Denoting by $t = |x - y|^2$, let us introduce $g(x, y)$ as

$$g(x, y) = t^{\gamma/2}(\Phi(x) + \Phi(y) + A).$$

We compute now $\Xi g(x, y) = tr(\Sigma(x, y)D^2 g(x, y))$. We have

$$D_x g = \gamma t^{\gamma/2-1}(x - y)(\Phi(x) + \Phi(y) + A) + t^{\gamma/2} D_x \Phi(x)$$

$$D_y g = \gamma t^{\gamma/2-1}(y - x)(\Phi(x) + \Phi(y) + A) + t^{\gamma/2} D_y \Phi(y)$$

$$D_{xx}^2 g = \gamma(\gamma - 2)t^{\gamma/2-2}(x - y) \otimes (x - y)(\Phi(x) + \Phi(y) + A) +$$
$$\gamma t^{\gamma/2-1} I(\Phi(x) + \Phi(y) + A) + 2\gamma t^{\gamma/2-1}(x - y) \otimes D_x \Phi(x) + t^{\gamma/2} D_{xx}^2 \Phi(x)$$

$$D_{yy}^2 g = \gamma(\gamma - 2)t^{\gamma/2-2}(y - x) \otimes (y - x)(\Phi(x) + \Phi(y) + A) +$$
$$\gamma t^{\gamma/2-1} I(\Phi(x) + \Phi(y) + A) + 2\gamma t^{\gamma/2-1}(y - x) \otimes D_y \Phi(y) + t^{\gamma/2} D_{yy}^2 \Phi(y)$$

$$D_{xy}^2 g = -\gamma(\gamma - 2)t^{\gamma/2-2}(x - y) \otimes (x - y)(\Phi(x) + \Phi(y) + A) -$$
$$\gamma t^{\gamma/2-1} I(\Phi(x) + \Phi(y) + A) + \gamma t^{\gamma/2-1}(x - y) \otimes (D_y \Phi(y) - D_x \Phi(x)).$$

Denoting by A_{ij} the 2×2 minor of Σ we have that

$$\Xi g(x, y) = tr(\Sigma(x, y)D^2 g(x, y)) =$$
$$tr(A_{11} D_{xx}^2 g + A_{12}(D_{xy}^2 g)^T + A_{12} D_{xy}^2 g + A_{22} D_{yy}^2 g).$$

Using the explicit derivatives written here above and the definition of Φ we obtain

$$\Xi g(x,y) = tr(\Sigma(x,y)D^2 g(x,y)) =$$
$$\gamma(\gamma-2)t^{\gamma/2-2}(\Phi(x) + \Phi(y) + A)(x_2 - y_2)^2(x_1 - y_1)^2 +$$
$$t^{\gamma/2}(\Delta_{Gx}\Phi(x) + \Delta_{Gy}\Phi(y)) + \gamma t^{\gamma/2-1}(\Phi(x) + \Phi(y) + A)(x_1 - y_1)^2 +$$
$$+4\gamma t^{\gamma/2-1}(x_1 - y_1)(x_2 - y_2)(x_1 x_2 + y_1 y_2)$$

where we denoted by $\Delta_G u(z) := tr(\sigma(z)\sigma^T(z)u(z))$, i.e. the horizontal Grushin Laplace operator.

We note that, by elementary calculations, it is possible to find a constant L_α such that

$$-\Delta_G \Phi(z) + \alpha z D_z \Phi(z) \geq 2\alpha \Phi(z) - L_\alpha. \tag{16}$$

Now we write the equation

$$\delta g(x,y) - \Xi g(x,y) + \alpha x D_x g + \alpha y D_y g = \delta t^{\gamma/2}(\Phi(x) + \Phi(y) + A) +$$
$$t^{\gamma/2}(-\Delta_{Gx}\Phi(x) + \alpha x D_x \Phi + (-\Delta_{Gy}\Phi(y) + \alpha y D_y \Phi)) -$$
$$\gamma(\gamma-2)t^{\gamma/2-2}(\Phi(x) + \Phi(y) + A)(x_2 - y_2)^2(x_1 - y_1)^2$$
$$-\gamma t^{\gamma/2-1}(\Phi(x) + \Phi(y) + A)(x_1 - y_1)^2 -$$
$$4\gamma t^{\gamma/2-1}(x_1 - y_1)(x_2 - y_2)(x_1 x_2 + y_1 y_2) +$$
$$\alpha\gamma t^{\gamma/2-1}(\Phi(x) + \Phi(y) + A)(x - y)^2 \geq \delta t^{\gamma/2}(\Phi(x) + \Phi(y) + A) +$$
$$t^{\gamma/2}(2\alpha(\Phi(x) + \Phi(y)) - 2L_\alpha) - \gamma t^{\gamma/2-1}(\Phi(x) + \Phi(y) + A)(x_1 - y_1)^2 -$$
$$4\gamma t^{\gamma/2-1}(x_1 - y_1)(x_2 - y_2)(x_1 x_2 + y_1 y_2) +$$
$$\alpha\gamma t^{\gamma/2-1}(\Phi(x) + \Phi(y) + A)(x - y)^2,$$

where in the last inequality we used (16). Hence by the definition of $t = |x - y|^2$ we have

$$\delta g(x,y) - \Xi g(x,y) + \alpha x D_x g + \alpha y D_y g \geq$$
$$\delta|x - y|^\gamma(\Phi(x) + \Phi(y) + A) +$$
$$|x - y|^\gamma(2\alpha(\Phi(x) + \Phi(y)) - 2L_\alpha) - \gamma|x - y|^{\gamma-2}(\Phi(x) + \Phi(y) + A)(x_1 - y_1)^2 -$$
$$4\gamma|x - y|^{\gamma-2}(x_1 - y_1)(x_2 - y_2)(x_1 x_2 + y_1 y_2) +$$
$$+\alpha\gamma|x - y|^{\gamma-2}I(\Phi(x) + \Phi(y) + A)(x - y)^2.$$

Recall that $\tilde{g}(x,y) = C_F g(x,y)$, hence $\tilde{g}$ satisfies:

$$\delta\tilde{g}(x,y) - \Xi\tilde{g}(x,y) + \alpha x D_x \tilde{g} + \alpha y D_y \tilde{g} \geq$$
$$C_F|x - y|^\gamma\Big(2\alpha(\Phi(x) + \Phi(y)) - 2L_\alpha + (\delta - \gamma)(\Phi(x) + \Phi(y) + A) +$$
$$+\alpha\gamma(\Phi(x) + \Phi(y) + A) - 4\gamma(x_1 x_2 + y_1 y_2)\Big).$$

Hence

$$\delta\tilde{g}(x,y) - \Xi\tilde{g}(x,y) + \alpha x D_x g + \alpha y D_y \tilde{g} \geq$$
$$C_F|x - y|^\gamma\Big((\Phi(x) + \Phi(y))(\delta + (2\alpha - \gamma) + \gamma\alpha)) +$$
$$A(\delta + (\alpha - 1)\gamma) - 2\gamma|x_1 x_2 + y_1 y_2| - 2L_\alpha\Big).$$

(We used that $|(x_1 - y_1)(x_2 - y_2)| \leq \frac{1}{2}(x - y)^2$.) Since $\alpha > 1$ and $\gamma \in (0, 1]$,

$$\delta \tilde{g}(x, y) - \Xi \tilde{g}(x, y) + \alpha x D_x g + \alpha y D_y \tilde{g} \geq$$

$$C_F |x - y|^\gamma \bigg((\Phi(x) + \Phi(y)) +$$

$$\gamma \alpha (\Phi(x) + \Phi(y)) + A(\delta + (\alpha - 1)\gamma) - 4\gamma(x_1 x_2 + y_1 y_2) - 2L_\alpha \bigg) >$$

$$C_F |x - y|^\gamma (\Phi(x) + \Phi(y)).$$

The last inequality is obtained noting that:
1) Since $\alpha > 1$ we can find $K_\alpha > 0$ such that $\alpha \Phi(x) - 4\gamma x_1 x_2 > -K_\alpha$.
2) Since $\alpha > 1$ we can choose A sufficiently large such that $A(\delta + (\alpha - 1)\gamma) - 2L_\alpha - K_\alpha > 0$.
Hence we obtain a contradiction of (15). $\qquad\square$

Remark 5. Note that if we consider a drift term of the type: $b(y) = (-\alpha_1 y_1, -\alpha_2 y_2)$ we can obtain the same result as before taking $\gamma = 1$ with $\alpha_1 > 1$ and $\alpha_2 > 0$. The calculations are tedious and we omit them.

3.2. **The convergence result.**

Theorem 3.2. *Assume $\alpha > 1$ and that, for $F(\cdot) = -H(x, \cdot, p, X, 0)$*

$$F, \frac{\partial F}{\partial y_2} \text{ and } \frac{\partial^2 F}{\partial y_2^2} \text{ are bounded Lipschitz continuous functions.} \qquad (17)$$

Then, the solution V^ϵ of (5) converges locally uniformly in $(0, T) \times \mathbb{R}^n \times \mathbb{R}^2$ to the unique viscosity solution V of (6) where $\overline{H}$ and $\overline{g}$ are defined in (7)-(8).

Proof. The arguments of the proof are analogous to those of [13, Theorem 2.1]; we only sketch them.

1. Well posedness of problem (5) and growth properties of V^ϵ.

   Proposition 3. *For any $\epsilon > 0$ there exists a unique continuous viscosity solution V^ϵ to problem (5) such that*

   $$|V^\epsilon(t, x, y)| \leq C_0(1 + |x|), \quad \forall (t, x, y) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^2$$

   for some positive constant C_0 independent on ϵ. In particular $\{V^\epsilon\}_\epsilon$ is a family of locally equibounded functions.

   Proof. The proof is the same as [13, Proposition 2.1]. $\qquad\square$

2. The cell problem.
   Let us consider the sequence of solutions of the approximated cell problem (10) $\{u_\delta\}_\delta$. Using Proposition 1 we can define (at least for a subsequence) the $\lim_{\delta \to 0}(u_\delta(y) - u_\delta(0)) := w(y)$ and using the Lemma 3.1 $\lim_{\delta \to 0} -\delta u_\delta(0) =: \lambda$.
   Thanks to Proposition 1 w is a global Lipschitz function and using the stability properties of viscosity solutions (w, λ) is a solution of the ergodic problem (9).
   Moreover:

Proposition 4. *The constant* $\lambda = -\int_{\mathbb{R}^2} H(\overline{x}, y, \overline{p}, \overline{X}, 0) dm(y)$ *(m is the invariant measure founded in Theorem 2.1) is the unique constant such that the cell problem* (9) *admits a solution w with an at most linear growth at infinity. Moreover w is globally Lipschitz continuous, satisfies*

$$|w(y) - w(0)| \leq C \left[1 + \log(y_1^4 + y_2^2 + 1) \right] \qquad \forall y \in \mathbb{R}^2 \tag{18}$$

and it is unique up to an additive constant within the function with an at most linear growth at infinity.

Proof. We refer the reader to Remark 3 to characterize λ.

Estimate (18) follows from an analogous estimate for u_δ that can be proved as in [13, Lemma 3.4] taking as supersolution of (10) the function $g(y_1, y_2) = C_1 \log(y_1^4 + y_2^2)$ which satisfies

$$\delta g - tr(\sigma\sigma^t D^2 g) + \alpha y Dg \geq C_1 \left(\frac{2y_1^6 - 10y_1^2 y_2^2}{(y_1^4 + y_2^2)^2} + \alpha \frac{4y_1^4 + y_2^2}{y_1^4 + y_2^2} \right) \geq F(y),$$

for $y \in \mathbb{R}^2 \setminus B_R$, with suitable C_1 and R sufficiently large. Hence repeating the same argument as in [13, Lemma 3.4] we get the result. $\square$

3. C^2-regularity of the corrector.

Proposition 5. *Let w be the solution of the cell problem* (9) *founded in Proposition 4. Then* $w \in C_{loc}^{2,\beta}(\mathbb{R}^2)$, *for some* $\beta \in (0, 1)$.

Proof. In this proof, β denotes a constant which may change from line to line. The corrector w solves

$$- tr(\sigma(y)\sigma^T(y) D^2 w(y)) + \alpha y Dw(y) = G(y) \tag{19}$$

with $G(y) := \lambda - H(\overline{x}, y, \overline{p}, \overline{X}, 0)$. First let us get the global Lipschitz continuity of $\frac{\partial w}{\partial y_2}$ and $\frac{\partial^2 w}{\partial y_2^2}$. Deriving equation (19) with respect to y_2 (remark that this is possible because G is regular enough thanks to (17)) we obtain that the function $u := \frac{\partial w}{\partial y_2}$ is bounded by Proposition 1 and it solves in the sense of distributions

$$- tr(\sigma\sigma^T D^2 u) + \alpha y Du + \alpha u = \frac{\partial G}{\partial y_2}. \tag{20}$$

From Proposition 1 and [13, Lemma 3.5], we get that $\frac{\partial w}{\partial y_2}$ is globally Lipschitz continuous in $\mathbb{R}^2$. Deriving again equation (20) with respect to y_2 we obtain that the function $\frac{\partial^2 w}{\partial y_2^2}$ is globally Lipschitz continuous in $\mathbb{R}^2$.

Using the global Lipschitz continuity of $\frac{\partial w}{\partial y_2}$ and Proposition 1 in (19), we infer: $\frac{\partial^2 w}{\partial y_1^2} \in L_{loc}^\infty$. Again, by the Lipschitz continuity of $\frac{\partial w}{\partial y_2}$, we obtain $\Delta w \in L_{loc}^\infty$; by standard elliptic theory, $Dw \in C_{loc}^{0,\beta}$. Using the global Lipschitz continuity of $\frac{\partial^2 w}{\partial y_2^2}$ in (19), we get $\frac{\partial^2 w}{\partial y_1^2} \in C_{loc}^{0,\beta}$. Again, by the Lipschitz continuity of $\frac{\partial^2 w}{\partial y_2^2}$, we have $\Delta w \in C_{loc}^{0,\beta}$. Applying standard theory, we accomplish the proof. $\square$

4. Conclusion.

We adapt the classical perturbed test function method (see [1, 6, 9]) to prove the convergence. To this end, we argue as in [13, Theorem 2.1] using

the Liouville property for $\mathcal{L}$, the regularity of the corrector and the existence of a Lyapunov function $(W(y) = y_1^2 + y_2^2)$.

$\square$

REFERENCES

[1] O. Alvarez and M. Bardi, Ergodicity, stabilization, and singular perturbations for Bellman-Isaacs equations, *Mem. Amer. Math. Soc.*, **204** (2010), vi+77 pp.

[2] M. Bardi, A. Cesaroni and L. Manca, Convergence by viscosity methods in multiscale financial models with stochastic volatility, *SIAM J. Financial Math.*, **1** (2010), 230–265.

[3] A. Bensoussan, *Perturbation Methods in Optimal Control*, Wiley/Gauthier-Villars Series in Modern Applied Mathematics. John Wiley & Sons, Chichester; Gauthier-Villars, Montrouge, 1988.

[4] I. Capuzzo Dolcetta and A. Cutrì On the Liouville property for the sub-Laplacians, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **25** (1997), 239–256.

[5] M. G. Crandall, H. Ishii and P. L. Lions, User's guide to viscosity solutions of second-order partial differential equations, *Bull. Amer. Math. Soc.*, **27** (1992), 1–67.

[6] L. C. Evans, The perturbed test-function method for viscosity solutions of nonlinear PDE, *Proc. Roy. Soc. Edinburgh Sect. A*, **111** (1989), 359–375.

[7] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, Berlin, 1993.

[8] Y. Fujita, H. Ishii and P. Loreti, Asymptotic solutions of viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator, *Comm. Partial Differential Equations*, **31** (2006), 827–848.

[9] D. Ghilli, Viscosity methods for large deviations estimates of multi scale stochastic processes, *ESAIM Control Optim. Calc. Var.*, to appear.

[10] H. Ishii and P.-L. Lions, Viscosity solution of fully nonlinear second-order elliptic partial differential equations, *J. Differential Equations*, **83** (1990), 26–78.

[11] P. L. Lions and M. Musiela, *Ergodicity of Diffusion Processes*, unpublished.

[12] P. Mannucci, C. Marchi and N. Tchou, The ergodic problem for some subelliptic operators with unbounded coefficients, *NoDEA Nonlinear Differential Equations Appl.*, **23** (2016), Art. 47, 26 pp.

[13] P. Mannucci, C. Marchi and N. A. Tchou, Singular perturbations for a subelliptic operator, *ESAIM Control Optim. Calc. Var.*, to appear.

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