Stable H-minimal hypersurfaces

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Abstract

We prove some stability results for smooth H-minimal hypersurfaces immersed in a sub-Riemannian k-step Carnot group $\mathbb{G}$. The main tools are the formulas for the 1st and 2nd variation of the H-perimeter measure σ_H^{n-1}.
Key words and phrases: Carnot groups; H-minimal hypersurfaces; 1st and 2nd variation of the H-perimeter; stability; characteristic set
Mathematics Subject Classification: 49Q15, 46E35, 22E60.

Contents

1. Introduction

In classical Differential Geometry a minimal (hyper)surface of $\mathbb{R}^n$ (or, more generally, of a Riemannian manifold $(M^n, \langle\cdot,\cdot\rangle)$) is a smooth codimension one submanifold having zero mean curvature. We recall that the Riemannian mean curvature $\mathcal{H}_{\mathcal{R}}$ of a hypersurface S is the trace of its 2nd fundamental form $B_{\mathcal{R}}$, which is the $\mathbf{C}^\infty$-bilinear form defined as $B_{\mathcal{R}}(X, Y) := \langle\nabla_X Y, \nu\rangle$ for every $X, Y \in \mathfrak{X}(TS) := \mathbf{C}^\infty(S, TS)$, where ∇ denotes the Levi-Civita connection on the ambient space (either $\mathbb{R}^n$ or M) and ν is the unit normal vector along S. Note that $\mathcal{H}_{\mathcal{R}} = -div_{TS}\,\nu$. The crucial fact here is that minimal hypersurfaces turn out to be critical points of the Riemannian $(n-1)$-dimensional volume $\sigma_{\mathcal{R}}^{n-1}$. In this setting, studying stability of a minimal hypersurface S means to study conditions under which S turns out to be a minimum of the functional $\sigma_{\mathcal{R}}^{n-1}$. Hence, it becomes important to study the 2nd variation of $\sigma_{\mathcal{R}}^{n-1}$ and, in order to avoid boundary contributions, we only consider compactly supported normal variations of S. For an introduction to these topics in the Euclidean and/or Riemannian setting we refer the reader to the surveys by Chern [19], Lawson [44] and Osserman [57]; see also Simons' paper [64]. Finally, for some results concerning stability of minimal and CMC hypersurfaces, we would like to mention the papers [9], [10], [25], [28] and [66].

That of Minimal Surfaces is one of the great chapters of the XX century Mathematics, above all, because was a rich source of entirely new ideas and theories such as that of Currents, introduced by Federer and Fleming [27] (see Federer's fundamental treatise [26]), that of Sets of Finite Perimeter

[1]F. M. has been partially supported by the Fondazione CaRiPaRo Project "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems".
[2]I would like to thank the anonymous referee of this paper for helpful comments and criticism.

created by De Giorgi and its school starting from the pioneering work of Caccioppoli (see the book by Giusti [35] or [3]), and that of Varifolds, heavily inspired by Almgreen and developed by Allard in [1, 2]. A highly recommended introduction for these topics is, of course, the book by Simon [63]; see also the survey by Bombieri [13] and Morgan's book [56].

In this paper, we shall study some of these problems, in the sub-Riemannian setting of Carnot groups. We recall that a *sub-Riemannian manifold* is a smooth n-dimensional manifold M, endowed with a non-integrable distribution $H \subset TM$ of h-planes, called the *horizontal bundle*, on which a (positive definite) metric g_H is given. The horizontal bundle H satisfies the *Hörmander condition* and this implies the validity of Chow theorem so that, different points can always be joined by using *horizontal curves* (i.e. curves that are everywhere tangent to H). The idea is simply that, in connecting two points, we are only allowed to follow horizontal paths joining them. The *CC-distance* d_H, is then defined by minimizing the g_H-length of horizontal curves connecting two given points: this is the distance used in sub-Riemannian geometry. As an introduction to these topics, we refer the reader to Gromov [37], Montgomery [54], Pansu [58, 59], Strichartz [68]. In this context, Carnot groups play a role similar to Euclidean spaces in Riemannian geometry. They serve as a model for the tangent space of a SR manifold and, further, represent a wide class of examples of sub-Riemannian geometries. By definition, a *k-step Carnot group* $\mathbb{G}$ is a n-dimensional, connected, simply connected and nilpotent Lie group (with respect to a group law $\bullet$ which is polynomial) having a *k-step stratified Lie algebra* $\mathfrak{g} \cong \mathbb{R}^n$. This means that $\mathfrak{g}$ splits into a direct sum of vector subspaces of $\mathbb{R}^n$ satisfying suitable commuting relations.

More precisely, we have $\mathfrak{g} = H_1 \oplus H_2 \oplus \cdots \oplus H_k$, $[H_1, H_i] = H_{i+1}$ for every $i = 1, ..., k-1$ and $[H_1, H_i] = 0$ for every $i \geq k$, where $[\cdot,\cdot]$ denote Lie brackets. We assume that $h_i = \dim H_i \, (i = 1, ..., k)$ so that $n = \sum_{i=1}^{k} h_i$. The stratification of $\mathfrak{g}$ can be seen as the algebraic counterpart of the Hörmander condition.

We recall that Carnot groups are homogeneous groups, in the sense that they admit a family of positive anisotropic dilations modeled on the stratification; see [67]. This richness of geometric structures, makes interesting the study of Geometric Measure Theory in Carnot groups; see, for instance, [4], [5], [6], [8], [20], [33], [29, 30, 31, 32], [50, 51, 52], [47, 48, 49], [55] and bibliographies therein. We also cite [12], [16, 17], [18], [23, 24], [34], [60], [41], [61], [62] for many important results concerning H-minimal and/or constant horizontal mean curvature (hyper)surfaces of the Heisenberg group. Nevertheless, here we have to remark that not much is known about the geometry of smooth H-minimal hypersurfaces in general groups.

The aim of this paper, which is somehow a continuation of [52], is studying the stability of smooth H-minimal hypersurfaces immersed in k-step Carnot groups. Let us briefly describe our results.

In Section 1.1, we will fix notation and main definitions concerning Carnot groups. We will use a left invariant frame $\underline{X} := \{X_1, ..., X_n\}$ on $\mathfrak{g} \cong T\mathbb{G}$ adapted to the stratification and we will fix a Riemannian metric $\langle\cdot,\cdot\rangle$ making $\underline{X}$ orthonormal. This frame satisfies some non-trivial commuting relations encoded by the so-called structural constants $C^{\mathfrak{g}\,r}_{i,j} := \left\langle [X_i, X_j], X_r \right\rangle \; \forall\, i, j, r = 1, ..., n$. Note also that the (uniquely determined) left invariant Levi Civita connection ∇ can be expressed in terms of structural constants. The projection of ∇ onto the horizontal H is denoted by ∇^H and called horizontal connection.

In Section 1.2 we recall basic facts about immersed hypersurfaces endowed with the H-perimeter measure σ^{n-1}_H. Note that $\sigma^{n-1}_H = |\mathcal{P}_H \nu|\, \sigma^{n-1}_{\mathcal{R}}$, where $\sigma^{n-1}_{\mathcal{R}}$ is the $(n-1)$-dimensional Riemannian measure, ν is the unit (Riemannian) normal along S and $\mathcal{P}_H$ is the projection onto H. Furthermore, the tangent bundle TS inherits the stratification of $T\mathbb{G} \cong \mathfrak{g}$. Let $\nu_H = \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}$ be the unit horizontal normal along S and let HS be the horizontal tangent sub-bundle of TS, which is $(h-1)$-dimensional at each non-characteristic point $p \in S \setminus C_S$, where $C_S := \{p \in S : |\mathcal{P}_H \nu| = 0\}$ is the characteristic set. It turns out that $H = HS \oplus span_{\mathbb{R}}\{\nu_H\}$ at each $p \in S \setminus C_S$. This allows us to define the horizontal 2nd fundamental form by setting $B_H(X, Y) := \left\langle \nabla^H_X Y, \nu_H \right\rangle$. However, this object is not symmetric, in general. Thus it can be decomposed in its symmetric and skew-symmetric parts, i.e. $B_H = S_H + A_H$.

In Section 2 we will discuss some divergence-type formulas, which are very important tools. In particular, these results will enable us to define the horizontal tangential operators $\mathcal{D}_{HS}$ and $\mathcal{L}_{HS}$, which are analogous, in this SR setting, to tangential divergence div_{TS} and Laplacian Δ_{TS}. An important fact is the validity of the formula

$$-\int_S \varphi \mathcal{L}_{HS}\varphi\, \sigma_H^{n-1} = \int_S |grad_{HS}\varphi|^2\, \sigma_H^{n-1}$$

for every compactly supported function $\varphi \in \mathbf{C}^2_{HS}(S \setminus C_S) \cap W^{1,2}_{HS}(S; \sigma_H^{n-1})$; see, for instance, Corollary 2.7 and Remark 2.8. We also stress that the previous formula holds true (a fortiori) when $\varphi \in \mathbf{C}^2(S)$. In the same section, we preliminarily discuss the basic calculations needed to prove the 1st variation formula for the H-perimeter σ_H^{n-1}.

Section 3 contains some important technical tools: adapted frames, connection 1-forms and lemmata concerning the horizontal 2nd fundamental form B_H. This material is then used in Section 4 to discuss and prove the variational formulas for σ_H^{n-1}. The presentation here is slightly different from [52]. In fact, we have tried to simplify the original proofs. More importantly, we have corrected a mistake that has caused the loss of some divergence-type terms in the variational formulas proved there; see Remark 2.12. Furthermore, we have extended the formulas to the characteristic case.

We say that a hypersurface S of class $\mathbf{C}^2$ is *H-minimal* if its horizontal mean curvature $\mathcal{H}_H$ is zero at each non-characteristic $p \in S \setminus C_S$. It is important to remark that, in general, we have to distinguish the notion of H-minimal from that of being "critical point"of the the H-perimeter functional σ_H^{n-1}. Let us explain this fact in more detail. Roughly speaking, the formula expressing the 1st variation of σ_H^{n-1} can easily be written by using the notion of Lie derivative of a differential form; see Section 2 and Section 4. The "infinitesimal"1st variation of σ_H^{n-1} turns out to be given by Lie derivative of σ_H^{n-1}. We have

$$\mathcal{L}_W \sigma_H^{n-1} = \left(-\mathcal{H}_H \langle W, \nu\rangle + div_{TS}\left(W^\top |\mathcal{P}_H \nu| - \langle W, \nu\rangle \nu_H^\top\right)\right) \sigma_{\mathcal{R}}^{n-1},$$

where $\mathcal{L}_W \sigma_H^{n-1}$ denotes the Lie derivative of σ_H^{n-1} with respect to the initial velocity W of the variation. The symbols $W^\perp$, $W^\top$ denote the normal and tangential components of W, respectively. If $\mathcal{H}_H$ is $L^1_{loc}(S; \sigma_{\mathcal{R}}^{n-1})$, the function $\mathcal{L}_W \sigma_H^{n-1}$ turns out to be integrable on S and the integral of $\mathcal{L}_W \sigma_H^{n-1}$ on S gives the 1st variation of σ_H^{n-1}. Note however that the third term in the previous formula depends on the normal component of W. In general, this term cannot be integrated on the boundary; see Theorem 4.6. We stress that this term was omitted in [52]. This can be done only under further assumptions on the characteristic set; see Corollary 4.8. In this case, the notion of H-minimality and that of being "critical point"of σ_H^{n-1} are coincident.

The formula for 2nd variation of σ_H^{n-1}, which is one of the main results of this paper, will be obtained as a result of a long calculation; see Theorem 4.13. This formula will be proved under some more thecnical assumptions. Mainly, they concern integrability of some geometric quantities but, for a precise statement, we refer the reader to Section 4.

Remark 1.1. *In the case of the Heisenberg group $\mathbb{H}^1$, the 1st variation formula characteristic surfaces of class $\mathbf{C}^2$ was first obtained by Ritoré and Rosales in* [62]. *Furthermore, we also stress that Hurtado, Ritoré and Rosales* [41] *have proved a formula for the 2nd variation which is very similar to that stated in Theorem 4.13. We also quote* [40], *for similar results in a very general sub-Riemannian setting.*

Using compactly supported variations together with suitable assumptions on the characteristic set, the formula takes the following simpler form

$$II_S(W, \sigma_H^{n-1}) = \int_S \left(|grad_{HS}\, w|^2 - w^2 \mathcal{B}_{TS}\right) \sigma_H^{n-1},$$

where W is the variation vector and $w = \frac{\langle W, \nu\rangle}{|\mathcal{P}_H \nu|}$; see Corollary 4.15. Here we have used the symbol $\mathcal{B}_{TS}$ to denote the following quantity

$$\mathcal{B}_{TS} := \underbrace{\|S_H\|^2_{\mathrm{Gr}} + \|A_H\|^2_{\mathrm{Gr}}}_{=\|B_H\|^2_{\mathrm{Gr}}} + \sum_{\alpha \in I_V} \left\langle \left(2 grad_{HS}(\varpi_\alpha) - C(\varpi)\tau_\alpha^{TS}\right), C^\alpha \nu_H \right\rangle;$$

for the notation, see Definition 1.11 and Definition 1.13 in Section 1.2. This expression involves the matrices of the structural constants and geometric quantities such as the horizontal 2nd fundamental form B_H and the vertical vector field ϖ, defined as

$$\varpi := \frac{\mathcal{P}_V \nu}{|\mathcal{P}_H \nu|} = \sum_{\alpha=h+1}^{n} \varpi_\alpha X_\alpha,$$

where $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|}$. This vector, which represents a "weighed"vertical projection of the (Riemannian) unit normal ν along S, plays an important role in this context.

In Section 5 we will state some further geometric identities for constant horizontal mean curvature hypersurfaces. In particular, we shall find some explicit solutions to the equation

$$\mathcal{L}_{HS}\,\varphi + \varphi\,\mathcal{B}_{TS} = 0.$$

This is a key-point of this paper and, using this fact, our main stability inequality will follow by adapting a standard argument in the Riemannian setting; see, e.g. [28]. In Section 6 we will prove the following:

Theorem 1.2. *Let $S \subset \mathbb{G}$ be a H-minimal hypersurface of class $\mathbf{C}^3$. If there exists $\alpha \in I_V = \{h+1, ..., n\}$ such that either $\varpi_\alpha > 0$ or $\varpi_\alpha < 0$ on S, then each non-characteristic domain $\Omega \subset S$ is stable.*

An immediate application of the previous result is contained in the next:

Corollary 1.3. *Let $S \subset \mathbb{G}$ be a complete H-minimal hypersurface of class $\mathbf{C}^3$. If S is a graph with respect to some given vertical direction, then each non-characteristic domain $\Omega \subset S$ is stable.*

An analysis of some (more or less simple) examples is given in Section 6.1, in order to illustrate our results; see, more precisely, Corollary 6.9, Corollary 6.10, and Corollary 6.12.

Finally, in Section 7 we will obtain a completely different stability result, which is based on a Sobolev-type inequality recently proved in [53]. The following theorem generalizes an idea by Spruck [66]:

Theorem 1.4. *Let $S \subset \mathbb{G}$ be a H-minimal hypersurface of class $\mathbf{C}^3$ satisfying the assumptions made in Corollary 4.15. There exists a dimensional constant C_0 such that if*

$$\int_S |\mathcal{B}_{TS}|^{\frac{Q-1}{2}}\,\sigma_H^{n-1} < C_0,$$

then S is strictly stable.

1.1. **Carnot groups.** A *k-step Carnot group* $(\mathbb{G}, \bullet)$ is a connected, simply connected, nilpotent and stratified Lie group (with respect to a group law $\bullet$) so that its Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$ is a direct sum of slices $\mathfrak{g} = H_1 \oplus ... \oplus H_k$ such that $[H_1, H_{i-1}] = H_i \quad (i = 2, ..., k)$, $H_{k+1} = \{0\}$. Let 0 be the identity of $\mathbb{G}$ and $\mathfrak{g} \cong T_0\mathbb{G}$. Let $h_i := \dim H_i$ for $i = 1, ..., k$ and $h_1 := h$. Moreover set $H := H_1$ and $V := H_2 \oplus ... \oplus H_k$. Note that H and V are smooth subbundles of $T\mathbb{G}$ called *horizontal* and *vertical*, respectively. The horizontal bundle H is generated by a frame $\underline{X_H} := \{X_1, ..., X_h\}$ of left-invariant vector fields, which can be completed to a global graded, left-invariant frame $\underline{X} := \{X_1, ..., X_n\}$ for $T\mathbb{G}$. We also stress that the standard basis $\{\mathrm{e}_i : i = 1, ..., n\}$ of $\mathbb{R}^n$ can be relabeled to be *graded* or *adapted to the stratification*. Any left-invariant vector field of $\underline{X}$ satisfies $X_i(x) = L_{x*}\mathrm{e}_i\,(i = 1, ..., n)$, where L_{x*} denotes the differential of the left-translation at $x \in \mathbb{G}$. We fix a Euclidean metric on $\mathfrak{g} = T_0\mathbb{G}$ which makes $\{\mathrm{e}_i : i = 1, ..., n\}$ an orthonormal basis; this metric extends to the whole tangent bundle by left-translations and makes $\underline{X}$ an orthonormal left-invariant frame. We shall denote by $g = \langle\cdot,\cdot\rangle$ this metric and assume that $(\mathbb{G}, g)$ is a Riemannian manifold.

We shall use the so-called *exponential coordinates of 1st kind* so that $\mathbb{G}$ will be identified with its Lie algebra $\mathfrak{g}$, via the (Lie group) exponential map $exp : \mathfrak{g} \longrightarrow \mathbb{G}$.

A *sub-Riemannian metric* g_H is a symmetric positive bilinear form on the horizontal bundle H. The *CC-distance* $d_H(x, y)$ between $x, y \in \mathbb{G}$ is given by

$$d_H(x, y) := \inf \int \sqrt{g_H(\dot\gamma, \dot\gamma)}\,dt,$$

where the infimum is taken over all piecewise-smooth horizontal paths γ joining x to y. From now on, we shall choose $g_H := g_{|H}$.

We recall that Carnot groups are *homogeneous groups*, i.e. they admit a one-parameter group of automorphisms $\delta_t : \mathbb{G} \longrightarrow \mathbb{G}$ for any $t \geq 0$. By definition, one has $\delta_t x := exp\left(\sum_{j,i_j} t^j x_{i_j} \mathrm{e}_{i_j}\right)$, for every $x = exp\left(\sum_{j,i_j} x_{i_j} \mathrm{e}_{i_j}\right) \in \mathbb{G}$. The *homogeneous dimension* of $\mathbb{G}$ is the integer $Q := \sum_{i=1}^{k} i\, h_i$ coinciding with the *Hausdorff dimension* of $(\mathbb{G}, d_H)$ as a metric space; see [37], [54].

The *structural constants* of $\mathfrak{g}$ associated with $\underline{X}$ are defined by $C^{\mathfrak{g}\,r}_{\ ij} := \left\langle [X_i, X_j], X_r \right\rangle$, $i, j, r = 1, ..., n$. They are skew-symmetric and satisfy Jacobi's identity. The stratification hypothesis on $\mathfrak{g}$ can be restated as follows:

$$X_i \in H_l,\ X_j \in H_m \Longrightarrow [X_i, X_j] \in H_{l+m} \tag{1}$$

and so if $i \in I_{H_s}$ and $j \in I_{H_r}$, then

$$C^{\mathfrak{g}\,m}_{\ ij} \neq 0 \Longrightarrow m \in I_{H_{s+r}}. \tag{2}$$

Later, we will set

- $C^{\alpha}_{H} := [C^{\mathfrak{g}\,\alpha}_{\ ij}]_{i,j=1,...,h} \in \mathcal{M}_{h\times h}(\mathbb{R}) \qquad \forall\ \alpha = h+1, ..., h+h_2;$
- $C^{\alpha} := [C^{\mathfrak{g}\,\alpha}_{\ ij}]_{i,j=1,...,n} \in \mathcal{M}_{n\times n}(\mathbb{R}) \qquad \forall\ \alpha = h+1, ..., n.$

We now introduce the left-invariant co-frame $\underline{\omega} := \{\omega_i : i = 1, ..., n\}$ dual to $\underline{X}$, i.e. $\omega_i = X_i^*$ for every $i = 1, ..., n$. In particular, note that the *left-invariant 1-forms* ω_i are uniquely determined by

$$\omega_i(X_j) = \left\langle X_i, X_j \right\rangle = \delta_i^j \qquad \forall\ i, j = 1, ..., n,$$

where δ_i^j denotes the Kronecker delta.

Let ∇ denote the (unique) left-invariant Levi-Civita connection on $\mathbb{G}$ associated with the left-invariant metric $g = \langle\cdot,\cdot\rangle$. It turns out that

$$\nabla_{X_i} X_j = \frac{1}{2}\sum_{r=1}^{n} (C^{\mathfrak{g}\,r}_{\ ij} - C^{\mathfrak{g}\,i}_{\ jr} + C^{\mathfrak{g}\,j}_{\ ri}) X_r \qquad \forall\ i, j = 1, ..., n.$$

If $X, Y \in \mathfrak{X}(H) := \mathbf{C}^{\infty}(\mathbb{G}, H)$, we shall set $\nabla^H_X Y := \mathcal{P}_H(\nabla_X Y)$. The operation ∇^H is a partial connection called *H-connection*. We stress that ∇^H is *flat, compatible with the metric* g_H and *torsion-free* (i.e. $\nabla^H_X Y - \nabla^H_Y X - \mathcal{P}_H[X, Y] = 0\ \forall\ X, Y \in \mathfrak{X}(H)$); see [52] and references therein.

Notation 1.5. *Let $X \in \mathfrak{X}^1(T\mathbb{G}) = \mathbf{C}^1(\mathbb{G}, T\mathbb{G})$. We shall denote by $\mathcal{J}_{\mathcal{R}} X$ the Jacobian matrix of X computed with respect to the left invariant frame $\underline{X} = \{X_1, ..., X_n\}$. Moreover, let $X \in \mathfrak{X}^1(H) = \mathbf{C}^1(\mathbb{G}, H)$. We shall denote by $\mathcal{J}_H X$ the horizontal Jacobian matrix of X computed with respect to the horizontal left invariant frame $\underline{X_H} = \{X_1, ..., X_h\}$.*

Remark 1.6 (Horizontal curvature tensor R_H)**.** *The flatness of ∇^H implies that horizontal curvature tensor R_H is identically zero, where we recall that*

$$\mathrm{R}_H(X, Y)Z := \nabla^H_Y \nabla^H_X Z - \nabla^H_X \nabla^H_Y Z - \nabla^H_{[Y,X]_H} Z \qquad \forall\ X, Y, Z \in \mathfrak{X}(H).$$

Horizontal gradient and horizontal divergence operators will be denoted by $grad_H$ and div_H.

A continuous distance $\varrho : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}_+ \cup \{0\}$ is called *homogeneous* if one has

$$\varrho(x, y) = \varrho(z \bullet x, z \bullet y) \quad \forall\ x,\ y,\ z \in \mathbb{G}; \qquad \varrho(\delta_t x, \delta_t y) = t\varrho(x, y) \quad \forall t \geq 0.$$

We recall a fundamental example.

Example 1.7 (Heisenberg groups $\mathbb{H}^n$)**.** *The Lie algebra $\mathfrak{h}_n \cong \mathbb{R}^{2n+1}$ of the n-th Heisenberg group can be defined by using a left-invariant frame $\underline{Z} := \{X_1, Y_1, ..., X_i, Y_i, ..., X_n, Y_n, T\}$, where, at each point $p = exp(x_1, y_1, x_2, y_2, ..., x_n, y_n, t) \in \mathbb{H}^n$, we have set: $X_i(p) := \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial t}$, $Y_i(p) := \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial t}$ for every $i = 1, ..., n$; $T(p) := \frac{\partial}{\partial t}$. One has $[X_i, Y_i] = T$ for every $i = 1, ..., n$, and all other commutators vanish, so that T is the center of $\mathfrak{h}_n$ and $\mathfrak{h}_n$ turns out to be nilpotent and stratified of step 2, i.e. $\mathfrak{h}_n = H \oplus H_2$. Finally, the structural constants of $\mathfrak{h}_n$ are described by the skew-symmetric $(2n \times 2n)$-matrix*

$$C_H^{2n+1} := \begin{vmatrix} 0 & 1 & \cdot & 0 & 0 \\ -1 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 1 \\ 0 & 0 & \cdot & -1 & 0 \end{vmatrix}$$

associated with the skew-symmetric bilinear map $\Gamma_H : H \times H \longrightarrow \mathbb{R}$ *given by* $\Gamma_H(X, Y) = \langle [X, Y], T\rangle$ *for every* $X, Y \in H$.

1.2. **Hypersurfaces and measures.** The Riemannian left-invariant volume form on $\mathbb{G}$ is defined as $\sigma^n_{\mathcal{R}} := \bigwedge_{i=1}^n \omega_i \in \bigwedge^n(T^*\mathbb{G})$. The measure $\sigma^n_{\mathcal{R}}$ is the Haar measure of $\mathbb{G}$ and equals the push-forward of the usual n-dimensional Lebesgue measure $\mathcal{L}^n$ on $\mathfrak{g} \cong \mathbb{R}^n$. Now let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^1$. We say that $x \in S$ is a *characteristic point* if $\dim H_x = \dim(H_x \cap T_xS)$. The *characteristic set* of S is given by $C_S := \{x \in S : \dim H_x = \dim(H_x \cap T_xS)\}$. Note that $x \in S$ is non-characteristic if, and only if, H is transversal to S at x. It turns out that the $(Q-1)$-dimensional CC Hausdorff measure of the characteristic set C_S vanishes, i.e. $\mathcal{H}^{Q-1}_{CC}(C_S) = 0$. Moreover, under further regularity assumptions, it is possible to show much more. For instance, if S is of class $\mathbf{C}^2$, then the $(n-1)$-dimensional Riemmanian Hausdorff measure of C_S is zero; see [11].

Let ν denote the unit normal vector along S. The $(n-1)$-dimensional Riemannian measure along an immersed hypersurface S can be defined in a canonical way by setting $\sigma^{n-1}_{\mathcal{R}} \llcorner S := (\nu \lrcorner \sigma^n_{\mathcal{R}})|_S$, where $\lrcorner$ denotes the "contraction" operator on differential forms; see Lee's book [45], pp. 334-346. We recall that $\lrcorner : \bigwedge^k(T^*\mathbb{G}) \to \bigwedge^{k-1}(T^*\mathbb{G})$ is defined, for $X \in T\mathbb{G}$ and $\alpha \in \bigwedge^k(T^*\mathbb{G})$, by setting

$$(X \lrcorner \alpha)(Y_1, ..., Y_{k-1}) := \alpha(X, Y_1, ..., Y_{k-1}).$$

Example 1.8. *Let* $(\mathbb{R}^3, \langle\cdot,\cdot\rangle)$ *be the Euclidean* 3*-space, endowed with its standard basis* $\mathrm{e}_1 = (1, 0, 0)$, $\mathrm{e}_2 = (0, 1, 0)$, $\mathrm{e}_3 = (0, 0, 1)$. *The corresponding dual basis of the cotangent bundle is then given by* $\mathrm{e}^*_i = dx_i$, $i = 1, 2, 3$. *Obviously, the canonical volume form of* $\mathbb{R}^3$ *is* $\sigma^3_{\mathcal{R}} = dx_1 \wedge dx_2 \wedge dx_3$. *So if* $S \subset \mathbb{R}^3$ *is a smooth immersed surface oriented by its outward-pointing unit normal vector* ν, *then*

$$\nu \lrcorner \sigma^3_{\mathcal{R}} = \nu_1 dx_2 \wedge dx_3 - \nu_2 dx_1 \wedge dx_3 + \nu_3 dx_1 \wedge dx_2,$$

and the restriction[3] *of this* 2*-form to* S *is nothing but the canonical surface measure.*

At each non-characteristic point of S the *unit* H*-normal* along S is the normalized projection of ν onto H and we shall set

$$\nu_H := \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}.$$

The H*-perimeter form* is the $(n-1)$-differential form $\sigma^{n-1}_H \in \bigwedge^{n-1}(T^*S)$ defined by

$$\sigma^{n-1}_H \llcorner S := (\nu_H \lrcorner \sigma^n_{\mathcal{R}})|_S.$$

If $C_S \neq \emptyset$ we extend σ^{n-1}_H to the whole of S by setting $\sigma^{n-1}_H \llcorner C_S = 0$.

Remark 1.9. *It is very important to note that*

$$\sigma^{n-1}_H \llcorner S = |\mathcal{P}_H \nu|\, \sigma^{n-1}_{\mathcal{R}} \llcorner S. \tag{3}$$

This follows from the well-known formula $X \lrcorner \sigma^n_{\mathcal{R}} = \langle X, \nu\rangle\, \sigma^{n-1}_{\mathcal{R}}$ *for any* $X \in T\mathbb{G}$. *In particular, note that* $C_S = \{x \in S : |\mathcal{P}_H \nu(x)| = 0\}$.

The differential form σ^{n-1}_H, which equals the "variational" H-perimeter on smooth hypersurfaces, will be later called *H-perimeter form*; see [52].

Let $\mathcal{S}^{Q-1}_{CC}$ denote the $(Q-1)$-dimensional spherical Hausdorff measure associated with the CC-distance d_H. Then $\sigma^{n-1}_H(S \cap B) = k(\nu_H)\, \mathcal{S}^{Q-1}_{CC} \llcorner (S \cap B)$ for all $B \in \mathcal{B}or(\mathbb{G})$, where the density $k(\nu_H)$, called *metric factor*, depends on ν_H; see [47]. The *horizontal tangent bundle* $HS \subset TS$ and the *horizontal normal*

[3] By the restriction of a form to a submanifold we mean its image under the pullback map induced by the inclusion.

bundle $\nu_{_H} S$ split the horizontal bundle H into an orthogonal direct sum, i.e. $H = \nu_{_H} \oplus HS$. We also recall that the stratification of $\mathfrak{g}$ induces a stratification of $TS := \oplus_{i=1}^{k} H_i S$, where $HS := H_1 S$; see [37].

Remark 1.10. *We have* $\dim H_p S = \dim H - 1 = h - 1$ *at each point* $p \in S \setminus C_S$. *Furthermore, note that the definition of* HS *makes sense even if* $p \in C_S$, *but in such a case* $\dim H_p S = \dim H_p = 2n$.

For the sake of simplicity, in the rest of this section we shall assume, unless otherwise mentioned, that $S \subset \mathbb{G}$ is a non-characteristic hypersurface of class $\mathbf{C}^2$. So let ∇^{TS} be the induced connection on S from ∇. The tangential connection ∇^{TS} induces a partial connection on HS defined by

$$\nabla^{HS}_X Y := \mathcal{P}_{HS}\left(\nabla^{TS}_X Y\right) \quad \forall\ X, Y \in \mathfrak{X}^1(HS) := \mathbf{C}^1(S, HS).$$

It turns out that $\nabla^{HS}_X Y = \nabla^{H}_X Y - \left\langle \nabla^{H}_X Y, \nu_{_H}\right\rangle \nu_{_H}$. In the sequel, HS-gradient and HS-divergence will be denoted, respectively, by $grad_{HS}$ and div_{HS}.

By definition, the horizontal 2nd fundamental form of S is the bilinear map given by

$$B_H(X,Y) := \left\langle \nabla^{H}_X Y, \nu_{_H}\right\rangle \quad \forall\ X, Y \in \mathfrak{X}^1(HS).$$

The horizontal mean curvature $\mathcal{H}_H$ is the trace of B_H, i.e. $\mathcal{H}_H := \mathrm{Tr} B_H = -div_H \nu_{_H}$. The torsion T_{HS} of the HS-connection ∇^{HS} is given by

$$\mathrm{T}_{HS}(X,Y) := \nabla^{HS}_X Y - \nabla^{HS}_Y X - \mathcal{P}_H[X,Y] \quad \forall\ X, Y \in \mathfrak{X}^1(HS).$$

There is a non-zero torsion because, in general, B_H is *not symmetric* in general. Hence it can be regarded as a sum of two matrices, i.e. $B_H = S_H + A_H$, where S_H is symmetric and A_H skew-symmetric.

Definition 1.11. *The* principal horizontal curvatures κ_j *of* S, $j \in I_{HS}$, *are the eigenvalues of* S_H, *i.e. eigenvalues of the symmetric part of the horizontal 2nd fundamental form* B_H. *Note that* $\mathcal{H}_H = \sum_{j \in I_{HS}} \kappa_j$. *We also define some important geometric objects:*

- $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|} \quad \forall\ \alpha = h+1, ..., n;$
- $\varpi_{H_2} := \frac{\mathcal{P}_{H_2}\nu}{|\mathcal{P}_H \nu|} = \sum_{\alpha \in I_{H_2}} \varpi_\alpha X_\alpha;$
- $\varpi := \frac{\mathcal{P}_V \nu}{|\mathcal{P}_H \nu|} = \sum_{\alpha \in I_V} \varpi_\alpha X_\alpha;$
- $C_H(\varpi_{H_2}) := \sum_{\alpha \in I_{H_2}} \varpi_\alpha C^\alpha_H;$
- $C(\varpi) := \sum_{\alpha \in I_V} \varpi_\alpha C^\alpha.$

Finally, we shall denote by $C_{HS}(\varpi_{H_2})$ *the restriction to the subspace* HS *of the linear operator* $C_H(\varpi_{H_2})$.

These objects play an important role in the horizontal geometry of immersed hypersurfaces. In particular, we have to stress that $A_H = \frac{1}{2} C_{HS}(\varpi_{H_2})$; see [52]. Moreover, for every $X,\ Y \in \mathfrak{X}^1_{HS}$ we have

$$\mathrm{T}_{HS}(X,Y) = \langle [X,Y], \varpi\rangle \nu_{_H} = -\langle C_{HS}(\varpi_{H_2}) X, Y\rangle.$$

Example 1.12 (Heisenberg group)**.** *We have* $\varpi := \varpi_T = \frac{\langle \nu, T\rangle}{|\mathcal{P}_H \nu|}$ *and* $C_H(\varpi_{H_2}) = \varpi C^{2n+1}_H$; *see Example 1.7. An elementary computation shows that the skew-symmetric part* A_H *of the horizontal 2nd fundamental form* B_H *is given by* $A_H = \varpi \frac{C^{2n+1}_{HS}}{2}$, *where* $C^{2n+1}_{HS} = C^{2n+1}_H|_{HS}$. *Since* $\|C^{2n+1}_{HS}\|^2_{\mathrm{Gr}} = 2(n-1)$, *it follows that* $\|B_H\|^2_{\mathrm{Gr}} = \|S_H\|^2_{\mathrm{Gr}} + \frac{n-1}{2}\varpi^2$.

Definition 1.13. *Let* $U \subseteq \mathbb{G}$ *be an open set and let* $\mathcal{U} := S \cap U$. *We call* adapted frame to $\mathcal{U}$ on U *any o.n. frame* $\underline{\tau} := \{\tau_1, ..., \tau_n\}$ *on* U *such that* $\tau_1|_{\mathcal{U}} := \nu_{_H}$, $H_p\mathcal{U} = \mathrm{span}\{(\tau_2)_p, ..., (\tau_h)_p\}\ \forall\ p \in \mathcal{U}$, $\tau_\alpha := X_\alpha$. *Furthermore, we set* $\tau^{TS}_\alpha := \tau_\alpha - \varpi_\alpha \tau_1$ *for every* $\alpha \in I_V$. *We stress that* $HS^\perp = \mathrm{span}_{\mathbb{R}}\{\tau^{TS}_\alpha : \alpha \in I_V\}$, *where* $HS^\perp$ *denotes the orthogonal complement of* HS *in* TS, *i.e.* $TS = HS \oplus HS^\perp$.

Note also that

$$\underline{\tau} = \{\underbrace{\tau_1}_{=\nu_H}, \underbrace{\tau_2, ..., \tau_h}_{\text{o.n. basis of } HS}, \underbrace{\tau_{h+1}, ..., \tau_n}_{\text{o.n. basis of } V}\}.$$

Notation 1.14. *Let* $n_i := \sum_{j=1}^{i} h_j$. *We set* $I_H = \{1, 2, ..., h\}$, $I_{H_i} = \{n_{i-1} + 1, ..., n_i\}$, $I_V = \{h + 1, ..., n\}$ *and* $I_{HS} := \{2, 3, ..., h\}$.

Every adapted orthonormal frame to a hypersurface is a graded frame. In particular, given an adapted frame $\underline{\tau}$ for $\mathcal{U}$ on U, then at every $p \in \mathcal{U}$, the linear change of coordinates from the fixed left-invariant o.n. frame $\underline{X}$ to the adapted one $\underline{\tau}$ is given by the orthogonal matrix $A(p) = [A_i^j(p)]_{i,j=1,...,n} \in \mathbf{O}_n(\mathbb{R})$ such that $\tau_i(p) = \sum_{j=1}^{n} A_i^j(p) X_j$ for all $i = 1, ..., n$.

Let $\underline{\phi} := \{\phi_1, ..., \phi_n\}$ be the dual co-frame of $\underline{\tau}$, i.e. $\phi_i(\tau_j) = \delta_i^j \ \forall\ i, j = 1, ..., n$, where δ_i^j denotes the Kroneker delta. Clearly, $\underline{\phi}$ satisfies the Cartan's structural equations:

$$\text{(4)} \qquad \text{(I)} \quad d\phi_i = \sum_{j=1}^{n} \phi_{ij} \wedge \phi_j, \qquad \text{(II)} \quad d\phi_{jk} = \sum_{l=1}^{n} \phi_{jl} \wedge \phi_{lk} - \Phi_{jk} \qquad \forall\ i, j, k = 1, ..., n,$$

where $\phi_{ij}(X) := \left\langle \nabla_X \tau_j, \tau_i \right\rangle$ denote the *connection 1-forms* of $\underline{\phi}$ and Φ_{jk} denote the *curvature 2-forms*, defined by $\Phi_{jk}(X, Y) := \phi_k(\mathrm{R}(X, Y)\tau_j) \ \forall\ X, Y \in \mathfrak{X}(\mathbb{G})$, where R is the *Riemannian curvature tensor*, i.e.

$$\mathrm{R}(X, Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z \qquad \forall\ X, Y, Z \in \mathfrak{X}(\mathbb{G}).$$

We have a basic identity between connection 1-forms and structural constants of $\underline{\tau}$, i.e.

$$\text{(5)} \qquad C_{ij}^k := \left\langle [\tau_i, \tau_j], \tau_k \right\rangle = \phi_{jk}(\tau_i) - \phi_{ik}(\tau_j) \qquad \forall\ i, j, k = 1, ..., n.$$

This can be proved by using the fact that ∇ is torsion-free.

Definition 1.15. *[Hyperplanes] A* vertical hyperplane $\mathcal{I}$ *is the zero-set of some linear homogeneous polynomial on* $\mathbb{G}$ *of homogeneous degree* 1. *A* non-vertical hyperplane $\mathcal{I}$ *is the zero-set of some linear polynomial on* $\mathbb{G}$ *of homogeneous degree greater than or equal to* 2.

Clearly, hyperplanes are $(n-1)$-dimensional vector subspaces of $\mathfrak{g}$. Vertical hyperplanes are very important objects because of the intrinsic rectifiability theory developed by Franchi, Serapioni and Serra Cassano in 2-step Carnot groups; see [29, 30, 31, 32]. They turn out to be ideals of the Lie algebra $\mathfrak{g}$ and may be thought of as generalized tangent spaces to sets of finite H-perimeter (in the variational sense); see also [6]. We stress that the definition of "non-vertical hyperplane" will be useful for later purposes.

2. Divergence formulas

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface. Assume first that S is non-characteristic. We denote by $\mathbf{C}_{HS}^i(S)$, $(i = 1, 2)$ the space of functions whose HS-derivatives up to the i-th order are continuous on S. Analogously, for any open subset $\mathcal{U} \subseteq S$, we set $\mathbf{C}_{HS}^i(\mathcal{U})$, to denote the space of functions whose HS-derivatives up to the i-th order are continuous on $\mathcal{U}$. Note that the previous definition extends to the case in which $C_S \neq \emptyset$ by requiring that all HS-derivatives up to the i-th order are continuous on C_S.

Remark 2.1. *The notions concerning the* HS*-connection* ∇^{HS}*, the horizontal 2nd fundamental form* B_H *and the torsion* T_{HS}*, can also be reformulated by replacing the vector space* $\mathfrak{X}^1(HS) = \mathbf{C}^1(S, HS))$ *with the larger one* $\mathfrak{X}_{HS}^1(HS) := \mathbf{C}_{HS}^1(S, HS)$.

Definition 2.2 (HS-differential operators)**.** *Let* $\mathcal{D}_{HS} : \mathfrak{X}_{HS}^1(HS) \longrightarrow \mathbf{C}(S)$ *be the 1st order differential operator given by*

$$\mathcal{D}_{HS} X := div_{HS} X + \langle C_H(\varpi_{H_2})\nu_H, X\rangle \qquad \forall\ X \in \mathfrak{X}_{HS}^1(HS).$$

Moreover, let $\mathcal{D}_{HS} : \mathbf{C}_{HS}^2(S) \longrightarrow \mathbf{C}(S)$ *be the 2nd order differential operator defined as*

$$\mathcal{L}_{HS}\varphi := \Delta_{HS}\varphi + \langle C_H(\varpi_{H_2})\nu_H, grad_{HS}\varphi\rangle \qquad \forall\ \varphi \in \mathbf{C}_{HS}^2(S).$$

Note that $\mathcal{D}_{HS}(\varphi X) = \varphi\mathcal{D}_{HS} X + \langle grad_{HS}\,\varphi, X\rangle$ for every $X \in \mathfrak{X}^1(HS)$ and every $\varphi \in \mathbf{C}^1_{HS}(S)$. Moreover $\mathcal{L}_{HS}\,\varphi = \mathcal{D}_{HS}(grad_{HS}\,\varphi)$ for every $\varphi \in \mathbf{C}^2_{HS}(S)$.

It is not difficult to see that the operators Δ_{HS} and $\mathcal{L}_{HS}$ naturally extend to horizontal vector fields. These extensions will be denoted by $\overrightarrow{\Delta_{HS}}$ and $\overrightarrow{\mathcal{L}_{HS}}$. We remark that

$$\overrightarrow{\mathcal{L}_{HS}}X = \overrightarrow{\Delta_{HS}}X + (\mathcal{J}_{HS}\,X)\,C_H(\varpi_{H_2})\nu_H$$

for every $X \in \mathbf{C}^2_{HS}(S \setminus C_S, HS)$, where $\mathcal{J}_{HS}\,X$ denotes the HS-Jacobian matrix of the horizontal tangent vector field X.

The above definitions are somehow motivated by Theorem 3.17 in [52].

Theorem 2.3. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth compact non-characteristic hypersurface having -piecewise-$\mathbf{C}^1$-smooth boundary ∂S. Then*

$$\int_S \mathcal{D}_{HS}\,X\,\sigma^{n-1}_H = \int_{\partial S} \langle X, \eta_{HS}\rangle\,\sigma^{n-2}_H \qquad \forall\ X \in \mathfrak{X}^1(HS),$$

where σ^{n-2}_H denotes a $(Q-2)$-homogeneous measure on the boundary ∂S; see Remark 2.4.

As a consequence, the following integral formula holds

$$\int_S \mathcal{D}_{HS}\,X\,\sigma^{n-1}_H = \int_S (div_{HS}\,X + \langle C_H(\varpi_{H_2})\nu_H, X\rangle)\,\sigma^{n-1}_H = 0$$

for every $X \in \mathbf{C}^1_0(S, HS)$.

Here above we have used a homogeneous measure σ^{n-2}_H, which plays the role of the intrinsic Hausdorff measure on $(n-2)$-dimensional submanifolds of $\mathbb{G}$.

Remark 2.4 (The measure σ^{n-2}_H)**.** *Let $\eta \in \mathfrak{X}(TS)$ be a unit normal vector orienting ∂S. Furthermore, let $\eta_{HS} := \frac{\mathcal{P}_{HS}\,\eta}{|\mathcal{P}_{HS}\,\eta|}$ be the* unit HS-normal *of ∂S. By definition, we set $\sigma^{n-2}_H \llcorner \partial S := \left(\eta_{HS} \lrcorner\, \sigma^{n-1}_H\right)|_{\partial S}$. Exactly as for the H-perimeter σ^{n-1}_H, the measure σ^{n-2}_H, which turns out to be $(Q-2)$-homogeneous with respect to Carnot dilations, can be represented in terms of the Riemannian measure $\sigma^{n-2}_{\mathcal{R}}$. We stress that $\sigma^{n-2}_H \llcorner \partial S = |\mathcal{P}_H\,\nu|\,|\mathcal{P}_{HS}\,\eta|\,\sigma^{n-2}_{\mathcal{R}} \llcorner \partial S$.*

Stokes formula is concerned with integrating a k-form over a k-dimensional manifold with boundary. A common way to state this fundamental result is the following.

Proposition 2.5. *Let M be an oriented k-dimensional manifold of class $\mathbf{C}^2$ with boundary ∂M. Then $\int_M d\alpha = \int_{\partial M} \alpha$ for every compactly supported $(k-1)$-form α of class $\mathbf{C}^1$.*

One requires M to be of class $\mathbf{C}^2$ for a technical reason concerning "pull-back" of differential forms. Without much effort, it is possible to extend Proposition 2.5 to the following cases:

- ($\star$) *$\overline{M}$ is of class $\mathbf{C}^1$ and α is a $(k-1)$-form such that α and $d\alpha$ are continuous;*
- ($\spadesuit$) *$\overline{M}$ is of class $\mathbf{C}^1$ and α is a $(k-1)$-form such that α, $d\alpha \in L^\infty(M)$ and $\iota^*_M\alpha \in L^\infty(\partial M)$, where $\iota_M : \partial M \longrightarrow \overline{M}$ is the natural inclusion.*

More general versions of Stokes formula are available in literature, see, for instance, [26]; for a more detailed discussion, we refer the reader to the book by Taylor [69].

We have here to remark that either condition ($\star$) or ($\spadesuit$) can be used to extend the horizontal integration by parts formulas to vector fields (and functions) possibly singular at the characteristic set C_S.

Definition 2.6. *Let $X \in \mathbf{C}^1(S \setminus C_S, HS)$ and set $\alpha_X := (X \lrcorner\, \sigma^{n-1}_H)|_S$. We say that X is* admissible (for the horizontal divergence formula) *if the differential forms α_X and $d\alpha_X$ satisfy either condition ($\star$) or ($\spadesuit$) on S. We say that $\phi \in \mathbf{C}^2_{HS}(S \setminus C_S)$ is* admissible *if $grad_{HS}\,\phi$ is admissible for the horizontal divergence formula. More generally, let $X \in \mathbf{C}^1(S \setminus C_S, TS)$ and set $\alpha_X := (X \lrcorner\, \sigma^{n-1}_H)|_S$. Then, we say that X is* admissible (for the Riemannian divergence formula) *whenever α_X and $d\alpha_X$ satisfy either condition ($\star$) or ($\spadesuit$) on S.*

For instance, condition ($\star$) requires that α_X and $d\alpha_X$ must be continuous on S. We stress that X is of class $\mathbf{C}^1$ out of C_S, but may be singular at C_S.

Using Definition 2.6 and Theorem 2.3 yields the following:

Corollary 2.7. *Let $S \subset \mathbb{G}$ be a compact $\mathbf{C}^2$ hypersurface with -piecewise- $\mathbf{C}^1$-smooth boundary ∂S. Then*

(i) $\int_S \mathcal{D}_{HS} X\, \sigma_H^{n-1} = \int_{\partial S} \langle X, \eta_{HS} \rangle\, \sigma_H^{n-2}$ *for every admissible* $X \in \mathbf{C}^1(S \setminus C_S, HS)$*;*

(ii) $\int_S \mathcal{L}_{HS}\, \phi\, \sigma_H^{n-1} = \int_{\partial S} \langle grad_{HS}\, \phi, \eta_{HS} \rangle\, \sigma_H^{n-2}$ *for every admissible* $\phi \in \mathbf{C}^2_{HS}(S \setminus C_S)$*;*

(iii) *if* $\partial S = \emptyset$*, then*

$$-\int_S \varphi \mathcal{L}_{HS}\, \varphi\, \sigma_H^{n-1} = \int_S |grad_{HS}\, \varphi|^2\, \sigma_H^{n-1} \tag{6}$$

for every function $\varphi \in \mathbf{C}^2_{HS}(S \setminus C_S)$ *such that* φ^2 *is admissible.*

Note that formula 6 holds true even if $\partial S \neq \emptyset$, but in this case we have to use compactly supported functions on S.

Remark 2.8. *Let $\varphi \in \mathbf{C}^2_{HS}(S \setminus C_S)$. Then, it is possible to show that φ^2 is admissible if, and only if,*

$$\varphi \in W^{1,2}_{HS}(S, \sigma_H^{n-1}) = \{\varphi \in L^2(S, \sigma_H^{n-1}) : |grad_{HS}\, \varphi| \in L^2(S, \sigma_H^{n-1})\}.$$

We do not prove this fact here; we just say that the "necessity" part is obvious.

Example 2.9 (Heisenberg group; see Example 1.12)**.** *One has $\mathcal{D}_{HS}(X) := div_{HS}\, X + \varpi \left\langle C_H^{2n+1} \nu_H, X \right\rangle$ for every $X \in \mathfrak{X}^1(HS)$ and $\mathcal{L}_{HS}\, \varphi := \mathcal{D}_{HS}(grad_{HS}\, \varphi) = \Delta_{HS}\, \varphi + \varpi \langle C_H^{2n+1} \nu_H, grad_{HS}\, \varphi \rangle$ for every $\varphi \in \mathbf{C}^2_{HS}(S)$.*

Notation 2.10. *Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^i$, $i \geq 2$. Let $X \in T\mathbb{G}$ and let ν be the outward-pointing unit normal vector along S. Hereafter, we shall denote by $X^\perp$ and $X^\top$ the standard decomposition of X into its normal and tangential components, i.e. $X^\perp = \langle X, \nu \rangle \nu$ and $X^\top = X - X^\perp$.*

We now make a simple (but fundamental) calculation.

Lemma 2.11. *If $X \in \mathfrak{X}^1(T\mathbb{G})$, then $(X \lrcorner\, \sigma_H^{n-1})|_S = \left(\left(X^\top |\mathcal{P}_H \nu| - \langle X, \nu \rangle \nu_H^\top\right) \lrcorner\, \sigma_{\mathcal{R}}^{n-1}\right) \llcorner S$. Moreover, at each non-characteristic point of S, we have*

$$d(X \lrcorner\, \sigma_H^{n-1})|_S = div_{TS} \left(X^\top |\mathcal{P}_H \nu| - \langle X, \nu \rangle \nu_H^\top\right) \sigma_{\mathcal{R}}^{n-1} \llcorner S.$$

Proof. We have

$$\begin{aligned} d(X \lrcorner\, \sigma_H^{n-1})|_S &= (X \lrcorner\, \nu_H \lrcorner\, \sigma_{\mathcal{R}}^n)|_S \\ &= d\left(\left(X^\top + X^\perp\right) \lrcorner \left(\nu_H^\top + \nu_H^\perp\right) \lrcorner\, \sigma_{\mathcal{R}}^n\right)\Big|_S \\ &= d\left(X^\top \lrcorner\, \nu_H^\perp \lrcorner\, \sigma_{\mathcal{R}}^n\right)\Big|_S + d\left(\nu_H^\top \lrcorner\, X^\perp \lrcorner\, \sigma_{\mathcal{R}}^n\right)\Big|_S \\ &= d\left(X^\top \lrcorner\, \sigma_H^{n-1}\right)\Big|_S + d\left(\nu_H^\top \lrcorner \langle X, \nu \rangle \sigma_{\mathcal{R}}^{n-1}\right)\Big|_S \\ &= div_{TS} \left(X^\top |\mathcal{P}_H \nu| - \langle X, \nu \rangle \nu_H^\top\right) \sigma_{\mathcal{R}}^{n-1} \llcorner S. \end{aligned}$$

□

Remark 2.12. *The previous calculation corrects a mistake in* [52]*, where the normal component of the vector field X is omitted and this has caused the loss of some divergence-type terms in the variational formulas proved there.*

We would like to stress that the importance of the previous calculation in the development of this paper comes from the well-known Cartan's identity for the Lie derivative of a differential form; see [14], [45]. More precisely, let M be a smooth manifold, let $\omega \in \Lambda^k(T^*M)$ be a differential k-form on M and let $X \in \mathfrak{X}(TM)$ be a differentiable vector field on M, with associated flow $\phi_t : M \longrightarrow M$. We recall that the

Lie derivative of ω with respect to X, is defined by $\mathcal{L}_X\omega := \frac{d}{dt}\phi_t^*\omega\big|_{t=0}$, where $\phi_t^*\omega$ denotes the pull-back of ω by ϕ_t. Then, Cartan's identity says that

$$\mathcal{L}_X\omega = (X \lrcorner\, d\omega) + d(X \lrcorner\, \omega). \tag{7}$$

This formula is a very useful tool in proving variational formulas, not only for the case of Riemannian volume forms, for which we refer the reader to Spivak's book [65] (see Ch. 9, pp. 411-426 and 513-535), but even for more general functionals; see, for instance, [38], [36]. In Section 4, we shall apply this method to write down the 1st and 2nd variation formulas for the H-perimeter measure σ_H^{n-1}. But let us say something more about the 1st variation formula. So let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^2$. We remark that the Lie derivative of σ_H^{n-1} with respect to X can be calculated elementarily as follows. We begin with the first term in formula (7). We have

$$X \lrcorner\, d\sigma_H^{n-1} = X \lrcorner\, d(\nu_H \lrcorner\, \sigma_{\mathcal{R}}^n) = X \lrcorner \left(div\, \nu_H \sigma_{\mathcal{R}}^n\right) = \langle X, \nu\rangle\, div\, \nu_H\, \sigma_{\mathcal{R}}^{n-1}.$$

Note that $div\, \nu_H = div_H\, \nu_H = -\mathcal{H}_H$. More precisely

$$div\, \nu_H = \sum_{i=1}^{n} \langle \nabla_{X_i}\nu_H, X_i\rangle = \sum_{i=1}^{h} X_i(\nu_{H\,i}) = div_H\, \nu_H = -\mathcal{H}_H.$$

The second term in formula (7) has been already computed in Lemma 2.11. Thus, we can conclude that

$$\mathcal{L}_X\sigma_H^{n-1} = \left(-\mathcal{H}_H\,\langle X, \nu\rangle + div_{TS}\left(X^\top|\mathcal{P}_H\nu| - \langle X, \nu\rangle \nu_H^\top\right)\right)\sigma_{\mathcal{R}}^{n-1}, \tag{8}$$

at each non-characteristic point of S. We will return on this point in Section 4.

Remark 2.13. *Roughly speaking, formula* (8) *expresses the "infinitesimal" 1st variation of the measure* σ_H^{n-1}. *However, in general, in order to integrate the function* $\mathcal{L}_X\sigma_H^{n-1}$ *over any* $\mathbf{C}^2$ *hypersurface* S *with -or without- boundary we have to require that* $\mathcal{H}_H$ *be locally integrable on* S, *with respect to the Riemannian measure* $\sigma_{\mathcal{R}}^{n-1}$. *This is because, in general* $\mathcal{H}_H$ *fails to be integrable locally around the characteristic* C_S; *see* [24]. *Moreover, note that hypothesis, implies the integrability of the function* $\mathcal{L}_X\sigma_H^{n-1}$; *see Remark 4.7. If* $C_S = \emptyset$ *this condition is automatically satisfied because, in general, if* S *is of class* $\mathbf{C}^2$, *then* $\mathcal{H}_H \in \mathbf{C}(S \setminus C_S)$.

Remark 2.14 (Riemannian case)**.** *We would like to stress the analogy with the 1st variation of* $\sigma_{\mathcal{R}}^{n-1}$ *for a hypersurface* S *of class* $\mathbf{C}^i$, $i \geq 1$, *immersed in the Euclidean space* $\mathbb{R}^n$. *It is well-known that the 1st variation formula is given by* $I_S(\sigma_{\mathcal{R}}^{n-1}) = \int_S div_{TS}\, W\, \sigma_{\mathcal{R}}^{n-1}$; *see Simon's book* [63], *Ch. 2, § 9, pp. 48-53. In the* $\mathbf{C}^1$ *case, the variation vector* W *cannot be decomposed in its normal and tangential parts, hereafter denoted as* $W^\perp$, *and* $W^\top$, *respectively. Obviously, this can be done if* S *is of class* $\mathbf{C}^2$. *In this case, one has*

$$I_S(\sigma_{\mathcal{R}}^{n-1}) = \int_S div_{TS}\, W\, \sigma_{\mathcal{R}}^{n-1} = \int_S \left(\langle W^\perp, \nu\rangle\, div_{TS}\, \nu + div_{TS}\, W^\top\right)\sigma_{\mathcal{R}}^{n-1}.$$

Note that, by definition, one has $-\mathcal{H}_{\mathcal{R}} = div_{TS}\, \nu$. *Hence, we have two contributions. The first is given by* $-\int_S \mathcal{H}_{\mathcal{R}}\langle W^\perp, \nu\rangle\, \sigma_{\mathcal{R}}^{n-1}$ *and only depends on the normal component of the variation vector* W. *The second, by Stokes' formula, can be transformed in a boundary integral*[4]. *This is given by* $\int_{\partial S}\langle W^\top, \eta\rangle\, \sigma_{\mathcal{R}}^{n-2}$ *and it really depends only on the tangential component of* W.

3. Some technical preliminaries about the connection 1-forms

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and let $U \subset \mathbb{G}$ be an open set having non-empty intersection with S and such that $\mathcal{U} := U \cap S$ is non-characteristic. We start with an elementary calculation.

Lemma 3.1. *One has* $div_{TS}\, \nu_H = -\mathcal{H}_H - \langle C(\mathcal{P}_V)\nu_H, \mathcal{P}_V\nu\rangle$, *where* $C(\mathcal{P}_V) := \sum_{\alpha \in I_V} \nu_\alpha C^\alpha$.

[4] In this case, we further assume that ∂S is a $(n-2)$-dimensional submanifold of class $\mathbf{C}^1$ oriented by the outward-pointing unit normal vector η.

Proof. We have $div_{TS}\,\nu_{_H} = div\,\nu_{_H} - \langle \nabla_{\nu}\nu_{_H}, \nu\rangle$. Since $div\,\nu_{_H} = -\mathcal{H}_H$, the thesis follows from

$$\langle \nabla_{\nu}\nu_{_H}, \nu\rangle = \sum_{j\in I_H}\sum_{\alpha,\beta\in I_V} \nu_{_H\,j}\nu_{\alpha}\nu_{\beta}\langle \nabla_{X_\alpha}X_j, X_\beta\rangle = \sum_{j\in I_H}\sum_{\alpha,\beta\in I_V} \nu_{_H\,j}\nu_{\alpha}\nu_{\beta}\frac{\left(C^{\mathfrak{g}\beta}_{\ \alpha j} + C^{\mathfrak{g}\alpha}_{\ \beta j}\right)}{2} = \langle C(\mathcal{P}_V\,\nu)\nu_{_H}, \mathcal{P}_V\,\nu\rangle\,.$$

$\square$

Remark 3.2. *We have*

$$-\mathcal{H}_H = div_H\,\nu_{_H} = div_H\left(\frac{\mathcal{P}_H\,\nu}{|\mathcal{P}_H\,\nu|}\right) = \frac{div_H(\mathcal{P}_H\,\nu) - \langle grad_{_H}|\mathcal{P}_H\,\nu|, \nu_{_H}\rangle}{|\mathcal{P}_H\,\nu|}. \tag{9}$$

Since $|\mathcal{P}_H\,\nu|$ is Lipschitz continuous, it follows that $\mathcal{H}_H \in L^1_{loc}(S\,;\sigma^{n-1}_H)$, but not necessarily $L^1_{loc}(S\,;\sigma^{n-1}_{\mathcal{R}})$. Note also that the last condition follows by assuming $\frac{1}{|\mathcal{P}_H\,\nu|}\in L^1_{loc}(S\,;\sigma^{n-1}_{\mathcal{R}})$.

Lemma 3.3. *The following identities hold:*

(i) $\phi_{1i}(\tau_j) = \phi_{1j}(\tau_i) + \left\langle C_H(\varpi_{H_2})\tau_i, \tau_j\right\rangle \quad \forall\ i,j\in I_{HS}$;

(ii) $\phi_{1i}(\tau^{TS}_{\alpha}) = \tau_i(\varpi_\alpha) + \frac{1}{2}\left\langle C^{\alpha}_H\tau_1, \tau_i\right\rangle - \langle C(\varpi)\tau^{TS}_{\alpha}, \tau_i\rangle \quad \forall\ i\in I_H\ \forall\ \alpha\in I_V$;

(iii) $\phi_{i\alpha}(\tau_j) = \phi_{j\alpha}(\tau_i) + \left\langle C^{\alpha}_H\tau_i, \tau_j\right\rangle \quad \forall\ i,j\in I_H\ \forall\ \alpha\in I_V$;

(iv) $\tau^{TS}_{\alpha}(\varpi_\beta) - \tau^{TS}_{\beta}(\varpi_\alpha) = \left\langle C(\varpi)\tau^{TS}_{\beta}, \tau^{TS}_{\alpha}\right\rangle \quad \forall\ \alpha,\beta\in I_V$;

(v) $\phi_{i\alpha}(\tau_\alpha) = 0 \quad \forall\ i\in I_H\ \forall\ \alpha\in I_V$;

(vi) $\phi_{\alpha i}(\tau_i) = 0 \quad \forall\ i\in I_H\ \forall\ \alpha\in I_V$;

(vii) $\phi_{i\alpha}(\tau_j) = \frac{1}{2}\left\langle C^{\alpha}_H\tau_i, \tau_j\right\rangle \quad \forall\ i,j\in I_H\ \forall\ \alpha\in I_V$.

Proof. By direct computation using the fact that the Lie brackets of tangent vector fields along S is still tangent; for a detailed proof, see [52]. $\square$

Lemma 3.4. *The matrix of the linear operator B_H can be written out as a sum of two matrices, one symmetric and the other skew-symmetric, i.e. $B_H = S_H + A_H$, where the skew-symmetric matrix A_H is given by $A_H = \frac{1}{2}C_H(\varpi_{H_2})|_{HS}$.*

Proof. It is sufficient to apply (i) of Lemma 3.3. $\square$

Lemma 3.5. *One has* $\mathrm{Tr}\left(B^2_H\right) = \|S_H\|^2_{\mathrm{Gr}} - \|A_H\|^2_{\mathrm{Gr}} = \sum_{j,k\in I_{HS}}\phi_{1k}(\tau_j)\phi_{1j}(\tau_k)$.

Proof. We have

$$\begin{aligned}
\sum_{j,k\in I_{HS}}\phi_{1k}(\tau_j)\phi_{1j}(\tau_k) &= \sum_{j,k\in I_{HS}}\left\langle \nabla_{\tau_j}\tau_1, \tau_k\right\rangle\left\langle \nabla_{\tau_k}\tau_1, \tau_j\right\rangle\\
&= \sum_{j,k\in I_{HS}}(B_H)_{kj}(B_H)_{jk}\\
&= \mathrm{Tr}\left(B^2_H\right)\\
&= \sum_{j\in I_{HS}}\left\langle B_H\tau_j, B^{\mathrm{Tr}}_H\tau_j\right\rangle\\
&= \sum_{j\in I_{HS}}\left\langle (S_H + A_H)\,\tau_j, (S_H - A_H)\,\tau_j\right\rangle\\
&= \|S_H\|^2_{\mathrm{Gr}} - \|A_H\|^2_{\mathrm{Gr}}.
\end{aligned}$$

$\square$

Lemma 3.6. *One has* $\sum_{\alpha\in I_V}\varpi_\alpha\mathcal{D}_{HS}\left(C^{\alpha}_H\tau_1\right) = 2\|A_H\|^2_{\mathrm{Gr}} + |C_H(\varpi_{H_2})\tau_1|^2$.

Proof. We have

$$\begin{aligned}\mathcal{D}_{HS}\,(C^\alpha_H\tau_1) &= \sum_{j\in I_{HS}}\left\langle \nabla_{\tau_j}C^\alpha_H\tau_1,\tau_j\right\rangle + \langle C^\alpha_H\tau_1, C_H(\varpi_{H_2})\tau_1\rangle\\ &= -\sum_{j\in I_{HS}}\left\langle \nabla_{\tau_j}\tau_1, C^\alpha_H\tau_j\right\rangle + \langle C^\alpha_H\tau_1, C_H(\varpi_{H_2})\tau_1\rangle \quad \text{(by linearity and skew-symmetry)}\\ &= -\sum_{j\in I_{HS}}\left\langle \nabla_{\tau_j}\tau_1, C^\alpha_{HS}\tau_j\right\rangle + \langle C^\alpha_H\tau_1, C_H(\varpi_{H_2})\tau_1\rangle,\end{aligned}$$

where $C^\alpha_{HS} := C^\alpha_H|_{HS}$. Since $\left\langle \nabla_{\tau_j}\tau_1, C^\alpha_{HS}\tau_j\right\rangle = -B_H(\tau_j, C^\alpha_{HS}\tau_j)\ \forall\, j\in I_{HS}$, it follows that

$$\begin{aligned}\sum_{\alpha\in I_V}\varpi_\alpha\mathcal{D}_{HS}\,(C^\alpha_H\tau_1) &= \sum_{\alpha\in I_V}\varpi_\alpha\sum_{j\in I_{HS}} B_H(\tau_j, C^\alpha_{HS}\tau_j) + |C_H(\varpi_{H_2})\tau_1|^2\\ &= \varpi_\alpha\sum_{j\in I_{HS}} B_H(\tau_j, C_{HS}(\varpi_{H_2})\tau_j) + |C_H(\varpi_{H_2})\tau_1|^2,\end{aligned}$$

where $C_{HS}(\varpi_{H_2}) = C_H(\varpi_{H_2})|_{HS} = 2A_H$; see Lemma 3.4. Therefore

$$\begin{aligned}\sum_{\alpha\in I_V}\varpi_\alpha\mathcal{D}_{HS}\,(C^\alpha_H\tau_1) &= 2\sum_{j\in I_{HS}} B_H(\tau_j, A_H\tau_j) + |C_H(\varpi_{H_2})\tau_1|^2\\ &= 2\sum_{j\in I_{HS}}\left\langle (S_H + A_H)\tau_j, A_H\tau_j\right\rangle + |C_H(\varpi_{H_2})\tau_1|^2\\ &= 2\|A_H\|^2_{\mathrm{Gr}} + |C_H(\varpi_{H_2})\tau_1|^2,\end{aligned}$$

where we have used the elementary identity $\sum_{j\in I_{HS}}\left\langle S_H\tau_j, A_H\tau_j\right\rangle = 0$. Let us prove the last identity. For every $j\in I_{HS}$ one has

$$\begin{aligned}\left\langle S_H\tau_j, A_H\tau_j\right\rangle &= \frac{1}{4}\left\langle \left(B_H + B_H^{\mathrm{Tr}}\right)\tau_j, \left(B_H - B_H^{\mathrm{Tr}}\right)\tau_j\right\rangle\\ &= \frac{1}{4}\left(\left\langle B_H\tau_j, B_H\tau_j\right\rangle - \left\langle B_H^{\mathrm{Tr}}\tau_j, B_H^{\mathrm{Tr}}\tau_j\right\rangle\right).\end{aligned}$$

By summing over $j\in I_{HS}$ we get $\mathrm{Tr}\,(S_H(\,\cdot\,, A_H\,\cdot)) = \|B_H\|^2_{\mathrm{Gr}} - \|B_H^{\mathrm{Tr}}\|^2_{\mathrm{Gr}} = 0$. □

We now recall some identities involving the (Riemannian) curvature 2-forms Φ_{IJ} associated with the orthonormal co-frame $\underline{\phi}$ (dual of $\underline{\tau}$) which can be found in [52]. In particular, we will compute the quantity $\sum_{j\in I_{HS}}\Phi_{1j}(X,\tau_j) = \sum_{j\in I_{HS}}\left\langle \mathrm{R}(X,\tau_j)\tau_1,\tau_j\right\rangle$ for any $X\in\nu_H S$, which is nothing but the Ricci curvature for the partial HS-connection ∇^{HS}.

Lemma 3.7. *We have:*

(i) $\left\langle \mathrm{R}(\tau_i,\tau_j)\tau_h,\tau_k\right\rangle = -\frac{3}{4}\sum_{\alpha\in I_{H_2}}\left\langle C^\alpha_H\tau_i,\tau_j\right\rangle\left\langle C^\alpha_H\tau_h,\tau_k\right\rangle \qquad \forall\ i\ j,h,k\in I_H$;

(ii) $\left\langle \mathrm{R}(\tau_\beta,\tau_i)\tau_j,\tau_k\right\rangle = -\frac{1}{4}\sum_{\alpha\in I_{H_2}}\left\langle C^\alpha_H\tau_j,\tau_k\right\rangle\left\langle C^\beta\tau_\alpha,\tau_i\right\rangle \qquad \forall\ i,j,k\in I_H,\ \beta\in I_{H_3}$.

Lemma 3.8. *For every $X = X_H + X_V\in\mathfrak{X}(\mathbb{G})$, $X \pitchfork S$, one has*

$$\sum_{j\in I_{HS}}\Phi_{1j}(X,\tau_j) = -\frac{3}{4}\sum_{\alpha\in I_{H_2}}\langle C^\alpha_H\nu_H, C^\alpha_H X_H\rangle - \frac{1}{4}\sum_{\alpha\in I_{H_2}}\sum_{\beta\in I_{H_3}} x_\beta\left\langle C^\alpha_H\nu_H, C^\beta\tau_\alpha\right\rangle.$$

Proof. Using Lemma 3.7. □

Lemma 3.9. *Let $\underline{\tau} = \{\tau_1,...,\tau_n\}$ be an adapted orthonormal frame for $\mathcal{U}\subseteq S$ on U and fix $p_0\in\mathcal{U}$. Then, we can always choose $\underline{\tau}$ so that the connection 1-forms $\underline{\phi} = \{\phi_1,...,\phi_n\}$ satisfy $\phi_{ij}(p_0) = 0$ whenever $i,j\in I_{HS} = \{2,...,h\}$.*

Proof. The proof follows by using a Riemannian geodesic frame. So let $\underline{\xi} = \{\xi_1, ..., \xi_n\}$ be a o.n. frame on U adapted to $\mathcal{U} = U \cap S$ satisfying $\xi_1(p) = \nu(p)$ and such that $T_pS = \mathrm{span}_{\mathbb{R}}\{\xi_2(p), ..., \xi_n(p)\}$ for every $p \in \mathcal{U}$. Let $\underline{\varepsilon} = \{\varepsilon_1, ..., \varepsilon_n\}$ denote its dual co-frame.

Claim 3.10. *It is always possible to choose another o.n. frame $\underline{\widetilde{\xi}}$ on U adapted to $\mathcal{U}$ satisfying:*

(i) $\underline{\widetilde{\xi}}(p_0) = \underline{\xi}(p_0)$;

(ii) *Let* $\widetilde{\varepsilon}_{ij} := \left\langle \nabla\widetilde{\xi}_i, \widetilde{\xi}_j \right\rangle$ $(i, j = 1, ..., n)$ *denote the connection 1-forms of* $\underline{\widetilde{\xi}}$. *Then, one has* $\widetilde{\varepsilon}_{ij}(p_0) = 0$ *for every* $i, j = 2, ..., n$.

Clearly $\underline{\widetilde{\xi}^S} = \{\widetilde{\xi}_2, ..., \widetilde{\xi}_n\}$ is a tangent orthonormal frame for $\mathcal{U}$. The proof of this claim is standard; see, for instance, [65], pag. 517-519, eq.(17).

So let us assume that $\xi_i(p_0) = \tau_i(p_0)$ for every $i \in I_{HS}$. In particular, we have

$$\widetilde{\varepsilon}_{ij}(X_{p_0}) = \left\langle \nabla_{X_{p_0}}\widetilde{\xi}_i, \widetilde{\xi}_j \right\rangle (p_0) = 0 \qquad \forall\ i, j \in I_{HS},\ \forall\ X \in \mathfrak{X}^1(TS).$$

By extending the orthonormal frame $\{\widetilde{\xi}_2, ..., \widetilde{\xi}_h\}$ for the horizontal tangent space to a full adapted frame $\underline{\tau}$ in the sense of Definition 1.13, the thesis easily follows. $\square$

The following notion will be very useful throughout the proof of Lemma 5.5.

Definition 3.11. *Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^i$ $(i \geq 2)$. We say that a function $f : \mathbb{G} \longrightarrow \mathbb{R}$ of class $\mathbf{C}^i$ is a defining function for S if $S = \{x \in \mathbb{G} : f = 0\}$ and $\mathit{grad} f \neq 0$ for all $x \in S$. Furthermore, we say that f is a* normalized defining function for S (abbreviated as NDF) *if, and only if, $|\mathit{grad}_H f| = 1$ for all $x \in S \setminus C_S$.*

Remark 3.12. *Some remarks are in order. First, it is not difficult to see that, given a defining function f for S, then a NDF $\widetilde{f}$ for S can simply be defined by dividing f by the magnitude of its horizontal gradient $|\mathit{grad}_H f|$, i.e.*

$$\mathit{grad}\widetilde{f}(p) = \mathit{grad}\left(\frac{f}{|\mathit{grad}_H f|}\right)(p) = \frac{\mathit{grad} f}{|\mathit{grad}_H f|}(p) = \nu_H(p) + \varpi(p) \qquad \forall\ p \in S \setminus C_S.$$

Note that the NDF $\widetilde{f}$ is one order of differentiability less smooth than f. This is what happens also in the Euclidean case; see the book by Krantz and Parks [43] *and references therein. However, at least for 2-step Carnot groups, a normalized defining function of class $\mathbf{C}^i$ for every hypersurface S of class $\mathbf{C}^i$ $(i \geq 2)$, is given by the (signed) CC-distance function from S; see* [7].

We end this section with a lemma, which will be important in the sequel.

Let S be as above, let $p_0 \in S$ and assume that, locally around p_0, S is the level set of a function $f : U \subset \mathbb{G} \longrightarrow \mathbb{R}$. We easily see that, locally around p_0, $Xf = 0$ for every $X \in \mathfrak{X}(TS)$. In particular, $\tau_\alpha^{TS}(f) = 0$ for every $\alpha \in I_V$. As a consequence, by using an adapted frame $\underline{\tau}$, one has $\tau_\alpha(f) = \varpi_\alpha \tau_1(f)$ for every $\alpha \in I_V$. A normal vector along S in a neighborhood of p_0 is given by $\mathcal{N} := \tau_1(f)\tau_1 + \sum_{\alpha \in I_V} \tau_\alpha(f)\tau_\alpha$ and we have $\nu = \frac{\mathcal{N}}{|\mathcal{N}|}$.

Lemma 3.13. *The following identities hold:*

(i) $\phi_{1j}(\tau_1) = \frac{\tau_j(\tau_1(f))}{\tau_1(f)} - \left\langle C_H(\varpi_{H_2})\tau_1, \tau_j \right\rangle \quad \forall\ j \in I_{HS}$;

(ii) $\phi_{1j}(\tau_\alpha) = \frac{1}{2}\left\langle C_H^\alpha \tau_1, \tau_j \right\rangle - \left\langle C(\varpi)\tau_\alpha, \tau_j \right\rangle + \frac{\tau_j(\tau_\alpha(f))}{\tau_1(f)} \quad \forall\ j \in I_{HS}\ \forall\ \alpha \in I_V$.

Proof. We have

$$[\tau_1, \tau_j] = \left\langle [\tau_1, \tau_j], \tau_1 \right\rangle \tau_1 + \sum_{k \in I_{HS}} \left\langle [\tau_1, \tau_j], \tau_k \right\rangle \tau_k + \sum_{\alpha \in I_V} \left\langle [\tau_1, \tau_j], \tau_\alpha \right\rangle \tau_\alpha.$$

Therefore

$$[\tau_1, \tau_j](f) = -\tau_j(\tau_1(f)) = \left\langle [\tau_1, \tau_j], \tau_1 \right\rangle \tau_1(f) + \sum_{\alpha\in I_V} \left\langle [\tau_1, \tau_j], \tau_\alpha \right\rangle \tau_\alpha(f)$$

and this implies

$$C^1_{1j} = \phi_{1j}(\tau_1) = \left\langle [\tau_1, \tau_j], \tau_1 \right\rangle = \frac{\tau_j(\tau_1(f))}{\tau_1(f)} - \sum_{\alpha\in I_V} \frac{\tau_\alpha(f)}{\tau_1(f)} \left\langle C^\alpha_H \tau_1, \tau_j \right\rangle, \tag{10}$$

where we have used the identity $C^\alpha_{1j} = -\left\langle C^\alpha_H \tau_1, \tau_j \right\rangle$.

This proves (i). Analogously, in order to prove (ii), we have

$$[\tau_\alpha, \tau_j] = \left\langle [\tau_\alpha, \tau_j], \tau_1 \right\rangle \tau_1 + \sum_{k\in I_{HS}} \left\langle [\tau_\alpha, \tau_j], \tau_k \right\rangle \tau_k + \sum_{\beta\in I_V} \left\langle [\tau_\alpha, \tau_j], \tau_\beta \right\rangle \tau_\beta,$$

from which we get

$$[\tau_\alpha, \tau_j](f) = -\tau_j(\tau_\alpha(f)) = \left\langle [\tau_\alpha, \tau_j], \tau_1 \right\rangle \tau_1(f) + \sum_{\beta\in I_V} \left\langle [\tau_\alpha, \tau_j], \tau_\beta \right\rangle \tau_\beta(f).$$

Thus

$$-\frac{\tau_j(\tau_\alpha(f))}{\tau_1(f)} = -\phi_{1j}(\tau_\alpha) + \phi_{1\alpha}(\tau_j) + \sum_{\beta\in I_V} \varpi_\beta C^1_{\alpha j},$$

where we have used the identity $C^1_{\alpha j} = \left\langle \nabla_{\tau_\alpha} \tau_j, \tau_1 \right\rangle - \left\langle \nabla_{\tau_j} \tau_\alpha, \tau_1 \right\rangle$. Finally, since $\phi_{1\alpha}(\tau_j) = \frac{1}{2} \left\langle C^\alpha_H \tau_1, \tau_j \right\rangle$ (see (vii) of Lemma 3.3), using $C^\beta_{\alpha j} = -\left\langle C^\beta \tau_\alpha, \tau_j \right\rangle$ it follows that

$$\phi_{1j}(\tau_\alpha) = \frac{1}{2} \left\langle C^\alpha_H \tau_1, \tau_j \right\rangle - \sum_{\beta\in I_V} \varpi_\beta \left\langle C^\beta \tau_\alpha, \tau_j \right\rangle + \frac{\tau_j(\tau_\alpha(f))}{\tau_1(f)} \tag{11}$$

and the thesis easily follows. □

4. Variational formulas for the H-perimeter σ^{n-1}_H

Below we will obtain the 1st and 2nd variation formulas for the H-perimeter measure σ^{n-1}_H on any "smooth" hypersurface $S \subset \mathbb{G}$. More precisely, we shall assume that S is of class $\mathbf{C}^2$, for the 1st variation, and that S is of class $\mathbf{C}^3$ for the 2nd variation. In particular, we stress that our formulas allow us to move the characteristic set C_S of S.

We stress that, in the case of the first Heisenberg group $\mathbb{H}^1$, a 1st variation formula for characteristic surfaces of class $\mathbf{C}^2$ was obtained by Ritoré and Rosales in [62]. Furthermore, Hurtado, Ritoré and Rosales [41] have proved a formula for the 2nd variation of σ^{n-1}_H that is very similar to that stated in Theorem 4.13 below; see also the unpublished preprint [40], where similar results are stated in a general sub-Riemannian setting.

Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^i$ ($i = 2, 3$), let $U \subset \mathbb{G}$ be a relatively compact open set having non-empty intersection with S and set $\mathcal{U} := U \cap S$. [The following calculations will be made for $\mathcal{U}$, which is a bounded open subset of S; in particular, we will often assume $\mathbf{C}^1$-regularity of $\partial\mathcal{U}$. If S is a compact hypersurface with boundary, the formulas obtained in the sequel will hold for S.]

Definition 4.1. *Let $\imath : \mathcal{U} \to \mathbb{G}$ denote the inclusion of $\mathcal{U} \subset S$ in $\mathbb{G}$ and let $\vartheta :]-\epsilon, \epsilon[\times \mathcal{U} \to \mathbb{G}$ be a map of class $\mathbf{C}^i$, $i = 2, 3$. We say that ϑ is a* variation *of $\imath$ if we have:*

(i) *every $\vartheta_t := \vartheta(t, \cdot) : \mathcal{U} \to \mathbb{G}$ is an immersion;*

(ii) $\vartheta_0 = \imath$.

Moreover, we say that ϑ keeps the boundary $\partial\mathcal{U}$ fixed *if:*

(iii) $\vartheta_t|_{\partial\mathcal{U}} = \imath|_{\partial\mathcal{U}}$ *for every $t \in]-\epsilon, \epsilon[$.*

The variation vector *of ϑ (i.e. its "initial velocity") is defined by* $W := \frac{\partial\vartheta}{\partial t}\Big|_{t=0} = \vartheta_* \frac{\partial}{\partial t}\Big|_{t=0}$.

We shall set $\widetilde{W} := \frac{\partial\vartheta}{\partial t} = \vartheta_*\frac{\partial}{\partial t}$ and assume that $\widetilde{W}$ is defined in a neighborhood of $\mathrm{Im}(\vartheta)$. For any "time" $t\in]-\epsilon,\epsilon[$, let ν^t be the unit normal vector along $\mathcal{U}_t := \vartheta_t(\mathcal{U})$ and let $(\sigma^{n-1}_{\mathcal{R}})_t$ be the Riemannian measure on $\mathcal{U}_t$. We assume that $f : U \longrightarrow \mathbb{R}$ is a local equation for the hypersurface S near $p_0\in S$ and that $f_t :]-\epsilon,\epsilon[\times U \longrightarrow \mathbb{R}$ is a family of $\mathbf{C}^i$ functions ($i = 2,3$) satisfying $f_0 = f$ and $f_t(\vartheta_t(x)) = t$ for every $t\in]-\epsilon,\epsilon[$. In other words, the hypersurfaces $\mathcal{U}_t$ are level sets of a defining function f_t and one has $\left\langle\nabla f_t, \widetilde{W}\right\rangle = 1$. Choose an orthonormal frame $\underline{\tau}$ on $U\subset\mathbb{G}$ satisfying:

$$\tau_1|_{\mathcal{U}_t} = \nu^t_H;\qquad HT_p\mathcal{U}_t = \mathrm{span}\{(\tau_2)_p,...,(\tau_h)_p\}\quad\forall\ p\in\mathcal{U}_t;\qquad \tau_\alpha = X_\alpha \tag{12}$$

for every $t\in]-\epsilon,\epsilon[$. Furthermore, let $\underline{\phi} := \{\phi_1,...,\phi_n\}$ be the dual co-frame of $\underline{\tau}$ (i.e. $\phi_i(\tau_j) = \delta_i^j$ for all $i,j = 1,...,n$). So, we have $\tau^{TS}_\alpha f_t = 0$; see Definition 1.13. This implies $\tau_\alpha(f_t) = \varpi^t_\alpha\tau_1(f_t)$, where $\varpi^t_\alpha := \frac{\nu^t_\alpha}{|\mathcal{P}_H\nu^t|}$. Moreover, since $\left\langle\nabla f_t, \widetilde{W}\right\rangle = 1$, we have $\widetilde{w}_1\tau_1(f_t) + \sum_{\alpha\in I_V}\widetilde{w}_\alpha\tau_\alpha(f_t) = 1$, where $\widetilde{w}_1 = \left\langle\widetilde{W},\tau_1\right\rangle$ and $\widetilde{w}_\alpha = \left\langle\widetilde{W},\tau_\alpha\right\rangle$. Therefore

$$\tau_1(f_t)\left(\widetilde{w}_1 + \sum_{\alpha\in I_V}\widetilde{w}_\alpha{\varpi^t}_\alpha\right) = 1.$$

Setting $w_t = \frac{\langle\widetilde{W},\nu^t\rangle}{|\mathcal{P}_H\nu^t|}$ it follows that $\tau_1(f_t) = \dfrac{1}{w_t}$ and $\tau_\alpha(f_t) = \dfrac{\varpi^t_\alpha}{w_t}$.

The following technical result will be used in the proof of the 2nd variation of σ^{n-1}_H.

Lemma 4.2. *Under the previous assumptions, we have:*

(i) $\mathcal{P}_{HS_t}(\nabla_{\tau_1}\tau_1) = -\left(\frac{grad_{HS_t}w_t}{w_t} + C_H(\varpi^t_{H_2})\tau_1\right)$;

(ii) $\mathcal{P}_{HS_t}(\nabla_{\tau_\alpha}\tau_1) = \frac{1}{2}C^\alpha_H\tau_1 - C(\varpi^t)\tau_\alpha + grad_{HS_t}\varpi^t_\alpha - \varpi^t_\alpha\frac{grad_{HS_t}w_t}{w_t}\qquad\forall\ \alpha\in I_V$.

Proof. By applying (i) of Lemma 3.13 we get that $\phi_{1j}(\tau_1) = -\frac{\tau_j(w_t)}{w_t} - \left\langle C_H(\varpi^t_{H_2})\tau_1,\tau_j\right\rangle$. Furthermore, (ii) of Lemma 3.13 implies

$$\phi_{1j}(\tau_\alpha) = \frac{1}{2}\left\langle C^\alpha_H\tau_1,\tau_j\right\rangle - \left\langle C(\varpi^t)\tau_\alpha,\tau_j\right\rangle + \tau_j(\varpi^t_\alpha) - \varpi^t_\alpha\frac{\tau_j(w_t)}{w_t}\qquad\forall\ \alpha\in I_V. \tag{13}$$

This achieves the proof. □

General remarks. In order to discuss the variational formulas of σ^{n-1}_H, let us set

$$(\sigma^{n-1}_H)_t \llcorner\ \mathcal{U}_t = (\tau_1 \lrcorner\ \phi_1\wedge...\wedge\phi_n)|_{\mathcal{U}_t} = (\phi_2\wedge...\wedge\phi_n)|_{\mathcal{U}_t}.$$

We also set $\Gamma(t) := \vartheta^*_t(\phi_2\wedge...\wedge\phi_n)$. Note that $\Gamma :]-\epsilon,\epsilon[\times\mathcal{U} \longrightarrow \Lambda^{n-1}(T^*\mathcal{U})$ defines a 1-parameter family of differential $(n-1)$-forms on $\mathcal{U}$.

Remark 4.3. *By definition, the* 1st and 2nd variation formulas of σ^{n-1}_H along $\mathcal{U}$ *are given by*

$$I_{\mathcal{U}}(\sigma^{n-1}_H) := \frac{d}{dt}\left(\int_{\mathcal{U}}\Gamma(t)\right)\Bigg|_{t=0},\qquad II_{\mathcal{U}}(\sigma^{n-1}_H) := \frac{d^2}{dt^2}\left(\int_{\mathcal{U}}\Gamma(t)\right)\Bigg|_{t=0}. \tag{14}$$

So we have a natural question: is it possible to bring the -time- derivatives inside the integral sign? Note that the answer is "yes"if we assume that $\overline{\mathcal{U}}$ is non-characteristic. Indeed, in such a case it is not difficult[5] *to show that there exists $\epsilon > 0$ such that the 1-parameter family $\Gamma(\cdot)$ of differential $(n-1)$-forms on $\mathcal{U}$ is of class $\mathbf{C}^{i-1}$ on $]-\epsilon,\epsilon[\times\mathcal{U}$. This allows us to estimate, uniformly in time, both differential $(n-1)$-forms $\dot{\Gamma}(t)$ and $\ddot{\Gamma}(t)$. However, when $\mathcal{U}$ has a non-empty characteristic set, i.e. $C_{\mathcal{U}}\neq\emptyset$, the answer is "no", in general. We return on this point later in this section; see Remark 2.14.*

[5]Actually, since $grad_H f_t\neq 0$ at $t = 0$, there must exist $\epsilon > 0$ such that $grad_H f_t\neq 0$ for all $t\in]-\epsilon,\epsilon[$ and hence $\nu^t_H = \frac{grad_H f_t}{|grad_H f_t|}$, which is the unit H-normal along $\mathcal{U}_t = \vartheta_t(\mathcal{U})$, turns out to be of class $\mathbf{C}^{i-1}$, $i = 2,3$. This implies that $(\sigma^{n-1}_H)_t$ is $\mathbf{C}^{i-1}$-smooth. Therefore $\Gamma(t) = \vartheta^*_t(\sigma^{n-1}_H)_t$ is $\mathbf{C}^{i-1}$-smooth.

Warning 4.4. *Preliminarily, we need the following assumptions:*

(A_1) *if $\mathcal{U}$ is of class $\mathbf{C}^2$ there exists a locally integrable differential $(n-1)$-form $\Phi_1 \in \Lambda^{n-1}(T^*\mathcal{U})$, such that*
$$|\dot{\Gamma}(t)(\mathrm{t}_1, ..., \mathrm{t}_{n-1})| \leq |\Phi_1(\mathrm{t}_1, ..., \mathrm{t}_{n-1})|$$
for every orthonormal basis $\underline{\mathrm{t}} = \{\mathrm{t}_1, ..., \mathrm{t}_{n-1}\}$ of $T\mathcal{U}$.

(A_2) *if $\mathcal{U}$ is of class $\mathbf{C}^3$ there exist locally integrable differential $(n-1)$-forms $\Phi_1, \Phi_2 \in \Lambda^{n-1}(T^*\mathcal{U})$, such that*
$$\begin{aligned} |\dot{\Gamma}(t)(\mathrm{t}_1, ..., \mathrm{t}_{n-1})| &\leq |\Phi_1(\mathrm{t}_1, ..., \mathrm{t}_{n-1})| \\ |\ddot{\Gamma}(t)(\mathrm{t}_1, ..., \mathrm{t}_{n-1})| &\leq |\Phi_2(\mathrm{t}_1, ..., \mathrm{t}_{n-1})| \end{aligned}$$
for every orthonormal basis $\underline{\mathrm{t}} = \{\mathrm{t}_1, ..., \mathrm{t}_{n-1}\}$ of $T\mathcal{U}$.

1st variation. We first note that
$$\int_{\mathcal{U}} \Gamma(t) = \int_{\mathcal{U}} \vartheta_t^*(\sigma_H^{n-1})_t = \int_{\mathcal{U}} |\mathcal{P}_{H_t}\nu^t|\,\mathcal{J}ac\,\vartheta_t\,\sigma_{\mathcal{R}}^{n-1},$$
where $\mathcal{J}ac\,\vartheta_t$ denotes the usual Jacobian of the map ϑ_t; see [63], Ch. 2, § 8, pp. 46-48. Indeed, by definition, we have $(\sigma_H^{n-1})_t = |\mathcal{P}_{H_t}\nu^t|(\sigma_{\mathcal{R}}^{n-1})_t$ and hence the previous formula follows from the well-known Area formula of Federer; see [26] or [63]. Let us set $f :]-\epsilon, \epsilon[\times\mathcal{U} \longrightarrow \mathbb{R}$,

$$f(t, x) := |\mathcal{P}_{H_t}\nu^t(x)|\,\mathcal{J}ac\,\vartheta_t(x). \tag{15}$$

In this case, we also set $C_{\mathcal{U}} := \{x \in \mathcal{U} : |\mathcal{P}_{H_t}\nu^t(x)| = 0\}$. With this notation, our original question can be solved by applying to f the Theorem of Differentiation under the integral; see [42], Corollary 1.2.2, p.124. More precisely, let us compute

$$\begin{aligned} \frac{df}{dt} &= \frac{d\,|\mathcal{P}_{H_t}\nu^t|}{dt}\mathcal{J}ac\,\vartheta_t + |\mathcal{P}_{H_t}\nu^t|\frac{d\,\mathcal{J}ac\,\vartheta_t}{dt} \\ &= \left\langle \widetilde{W}, grad\,|\mathcal{P}_{H_t}\nu^t|\right\rangle \mathcal{J}ac\,\vartheta_t + |\mathcal{P}_{H_t}\nu^t|\frac{d\,\mathcal{J}ac\,\vartheta_t}{dt} \\ &= \left(\left\langle \widetilde{W}^{\perp}, grad\,|\mathcal{P}_{H_t}\nu^t|\right\rangle + \left\langle \widetilde{W}^{\top}, grad\,|\mathcal{P}_{H_t}\nu^t|\right\rangle + |\mathcal{P}_{H_t}\nu^t| div_{T\mathcal{U}_t}\widetilde{W}\right)\mathcal{J}ac\,\vartheta_t \\ &= \left(\left\langle \widetilde{W}^{\perp}, grad\,|\mathcal{P}_{H_t}\nu^t|\right\rangle + div_{T\mathcal{U}_t}\left(\widetilde{W}|\mathcal{P}_{H_t}\nu^t|\right)\right)\mathcal{J}ac\,\vartheta_t, \end{aligned} \tag{16}$$

where we have used the very definition of tangential divergence and the well-known calculation of $\frac{d\,\mathcal{J}ac\,\vartheta_t}{dt}$, which can be found in Chavel's book [15]; see Ch.2, p.34. Now since $|\mathcal{P}_{H_t}\nu^t|$ is a Lipschitz continuous function, it follows that $\frac{df}{dt}$ is bounded on $\mathcal{U}\setminus C_{\mathcal{U}}$ and so lies to $L^1_{loc}(\mathcal{U}; \sigma_{\mathcal{R}}^{n-1})$. Therefore, we can pass the time-derivative through the integral sign. This shows that: *condition* (A_1) *in Warning 4.4 is always satisfied.* In particular, we have proved the following 1st variation formula:

$$I_{\mathcal{U}}(\sigma_H^{n-1}) = \int_{\mathcal{U}} \dot{\Gamma}(0) = \int_{\mathcal{U}} \left(\left\langle W^{\perp}, grad\,|\mathcal{P}_H\nu_H|\right\rangle + div_{T\mathcal{U}}\left(W|\mathcal{P}_H\nu_H|\right)\right)\sigma_{\mathcal{R}}^{n-1}. \tag{17}$$

It follows from definitions that $\frac{df}{dt}$ can be regarded in terms of a Lie derivative of the differential $(n-1)$-form $(\sigma_H^{n-1})_t$ with respect to the variation vector $\widetilde{W}$. More precisely, we have

$$\frac{df}{dt} = \vartheta_t^*\mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t. \tag{18}$$

Strictly speaking, these calculations are valid at each non-characteristic point of $\mathcal{U}_t$, for any $t \in]-\epsilon, \epsilon[$.

Remark 4.5. *Note that formula* (18) *can be proved exactly as in Spivak's book* [65], *Ch. 9, p. 420. As already mentioned at the end of Section 2, this fact allows us to use some standard tools in Differential Geometry such as the Cartan's magic formula. In this way, another expression for the integrand $\dot{\Gamma}(0)$ can easily be derived. Actually this has been already done; see formula* (8). *Below we will prove this in another way. Nevertheless, we have to stress that this new expression it is not necessarily in L^1_{loc} with* *respect to the Riemannian measure $\sigma_{\mathcal{R}}^{n-1}$. For this reason we will need a further integrability condition on the horizontal mean curvature $\mathcal{H}_H$; see also Remark 2.13.*

More precisely, we have

$$\dot{\Gamma}(0) = \imath^*\left(\mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t\right) = \imath^*\left(\mathcal{L}_{\widetilde{W}}(\phi_2 \wedge ... \wedge \phi_n)\right).$$

By Cartan's formula

$$\mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t = \widetilde{W} \lrcorner\, d\,(\sigma_H^{n-1})_t + d\,(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t)$$

and hence

$$\dot{\Gamma}(0) = \imath^*\left(\widetilde{W} \lrcorner\, d\,(\sigma_H^{n-1})_t + d\left(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\right)\right). \tag{19}$$

By applying the 1st structure equation of the co-frame $\underline{\phi}$ (see formula (4)) we have

$$d\,(\sigma_H^{n-1})_t = \sum_{j=2}^{n}(-1)^j \phi_2 \wedge ... \wedge d\,\phi_j \wedge ... \wedge \phi_n = \sum_{j\in I_{HS}} \phi_{1j}(\tau_j)\,\phi_1 \wedge ... \wedge \phi_n = -(\mathcal{H}_H)_t(\sigma_{\mathcal{R}}^n)_t,$$

where have set $(\mathcal{H}_H)_t := -\sum_{j\in I_{HS}} \phi_{1j}(\tau_j) = \sum_{j\in I_{HS}} \left\langle \nabla^H_{\tau_j}\tau_j, \nu^t_H \right\rangle$, to denote the horizontal mean curvature of $\mathcal{U}_t$. Note also that we have used (v) in Lemma 3.3.

The calculation of the second term has been discussed in detail in Section 2; see Lemma 2.11. More precisely, we have

$$d\left(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t\right) = div_{T\mathcal{U}_t}\left(\widetilde{W}^\top |\mathcal{P}_{H_t}\nu^t| - \langle \widetilde{W}, \nu^t\rangle {\nu^t_H}^\top\right)(\sigma_{\mathcal{R}}^{n-1})_t.$$

Therefore, under the previous assumptions, we have proved that

$$\mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t = \left(-(\mathcal{H}_H)_t\langle \widetilde{W}, \nu^t\rangle + div_{T\mathcal{U}_t}\left(\widetilde{W}^\top |\mathcal{P}_{H_t}\nu^t| - \langle \widetilde{W}, \nu^t\rangle {\nu^t_H}^\top\right)\right)(\sigma_{\mathcal{R}}^{n-1})_t. \tag{20}$$

Finally, the desired formula follows by setting $t = 0$; see formula (8). We have the following:

Theorem 4.6 (1st variation of σ_H^{n-1}). *Let $S \subset \mathbb{G}$ be a compact $\mathbf{C}^2$-smooth hypersurface with, or without, boundary and let $\vartheta :]-\epsilon, \epsilon[\times S \to \mathbb{G}$ be a $\mathbf{C}^2$ variation of S. Let $W = \frac{d\vartheta_t}{dt}\big|_{t=0}$ be the variation vector field and let $W^\perp$ and $W^\top$ be the normal and tangential components of W along S, respectively. Then*

$$I_S(\sigma_H^{n-1}) = \int_S \left(\left\langle W^\perp, grad\,|\mathcal{P}_H\nu_H|\right\rangle + div_{TS}\,(W|\mathcal{P}_H\nu_H|)\right)\sigma_{\mathcal{R}}^{n-1}. \tag{21}$$

Set $w := \frac{\langle W^\perp, \nu\rangle}{|\mathcal{P}_H\nu|}$. If $\mathcal{H}_H \in L^1_{loc}(S;\sigma_{\mathcal{R}}^{n-1})$, then

$$\begin{aligned} I_S(W, \sigma_H^{n-1}) &= \int_S -\mathcal{H}_H\, w\,\sigma_H^{n-1} + \int_S div_{TS}\left(W^\top|\mathcal{P}_H\nu| - \langle W, \nu\rangle \nu_H^\top\right)\sigma_{\mathcal{R}}^{n-1} && (22)\\ &= \int_S \left(-\mathcal{H}_H\langle W^\perp, \nu\rangle + div_{TS}\left(W^\top|\mathcal{P}_H\nu| - \langle W^\perp, \nu\rangle\nu_H^\top\right)\right)\sigma_{\mathcal{R}}^{n-1}. && (23)\end{aligned}$$

Proof. Formula (21) is nothing but formula (17). Furthermore, let us set $t = 0$ in formula (20). If $\mathcal{H}_H \in L^1_{loc}(S;\sigma_{\mathcal{R}}^{n-1})$, then we can integrate this formula over S. Indeed, under such an assumption, all terms in the formula above turn out to be in $L^1(S;\sigma_{\mathcal{R}}^{n-1})$. In this case, we have

$$I_S(\sigma_H^{n-1}) = \int_S \dot{\Gamma}(0) = \int_S \mathcal{L}_{\widetilde{W}}(\sigma_H^{n-1})_t\big|_{t=0} = \int_S \left(-\mathcal{H}_H\langle W, \nu\rangle + div_{TS}\left(W^\top|\mathcal{P}_H\nu| - \langle W, \nu\rangle\nu_H^\top\right)\right)\sigma_{\mathcal{R}}^{n-1}.$$

□

Remark 4.7. *The divergence terms in the previous formulas* (22) *and* (23) *require a short comment. For what concerns the term $div_{TS}\,(W^\top|\mathcal{P}_H\nu|)$, note that $W^\top \in \mathfrak{X}^1(TS) = \mathbf{C}^1(S, TS)$ and that $|\mathcal{P}_H\nu|$ is Lipschitz continuous. Thus, the first divergence-type term can be integrated over all of S. Moreover, if $\mathcal{H}_H \in L^1_{loc}(S;\sigma_{\mathcal{R}}^{n-1})$, the second term $div_{TS}\left(\langle W, \nu\rangle\nu_H^\top\right)$ belongs to $L^1(S;\sigma_{\mathcal{R}}^{n-1})$. In fact, one has*

$$div_{TS}\left(\langle W, \nu\rangle\nu_H^\top\right) = div_{TS}\,(\langle W, \nu\rangle\,(\nu_H - |\mathcal{P}_H\nu|\nu))$$

and the claim easily follows by using Lemma 3.1.

Corollary 4.8. *Let the assumptions of Theorem 4.6 hold. Let ∂S be of class $\mathbf{C}^1$ and let η be the outward-pointing unit normal along ∂S. If $C_S \neq \emptyset$, we shall also assume that there exists a family $\{\mathcal{U}_\delta\}_{\delta>0}$ of open subsets of S such that:*

(i) $C_S \Subset \mathcal{U}_\delta$,
(ii) $\sigma_{\mathcal{R}}^{n-1}(\mathcal{U}_\delta) \longrightarrow 0$ *as long as* $\delta \to 0$,
(iii) $\partial\mathcal{U}_\delta$ *is of class* $\mathbf{C}^1$ *and* $\sigma_{\mathcal{R}}^{n-2}(\partial\mathcal{U}_\delta) \longrightarrow 0$ *as long as* $\delta \to 0$.

Then, the vector field $Y := W^\top|\mathcal{P}_H\nu| - \langle W^\perp, \nu\rangle \nu_H^\top$ is admissible (for the Riemannian divergence formula); see Definition 2.6. Furthermore, we have

$$I_S(W, \sigma_H^{n-1}) = \int_S -\mathcal{H}_H w\, \sigma_H^{n-1} + \int_{\partial S} \left\langle \left(W^\top|\mathcal{P}_H\nu| - \langle W, \nu\rangle \nu_H^\top\right), \eta\right\rangle \sigma_{\mathcal{R}}^{n-2}. \tag{24}$$

Proof. We just have to prove the first statement. We start from formula (23). We have

$$\begin{aligned}\int_S div_{TS} \underbrace{\left(W^\top|\mathcal{P}_H\nu| - \langle W, \nu\rangle \nu_H^\top\right)}_{=Y} \sigma_{\mathcal{R}}^{n-1} &= \int_{(S\setminus\mathcal{U}_\delta)\cup\mathcal{U}_\delta} div_{TS}\, Y\sigma_{\mathcal{R}}^{n-1}\\ &= \underbrace{\int_{S\setminus\mathcal{U}_\delta} div_{TS}\, Y\sigma_{\mathcal{R}}^{n-1}}_{=A} + \underbrace{\int_{\mathcal{U}_\delta} div_{TS}\, Y\sigma_{\mathcal{R}}^{n-1}}_{=B}.\end{aligned}$$

Under our current assumptions, we have that $B \longrightarrow 0$ as long as $\delta \to 0$. Furthermore, by applying Stokes' formula, we get that

$$A = \int_{\partial S} \langle Y, \eta\rangle\, \sigma_{\mathcal{R}}^{n-2} - \int_{\partial\mathcal{U}_\delta} \langle Y, \eta^+\rangle\, \sigma_{\mathcal{R}}^{n-2},$$

where η^+ denotes outward-pointing unit normal along $\partial\mathcal{U}_\delta$. Since $Y \in \mathbf{C}^1(S \setminus C_S)$, it follows that $Y|_{\partial\mathcal{U}_\delta}$ is bounded. The thesis follows from (ii). □

2nd variation. We will regard this proof as a continuation of the proof of the 1st variation formula. From now on, we assume $\mathcal{U}$ and S to be of class $\mathbf{C}^3$. Moreover, the boundary $\partial\mathcal{U}$ (or, ∂S when S is compact) is assumed to be of class $\mathbf{C}^1$. We also recall that, for the 2nd variation formula, the variation ϑ is assumed to be of class $\mathbf{C}^3$ on $]-\epsilon, \epsilon[\times\mathcal{U}$.

First, let us compute the second time-derivative of the function $f(t, x)$; see (15). To this end we begin with formula (16). We have

$$\begin{aligned}\frac{d^2 f}{dt^2} &= \frac{d}{dt}\left[\frac{d\,|\mathcal{P}_{H_t}\nu^t|}{dt}\mathcal{J}ac\,\vartheta_t + |\mathcal{P}_{H_t}\nu^t|\frac{d\,\mathcal{J}ac\,\vartheta_t}{dt}\right]\\ &= \frac{d^2\,|\mathcal{P}_{H_t}\nu^t|}{dt^2}\mathcal{J}ac\,\vartheta_t + 2\frac{d\,|\mathcal{P}_{H_t}\nu^t|}{dt}\frac{d\,\mathcal{J}ac\,\vartheta_t}{dt} + |\mathcal{P}_{H_t}\nu^t|\frac{d^2\,\mathcal{J}ac\,\vartheta_t}{dt^2}.\end{aligned}$$

At a first glance, it is clear that only the first term is not bounded near the characteristic set $C_{\mathcal{U}}$. More precisely, it is elementary to see that

$$\frac{d^2\,|\mathcal{P}_{H_t}\nu^t|}{dt^2} = \frac{\left|\frac{d\mathcal{P}_{H_t}\nu^t}{dt}\right|^2 - \left\langle\frac{d\mathcal{P}_{H_t}\nu^t}{dt}, \nu_H^t\right\rangle^2}{|\mathcal{P}_{H_t}\nu^t|} + \left\langle\frac{d^2\,\mathcal{P}_{H_t}\nu^t}{dt^2}, \nu_H^t\right\rangle.$$

This shows that, in order to differentiate under the integral sign, we need the following further hypothesis:

(A_3) *for every $t \in]-\epsilon, \epsilon[$ one has* $\frac{1}{|\mathcal{P}_{H_t}\nu^t|} \in L^1_{loc}\left(\mathcal{U}_t; (\sigma_{\mathcal{R}}^{n-1})_t\right)$.

Remark 4.9. *Using (A_3) it is not difficult to show the validity of (A_2) in Warning 4.4. Note that unlike the 1st variation, the 2nd variation cannot be computed without the previous assumption, if we allow the hypersurface to have characteristic points.*

Hereafter, we will continue our proof of the 2nd variation of σ_H^{n-1} with the calculation of $\ddot{\Gamma}(t)$ at a fixed non-characteristic point $p_0 \in \mathcal{U} \setminus C_{\mathcal{U}}$. To this end, we start from the following formula:

$$\ddot{\Gamma}(t) = \vartheta_t^* \left(\mathcal{L}_{\widetilde{W}} \left(\widetilde{W} \lrcorner\, d(\sigma_H^{n-1})_t \right) + \mathcal{L}_{\widetilde{W}}\, d \left(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t \right) \right). \tag{25}$$

In other words, as already said, the 2nd time-derivative of $\Gamma(t)$ can still be computed as a Lie derivative. Moreover, since $d \circ \mathcal{L} = \mathcal{L} \circ d$, we have

$$\ddot{\Gamma}(t) \;=\; \vartheta_t^* \left(\underbrace{\mathcal{L}_{\widetilde{W}} \left(\widetilde{W} \lrcorner\, d(\sigma_H^{n-1})_t \right)}_{=:A} + d\, \underbrace{\mathcal{L}_{\widetilde{W}} \left(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t \right)}_{=:B} \right). \tag{26}$$

The calculation of $A = \mathcal{L}_{\widetilde{W}} \left(\widetilde{W} \lrcorner\, d(\sigma_H^{n-1})_t \right)$ is the "hard" part of the 2nd variation formula and will be done below. So let us preliminarily consider the quantity $B = \mathcal{L}_{\widetilde{W}} \left(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t \right)$. In this calculation we will use the following general identity for Lie derivatives:

$$\mathcal{L}_Z (Y \lrcorner\, \omega) = [Z, Y] \lrcorner\, \omega + Y \lrcorner\, \mathcal{L}_Z \omega; \tag{27}$$

see [65], Ch. 9, p. 515. We have

$$\begin{aligned}
B &= \mathcal{L}_{\widetilde{W}} \left(\widetilde{W} \lrcorner (\sigma_H^{n-1})_t \right) \\
&= \mathcal{L}_{\widetilde{W}} \left(\underbrace{\left(\widetilde{W}^{\top} |\mathcal{P}_{H_t} \nu^t| - \langle \widetilde{W}, \nu^t \rangle {\nu_H^t}^{\top} \right)}_{=:\widetilde{Y}} \lrcorner (\sigma_{\mathcal{R}}^{n-1})_t \right) \qquad \text{(by Lemma 2.11)} \\
&= [\widetilde{W}, \widetilde{Y}] \lrcorner (\sigma_{\mathcal{R}}^{n-1})_t + \widetilde{Y} \lrcorner\, \mathcal{L}_{\widetilde{W}} (\sigma_{\mathcal{R}}^{n-1})_t \qquad \text{(by 27)} \\
&= [\widetilde{W}, \widetilde{Y}]^{\top} \lrcorner (\sigma_{\mathcal{R}}^{n-1})_t + \widetilde{Y} \lrcorner \underbrace{\left(-\langle \widetilde{W}, \nu^t \rangle (\mathcal{H}_{\mathcal{R}})_t + div_{TU_t} \left(\widetilde{W}^{\top} \right) \right)}_{=:g_t} (\sigma_{\mathcal{R}}^{n-1})_t \qquad \text{(by the 1st variation of } (\sigma_{\mathcal{R}}^{n-1})_t) \\
&= \left([\widetilde{W}, \widetilde{Y}]^{\top} + g_t \widetilde{Y} \right) \lrcorner (\sigma_{\mathcal{R}}^{n-1})_t.
\end{aligned}$$

Therefore, the second term in formula (26), i.e. dB, is given by

$$dB = d \left\{ \left([\widetilde{W}, \widetilde{Y}]^{\top} + g_t \widetilde{Y} \right) \lrcorner (\sigma_{\mathcal{R}}^{n-1})_t \right\} = div_{TU_t} \left([\widetilde{W}, \widetilde{Y}]^{\top} + g_t \widetilde{Y} \right) (\sigma_{\mathcal{R}}^{n-1})_t. \tag{28}$$

Step 0. [Divergence-type terms]. Set $t = 0$. First, note that $[\widetilde{W}, \widetilde{Y}]^{\top}\big|_{t=0}$ is a vector field of class $\mathbf{C}^1$ out of $C_{\mathcal{U}}$. We also stress that

$$[\widetilde{W}, \widetilde{Y}]^{\top}\big|_{t=0} = [\widetilde{W}, \widetilde{W}^{\top}]^{\top}\big|_{t=0} - W(\langle W, \nu \rangle) \nu_H^{\top} - \langle W, \nu \rangle [\widetilde{W}, {\nu_H^t}^{\top}]^{\top}\big|_{t=0}.$$

We see that the second and third terms in this formula are not defined at $C_{\mathcal{U}}$. The second term is the product of a $\mathbf{C}^1$ function times the vector field $\nu_H^{\top}$. Furthermore, note that

$$[\widetilde{W}, {\nu_H^t}^{\top}]^{\top}\big|_{t=0} = \left[\widetilde{W}, \left(\nu_H^t - |\mathcal{P}_{H_t} \nu^t| \nu^t \right) \right]^{\top} \Big|_{t=0} = [\widetilde{W}, \nu_H^t]^{\top}\big|_{t=0} - |\mathcal{P}_H \nu|\, [\widetilde{W}, \nu^t]\big|_{t=0}.$$

By using the very definition of $\nu_H^t = \frac{\mathcal{P}_{H_t} \nu^t}{|\mathcal{P}_{H_t} \nu^t|}$, it can easily be seen that $[\widetilde{W}, \nu_H^t]^{\top}\big|_{t=0}$ can be estimated near the characteristic set $C_{\mathcal{U}}$ by (a constant times) the function $\frac{1}{|\mathcal{P}_H \nu|^2}$. By continuing this argument, it is easy to realize that the tangential divergence of $[\widetilde{W}, \widetilde{Y}]^{\top}$, at $t = 0$, can now be estimated by (a constant times) the function $\frac{1}{|\mathcal{P}_H \nu|^3}$, locally around $C_{\mathcal{U}}$. An analogous (but simpler) argument can be repeated for the second vector divergence-type term in formula (28). In fact, since the function g_t is of class $\mathbf{C}^1$ on $\mathcal{U}_t$ for all $t \in]-\epsilon, \epsilon[$, we easily see that $div_{TU}(g_0 Y)$ can be estimated near the characteristic set $C_{\mathcal{U}}$ by (a constant times) the function $\frac{1}{|\mathcal{P}_H \nu|^2}$.

Remark 4.10. *The previous estimates show that, in order to integrate on $\mathcal{U}$ the divergence-type term dB, we need a further condition. More precisely, we shall assume that the function $\frac{1}{|\mathcal{P}_H\nu|^3}$ is integrable on $\mathcal{U}$, with respect to the Riemannian measure $\sigma^{n-1}_{\mathcal{R}}$. This condition can also be formulated in terms of the H-perimeter measure. In fact it is equivalent to require that $\frac{1}{|\mathcal{P}_H\nu|^2}\in L^2(\mathcal{U};\sigma^{n-1}_H)$. Here, we also stress that this assumption will be necessary in order to ensure integrability on $\mathcal{U}$ of the term A in formula* (26).

Step 1. We start with the calculation of the term A in formula (26).

Warning 4.11. *In order to simplify our calculations, hereafter we shall assume that $\mathcal{H}_H$* is constant.

Remark 4.12. *We stress that if $\mathcal{H}_H$ = const., then $\mathcal{L}_X\mathcal{H}_H=0$ for all $X\in\mathfrak{X}^1(TS)$. In particular, if W denotes the variation vector of ϑ_t, we have $\imath^*\left(\mathcal{L}_{\widetilde{W}_{HS}}(\mathcal{H}_H)_t\right)=\mathcal{L}_{W_{HS}}\mathcal{H}_H=0$. Analogously, we have $\imath^*\left(\mathcal{L}_{\tau^{TS}_\alpha}(\mathcal{H}_H)_t\right)=\mathcal{L}_{\tau^{TS}_\alpha}\mathcal{H}_H=0$ for all $\alpha\in I_V$. Hence*

$$\imath^*\left(\mathcal{L}_{\tau_\alpha}(\mathcal{H}_H)_t\right)=\imath^*\left(\mathcal{L}_{\varpi^t_\alpha\nu^t_H}(\mathcal{H}_H)_t\right)\qquad\forall\ \alpha\in I_V. \tag{29}$$

If $t=0$, we have

$$\begin{aligned}
A|_{t=0} &= \imath^*\left(\mathcal{L}_{\widetilde{W}}\left(-w_t(\mathcal{H}_H)_t(\sigma^{n-1}_H)_t\right)\right)\\
&= \left(-w\mathcal{H}_H\mathcal{L}_{\widetilde{W}}(\sigma^{n-1}_H)_t-W(w)\mathcal{H}_H-w\,\imath^*\left(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H)_t\right)\right)\sigma^{n-1}_H\\
&= -w\mathcal{H}_H\left(-\mathcal{H}_H\langle W,\nu\rangle+div_{T\mathcal{U}}\left(W^\top|\mathcal{P}_H\nu|-\langle W,\nu\rangle\nu_H^\top\right)\right)\sigma^{n-1}_{\mathcal{R}}-\left(W(w)\mathcal{H}_H+w\,\imath^*\left(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H)_t\right)\right)\sigma^{n-1}_H\\
&= \left(\mathcal{H}_H^2w^2-W(w)\mathcal{H}_H-\imath^*\left(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H)_t\right)\right)\sigma^{n-1}_H-w\mathcal{H}_H\,div_{T\mathcal{U}}\left(W^\top|\mathcal{P}_H\nu|-\langle W,\nu\rangle\nu_H^\top\right)\sigma^{n-1}_{\mathcal{R}},
\end{aligned}$$

where we have used the 1st variation of σ^{n-1}_H.

Step 2. Setting $W_\pitchfork:=w_1\nu_H+W_V$, where $W_V=\sum_{\alpha\in I_V}w_\alpha\tau_\alpha$, we get that

$$\begin{aligned}
\imath^*\left(\mathcal{L}_{\widetilde{W}}(\mathcal{H}_H)_t\right) &= \imath^*\left(\mathcal{L}_{\widetilde{W}_{HS}}(\mathcal{H}_H)_t\right)+\imath^*\left(\mathcal{L}_{\widetilde{W}_\pitchfork}(\mathcal{H}_H)_t\right)\\
&= \imath^*\left(\mathcal{L}_{\widetilde{W}_\pitchfork}(\mathcal{H}_H)_t\right)\qquad\text{(by Remark 4.12)}\\
&= \imath^*\left(\mathcal{L}_{\widetilde{w}_1\nu^t_H}(\mathcal{H}_H)_t\right)+\sum_{\alpha\in I_V}\imath^*\left(\mathcal{L}_{\widetilde{w}_\alpha\tau_\alpha}(\mathcal{H}_H)_t\right)\\
&= \imath^*\left(\mathcal{L}_{\widetilde{w}_1\nu^t_H}(\mathcal{H}_H)_t\right)+\sum_{\alpha\in I_V}\imath^*\left(\mathcal{L}_{\widetilde{w}_\alpha\varpi^t_\alpha\nu^t_H}(\mathcal{H}_H)_t\right)\qquad\text{(by (29))}\\
&= \imath^*\left(\mathcal{L}_{w_t\nu^t_H}(\mathcal{H}_H)_t\right).
\end{aligned}\tag{30}$$

Step 3. From Step 2, we see that it remains to calculate $\mathcal{L}_{w_t\nu^t_H}(\mathcal{H}_H)_t=w_t\frac{\partial(\mathcal{H}_H)_t}{\partial\nu^t_H}$. This will be done by using an adapted frame $\underline{\tau}=\{\tau_1,...,\tau_n\}$ to $\mathcal{U}$ which satisfies Lemma 3.9 at $p_0\in\mathcal{U}$. (Recall that $\tau_1(x)=\nu_H(x)$ for every $x\in\mathcal{U}$). We have

$$\begin{aligned}
-\frac{\partial(\mathcal{H}_H)_t}{\partial\nu^t_H} &= \sum_{j\in I_{HS}}\frac{\partial}{\partial\tau_1}\left\langle\nabla_{\tau_j}\tau_1,\tau_j\right\rangle\\
&= \sum_{j\in I_{HS}}\left(\left\langle\nabla_{\tau_1}\nabla_{\tau_j}\tau_1,\tau_j\right\rangle+\left\langle\nabla_{\tau_j}\tau_1,\nabla_{\tau_1}\tau_j\right\rangle\right)\\
&= \sum_{j\in I_{HS}}\left(\left\langle\left(\nabla_{\tau_1}\nabla_{\tau_j}\tau_1\mp\nabla_{\tau_j}\nabla_{\tau_1}\tau_1\mp\nabla_{[\tau_1,\tau_j]}\tau_1\right),\tau_j\right\rangle+\sum_{k=2}^n\left\langle\nabla_{\tau_j}\tau_1,\tau_k\right\rangle\left\langle\nabla_{\tau_1}\tau_j,\tau_k\right\rangle\right)\\
&= \sum_{j\in I_{HS}}\left(-\Phi_{1j}(\tau_1,\tau_j)+\left\langle\nabla_{\tau_j}\nabla_{\tau_1}\tau_1,\tau_j\right\rangle+\left\langle\nabla_{[\tau_1,\tau_j]}\tau_1,\tau_j\right\rangle+\sum_{\alpha\in I_V}\left\langle\nabla_{\tau_j}\tau_1,\tau_\alpha\right\rangle\left\langle\nabla_{\tau_1}\tau_j,\tau_\alpha\right\rangle\right)
\end{aligned}$$

where we have used the definition of $\Phi_{1j}(\tau_1, \tau_j)$ and the fact (Lemma 3.9) that $\phi_{jk} = 0$ at $p_0 \in \mathcal{U}$ for every $j, k \in I_{HS}$. We have

$$\begin{aligned}\left\langle \nabla_{[\tau_1,\tau_j]}\tau_1, \tau_j\right\rangle &= C^1_{1j}\phi_{1j}(\tau_1) + \sum_{k\in I_{HS}} C^k_{1j}\phi_{1j}(\tau_k) + \sum_{\alpha\in I_V} C^\alpha_{1j}\phi_{1j}(\tau_\alpha)\\ &= -\left(\phi_{1j}(\tau_1)\right)^2 - \sum_{k\in I_{HS}} \phi_{1k}(\tau_j)\phi_{1j}(\tau_k) - \sum_{\alpha\in I_V}\left\langle C^\alpha_H\tau_1, \tau_j\right\rangle \phi_{1j}(\tau_\alpha).\end{aligned}$$

Moreover, using (vii) of Lemma 3.9 yields

$$\left\langle \nabla_{\tau_j}\tau_1, \tau_\alpha\right\rangle\left\langle \nabla_{\tau_1}\tau_j, \tau_\alpha\right\rangle = \phi_{1\alpha}(\tau_j)\phi_{j\alpha}(\tau_1) = -\frac{1}{4}\left\langle C^\alpha_H\tau_1, \tau_j\right\rangle^2.$$

Therefore, Lemma 3.8 implies that

$$-\frac{\partial(\mathcal{H}_H)_t}{\partial\nu^t_H} = \frac{1}{2}\sum_{\alpha\in I_{H_2}}|C^\alpha_H\tau_1|^2 + div_{HS_t}(\nabla_{\tau_1}\tau_1) - \sum_{j,k\in I_{HS}\ \alpha\in I_V}\left(\left(\phi_{1j}(\tau_1)\right)^2 + \phi_{1k}(\tau_j)\phi_{1j}(\tau_k) + \left\langle C^\alpha_H\tau_1, \tau_j\right\rangle\phi_{1j}(\tau_\alpha)\right).$$

Hence, from Lemma 3.5, Lemma 4.2 and formula (13) we get that

$$\begin{aligned}-\frac{\partial(\mathcal{H}_H)_t}{\partial\nu^t_H} &= \frac{1}{2}\sum_{\alpha\in I_{H_2}}|C^\alpha_H\tau_1|^2 - div_{HS_t}\left(\frac{grad_{HS_t}w_t}{w_t} + C_H(\varpi^t_{H_2})\tau_1\right)\\ &\quad - \left|\frac{grad_{HS_t}w_t}{w_t} + C_H(\varpi^t_{H_2})\tau_1\right|^2 + \|A^t_H\|^2_{\mathrm{Gr}} - \|S^t_H\|^2_{\mathrm{Gr}}\\ &\quad - \sum_{j\in I_{HS}\ \alpha\in I_V}\left\langle C^\alpha_H\tau_1, \tau_j\right\rangle\left(\frac{1}{2}\left\langle C^\alpha_H\tau_1, \tau_j\right\rangle - \left\langle C(\varpi^t)\tau_\alpha, \tau_j\right\rangle + \tau_j(\varpi^t_\alpha) - \varpi^t_\alpha\frac{\tau_j(w_t)}{w_t}\right).\\ &= -div_{HS_t}\left(\frac{grad_{HS_t}w_t}{w_t} + C_H(\varpi^t_{H_2})\tau_1\right) - \left|\frac{grad_{HS_t}w_t}{w_t} + C_H(\varpi^t_{H_2})\tau_1\right|^2 + \|A^t_H\|^2_{\mathrm{Gr}} - \|S^t_H\|^2_{\mathrm{Gr}}\\ &\quad + \sum_{j\in I_{HS}\ \alpha\in I_V}\left\langle C^\alpha_H\tau_1, \tau_j\right\rangle\left(\left\langle C(\varpi^t)\tau_\alpha, \tau_j\right\rangle - \tau_j(\varpi^t_\alpha) + \varpi^t_\alpha\frac{\tau_j(w_t)}{w_t}\right)\\ &= -div_{HS_t}\left(\frac{grad_{HS_t}w_t}{w_t} + C_H(\varpi^t_{H_2})\tau_1\right) - \left|\frac{grad_{HS_t}w_t}{w_t} + C_H(\varpi^t_{H_2})\tau_1\right|^2 + \|A^t_H\|^2_{\mathrm{Gr}} - \|S^t_H\|^2_{\mathrm{Gr}}\\ &\quad + \sum_{\alpha\in I_V}\left(\left\langle C^\alpha_H\tau_1, C(\varpi^t)\tau_\alpha\right\rangle - \left\langle C^\alpha_H\tau_1, grad_{HS_t}\varpi^t_\alpha\right\rangle\right) + \left\langle C_H(\varpi^t_{H_2})\tau_1, \frac{grad_{HS_t}w_t}{w_t}\right\rangle\\ &= -\frac{\Delta_{HS_t}w_t}{w_t} - div_{HS_t}\left(C_H(\varpi^t_{H_2})\tau_1\right) - \left|C_H(\varpi^t_{H_2})\tau_1\right|^2 - 2\left\langle\frac{grad_{HS_t}w_t}{w_t}, C_H(\varpi^t_{H_2})\tau_1\right\rangle + \|A^t_H\|^2_{\mathrm{Gr}} - \|S^t_H\|^2_{\mathrm{Gr}}\\ &\quad + \sum_{\alpha\in I_V}\left(\left\langle C^\alpha_H\tau_1, C(\varpi^t)\tau_\alpha\right\rangle - \left\langle C^\alpha_H\tau_1, grad_{HS_t}\varpi^t_\alpha\right\rangle\right) + \left\langle C_H(\varpi^t_{H_2})\tau_1, \frac{grad_{HS_t}w_t}{w_t}\right\rangle\\ &= -\frac{\Delta_{HS_t}w_t}{w_t} - div_{HS_t}\left(C_H(\varpi^t_{H_2})\tau_1\right) - \left|C_H(\varpi^t_{H_2})\tau_1\right|^2 - \left\langle\frac{grad_{HS_t}w_t}{w_t}, C_H(\varpi^t_{H_2})\tau_1\right\rangle + \|A^t_H\|^2_{\mathrm{Gr}} - \|S^t_H\|^2_{\mathrm{Gr}}\\ &\quad + \sum_{\alpha\in I_V}\left(\left\langle C^\alpha_H\tau_1, C(\varpi^t)\tau_\alpha\right\rangle - \left\langle C^\alpha_H\tau_1, grad_{HS_t}\varpi^t_\alpha\right\rangle\right)\\ &= -\frac{\mathcal{L}_{HS_t}w_t}{w_t} - \mathcal{D}_{HS_t}\left(C_H(\varpi^t_{H_2})\tau_1\right) + \|A^t_H\|^2_{\mathrm{Gr}} - \|S^t_H\|^2_{\mathrm{Gr}} + \sum_{\alpha\in I_V}\left(\left\langle C^\alpha_H\tau_1, C(\varpi^t)\tau_\alpha\right\rangle - \left\langle C^\alpha_H\tau_1, grad_{HS_t}\varpi^t_\alpha\right\rangle\right).\end{aligned}$$

Now we can achieve the proof of the 2nd variation of σ^{n-1}_H, under the assumptions previously made; see Warning 4.4, Remark 4.9 and Warning 4.11.

Step 4. If $\frac{1}{|\mathcal{P}_H\nu|^2}\in L^2\left(S\,;\sigma^{n-1}_H\right)$, then we have

$$
\begin{aligned}
II_{\mathcal{U}}(W,\sigma^{n-1}_H) &= \int_{\mathcal{U}}\Big((w\,\mathcal{H}_H)^2 - W(w)\mathcal{H}_H - w\,\iota^*\left(\mathcal{L}_{w_t\,\nu^t_H}(\mathcal{H}_H)_t\right)\Big)\,\sigma^{n-1}_H\\
&\quad+\int_{\mathcal{U}}\Big(div_{T\mathcal{U}}\left([\widetilde{W},\widetilde{Y}]^\top + g_0 Y\right) - w\mathcal{H}_H\,div_{T\mathcal{U}}\left(W^\top|\mathcal{P}_H\nu| - \langle W,\nu\rangle\nu_H^\top\right)\Big)\,\sigma^{n-1}_{\mathcal{R}}\\
&= \int_{\mathcal{U}}\bigg\{-W(w)\mathcal{H}_H + w^2\left((\mathcal{H}_H)^2 + \|A_H\|^2_{\mathrm{Gr}} - \|S_H\|^2_{\mathrm{Gr}}\right) - w\,\mathcal{L}_{HS}\,w\\
&\quad+w^2\Big[-\mathcal{D}_{HS}\,\left(C_H(\varpi_{H_2})\tau_1\right) + \sum_{\alpha\in I_V}\left(\langle C^\alpha_H\tau_1, C(\varpi)\tau_\alpha\rangle - \langle C^\alpha_H\tau_1, grad_{HS}\,\varpi_\alpha\rangle\right)\Big]\bigg\}\sigma^{n-1}_H\\
&\quad+\int_{\mathcal{U}}\Big(div_{T\mathcal{U}}\left([\widetilde{W},\widetilde{Y}]^\top + g_0 Y\right) - w\mathcal{H}_H\,div_{T\mathcal{U}}\left(W^\top|\mathcal{P}_H\nu| - \langle W,\nu\rangle\nu_H^\top\right)\Big)\,\sigma^{n-1}_{\mathcal{R}}\\
&= \int_{\mathcal{U}}\bigg\{-W(w)\mathcal{H}_H + w^2\left((\mathcal{H}_H)^2 + \|A_H\|^2_{\mathrm{Gr}} - \|S_H\|^2_{\mathrm{Gr}}\right) + |grad_{HS}\,w|^2 \qquad \text{(by formula (6))}\\
&\quad+w^2\sum_{\alpha\in I_V}\left(-\varpi_\alpha\mathcal{D}_{HS}\,(C^\alpha_H\tau_1) + \langle C^\alpha_H\tau_1, C(\varpi)\tau_\alpha\rangle - 2\,\langle C^\alpha_H\tau_1, grad_{HS}\,\varpi_\alpha\rangle\right)\bigg\}\,\sigma^{n-1}_H\\
&\quad+\int_{\mathcal{U}}\Big(div_{T\mathcal{U}}\left([\widetilde{W},\widetilde{Y}]^\top + g_0 Y\right) - w\mathcal{H}_H\,div_{T\mathcal{U}}\left(W^\top|\mathcal{P}_H\nu| - \langle W,\nu\rangle\nu_H^\top\right)\Big)\,\sigma^{n-1}_{\mathcal{R}}.
\end{aligned}
\tag{31}
$$

By applying Remark 2.8, it follows that if $\frac{1}{|\mathcal{P}_H\nu|^2}\in L^2(\mathcal{U},\sigma^{n-1}_H)$, then the function w^2 turns out to be admissible; see Definition 2.6. Finally, from Lemma 3.6 we get that

$$
\begin{aligned}
&\int_{\mathcal{U}}\bigg\{-W(w)\mathcal{H}_H + w^2\left((\mathcal{H}_H)^2 - \|A_H\|^2_{\mathrm{Gr}} - \|S_H\|^2_{\mathrm{Gr}}\right) + |grad_{HS}\,w|^2\\
&\quad+w^2\sum_{\alpha\in I_V}\Big[-|C_H(\varpi_{H_2})\tau_1|^2 + \langle C^\alpha_H\tau_1, C(\varpi)\tau_\alpha\rangle - 2\,\langle C^\alpha_H\tau_1, grad_{HS}\,\varpi_\alpha\rangle\Big]\bigg\}\,\sigma^{n-1}_H\\
=\;&\int_{\mathcal{U}}\bigg\{-W(w)\mathcal{H}_H + w^2\left((\mathcal{H}_H)^2 - \|A_H\|^2_{\mathrm{Gr}} - \|S_H\|^2_{\mathrm{Gr}}\right) + |grad_{HS}\,w|^2\\
&\quad-w^2\sum_{\alpha\in I_V}\left\langle\left(2\,grad_{HS}(\varpi_\alpha) - C(\varpi)\tau^{TS}_\alpha\right), C^\alpha\tau_1\right\rangle\bigg\}\,\sigma^{n-1}_H.
\end{aligned}
$$

Using the last identity in (31) yields the following:

Theorem 4.13 (2nd variation of σ^{n-1}_H)**.** *Let $S\subset\mathbb{G}$ be a compact $\mathbf{C}^3$-smooth hypersurface with, or without, boundary and let $\vartheta:]-\epsilon,\epsilon[\times S\to\mathbb{G}$ be a $\mathbf{C}^2$ variation of S. Let $W=\frac{d\vartheta_t}{dt}\Big|_{t=0}$ be the variation vector field and let $W^\perp$, $W^\top$ denote the normal and tangential components of W along S, respectively. Moreover, set $w:=\frac{\langle W^\perp,\nu\rangle}{|\mathcal{P}_H\nu|}$. We also assume that:*

(i) *for every $t\in]-\epsilon,\epsilon[$ one has*[6] $\frac{1}{|\mathcal{P}_{H_t}\nu^t|}\in L^1_{loc}\left(S_t;(\sigma^{n-1}_{\mathcal{R}})_t\right)$*, where $S_t=\vartheta_t(S)$;*
(ii) *the horizontal mean curvature $\mathcal{H}_H$ of S is constant;*
(iii) *the function $\frac{1}{|\mathcal{P}_H\nu|^2}\in L^2\left(S\,;\sigma^{n-1}_H\right)$.*

[6] Alternatively, we can assume the validity of (A_2) in Warning 4.4.

Then, the following formula holds:

$$
\begin{aligned}
II_S(W,\sigma^{n-1}_H) \;=\; & \int_S \Big\{ - W(w)\mathcal{H}_H + w^2\left((\mathcal{H}_H)^2 - \|A_H\|^2_{\mathrm{Gr}} - \|S_H\|^2_{\mathrm{Gr}}\right) + |grad_{HS}\, w|^2 \\
& -w^2 \sum_{\alpha\in I_V} \langle (2\, grad_{HS}(\varpi_\alpha) - C(\varpi)\tau^{TS}_\alpha), C^\alpha \nu_H\rangle \Big\}\,\sigma^{n-1}_H \qquad (32)\\
& + \int_S \Big\{ div_{TS}\left([\widetilde{W},\widetilde{Y}]^\top + g_0 Y\right)\Big|_{t=0} - w\mathcal{H}_H\, div_{TS}\left(W^\top|\mathcal{P}_H\nu| - \langle W,\nu\rangle\nu_H^\top\right)\Big\}\,\sigma^{n-1}_{\mathcal{R}}
\end{aligned}
$$

where $\widetilde{Y} := \widetilde{W}^\top|\mathcal{P}_{H_t}\nu^t| - \langle\widetilde{W},\nu^t\rangle\nu^t_H{}^\top$, $Y = \widetilde{Y}|_{t=0}$ *and* $g_0 = (-\langle W^\perp,\nu\rangle\mathcal{H}_{\mathcal{R}} + div_{TS}\, W^\top)$.

Proof. As already observed, the hypothesis (i) implies, in the general case[7], the possibility to differentiate under the integral sign the function $f(t,x)$ defined by formula (15). This has been done by using the machinery of differential forms. This way we have obtained the above formula by further assuming that $\mathcal{H}_H$ is constant. Nevertheless, exactly as in the case of the 1st variation formula, we have to take care of the existence of the involved integrals. The integrability of the divergence-type terms has been already discussed at Step 0. Moreover, it is not difficult to see that the condition $\frac{1}{|\mathcal{P}_H\nu|^2} \in L^2(S,\sigma^{n-1}_H)$ implies that the function w^2 turns out to be admissible; see Definition 2.6. Hence, using formula (6), we see that the function $-w\,\mathcal{L}_{HS}\, w$ can be integrated by parts, as previously done. Furthermore, a rather tedious (but completely elementary) analysis shows that the same condition implies that each term in the formula obtained is integrable over S. [Actually, the integral of each of these terms can be estimated, near the characteristic set C_S, by (a constant times) $\int_S \frac{1}{|\mathcal{P}_H\nu|^4}\,\sigma^{n-1}_H$.] This achieves the proof. □

Remark 4.14. *Note that the two integrals in formula* (32) *are computed with respect to the measures* σ^{n-1}_H *and* $\sigma^{n-1}_{\mathcal{R}}$*, respectively. In particular, we remark that, near the characteristic set* C_S*, each term of the first integrand can be estimated in terms of either* $\frac{1}{|\mathcal{P}_H\nu|^3}$ *or* $\frac{1}{|\mathcal{P}_H\nu|^4}$*. Analogously, near the characteristic set* C_S*, each term of the second integrand can be estimated in terms of either* $\frac{1}{|\mathcal{P}_H\nu|^2}$ *or* $\frac{1}{|\mathcal{P}_H\nu|^3}$*. Actually, all these calculations use the same idea*[8] *behind formula* (9)*; see Remark 3.2. An analogous argument was made at Step 0.*

Corollary 4.15. *Let the assumptions of Theorem 4.13 hold and assume that* ϑ *is compactly supported on* S*. Furthermore, let* S *be H-minimal, i.e.* $\mathcal{H}_H = 0$*. If* $C_S \neq \emptyset$*, we shall assume that there exists a family* $\{\mathcal{U}_\delta\}_{\delta>0}$ *of open subsets of* S *such that:*

(i) $C_S \Subset \mathcal{U}_\delta$*,*
(ii) $\sigma^{n-1}_{\mathcal{R}}(\mathcal{U}_\delta) \longrightarrow 0$ *as long as* $\delta \to 0$*,*
(iii) $\partial\mathcal{U}_\delta$ *is of class* $\mathbf{C}^1$ *and* $\sigma^{n-2}_{\mathcal{R}}(\partial\mathcal{U}_\delta) \longrightarrow 0$ *as long as* $\delta \to 0$*.*

Then, we have

$$
II_S(W,\sigma^{n-1}_H) = \int_S \left\{ |grad_{HS}\, w|^2 - w^2\left(\|A_H\|^2_{\mathrm{Gr}} + \|S_H\|^2_{\mathrm{Gr}} + \sum_{\alpha\in I_V} \langle (2\, grad_{HS}(\varpi_\alpha) - C(\varpi)\tau^{TS}_\alpha), C^\alpha\nu_H\rangle \right)\right\}\sigma^{n-1}_H .
$$

Proof. We have just to analyze the 2nd integral in formula (32). We already know that Y is admissible; see Corollary 4.8. Since g_0 is of class $\mathbf{C}^1$ on S and $g_0 = 0$ on ∂S, we can conclude that $g_0 Y$ is admissible and that $\int_S div_{TS}\,(g_0 Y)\,\sigma^{n-1}_{\mathcal{R}} = 0$. Furthermore, since $\mathcal{H}_H = 0$, the only thing to be proved is that $\int_S div_{TS}\left([\widetilde{W},\widetilde{Y}]^\top\big|_{t=0}\right)\sigma^{n-1}_{\mathcal{R}} = 0$. Equivalently, since $[\widetilde{W},\widetilde{Y}]^\top\big|_{t=0} = 0$ on ∂S, we have to show that $[\widetilde{W},\widetilde{Y}]^\top\big|_{t=0}$ is admissible. Under our assumptions, this can be done exactly as we have done in the proof of Corollary 4.8. This achieves the proof. □

---

[7]That is, $C_S \neq \emptyset$.

[8]That is, $X|\mathcal{P}_H\nu| = \frac{\langle \mathcal{J}_{\mathcal{R}}\mathcal{P}_H\nu, X\rangle}{|\mathcal{P}_H\nu|}$, for any $X \in \mathfrak{X}(\mathbb{G})$.

Notation 4.16. *For the sake of simplicity, we shall set:*

$$\mathcal{B}_{TS} := \underbrace{\|S_H\|^2_{\mathrm{Gr}} + \|A_H\|^2_{\mathrm{Gr}}}_{=\|B_H\|^2_{\mathrm{Gr}}} + \sum_{\alpha\in I_V} \langle (2grad_{HS}(\varpi_\alpha) - C(\varpi)\tau^{TS}_\alpha), C^\alpha\tau_1\rangle . \tag{33}$$

We stress that, unlike the Euclidean case where $\mathcal{B}_{TS} := \|B_{\mathcal{R}}\|^2_{\mathrm{Gr}}$, it is not necessarily true that $\mathcal{B}_{TS} \geq 0$; an example of this fact can be found in Section 6.2, in the case $n = 1$.

Remark 4.17. *In the Heisenberg group $\mathbb{H}^n$, one easily gets that*

$$\mathcal{B}_{TS} = \|S_H\|^2_{\mathrm{Gr}} - \left(2\frac{\partial\varpi}{\partial\nu_H^\perp} - \frac{n+1}{2}\varpi^2\right); \tag{34}$$

see Example 1.12.

5. Geometric identities for constant H-mean curvature hypersurfaces

Lemma 5.1. *Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^2$ and let $\phi \in \mathbf{C}^2(\mathbb{G})$. Then we have*

$$\Delta_{HS}\,\phi = \Delta_H\,\phi + \mathcal{H}_H\frac{\partial\phi}{\partial\nu_H} - \langle \mathrm{Hess}_H\,\phi\,\nu_H, \nu_H\rangle$$

at each non-characteristic point $x \in S \setminus C_S$.

Proof. First, note that we can use the invariant definition of the Laplacian on vector bundles; see, for instance, [14]. So we have

$$\begin{aligned}
\Delta_H\phi &= \sum_{i\in I_H}\left(\tau_i^{(2)} - \nabla^H_{\tau_i}\tau_i\right)(\phi)\\
&= \tau_1^{(2)}(\phi) - \left(\nabla^H_{\tau_1}\tau_1\right)(\phi) + \sum_{i\in I_{HS}}\left(\left(\tau_i^{(2)} - \nabla^{HS}_{\tau_i}\tau_i\right)(\phi) - \left\langle\nabla^H_{\tau_i}\tau_i, \nu_H\right\rangle\frac{\partial\phi}{\partial\nu_H}\right)\\
&= \tau_1^{(2)}(\phi) - \left(\nabla^H_{\tau_1}\tau_1\right)(\phi) + \Delta_{HS}\,\phi - \mathcal{H}_H\frac{\partial\phi}{\partial\nu_H}.
\end{aligned}$$

Now we claim that $\tau_1^{(2)}(\phi) - \left(\nabla^H_{\tau_1}\tau_1\right)(\phi) = \langle\mathrm{Hess}_H(\phi)\nu_H, \nu_H\rangle$. To prove this claim, set $\tau_1 = \sum_{i\in I_H} A^1_i X_i$ and compute

$$\tau_1^{(2)}(\phi) = \sum_{i\in I_H}\left(\tau_1(A^1_i X_i(\phi))\right) = \sum_{i,j\in I_H}\left(\tau_1(A^1_i)X_i(\phi) + A^1_iA^1_jX_j(X_i(\phi))\right).$$

Since $\nabla^H_{\tau_1}\tau_1 = \sum_{i,j\in I_H}\left(\tau_1(A^1_i)X_i + A^1_iA^1_j\underbrace{\nabla^H_{X_i}X_j}_{=0}\right)$, we get that

$$\tau_1^{(2)}(\phi) - \left(\nabla^H_{\tau_1}\tau_1\right)(\phi) = \sum_{i,j\in I_H} A^1_iA^1_jX_j(X_i(\phi)) = \langle\mathrm{Hess}_H(\phi)\nu_H, \nu_H\rangle,$$

as wished. □

Lemma 5.2. *Let $S \subset \mathbb{G}$ be a non-characteristic hypersurface of class $\mathbf{C}^2$. Suppose that the horizontal mean curvature $\mathcal{H}_H$ is constant. Then, the following identities hold:*

(i) $\sum_{i\in I_{HS}}\left\langle\nabla^H_{\tau_i}\nabla^H_{\tau_i}\nu_H, \nu_H\right\rangle = -\|B_H\|^2_{\mathrm{Gr}}$;

(ii) $\sum_{i\in I_{HS}}\left\langle\nabla^H_{\tau_i}\nabla^H_{\tau_i}\nu_H, \tau_k\right\rangle = -\Big(\left\langle\nabla^H_{\nu_H}\nu_H, C_{HS}(\varpi_{H_2})\tau_k\right\rangle + \sum_{\alpha\in I_V}\left\langle C^\alpha_H grad_{HS}\,\varpi_\alpha, \tau_k\right\rangle + \mathcal{H}_H\langle C_H(\varpi_{H_2})\nu_H, \tau_k\rangle - B_H(C_H(\varpi_{H_2})\nu_H, \tau_k)\Big) \quad \forall\ k \in I_{HS}$.

Proof. Throughout this proof, we shall make use of an adapted frame as in Lemma 3.9.
Proof of (i). Since $\langle \nu_{_H}, \nu_{_H}\rangle = 1$ we get that $\left\langle \nabla^H_{\tau_i}\nu_{_H}, \nu_{_H}\right\rangle = 0\ \forall\ i\in I_{HS}$. So, we have

$$\begin{aligned}\sum_{i\in I_{HS}}\left\langle \nabla^H_{\tau_i}\nabla^H_{\tau_i}\nu_{_H}, \nu_{_H}\right\rangle &= -\sum_{i\in I_{HS}}\left\langle \nabla^H_{\tau_i}\nu_{_H}, \nabla^H_{\tau_i}\nu_{_H}\right\rangle\\ = -\sum_{i,j,k\in I_{HS}}\left\langle \nabla^H_{\tau_i}\nu_{_H}, \tau_j\right\rangle\left\langle \nabla^H_{\tau_i}\nu_{_H}, \tau_k\right\rangle\left\langle \tau_j, \tau_k\right\rangle &= -\sum_{i,j\in I_{HS}}\left\langle \nabla^H_{\tau_i}\nu_{_H}, \tau_j\right\rangle^2 = -\|B_H\|^2_{\mathrm{Gr}}.\end{aligned}$$

Proof of (ii). Since $\langle \nu_{_H}, \tau_k\rangle = 0$ for any $k\in I_{HS}$ we get that $\left\langle \nabla^H_{\tau_i}\nu_{_H}, \tau_k\right\rangle = -\left\langle \nu_{_H}, \nabla^H_{\tau_i}\tau_k\right\rangle$ for every $i\in I_{HS}$. Therefore

$$\left\langle \nabla^H_{\tau_i}\nabla^H_{\tau_i}\nu_{_H}, \tau_k\right\rangle + \left\langle \nabla^H_{\tau_i}\nu_{_H}, \nabla^H_{\tau_i}\tau_k\right\rangle = -\left\langle \nabla^H_{\tau_i}\nu_{_H}, \nabla^H_{\tau_i}\tau_k\right\rangle - \left\langle \nu_{_H}, \nabla^H_{\tau_i}\nabla^H_{\tau_i}\tau_k\right\rangle.$$

Note that $\nabla^H_{\tau_i}\nu_{_H}\in HS$ and that, by our choice of the moving frame, we have $\left(\nabla^{HS}_{\tau_i}\tau_k\right)(p) = 0$. Hence

$$\begin{aligned}A_i := \left\langle \nabla^H_{\tau_i}\nabla^H_{\tau_i}\nu_{_H}, \tau_k\right\rangle &= -\left\langle \nu_{_H}, \nabla^H_{\tau_i}\nabla^H_{\tau_i}\tau_k\right\rangle\\ &= -\left\langle \nu_{_H}, \nabla^H_{\tau_i}\left(\nabla^H_{\tau_k}\tau_i + [\tau_i, \tau_k]_H\right)\right\rangle\\ &= -\left\langle \nu_{_H}, \nabla^H_{\tau_i}\nabla^H_{\tau_k}\tau_i\right\rangle - \left\langle \nu_{_H}, \nabla^H_{\tau_i}\left(\langle [\tau_i, \tau_k]_H, \nu_{_H}\rangle\, \nu_{_H}\right)\right\rangle \quad \text{(by Lemma 3.9)}\\ &= -\left\langle \nu_{_H}, \nabla^H_{\tau_i}\nabla^H_{\tau_k}\tau_i\right\rangle - \tau_i\left(\langle [\tau_i, \tau_k]_H, \nu_{_H}\rangle\right) \qquad \forall\ i,k\in I_{HS}.\end{aligned}$$

Now since $\langle [\tau_i, \tau_k]_H, \nu_{\mathcal{R}}\rangle = \langle [\tau_i, \tau_k], \nu_{\mathcal{R}}\rangle = 0$, we get

$$\langle [\tau_i, \tau_k], \nu_{_H}\rangle = -\sum_{\alpha\in I_V}\varpi_\alpha\,\langle [\tau_i, \tau_k], \tau_\alpha\rangle = -\sum_{\alpha\in I_V}\varpi_\alpha C^\alpha_{ik} = \sum_{\alpha\in I_V}\varpi_\alpha\,\langle C^\alpha_H\tau_i, \tau_k\rangle = \langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle \quad \forall\ i,k\in I_{HS}.$$

Hence $A_i = -\left\langle \nu_{_H}, \nabla^H_{\tau_i}\nabla^H_{\tau_k}\tau_i\right\rangle - \tau_i\left(\langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle\right)$. Using $\mathrm{R}_H = 0$ (see Remark 1.6 in Section 1.1) yields

$$\left\langle \nabla^H_{\tau_i}\nabla^H_{\tau_k}\tau_i, \nu_{_H}\right\rangle = \left\langle \nabla^H_{\tau_k}\nabla^H_{\tau_i}\tau_i, \nu_{_H}\right\rangle + \left\langle \nabla^H_{[\tau_i,\tau_k]_H}\tau_i, \nu_{_H}\right\rangle \qquad \forall\ i,k\in I_{HS}..$$

Therefore

$$\sum_{i\in I_{HS}} A_i = \underbrace{-\left\langle \nu_{_H}, \nabla^H_{\tau_k}\left(\sum_{i\in I_{HS}}\nabla^H_{\tau_i}\tau_i\right)\right\rangle}_{=:A} - \sum_{i\in I_{HS}}\left(\left\langle \nabla^H_{[\tau_i,\tau_k]_H}\tau_i, \nu_{_H}\right\rangle + \tau_i\left(\langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle\right)\right).$$

We claim that $A = 0$ at p. Indeed, since $\mathcal{H}_H = \left\langle \sum_{i\in I_{HS}}\nabla^H_{\tau_i}\tau_i, \nu_{_H}\right\rangle$ is assumed to be constant, we get that $\left\langle \sum_{i\in I_{HS}}\nabla^H_{\tau_i}\tau_i, \nabla^H_{\tau_k}\nu_{_H}\right\rangle = 0$ at p and the claim follows. Furthermore, since at the point $p\in S$, one has $[\tau_i, \tau_k]_H = \langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle\,\nu_{_H}\ \forall\ i,k\in I_{HS}$, it follows that

$$\begin{aligned}\sum_{i\in I_{HS}} A_i &= \sum_{i\in I_{HS}}\left(\langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle\left\langle \nabla^H_{\nu_{_H}}\nu_{_H}, \tau_i\right\rangle - \tau_i\left(\langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle\right)\right)\\ &= -\left(\left\langle \nabla^H_{\nu_{_H}}\nu_{_H}, C_{HS}(\varpi_{H_2})\tau_k\right\rangle + \sum_{i\in I_{HS}}\tau_i\left(\langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle\right)\right).\end{aligned}$$

Finally, (ii) follows from the next calculation:

$$\begin{aligned}\tau_i\left(\langle C_{HS}(\varpi_{H_2})\tau_i, \tau_k\rangle\right) &= \sum_{\alpha\in I_V}\left(\tau_i(\varpi_\alpha)\,\langle C^\alpha_H\tau_i, \tau_k\rangle + \varpi_\alpha\left(\left\langle C^\alpha_H\nabla^H_{\tau_i}\tau_i, \tau_k\right\rangle + \left\langle C^\alpha_H\tau_i, \nabla^H_{\tau_i}\tau_k\right\rangle\right)\right)\\ &= \sum_{\alpha\in I_V}\left(\tau_i(\varpi_\alpha)\,\langle C^\alpha_H\tau_i, \tau_k\rangle + \varpi_\alpha\left(-\left\langle \nabla^H_{\tau_i}\tau_i, \nu_{_H}\right\rangle\langle C^\alpha_H\tau_k, \nu_{_H}\rangle + \langle C^\alpha_H\tau_i, \nu_{_H}\rangle\left\langle \nabla^H_{\tau_i}\tau_k, \nu_{_H}\right\rangle\right)\right)\\ &= \sum_{\alpha\in I_V}\tau_i(\varpi_\alpha)\,\langle C^\alpha_H\tau_i, \tau_k\rangle + \mathcal{H}_H\,\langle C_H(\varpi_{H_2})\nu_{_H}, \tau_k\rangle - B_H(C_H(\varpi_{H_2}), \tau_k).\end{aligned}$$

□

Using (i) of Lemma 5.2, yields the following "folklore" result:

Proposition 5.3. *Let $S \subset \mathbb{G}$ be a non-characteristic hypersurface of class $\mathbf{C}^2$. Moreover, we suppose that the horizontal mean curvature $\mathcal{H}_H$ is constant. Then*

(i) $\left\langle \overrightarrow{\Delta_{HS}}\, \nu_{_H}, \nu_{_H} \right\rangle = -\|B_H\|^2_{\mathrm{Gr}}$;

(ii) $\overrightarrow{\Delta_{HS}}\, x_H = \mathcal{H}_H\, \nu_{_H}$.

Below we shall compute the HS-laplacian $\mathcal{L}_{HS}$ of the function $f_H := \langle V_H, \nu_{_H} \rangle$, where $V_H \in \mathfrak{X}(H)$ is a constant horizontal left invariant vector field.

Lemma 5.4. *Let $S \subset \mathbb{G}$ be a non-characteristic hypersurface of class $\mathbf{C}^2$. Moreover, we suppose that the horizontal mean curvature $\mathcal{H}_H$ is constant. Then*

$$-\mathcal{L}_{HS}\, f_H = f_H \|B_H\|^2_{\mathrm{Gr}} + \left\langle \nabla^H_{\nu_{_H}} \nu_{_H}, C_{HS}(\varpi_{H_2}) V_{HS} \right\rangle + \sum_{\alpha \in I_V} \langle C^\alpha_H\, grad_{HS}\, \varpi_\alpha, V_{HS} \rangle + \mathcal{H}_H \langle C_H(\varpi_{H_2}) \nu_{_H}, V_{HS} \rangle$$

at each non-characteristic point.

Proof. As above, we preliminarily fix a point $p \in S$ and choose a moving frame centered at p. We have

$$\begin{aligned} \Delta_{HS}\, f_H &= \sum_{i \in I_{HS}} \tau_i \tau_i(\langle V_H, \nu_{_H} \rangle) = \sum_{i \in I_{HS}} \tau_i \left(\left\langle V_H, \nabla^H_{\tau_i} \nu_{_H} \right\rangle \right) = \sum_{i \in I_{HS}} \left(\left\langle V_H, \nabla^H_{\tau_i} \nabla^H_{\tau_i} \nu_{_H} \right\rangle \right) \\ &= -\Bigg(f_H \|B_H\|^2_{\mathrm{Gr}} + \left\langle \nabla^H_{\nu_{_H}} \nu_{_H}, C_{HS}(\varpi_{H_2}) V_{HS} \right\rangle + \sum_{\alpha \in I_V} \langle C^\alpha_H\, grad_{HS}\, \varpi_\alpha, V_{HS} \rangle \\ &\qquad + \mathcal{H}_H \langle C_H(\varpi_{H_2}) \nu_{_H}, V_{HS} \rangle - B_H(C_H(\varpi_{H_2}) \nu_{_H}, V_{HS}) \Bigg), \end{aligned}$$

where we have used (i) and (ii) of Lemma 5.2. The thesis follows since

$$B_H(C_H \nu_{_H}, V_{HS}) = -\langle C_H \nu_{_H}, grad_{HS}\, f_H \rangle .$$

$\square$

A simple consequence of this lemma, at least from a "formal" point of view, is that, in general, the function f_H cannot be an eigenfunction of any linear eigenvalue problem of the type $\mathcal{L}_{HS}\, \varphi + \lambda \mathcal{B} \varphi = 0$, where $\mathcal{B}$ is some given smooth function on $S \setminus C_S$. This seems to be very different with respect to the Euclidean case where, for any constant vector field $V \in \mathbb{R}^n$, the function $f = \langle V, \nu \rangle$ is always a solution to the linear equation $\Delta_{TS}\, \varphi + \|B_{\mathcal{R}}\|^2_{\mathrm{Gr}} \varphi = 0$. Here Δ_{TS} is the Laplace-Beltrami operator on S and $B_{\mathcal{R}}$ is the 2nd fundamental form of S. This says that V is a *Killing field* for any constant mean curvature hypersurface $S \subset \mathbb{R}^n$; see [37]. Nevertheless, we have the following important:

Lemma 5.5. *Let $S \subset \mathbb{G}$ be a non-characteristic hypersurface of class $\mathbf{C}^2$. Moreover, we suppose that the horizontal mean curvature $\mathcal{H}_H$ is constant. Then*

$$-\mathcal{L}_{HS}\, \varpi_\alpha = \varpi_\alpha \mathcal{B}_{TS} \qquad \forall\ \alpha \in I_V$$

at each non-characteristic point.

Proof. For the sake of simplicity, we shall assume that f is a *normalized defining function for S;* see Definition 3.11. Let $\underline{\tau}$ be an adapted moving frame along S. We have $grad_H f = \tau_1$ (and hence $\tau_1(f) = 1$) and $\tau_\alpha f = \varpi_\alpha$ for every $\alpha \in I_V$. We stress that $\frac{\partial \varpi_\alpha}{\partial \tau_1} = X_\alpha \left(\frac{\partial f}{\partial \tau_1} \right) = X_\alpha(1) = 0$. Thus, using Lemma 5.1

yields

$$\begin{aligned}\Delta_{HS}\,\varpi_\alpha &= \Delta_H\,\varpi_\alpha - \langle \mathrm{Hess}_H(\varpi_\alpha)\tau_1,\tau_1\rangle\\ &= \Delta_H\,(\tau_\alpha f) - \langle \mathrm{Hess}_H\,(\tau_\alpha f)\,\tau_1,\tau_1\rangle\\ &= \tau_\alpha\,(\Delta_H(f)) - \langle \nabla_{\tau_\alpha}\,(\mathrm{Hess}_H(f))\,\tau_1,\tau_1\rangle\\ &= \varpi_\alpha\tau_1\,(\Delta_H(f)) - \langle \nabla_{\tau_\alpha}\,(\mathrm{Hess}_H(f))\,\tau_1,\tau_1\rangle\\ &= -\varpi_\alpha\tau_1\,(\mathcal{H}_H) - \langle \nabla_{\tau_\alpha}\,(\mathcal{J}_H\tau_1)\,\tau_1,\tau_1\rangle\,.\end{aligned}$$

Since $\langle(\mathcal{J}_H\tau_1)\,\tau_1,\tau_1\rangle = 0$, we get that $\langle\nabla_{\tau_\alpha}\,((\mathcal{J}_H\tau_1)\,\tau_1)\,,\tau_1\rangle = -\langle(\mathcal{J}_H\tau_1)\,\tau_1,\nabla_{\tau_\alpha}\tau_1\rangle$ and hence

$$\langle(\mathcal{J}_H\tau_1)\,\nabla_{\tau_\alpha}\tau_1,\tau_1\rangle + \langle\nabla_{\tau_\alpha}\,(\mathcal{J}_H\tau_1)\,\tau_1,\tau_1\rangle = -\langle(\mathcal{J}_H\tau_1)\,\tau_1,\nabla_{\tau_\alpha}\tau_1\rangle \qquad \forall\ \alpha\in I_V\,.$$

But since $\langle(\mathcal{J}_H\tau_1)\,\nabla_{\tau_\alpha}\tau_1,\tau_1\rangle = 0$, we obtain

$$\langle\nabla_{\tau_\alpha}\,(\mathcal{J}_H\tau_1)\,\tau_1,\tau_1\rangle = -\langle(\mathcal{J}_H\tau_1)\,\tau_1,\nabla_{\tau_\alpha}\tau_1\rangle = -\left\langle\nabla^H_{\tau_1}\tau_1, grad_H\,\varpi_\alpha\right\rangle.$$

By using (i) of Lemma 3.13, it follows that $\nabla^H_{\tau_1}\tau_1 = -C_H(\varpi_{H_2})\tau_1$ and so, by adding the quantity $\langle C_H(\varpi_{H_2})\tau_1, grad_{HS}\,\varpi_\alpha\rangle$, we finally get the identity $\mathcal{L}_{HS}\,\varpi_\alpha = -\varpi_\alpha\tau_1\,(\mathcal{H}_H)$. The quantity $\tau_1\,(\mathcal{H}_H)$ can now be calculated by repeating the calculations made in the proof of the 2nd variation formula. More precisely, we have

$$\begin{aligned}&-\tau_1\,(\mathcal{H}_H)\\ &= div_{HS}\,(C_H(\varpi_{H_2})\tau_1) - \left|C_H(\varpi_{H_2})\tau_1\right|^2 + \|A_H\|^2_{\mathrm{Gr}} - \|S_H\|^2_{\mathrm{Gr}} + \sum_{j\in I_{HS}\,\alpha\in I_V}\left\langle C^\alpha_H\tau_1,\tau_j\right\rangle\left(\left\langle C(\varpi)\tau_\alpha,\tau_j\right\rangle - \tau_j(\varpi_\alpha)\right)\\ &= -\|A_H\|^2_{\mathrm{Gr}} - \|S_H\|^2_{\mathrm{Gr}} - \sum_{\alpha\in I_V}\left\langle(2grad_{HS}\,(\varpi_\alpha) - C(\varpi)\tau^{TS}_\alpha)\,,C^\alpha\tau_1\right\rangle = -\mathcal{B}_{TS}\,.\end{aligned}$$

□

In Section 6.1, just as an exercise, we will reprove this identity for the class of non-vertical hyperplanes

$$\mathcal{I}_{\alpha'} := \left\{x = exp\left(\sum_{j=1}^n x_j\right)\in\mathbb{G} : x_{\alpha'} = 0\right\},$$

where $\alpha'\in I_V$; see Definition 1.15. However, for the sake of simplicity, this will be done only for Carnot groups of step 2. We recall that these hyperplanes are very different from the vertical ones and, for instance, they turn out to be characteristic at the identity $0\in\mathbb{G}$.

Now let us state an immediate consequence of the previous lemma. To this aim, let $V\in\mathfrak{X}(\mathbb{G})$ be a constant left invariant vector field.

Corollary 5.6. *Let $S\subset\mathbb{G}$ be a non-characteristic hypersurface of class $\mathbf{C}^2$. Moreover, we suppose that the horizontal mean curvature $\mathcal{H}_H$ is constant. Then the function $f_V := \langle V,\varpi\rangle$ satisfies the equation $-\mathcal{L}_{HS}\,f_V = f_V\mathcal{B}_{TS}$ at each non-characteristic point.*

6. Stability of H-minimal hypersurfaces

Definition 6.1 (Stability)**.** *Let $\mathbb{G}$ be a k-step Carnot group and let $S\subset\mathbb{G}$ be a H-minimal hypersurface of class $\mathbf{C}^3$, i.e. $\mathcal{H}_H = 0$.*

- (S_1) *Let $C_S = \emptyset$. We say that S is* stable *if $II_S(\sigma^{n-1}_H)\geq 0$ for every compactly supported variation $\vartheta_t :]-\epsilon,\epsilon[\times S\longrightarrow\mathbb{G}$ of class $\mathbf{C}^3$.*
- (S_2) *Let $C_S\neq\emptyset$. In this case, we assume that $\frac{1}{|\mathcal{P}_H\nu|^2}\in L^2_{loc}(S,\sigma^{n-1}_H)$ and that there exists a family $\{\mathcal{U}_\delta\}_{\delta>0}$ of open subsets of S such that:*
  - (i) *$C_S\Subset\mathcal{U}_\delta$,*
  - (ii) *$\sigma^{n-1}_{\mathcal{R}}(\mathcal{U}_\delta)\longrightarrow 0$ as long as $\delta\to 0$,*
  - (iii) *$\partial\mathcal{U}_\delta$ is of class $\mathbf{C}^1$ and $\sigma^{n-2}_{\mathcal{R}}(\partial\mathcal{U}_\delta)\longrightarrow 0$ as long as $\delta\to 0$.*

Under these assumptions, we say that S is stable *if $II_S(\sigma_H^{n-1}) \geq 0$ for every compactly supported variation $\vartheta_t :]-\epsilon, \epsilon[\times S \longrightarrow \mathbb{G}$ of class $\mathbf{C}^3$ such that $\frac{1}{|\mathcal{P}_{H_t}\nu^t|} \in L^1_{loc}\left(S_t; (\sigma_{\mathcal{R}}^{n-1})_t\right)$ for every $t \in]-\epsilon, \epsilon[$, where ν^t denotes the outward-pointing unit normal along $S_t = \vartheta_t(S)$; see Corollary 4.15.*

Remark 6.2. *We shall sometimes say that S is* strictly stable *when the stability inequality is strict. Furthermore, if $C_S \neq \emptyset$ but we consider only compactly supported variations on $S^* := S \setminus C_S$, then (S_1) in Definition 6.1 can be applied to any non-characteristic domain $\Omega \Subset S^*$.*

Lemma 6.3. *Let $S \subset \mathbb{G}$ be as in Definition 6.1 and let us consider the following* linear eigenvalue problem, *i.e.*

$$\begin{cases} \mathcal{L}_{HS}\,\varphi + \lambda\,\mathcal{B}_{TS}\,\varphi = 0 & on\ S\\ \varphi = 0 & on\ \partial S. \end{cases}$$

Under the previous assumptions, a sufficient condition for stability of S is that the first (non-trivial) eigenvalue λ_1 of this problem is greater than or equal to 1.

Proof. This is an immediate consequence of the horizontal Green formula (6); see Corollary 2.7. □

The next lemma generalizes a well-known result in the Riemannian setting; see [28].

Lemma 6.4. *Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^2$ and let $\Omega \subset S$ be a bounded domain. If there exists a smooth function $\psi > 0$ on Ω satisfying the equation $\mathcal{L}_{HS}\,\psi = q\psi$, then*

$$\int_\Omega \left(|grad_{HS}\,\varphi|^2 + q\varphi^2\right)\sigma_H^{n-1} \geq 0$$

for all smooth function φ compactly supported on Ω.

Proof. If $\psi > 0$ satisfies $\mathcal{L}_{HS}\,\psi = q\psi$ on Ω, let us define a new function $\phi := log\,\psi$. By an elementary calculation we see that $\mathcal{L}_{HS}\,\phi = q - |grad_{HS}\,\phi|^2$. More precisely, we have

$$\begin{aligned}
\mathcal{L}_{HS}\,\phi &= div_{HS}\,(grad_{HS}\,\phi) + \langle C_H(\varpi_{H_2})\nu_H, grad_{HS}\,\phi\rangle\\
&= div_{HS}\left(\frac{grad_{HS}\,\psi}{\psi}\right) + \left\langle C_H(\varpi_{H_2})\nu_H, \frac{grad_{HS}\,\psi}{\psi}\right\rangle\\
&= \left(\frac{\Delta_{HS}\,\psi}{\psi}\left\langle C_H(\varpi_{H_2})\nu_H, \frac{grad_{HS}\,\psi}{\psi}\right\rangle\right) - \frac{|grad_{HS}\,\psi|^2}{\psi^2}\\
&= \frac{\mathcal{L}_{HS}\,\psi}{\psi} - |grad_{HS}\,\phi|^2\\
&= q - |grad_{HS}\,\phi|^2.
\end{aligned}$$

So let φ be a smooth function with compact support on Ω. Multiplying by $-\varphi^2$ both sides of this equation and integrating by parts, yields

$$(35)\quad -\int_\Omega \varphi^2\left(q - |grad_{HS}\,\phi|^2\right)\sigma_H^{n-1} = -\int_\Omega \varphi^2\mathcal{L}_{HS}\,\phi\,\sigma_H^{n-1} = \int_\Omega 2\varphi\,\langle grad_{HS}\,\varphi, grad_{HS}\,\phi\rangle\,\sigma_H^{n-1},$$

where we have used Corollary 2.7. Since

$$2|\varphi\,\langle grad_{HS}\,\varphi, grad_{HS}\,\phi\rangle\,| \leq 2|\varphi||grad_{HS}\,\varphi||grad_{HS}\,\phi| \leq |\varphi|^2|grad_{HS}\,\phi|^2 + |grad_{HS}\,\varphi|^2,$$

the thesis follows by inserting the last inequality into (35) and by canceling the terms $\int_\Omega \varphi^2|grad_{HS}\,\phi|^2\,\sigma_H^{n-1}$. □

Remark 6.5. *Lemma 6.4 can be generalized to the case where ψ is smooth only on $S^* = S\setminus C_S$. However, in such a case, we have to restrict ourselves to the class of smooth compactly supported functions φ on Ω such that the function $\phi\,\varphi^2$ is admissible.*

As a consequence of Lemma 5.5 and Lemma 6.4, we can infer an interesting condition for stability.

Theorem 6.6. *Let $S \subset \mathbb{G}$ be a H-minimal hypersurface of class $\mathbf{C}^3$. If there exists $\alpha \in I_V$ such that either $\varpi_\alpha > 0$ or $\varpi_\alpha < 0$ on S, then each non-characteristic domain $\Omega \subset S$ turns out to be stable.*

Proof. By applying Lemma 6.4 to the function ϖ_α we immediately get the stability inequality

$$II_S(W, \sigma_H^{n-1}) \geq 0$$

for every non-zero compactly supported variation ϑ_t of S. □

We have the following reformulations of Theorem 6.6:

Corollary 6.7. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^3$-smooth H-minimal hypersurface. Let $V \in \mathfrak{X}(\mathbb{G})$ be a constant left invariant vector field and set $f_V = \langle V, \varpi\rangle$. If either $f_V > 0$ or $f_V < 0$, then each non-characteristic domain $\Omega \subset S$ is stable.*

Corollary 6.8. *Let $S \subset \mathbb{G}$ be a complete H-minimal hypersurface of class $\mathbf{C}^3$. If S is a graph with respect to some given vertical direction, then each non-characteristic domain $\Omega \subset S$ is stable.*

Below we shall study some (more or less simple) examples in order to illustrate some of our results.

6.1. **Examples.** Our first example, which is that of *vertical hyperplanes,* is the simplest one and, to the best of our knowledge, the only known in literature outside the Heisenberg group setting. Roughly speaking, vertical hyperplanes are, by definition, level-sets of linear homogeneous polynomial having (homogeneous) degree 1, which are ideals of the Lie algebra $\mathfrak{g}$.

We claim that they are (strictly) stable hypersurfaces. This immediately follows from the fact that $\mathcal{B}_{TS} = 0$. Hence, for any regular bounded domain $\mathcal{U}$ contained on a vertical hyperplane $\mathcal{I}$, we have $II_{\mathcal{U}}(W, \sigma_H^{n-1}) = \int_{\mathcal{U}} |grad_{HS}\, w|^2\, \sigma_H^{n-1} \geq 0$, with equality if, and only if, $w = 0$.

Corollary 6.9. *Let $\mathbb{G}$ be a k-step Carnot group. Any vertical hyperplane turns out to be a $\mathbf{C}^\infty$-smooth strictly stable H-minimal non-characteristic hypersurface.*

Now we analyze a completely different family of hyperplanes. From an intrinsic point of view, they are homogeneous "cones", which turn out to be characteristic at a single point. For the sake of simplicity, we just consider the case of 2-step Carnot groups. So we have $\mathfrak{g} = H \oplus V$ ($\dim H = h$, $\dim V = n - h$) and we may assume that

$$X_i(x) := \mathrm{e}_i + \frac{1}{2}\sum_{\alpha \in I_V} \langle C_H^\alpha \mathrm{e}_i, x_H\rangle\, \mathrm{e}_\alpha, \qquad X_\alpha = \mathrm{e}_\alpha$$

for every $i \in I_H = \{1, ..., h\}$ and every $\alpha \in I_V = \{h+1, ..., n\}$, where $\mathrm{e}_j = (0, ..., \underbrace{1}_{j-thplace}, ...0)$, $j = 1..., n$, is the j-th vector of the canonical basis of $\mathbb{R}^n \cong \mathfrak{g}$ and $x_H \equiv (x_1, ..., x_h)$ is the horizontal position vector. As usual, we identify vector fields and differential operators.

Fix $\alpha' \in I_V$ and consider the *non-vertical hyperplane* $\mathcal{I}_{\alpha'} := \left\{x = exp\left(\sum_j x_j\right) \in \mathbb{G} : x_{\alpha'} = 0\right\}$. We have $grad_H\, x_{\alpha'} = -\frac{1}{2} C_H^{\alpha'} x_H$ and so $\nu_{_H} = \frac{-C_H^{\alpha'} x_H}{|C_H^{\alpha'} x_H|}$. Moreover, $\varpi_\beta = 0$ for all $\beta \neq \alpha'$ and $\varpi_{\alpha'} = \frac{2}{|C_H^{\alpha'} x_H|}$. Since

$$div_H\left(C_H^{\alpha'} x_H\right) = \sum_{j \in I_H} \left\langle \nabla_{X_j} C_H^{\alpha'} x_H, X_j\right\rangle = \sum_{j \in I_H} \left\langle C_H^{\alpha'} X_j, X_j\right\rangle = 0$$

and

$$\left\langle grad_H\left(\frac{1}{|C_H^{\alpha'} x_H|}\right), C_H^{\alpha'} x_H\right\rangle = -\left\langle \frac{grad_H |C_H^{\alpha'} x_H|}{|C_H^{\alpha'} x_H|^2}, C_H^{\alpha'} x_H\right\rangle = \left\langle \frac{C_H^{\alpha'} \nu_{_H}}{|C_H^{\alpha'} x_H|}, \nu_{_H}\right\rangle = 0,$$

it follows that $\mathcal{H}_H = -div_H\, \nu_{_H} = 0$, i.e. $\mathcal{I}_{\alpha'}$ is *H-minimal*. The above calculation also shows that $grad_H\left(|C_H^{\alpha'} x_H|\right) = C_H^{\alpha'} \nu_{_H}$. Furthermore, we easily get that

$$-\mathcal{J}_H\, \nu_{_H} = \frac{C_H^{\alpha'} + \nu_{_H} \otimes C_H^{\alpha'} \nu_{_H}}{|C_H^{\alpha'} x_H|},$$

which, in turn, implies

$$B_H(\tau_i,\tau_j)=\left\langle\frac{C^{\alpha'}_{HS}}{|C^{\alpha'}_{H}x_H|}\tau_i,\tau_j\right\rangle=A_H(\tau_i,\tau_j)\qquad\forall\ i,j\in I_{HS}.$$

Therefore $S_H=\mathbf{0}_H$ (i.e. the 0-matrix on H) and $\|B_H\|^2_{\mathrm{Gr}}=\|A_H\|^2_{\mathrm{Gr}}=\frac{\varpi^2_{\alpha'}\|C^{\alpha'}_{HS}\|^2_{\mathrm{Gr}}}{4}$. So it remains to compute the quantity $\Upsilon:=-\sum_{\alpha\in I_V}\langle(2grad_{HS}(\varpi_\alpha)-C(\varpi)\tau^{TS}_\alpha),C^\alpha\tau_1\rangle$; see formula (33). Because of the 2-step assumption, we have

$$\Upsilon=-\sum_{\alpha\in I_V}\langle(2grad_{HS}(\varpi_\alpha)+\varpi_\alpha C(\varpi)\tau_1),C^\alpha\tau_1\rangle.$$

From the previous calculations, it follows that $\Upsilon=0$ and so $\mathcal{B}_{TS}=\|A_H\|^2_{\mathrm{Gr}}$. In other words, we have

$$II_{\mathcal{U}}(W,\sigma^{n-1}_H)=\int_{\mathcal{U}}\left(|grad_{HS}w|^2-w^2\|A_H\|^2_{\mathrm{Gr}}\right)\sigma^{n-1}_H=\int_{\mathcal{U}}\left(|grad_{HS}w|^2-w^2\frac{\varpi^2_{\alpha'}\|C^{\alpha'}_{HS}\|^2_{\mathrm{Gr}}}{4}\right)\sigma^{n-1}_H$$

for any non-characteristic bounded domain $\mathcal{U}\subset\mathcal{I}_{\alpha'}(\equiv S)$, where $\sigma^{n-1}_H\llcorner\mathcal{I}_{\alpha'}=\frac{|C^{\alpha'}_H x_H|}{2}\,d\mathcal{L}^{n-1}_{\mathrm{Eu}}\llcorner\mathcal{I}_{\alpha'}$ and

$$d\mathcal{L}^{n-1}_{\mathrm{Eu}}=dx_1\wedge dx_2\wedge\ldots\wedge\ldots\wedge\widehat{dx_{\alpha'}}\wedge\ldots\wedge dx_n.$$

It goes without saying that the previous formula holds true near the characteristic set only under the assumptions made in Corollary 4.15. In particular, we have to check that $\int_{\mathcal{U}}\frac{1}{|\mathcal{P}_H\nu|^4}\sigma^{n-1}_H<+\infty$, which is clearly equivalent to the next condition:

$$\int_{\mathcal{U}}\frac{1}{|C^{\alpha'}_H x_H|^3}\,d\mathcal{L}^{n-1}_{\mathrm{Eu}}\llcorner\mathcal{I}_{\alpha'}<+\infty.\tag{36}$$

Two remarks are in order:

- a necessary condition for the validity of (36) is that the dimension of H is ≥ 4, i.e.

$$h=\dim H\geq 4;\tag{37}$$

- in the Heisenberg group $\mathbb{H}^n$, the previous analysis reduces to the case of the horizontal hyperplane $\{p=exp(z,t)\in\mathbb{H}^n:t=0\}$ and, in this case, (37) is also sufficient for (36) to hold.

Now let us compute

$$\begin{aligned}\Delta_{HS}\left(\frac{1}{|C^{\alpha'}_H x_H|}\right)&=-div_{HS}\left(\frac{C^{\alpha'}_H\nu_H}{|C^{\alpha'}_H x_H|^2}\right)\\&=\frac{2|C^{\alpha'}_H\nu_H|^2}{|C^{\alpha'}_H x_H|^3}-\frac{div_{HS}(C^{\alpha'}_H\nu_H)}{|C^{\alpha'}_H x_H|^2}\\&=\frac{2|C^{\alpha'}_H\nu_H|^2}{|C^{\alpha'}_H x_H|^3}+\frac{\sum_{j,k\in I_{HS}}\left\langle\nabla^H_{\tau_j}\tau_1,\tau_k\right\rangle\left\langle C^{\alpha'}_H\tau_j,\tau_k\right\rangle}{|C^{\alpha'}_H x_H|^2}\\&=\frac{2|C^{\alpha'}_H\nu_H|^2-\|C^{\alpha'}_{HS}\|^2_{\mathrm{Gr}}}{|C^{\alpha'}_H x_H|^3}.\end{aligned}$$

From this computation and the very definition of $\mathcal{L}_{HS}$, it follows that

$$\mathcal{L}_{HS}\left(\frac{1}{|C^{\alpha'}_H x_H|}\right)=\Delta_{HS}\left(\frac{1}{|C^{\alpha'}_H x_H|}\right)+\left\langle C_H(\varpi_{H_2})\nu_H,grad_{HS}\left(\frac{1}{|C^{\alpha'}_H x_H|}\right)\right\rangle=-\frac{\|C^{\alpha'}_{HS}\|^2_{\mathrm{Gr}}}{|C^{\alpha'}_H x_H|^3},$$

which is equivalent to the equation $\mathcal{L}_{HS}\varpi_{\alpha'}=-\varpi_{\alpha'}\|A_H\|^2_{\mathrm{Gr}}$, as predicated by Lemma 5.5.

The previous discussion is summarized in the following:

Corollary 6.10. *Let $\mathbb{G}$ be a 2-step Carnot group and let $\mathcal{I}_{\alpha'}$ be a horizontal hyperplane. Then $\mathcal{I}_{\alpha'}$ is a $\mathbf{C}^{\infty}$-smooth H-minimal hypersurface. The only characteristic point of $\mathcal{I}_{\alpha'}$ is $0 \in \mathbb{G}$. Furthermore, any bounded domain $\mathcal{U} \Subset \mathcal{I}_{\alpha'} \setminus \{0\}$ turns out to be strictly stable.*

6.2. **An example in the Heisenberg group $\mathbb{H}^n$.** For the notation used in this section we refer the reader to Example 1.7 and Example 1.12. We recall that any point $p \in \mathbb{H}^n$ is identified with $(z, t) \in \mathbb{R}^{2n+1}$, where $z = (x_1, y_1, x_2, y_2, ..., x_n, y_n)$. We use the following further notation:

$$\overline{v}^{1,0} := (v_1, 0, v_2, 0, ..., v_n, 0) \in \mathbb{R}^{2n}, \qquad \overline{v}^{0,1} := (0, v_1, 0, v_2, 0, ..., 0, v_n) \in \mathbb{R}^{2n} \qquad \forall\ v = (v_1, v_2, ..., v_n) \in \mathbb{R}^n.$$

Using this notation yields $z = \overline{x}^{1,0} + \overline{y}^{0,1} \in \mathbb{R}^{2n}$, where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. In the sequel, we shall study the following *hyperbolic paraboloid:*

$$S := \left\{ p \equiv (z, t) \in \mathbb{H}^n : t = \frac{\left\|\overline{x}^{1,0}\right\|^2_{\mathbb{R}^n} - \left\|\overline{y}^{1,0}\right\|^2_{\mathbb{R}^n}}{4} \right\}, \tag{38}$$

First, note that $grad_{H} t = \frac{z^{\perp}}{2}$, where $z^{\perp} := -C_H^{2n+1} z$. Furthermore, a simple calculation shows that $grad_H \left(\frac{\|\overline{x}^{1,0}\|^2_{\mathbb{R}^n} - \|\overline{y}^{1,0}\|^2_{\mathbb{R}^n}}{4} \right) = \frac{1}{2}\left(\overline{x}^{1,0} - \overline{y}^{0,1}\right)$ and hence $\nu_{H} = \frac{-\overline{v}^{1,0} + \overline{v}^{0,1}}{|-\overline{v}^{1,0} + \overline{v}^{0,1}|}$, where we have set $v = x + y \in \mathbb{R}^n$. Therefore

$$\nu_{H} = \frac{\sqrt{2}}{2} \left(\frac{-\overline{v}^{1,0} + \overline{v}^{0,1}}{\sqrt{\rho^2 + 2\langle x, y\rangle_{\mathbb{R}^n}}} \right), \qquad \nu_{H}^{\perp} = -\frac{\sqrt{2}}{2} \left(\frac{\overline{v}^{1,0} + \overline{v}^{0,1}}{\sqrt{\rho^2 + 2\langle x, y\rangle_{\mathbb{R}^n}}} \right),$$

where $\sqrt{\rho^2 + 2\langle x, y\rangle_{\mathbb{R}^n}} = \|x + y\|_{\mathbb{R}^n}$ and $\rho := \sqrt{\|x\|^2_{\mathbb{R}^n} + \|y\|^2_{\mathbb{R}^n}}$. Clearly, the characteristic set C_S of S is the set of all points $p \equiv (z, t) \in S$ such that $\overline{x + y}^{1,0} = \overline{x + y}^{0,1} = 0 \in \mathbb{R}^{2n}$. Hence $p \equiv (z, t) \in C_S$ if, and only if $x_i = -y_i$ for every $i = 1, ..., n$. Since $X_i\left(\frac{1}{\|x+y\|_{\mathbb{R}^n}}\right) = Y_i\left(\frac{1}{\|x+y\|_{\mathbb{R}^n}}\right)$, we easily get that $div_H \nu_{H} = 0$, i.e. S turns out to be *H-minimal.*

We have $\varpi = \frac{\sqrt{2}}{\|x+y\|_{\mathbb{R}^n}}$ and $\frac{\partial \varpi}{\partial \nu_H^{\perp}} = \frac{2}{\|x+y\|^2_{\mathbb{R}^n}}$. In order to calculate the horizontal 2nd fundamental form B_H (and some of its invariants) we need the horizontal Jacobian matrix $\mathcal{J}_H \nu_{H} =: \left[a_{ij}\right]_{i,j \in I_H}$ of the H-normal ν_{H}. For the sake of simplicity, we treat the case $n = 2$, which corresponds to the 2nd Heisenberg group. The general case is completely analogous. We have

- $a := a_{11} = a_{12} = -\frac{\sqrt{2}}{2}\left(\frac{\|x+y\|^2_{\mathbb{R}^2} - (x_1+y_1)^2}{\|x+y\|^3_{\mathbb{R}^2}}\right), \quad b := a_{13} = a_{14} = -\frac{\sqrt{2}}{2}\left(\frac{-(x_1+y_1)(x_2+y_2)}{\|x+y\|^3_{\mathbb{R}^2}}\right);$
- $a_{2j} = -a_{1j}$ for every $j = 1, ..., 4$;
- $a_{31} = a_{32} = -\frac{\sqrt{2}}{2}\left(\frac{-(x_1+y_1)(x_2+y_2)}{\|x+y\|^3_{\mathbb{R}^2}}\right), \quad c := a_{33} = a_{34} = -\frac{\sqrt{2}}{2}\left(\frac{\|x+y\|^2_{\mathbb{R}^2} - (x_2+y_2)^2}{\|x+y\|^3_{\mathbb{R}^2}}\right);$
- $a_{4j} = -a_{3j}$ for every $j = 1, ..., 4$.

Equivalently, $\mathcal{J}_H \nu_{H} = \begin{bmatrix} a & a & b & b \\ -a & -a & -b & -b \\ b & b & c & c \\ -b & -b & -c & -c \end{bmatrix}$. It follows that $\nu_{H} \in \mathrm{K}er\mathcal{J}_H \nu_{H}$ and hence $B_H = -\mathcal{J}_H \nu_{H}$. By definition, we have $S_H = -\left(\frac{\mathcal{J}_H \nu_H + (\mathcal{J}_H \nu_H)^{\mathrm{Tr}}}{2}\right) = -\begin{bmatrix} a & 0 & b & 0 \\ 0 & -a & 0 & -b \\ b & 0 & c & 0 \\ 0 & -b & 0 & -c \end{bmatrix}$. So if $n = 2$, we have

$$\|B_H\|^2_{\mathrm{Gr}} = 4\left(a^2 + 2b^2 + c^2\right) = \frac{2}{\|x + y\|^2_{\mathbb{R}^n}}, \qquad \|S_H\|^2_{\mathrm{Gr}} = 2\left(a^2 + 2b^2 + c^2\right) = \frac{1}{\|x + y\|^2_{\mathbb{R}^n}} = \|A_H\|^2_{\mathrm{Gr}}.$$

In the general case, an analogous calculation gives $\|B_H\|^2_{\mathrm{Gr}} = \frac{2(n-1)}{\|x+y\|^2_{\mathbb{R}^n}}$ and $\|S_H\|^2_{\mathrm{Gr}} = \|A_H\|^2_{\mathrm{Gr}} = \frac{n-1}{\|x+y\|^2_{\mathbb{R}^n}}$. Therefore, using (34) yields

$$\begin{aligned}\mathcal{B}_{TS} &= \|S_H\|^2_{\mathrm{Gr}} - \left(2\frac{\partial\varpi}{\partial\nu_H^\perp} - \frac{n+1}{2}\varpi^2\right)\\ &= \frac{n-1}{\|x+y\|^2_{\mathbb{R}^n}} - \left(\frac{4}{\|x+y\|^2_{\mathbb{R}^n}} - \frac{n+1}{\|x+y\|^2_{\mathbb{R}^n}}\right)\\ &= \frac{2(n-2)}{\|x+y\|^2_{\mathbb{R}^n}}.\end{aligned}$$

So we have found that

$$II_{\mathcal{U}}(W,\sigma_H^{2n}) = \int_{\mathcal{U}}\left(|grad_{HS}\,w|^2 - w^2\frac{2(n-2)}{\|x+y\|^2_{\mathbb{R}^n}}\right)\sigma_H^{2n}, \tag{39}$$

for any non-characteristic bounded domain $\mathcal{U}\subset S$, where $\sigma_H^{2n} = \frac{\|x+y\|_{\mathbb{R}^n}}{\sqrt{2}}\,dz$ and we have set

$$dz = dx_1\wedge dy_1\wedge ...\wedge dx_n\wedge dy_n.$$

Remark 6.11 (Failure of $\int_{\mathcal{U}}\frac{1}{|\mathcal{P}_H\nu|^4}\,\sigma_H^{n-1} < +\infty$ for characteristic domains)**.** *In order to apply the 2nd variation formula for any characteristic domain $\mathcal{U}\subset S$, we need at least to check that*

$$\int_{\mathcal{U}}\frac{1}{\|x+y\|^3_{\mathbb{R}^n}}\,d\mathcal{L}^{2n}_{\mathrm{Eu}} < +\infty. \tag{40}$$

However, in general, this condition fails to hold if $C_{\mathcal{U}}\neq\emptyset$.

Lemma 5.1 says $\Delta_{HS}\,\varpi = \Delta_H\,\varpi - \langle \mathrm{Hess}_H\,\varpi\,\nu_H,\nu_H\rangle$. Since $grad_H\,\varpi = -\sqrt{2}\left(\frac{\overline{(x+y)}^{1,0}+\overline{(x+y)}^{0,1}}{\|x+y\|^3_{\mathbb{R}^n}}\right)$, we easily get that $\Delta_H\,\varpi = -\varpi\frac{(2n-3)}{\|x+y\|^2_{\mathbb{R}^n}}$. Furthermore

$$\langle \mathrm{Hess}_H\,\varpi\,\nu_H,\nu_H\rangle = \left\langle \begin{bmatrix} 1 & 1 & 0 & 0 & \dots\\ 1 & 1 & 0 & 0 & \dots\\ 0 & 0 & 1 & 1 & \dots\\ 0 & 0 & 1 & 1 & \dots\\ \dots & \dots & \dots & \dots & \dots\end{bmatrix}\nu_H,\nu_H\right\rangle = -\frac{\varpi}{\|x+y\|^2_{\mathbb{R}^n}}.$$

All together, we have shown that

$$\mathcal{L}_{HS}\,\varpi = \Delta_{HS}\,\varpi - \varpi\frac{\partial\varpi}{\partial\nu_H^\perp} = -\varpi\frac{2(n-2)}{\|x+y\|^2_{\mathbb{R}^n}},$$

which illustrates the content of Lemma 5.5.

Corollary 6.12. *Let $S\subset\mathbb{H}^n$ be the hypersurface defined by* (38)*. Then S is a $\mathbf{C}^\infty$-smooth H-minimal. Furthermore, one has $C_S = \{p = exp(z,t)\in S : x_i = -y_i,\ i = 1,...,n\}$. Finally, any bounded domain $\mathcal{U}\Subset S\setminus C_S$ is strictly stable.*

7. A different sufficient condition for stability

Below we shall generalize a weak-stability result for minimal m-dimensional sub-manifolds of the Euclidean space $\mathbb{R}^n$, formulated in the Seventies by Spruck; see [66]. This criterion is based on a tricky application of the 2nd variation of the Riemannian m-dimensional volume together with the Sobolev-type inequality for minimal sub-manifolds proved by Michael and Simon in [46].

We have already discussed the 2nd variation formula for σ_H^{n-1}. Moreover, we will need a Sobolev-type inequality analogous to that of Michael and Simon. This result has been recently obtained in [53].

Let $\mathbb{G}$ be a k-step Carnot group equipped with the homogeneous norm defined by

$$\|y\|_{\varrho} = \left(\sum_{i=1}^{k} C_i |y_{H_i}|^{\frac{\lambda}{i}}\right)^{\frac{1}{\lambda}} \tag{41}$$

for every $y = exp\left(\sum_{i=1}^{k} y_{H_i}\right)$, where $|y_{H_i}|$ is the Euclidean norm on $H_i \cong \mathbb{R}^{h_i}$, λ is a positive integer evenly divisible by $i = 1, ..., k$ and we suppose $C_1 = 1$ and $C_i > 0$ for every $i = 2, ..., k$.

Theorem 7.1 (see [53])**.** *Let $\mathbb{G}$ be a k-step Carnot group equipped with the homogeneous norm defined by* (41)*. Let $S \subset \mathbb{G}$ be a H-minimal hypersurface of class $\mathbf{C}^2$ and assume that there exists a family $\{\mathcal{U}_\delta\}_{\delta>0}$ of open subsets of S such that:*

(i) $C_S \Subset \mathcal{U}_\delta$,
(ii) $\sigma^{n-1}_{\mathcal{R}}(\mathcal{U}_\delta) \longrightarrow 0$ *as long as* $\delta \to 0$,
(iii) $\partial\mathcal{U}_\delta$ *is of class* $\mathbf{C}^1$ *and* $\sigma^{n-2}_{\mathcal{R}}(\partial\mathcal{U}_\delta) \longrightarrow 0$ *as long as* $\delta \to 0$.

Then there exists $C_1 > 0$ such that

$$\left(\int_S |\psi|^{\frac{Q-1}{Q-2}} \sigma^{n-1}_H\right)^{\frac{Q-2}{Q-1}} \leq C_1 \int_S |grad_{HS}\,\psi|\, \sigma^{n-1}_H. \tag{42}$$

for every $\psi \in \mathbf{C}^2_0(S)$. Furthermore, there exists $C_2 > 0$ such that

$$\|\psi\|_{L^{p^*}(S)} \leq C_2 \|grad_{HS}\,\psi\|_{L^p(S)} \tag{43}$$

for every $\psi \in \mathbf{C}^2_0(S)$, where we have set $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{Q-1}$, for every $p > 0$.

We use only the case $p = 2$.

Theorem 7.2. *Let $S \subset \mathbb{G}$ be a H-minimal hypersurface of class $\mathbf{C}^3$ satisfying the assumptions made in Corollary 4.15. There exists a dimensional constant C_0 such that if*

$$\int_S |\mathcal{B}_{TS}|^{\frac{Q-1}{2}}\, \sigma^{n-1}_H < C_0,$$

then S is strictly stable.

Proof. We begin by assuming that there exists $C_0 > 0$ such that

$$\int_S |\mathcal{B}_{TS}|^{\frac{Q-1}{2}}\, \sigma^{n-1}_H < C_0. \tag{44}$$

Now we argue by contradiction. So we have

$$\int_S |grad_{HS}\, w|^2 \sigma^{n-1}_H \leq \int_S w^2\, |\mathcal{B}_{TS}|\sigma^{n-1}_H \tag{45}$$

for some (smooth) test function $w \neq 0$. Then, using the above isoperimetric inequality (with $p = 2$) yields

$$\left(\int_S |w|^{\frac{2(Q-1)}{Q-3}} \sigma^{n-1}_H\right)^{\frac{Q-3}{2(Q-1)}} \leq C_2 \sqrt{\int_S |grad_{HS}\, w|^2 \sigma^{n-1}_H} \leq C_2 \sqrt{\int_S w^2\, |\mathcal{B}_{TS}|\sigma^{n-1}_H}. \tag{46}$$

By Hölder inequality we obtain

$$\int_S w^2\, |\mathcal{B}_{TS}|\sigma^{n-1}_H \leq \left(\int_S |\mathcal{B}_{TS}|^{\frac{Q-1}{2}}\, \sigma^{n-1}_H\right)^{\frac{2}{Q-1}} \left(\int_S w^{\frac{2(Q-1)}{Q-3}}\, \sigma^{n-1}_H\right)^{\frac{Q-3}{Q-1}},$$

which together with (46) gives us

$$\left(\int_S |w|^{\frac{2(Q-1)}{Q-3}}\, \sigma^{n-1}_H\right)^{\frac{Q-3}{2(Q-1)}} \leq C_2 \left(\int_S |\mathcal{B}_{TS}|^{\frac{Q-1}{2}}\, \sigma^{n-1}_H\right)^{\frac{1}{Q-1}} \left(\int_S w^{\frac{2(Q-1)}{Q-3}}\, \sigma^{n-1}_H\right)^{\frac{Q-3}{2(Q-1)}}.$$

Hence $\left(\int_S |\mathcal{B}_{TS}|^{\frac{Q-1}{2}}\sigma_H^{n-1}\right)^{\frac{1}{Q-1}} \geq \dfrac{1}{C_2}$. Set $C_0 := \frac{1}{(C_2)^{Q-1}}$. The previous argument shows that, under the assumption (44), it must be

$$\int_S |grad_{HS}\, w|^2\sigma_H^{n-1} > \int_S w^2\, |\mathcal{B}_{TS}|\sigma_H^{n-1} \tag{47}$$

for all test function $w \neq 0$. This achieves the proof. □

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