DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **35**, Number **9**, September **2015**

doi:10.3934/dcds.2015.35.4293

pp. **4293–4322**

HIGHER ORDER DISCRETE CONTROLLABILITY AND THE APPROXIMATION OF THE MINIMUM TIME FUNCTION

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Abstract. We give sufficient conditions to reach a target for a suitable discretization of a control affine nonlinear dynamics. Such conditions involve higher order Lie brackets of the vector fields driving the state and so the discretization method needs to be of a suitably high order as well. As a result, the discrete minimal time function is bounded by a fractional power of the distance to the target of the initial point. This allows to use methods based on Hamilton-Jacobi theory to prove the convergence of the solution of a fully discrete scheme to the (true) minimum time function, together with error estimates. Finally, we design an approximate suboptimal discrete feedback and provide an error estimate for the time to reach the target through the discrete dynamics generated by this feedback. Our results make use of ideas appearing for the first time in [3] and now extensively described in [12]. Numerical examples are presented.

1. **Introduction.** Let $S \subset \mathbb{R}^n$ be closed and consider the minimum time $T(\xi)$ to reach S subject to the controlled dynamics $\dot{x} = f(x,u)$, $u \in U \subset \mathbb{R}^m$, starting from $x(0) = \xi$. The work [1] opened the door to the approximation of the minimum time function T through numerical schemes for a suitable boundary value problem of Hamilton-Jacobi type. The first paper on this subject was [3], where a semidiscrete scheme was developed under the assumption of Lipschitz continuity of T in a neighborhood of S. Such requirement is equivalent to the so called Petrov controllability condition, which essentially amounts to saying that for all $x \in \partial S$ there exists a control u_x such that $f(x,u_x)$ makes a negative (bounded away from zero) scalar product with an external normal to S. Equivalently – forgetting for a moment the regularity to be imposed on S and f to say that – for every x close enough to S there exists a control u_x such that $f(x,u_x)$ *points towards* S.

Finer controllability conditions are well known in the literature, in particular when S is an equilibrium point of the dynamics and $f(x,u) = f_0(x) + g(x)u$. They are called *higher order* conditions, in the sense that for every x close enough to S there exists a Lie bracket of the vector fields which points towards S. If this Lie

2010 *Mathematics Subject Classification.* 49J52, 49M25, 49L20.

Key words and phrases. Higher order scheme, Lie brackets, error estimates, approximate feedback, minimum time.

This work was partially supported by CARIPARO Project "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems", and University of Padova research project "Some analytic and differential geometric aspects in Nonlinear Control Theory, with applications to Mechanics". The authors are also supported by the European Union under the 7th Framework Programme "FP7-PEOPLE-2010-IT", Grant agreement number 264735-SADCO, and the first author is a member of the GNAMPA group of INdAM. The second author is supported by a PhD fellowship for foreign students funded by Fondazione CARIPARO.

bracket can be approximated by admissible trajectories of the controlled dynamics, by switching between suitable controls $\pm u$, then one can prove that it is possible to reach S in finite time from a neighborhood, and T is Hölder continuous with a suitable exponent depending on the maximal order of the Lie brackets. This is the case, for example, if S is the origin and the dynamics is linear and satisfies the classical Kalman rank condition. More in general, in [17, 18], see also references therein, higher order controllability conditions were established for a rather general target and some classes of nonlinear, control affine, dynamics, and several examples were presented.

Taking into account the above discussion, we believe it is natural generalizing to the case of higher order controllability numerical methods which were established under the first order condition. To this aim, all the PDE tools were developed in [1, 3, 4]. The idea is first considering a one step discretization method for the dynamics, namely a discrete controlled dynamical system which approximates the given continuous time system. If one can reach the target S, subject to this approximate discrete dynamics, within a time which is bounded by a fractional power of the distance to S of the initial point, then the approximate time converges to the true one as the time discretization step tends to zero. What remains to do, then, is transferring to a suitable discrete approximate dynamics the controllability which holds for the continuous time system. In the k-th ($k \geq 1$) order case, at each step the gain in the distance to the target is a k-th power of the time length. Thus, the order of the numerical scheme must be at least $k+1$, in order not to destroy this gain. This is exactly what is done here: we prove that some controllability conditions on the original dynamics are also sufficient for a suitable one step discretization to reach the target and prove the desired estimate on the time (see Sections 4.1, 4.2, and 4.3). That given, a fully discrete approximation together with error estimates follows from well established arguments (see Sections 4.4 and 4.5).

The last part (Section 5) of the paper is devoted to the design of an approximate feedback. It is well known that the steepest descent feedback (i.e., the feedback $u(x)$ suggested by the dynamic programming equation, see, e.g., [7]) is - in general - discontinuous, and so the O.D.E. $\dot{x} = f(x, u(x))$ may not admit solutions. Moreover, it is well known that generalized solutions (of Krasovskiĭ or Filippov type) are not always satisfactory, as they even may not reach the target (see, e.g. [20]). Following a well established method (see, e.g., [9, 12, 10]), the idea is substituting the continuous time dynamical system with a discrete one: this way, the problem of existence of solutions is bypassed. The approximate feedback is obtained, as one can expect, by choosing a control which minimizes a discretized Hamiltonian. Of course the point is proving that this strategy is suboptimal. To this aim, in order to be sure to reach the desired target S, one needs to consider the problem of reaching a suitable shrinking of S. In Section 3 we show that higher order sufficient conditions for both discrete and continuous time controllability are indeed robust with respect to a shrinking of S, provided the target is regular enough. Essentially we allow S to be nonsmooth but rule out outward angles and inward cusps; technically, we require S to be wedged and to satisfy a uniform internal sphere condition.

In Section 6, two 2-dimensional numerical examples are presented.

2. **Preliminaries.**

Let $S \subset \mathbb{R}^n$ be a closed set and $\delta > 0$ be given. We set, for $x \in \mathbb{R}^n$, $d_S(x) = \min\{\|y - x\| : y \in S\}$ and

$$S_\delta = \{x \in \mathbb{R}^n : d_S(x) \leq \delta\}.$$

Denoting S^c as the complement of S, $S^c = \mathbb{R}^n \setminus S$, we also define

$$S_{-\delta} = \{x \in \mathbb{R}^n : d_{\overline{S^c}}(x) \geq \delta\}.$$

Of course $S_{-\delta}$ may be empty, but we will consider mainly cases where this behavior does not occur. The interior of S will be denoted by $\operatorname{int} S$.

We recall now some concepts of nonsmooth analysis which will be used throughout the paper. Among many reference books on the subject we choose to quote here [8], which also contains an introduction to control problems. We say that $v \in \mathbb{R}^n$ is a proximal normal to S at $x \in S$ if there exists $\sigma = \sigma(v, x) \geq 0$ such that

$$\langle v, y - x\rangle \leq \sigma \|y - x\|^2, \quad \forall y \in S.$$

The set of such vectors is the proximal normal cone to S at x, $N_S^P(x)$. The cone of limiting normals is denoted by $N_S^L(x)$, and consists of those $v \in \mathbb{R}^n$ for which there exist sequences $\{x_i\} \subset S$, and $\{v_i\}$, with $v_i \in N_S^P(x_i)$, such that $x_i \to x$ and $v_i \to v$. This cone never trivializes if $x \in \partial S$, the boundary of S. If S is convex, then $N_S^P = N_S^L = N_S$, the normal cone of Convex Analysis. The Clarke normal cone $N_S^C(x)$ equals the closed convex hull of $N_S^L(x)$.

Let $\Omega \subseteq \mathbb{R}^n$ be given and $f\colon \Omega \to \mathbb{R}$ be an upper semicontinous function. Then the hypograph of f, $hypo(f)$, is (locally) closed and one can define (super-) differentiability concepts, through normal cones to $hypo(f)$. In particular, the proximal superdifferential of f, $\partial^P f(x)$, is the set of those $v \in \mathbb{R}^n$ such that $(-v, 1) \in N^P_{hypo(f)}(x, f(x))$.

Our controllability results will be largely based on some properties of the *distance function*, in connection with suitable regularity assumptions on the target. We now recall such assumptions.

Definition 2.1. (1) Let $S \in \mathbb{R}^n$ be closed and let $\rho > 0$. We say that S satisfies a ρ–internal sphere condition if S is the union of closed spheres of radius ρ, i.e., for any $x \in S$ there exists y such that $x \in \overline{B_\rho(y)} \subset S$.
(2) We say that S has reach ρ if the inequality

$$\langle v, y - x\rangle \leq \frac{\|v\|}{2\rho} \|y - x\|^2$$

holds, for every $x \in S, v \in N_S^P(x), y \in S$.

Relations between the above concepts were studied in detail in [19]. We recall, in particular, that if S has reach ρ, then the closure of its complement, $\overline{S^c}$, satisfies a ρ–internal sphere condition, but the converse is not true in general. The main property of the distance we are going to use is its *semiconcavity*. We say that a function $f\colon \Omega \to \mathbb{R}$ is locally semiconcave if for every $x \in \Omega$ there exists a ball $B_r(x)$ and a positive constant C such that

$$\lambda f(y) + (1 - \lambda) f(y') \leq f(\lambda y + (1 - \lambda) y') + C \|y - y'\|^2 \tag{2.1}$$

for all $y, y' \in B_r(x)$ and all $\lambda \in [0, 1]$. Global semiconcavity means that the above inequality is satisfied by every $y, y' \in \Omega$ such that the segment $[y, y'] \subset \Omega$ with the same constant C. The constant C appearing in (2.1) is labeled as semiconcavity constant.

The following results are well known (see, e.g., Proposition 2.2.2 in [6] and Section 4 in [13]).

Proposition 2.2. *Let $S \subset \mathbb{R}^n$ be closed. Then the distance function d_S satisfies the following properties:*

(i) *d_S is locally semiconcave in $\mathbb{R}^n \setminus S$. More precisely, given a set $K \subset \mathbb{R}^n \setminus S$ such that $\inf_{x\in K} d(x,S) = \delta > 0$, d_S is semiconcave in K with semiconcavity constant equal to $\frac{1}{\delta}$.*

(ii) *If S satisfies a ρ- internal sphere condition, then d_S is semiconcave in $\overline{S^c}$ with semiconcavity constant $\frac{1}{\rho}$.*

(iii) *If S has reach $\rho > 0$, then d_S is differentiable, and ∇d_S is locally Lipschitz, in $S_\rho \setminus S$.*

Using the metric projection, i.e., the set

$$\pi_S(x) = \{y \in S : \|y - x\| = d_S(x)\},$$

it is possible to characterize the (super-)differential of the distance function (see, e.g, Corollary 3.4.5 in [6] and Section 4 in [13]). We have, for all $x \in \mathbb{R}^n \setminus S$,

$$\partial^P d_S(x) = \frac{x - \operatorname{co}\pi_S(x)}{d_S(x)},$$

where "co" denotes the convex hull. Moreover, if S has reach $\rho > 0$ and $x \in S_\rho \setminus S$, then $\pi_S(x)$ is a singleton and

$$\nabla d_S(x) = \frac{x - \pi_S(x)}{d_S(x)},$$

and $\nabla d_S(x) \in N_S^P(\pi_S(x))$.

We now are going to recall some basic notations of control theory. Consider the control system

$$\dot{y}(t) = f(y(t), u(t)) \tag{2.2}$$

with $u(t) \in U$ for a.e. t, $U \subset \mathbb{R}^m$ a compact set, together with the initial condition

$$y(0) = \xi. \tag{2.3}$$

Under standard assumptions, for any $u(\cdot)$ measurable and any ξ, the solution $y(\cdot, \xi, u)$ of (2.2 and (2.3) is unique and globally defined. Let $S \subset \mathbb{R}^n$ be a nonempty compact set, the target. For each measurable control $u(\cdot)$ and $\xi \notin S$ we set

$$t(u,\xi) = \min\{t : y(t,\xi,u) \in S\} \le +\infty. \tag{2.4}$$

We define the minimum time function to reach S from ξ as

$$T(\xi) = \inf\{t(u,\xi) : u(t) \in U \text{ a.e., } u(\cdot) \text{ measurable}\}.$$

Under standard assumptions, the infimum is attained, provided it is not $+\infty$. We set also

$$\mathcal{R} = \{\xi \in \mathbb{R}^n : T(\xi) < +\infty\},$$

the reachable set.

In the next section we will recall some sufficient conditions in order that $\mathcal{R}$ be a neighborhood of S and T is Hölder continuous in $\mathcal{R}$.

3. Continuous time controllability.

3.1. Lie brackets and estimates on bang-bang trajectories.

Consider the affine control system in $\mathbb{R}^n$

$$\dot{x} = f(x) + \sum_{i=1}^{M} g_i(x)u_i, \tag{3.1}$$

where $f, g_i\colon \mathbb{R}^n \to \mathbb{R}^n$ are C^∞-maps and $u_i \in [-1,1]$, $i = 1, ..., M$, together with the initial condition

$$x(0) = \xi. \tag{3.2}$$

For the sake of simplicity , we set $F(x,u) := f(x) + \sum_{i=1}^{M} g_i(x)u_i$ and denote by U all measurable functions from a real interval with values a.e. in $[-1,1]^M$. The standard assumptions on F and the target set S we need are the following:

Assumptions 3.1. *(1) f, g_i are $\mathcal{C}^\infty$ and all partial derivatives are Lipschitz with Lipschitz constant $L > 0$, $i = 1, ..., M$; moreover,*

$$\|f(y)\|, \|g_i(y)\| \leq K_0(1 + \|y\|)$$

for all $y \in \mathbb{R}^n$, where K_0 is a positive constant.
(2) S is compact.

Such assumptions will be always supposed to be satisfied in this sequel and we label them as *standard assumptions* on the dynamics.

Given the target $S \subset \mathbb{R}^n$, we will state some sufficient conditions in order to reach S from every ξ in a neighborhood in finite time and give an upper bound for the minimum time $T(\xi)$. Such conditions involve *Lie brackets* of the vector fields $f, g_1, ..., g_M$. We recall their definition for general C^1 vector fields X, Y. We set

$$[X,Y](x) = \nabla X(x)Y(x) - \nabla Y(x)X(x),$$

and higher order brackets are defined recursively, provided X, Y are smooth enough. Let now Φ_t^X and Φ_t^Y, $t \geq 0$, be the flows generated by the vector fields X and Y, namely $\Phi_t^X(\xi)$, respectively $\Phi_t^Y(\xi)$, are the solution at time t of the Cauchy problems

$$\dot{x} = X(x), \quad x(0) = \xi; \qquad \dot{x} = Y(x), \quad x(0) = \xi.$$

It is well known that $\Phi_t^X(\cdot)$ and $\Phi_t^Y(\cdot)$ are diffeomorphisms for all $t \geq 0$ small enough. The *formal Lie bracket* between Φ_t^X and Φ_t^Y is defined by setting

$$[\Phi^X, \Phi^Y]_t(\xi) = (\Phi_t^X)^{-1} \circ (\Phi_t^Y)^{-1} \circ (\Phi_t^X) \circ (\Phi_t^Y)(\xi)$$

The procedure may be iterated and the order of such iterations can defined by induction. If Φ is either $\Phi_t^X(\cdot)$ or $\Phi_t^Y(\cdot)$, then $ord(\Phi) = 1$; otherwise, if A and B are nested Lie brackets of $\Phi_t^X(\cdot)$ and $\Phi_t^Y(\cdot)$, we set $ord([A,B]) = ord(A) + ord(B)$. The power of a Lie bracket B, $pw(B)$, is set to 1 if B consists of a single diffeomorphism, while $pw([A,B]) = 2 \times pw(A) + 2 \times pw(B)$. The following classical result establishes a relation between the two types of Lie brackets.

Theorem 3.2. *Let $\{X_i\}_{i\in\mathbb{N}}$ be smooth vector fields and let B be a nested formal Lie bracket of order $\bar{k} \in \mathbb{N}$ of the corresponding flows $\left\{\Phi_t^{X_i}\right\}_{i\in\mathbb{N}}$, for $t > 0$ small enough, $B = B(\Phi^{X_{i_1}}, ..., \Phi^{X_{i_k}})$, $k \leq \bar{k}$. Then*

$$\frac{\partial^j}{\partial t^j} B(\Phi^{X_{i_1}}, ..., \Phi^{X_{i_k}}) \,|_{t=0} = 0, \quad \forall 1 \leq j < \bar{k},$$

$$\frac{1}{\bar{k}!} \frac{\partial^{\bar{k}}}{\partial t^{\bar{k}}} B(\Phi^{X_{i_1}}, ..., \Phi^{X_{i_k}}) \,|_{t=0} = B(X_{i_1}, ..., X_{i_k}).$$

In what follows we will consider iterated Lie brackets of the vector fields $f \pm g_i$, $i = 1, ..., M$, where f, g_i appear in (3.1), possibly with $f \equiv 0$. We denote by $\mathcal{L}$ the set of all iterated Lie brackets of the above vector fields.

Let B be such a non-vanishing Lie bracket with order k, $ord(B) = k$, and power $pw(B)$. Let $x_\xi^B(\cdot)$ be the trajectory of (3.1) and (3.2) corresponding to B, namely the trajectory which uses bang-bang controls ± 1, according to the vector fields appearing in B. We obtain immediately from Theorem 3.2 the following expansion:

$$x_\xi^B(pw(B)t) = \xi + B(\xi)t^k + o(t^k), \qquad t \to 0^+, \tag{3.3}$$

where for each compact C containing ξ, there exists $K_C > 0$ such that

$$\left\| o(t^k) \right\| \le K_C t^{k+1} \qquad \text{for all } t \text{ small enough.} \tag{3.4}$$

Now we proceed by applying the above approximation (3.3) to an estimate of the distance from the target of suitable trajectories of (3.1).

Proposition 3.3. *Let S be a closed set and let $\xi \notin S$. Let B be a non-vanishing Lie bracket of order k of the vector fields $f \pm g_i$, $i = 1, ..., M$. Let $x_\xi^B(\cdot)$ be the corresponding trajectory of (3.1) and (3.2). Let $t > 0$ and assume that $x_\xi^B(s) \notin S$ for all $s \in [0, t]$. Let $\zeta \in \partial^P d_S(\xi)$. Then we have, for every compact set C containing ξ,*

$$d_S\Big(x_\xi^B(pw(B)t)\Big) \le d_S(\xi) + \langle \zeta, B(\xi) \rangle \, t^k + K\left(t^{k+1} + \frac{t^{2k}}{d_S(\xi)}\right), \tag{3.5}$$

where K depends only on the constant K_C appearing in (3.4.

Moreover, if S satisfies a ρ-internal sphere condition, then $\frac{1}{d_S(\xi)}$ can be substituted by $\frac{1}{\rho}$, and $x_\xi^B(\cdot)$ may touch S.

Proof. Set $x_\xi^B(pw(B)t) = x_t$. By putting together Proposition 2.2 (i), (ii) and (3.3), (3.4) we obtain

$$\begin{aligned} d_S(x_t) &\le d_S(\xi) + \langle \zeta, x_t - \xi \rangle + \frac{K}{d_S(\xi)} \left\| x_t - \xi \right\|^2 \\ &\le d_S(\xi) + \langle \zeta, B(\xi) \rangle \, t^k + K' t^{k+1} + \frac{\bar{K} t^{2k}}{d_S(\xi)}, \end{aligned}$$

for suitable constants $K', \bar{K}$ satisfying the desired properties. ☐

Remark 3.4. The regularity requirements on f and g_i can be weakened if Lie brackets only up to a fixed order k are considered. Actually, in most of our results we need only that, for a given $k \in \mathbb{N}$, f and g_i, $i = 1, \ldots, M$, are of class $\mathcal{C}^k$ and all partial derivatives up to the order k are Lipschitz with the same constant.

3.2. **Hölder continuity of the minimum time function.** We state here two controllability results, proved in [18], which are at the basis of our results. We treat separately the case where the target S satisfies an internal and an external sphere condition.

We say that a Lie bracket B is *compatible* with the controlled dynamics (3.1) if the (direct and reversed) flows appearing in the formal Lie bracket of Theorem 3.2 are flows of (3.1). A simple sufficient condition ensuring compatibility is, of course, the drift f to be zero. More in general, compatibility can be seen as a time reversibility of the dynamics. In Section 4.3, controllability conditions which do not require time reversibility will be given for the case of second order Lie brackets.

Theorem 3.5 (see Corollaries 5.9 and 5.11 in [18])**.** *Let S be compact and let one of the two following assumptions be valid. Either,*

(IS) *let S be satisfying a ρ–internal sphere condition and assume there exist $\delta > 0$, $\mu > 0$, and $k \in \mathbb{N}$ such that for every $\xi \in S_{2\delta} \setminus S$ there exist $\zeta_\xi \in \partial^P d_S(\xi)$ and a compatible $B_\xi \in \mathcal{L}$, with $ord(B_\xi) \leq k$, enjoying the following property:*

$$\langle \zeta_\xi, B_\xi(\xi) \rangle \leq -\mu. \tag{3.6}$$

Or, alternatively,

(ES) *let S have reach $\rho > 0$ and assume there exist $0 < \delta < \frac{\rho}{2}$, $\mu > 0$, and $k \in \mathbb{N}$ such that for every $\xi \in S_\delta \setminus S$ there exists a compatible $B_\xi \in \mathcal{L}$, with $ord(B_\xi) \leq k$, enjoying the following property:*

$$\langle \nabla d_S(\xi), B_\xi(\xi) \rangle \leq -\mu. \tag{3.7}$$

Then the minimum time function to reach S from ξ subject to the dynamics (3.1), $T(\xi)$, is (finite and) Hölder continuous with exponent $\frac{1}{k}$ on S_δ. More precisely, there exists a constant Λ, depending only on ρ, δ, μ and on the vector fields $f, g_i, i = 1, ..., M$, such that for all $\xi_1, \xi_2 \in S_\delta$ it holds

$$|T(\xi_1) - T(\xi_2)| \leq \Lambda \|\xi_1 - \xi_2\|^{1/k}. \tag{3.8}$$

Remark 3.6. Observe that assumptions (IS) and (ES) are of a different nature, because an external sphere condition is assumed either on the closure of the complement of S (case (IS)), or on S (case (ES)). If S satisfies both an internal and an external sphere condition, then its boundary is of class $\mathcal{C}^{1,1}$.

In section 3, we will need to ensure that small time controllability holds not only with respect to S, but also to a suitable shrinking or enlargement of S. The following are the relevant statements. The first one requires, in addition to the internal sphere condition, a uniform external cone condition (see (3.10) in Theorem 3.7 below). Of course, such additional requirement is satisfied if the boundary of the target is of class $\mathcal{C}^{1,1}$.

Theorem 3.7. *Let the assumption (IS) hold and let k, $\rho, \delta, \mu > 0$ be as in (IS). Let $L_{\mathcal{L}}$ be the Lipschitz constant of all $B \in \mathcal{L}, ord(B) \leq k$, on S_δ and set*

$$C_B := \max\left\{\|B(x)\| : x \in S_\delta, B \in \mathcal{L}, ord(B) \leq k\right\}. \tag{3.9}$$

Assume furthermore that there exists $0 < \mu' < \frac{\mu}{2}$ such that

$$\max\left\{\, \|\zeta' - \zeta\| : \|\zeta\| = \|\zeta'\| = 1,\ \ \zeta, \zeta' \in N_S^C(x), x \in S \right\} < \frac{\mu'}{C_B}. \tag{3.10}$$

Then there exists $0 < \bar\sigma < \rho$, depending only on μ', C_B and $L_{\mathcal{L}}$, such that for all $0 < \sigma < \bar\sigma$ assumption (IS) holds for $S_{-\sigma}$. More precisely, for every $\xi \in S_\delta \setminus S_{-\sigma}$ there exist $\zeta_\xi \in \partial^P d_{S_{-\sigma}}(\xi)$ and $B_\xi \in \mathcal{L}$, with $ord(B_\xi) \leq k$, enjoying (3.6) with $\min\{\mu, \mu'\}$ in place of μ. Consequently, if the Lie brackets appearing in (IS) are compatible, the minimum time function to reach $S_{-\sigma}$ is finite and Hölder continuous with exponent $\frac{1}{k}$ on S_δ. Moreover, the constant Λ appearing in (3.8) can be chosen independently of σ.

Proof. We claim first that if $0 < \bar\sigma < \rho$, then $S_{-\bar\sigma}$ satisfies a uniform $(\rho - \bar\sigma)$-internal sphere condition. Indeed, recalling Corollaries 16 and 19 in [19], the external cone condition implies that $(\operatorname{int} S)^c$ has ρ-positive reach. Therefore, if $0 < \bar\sigma < \rho$, then $(\operatorname{int} S_{-\bar\sigma})^c$ has $(\rho - \bar\sigma)$-positive reach. Invoking again Corollaries 16 and 19 in [19] we obtain that $S_{-\bar\sigma}$ satisfies a uniform $(\rho - \bar\sigma)$-internal sphere condition.

Consequently, in order to prove that the minimum time function to reach $S_{-\sigma}$ ($0 < \sigma < \bar{\sigma}$) is Hölder on S_δ it is enough to establish the analogue of (3.6) for $S_{-\sigma}$ as claimed in the statement of the theorem.

To this aim, fix first $\xi \in S_\delta \setminus S$. We claim that

$$\partial^P d_S(\xi) = \partial^P d_{S_{-\sigma}}(\xi).$$

Indeed, set $\bar{\xi} = \pi_S(\xi)$. Since $\overline{S^c}$ has positive reach, S has at $\bar{\xi}$ both an internal and an external nontrivial proximal normal. Thus $\bar{\zeta} := \dfrac{\bar{\xi} - \xi}{d_S(\xi)}$ is the unique unit normal to $\overline{S^c}$ at $\bar{\xi}$. Define, for $0 \le t < \rho$, $\xi_t = \bar{\xi} + t\bar{\zeta}$. Observe that, by the internal ρ-positive reach condition, $\bar{\xi}$ is the unique projection of ξ_t onto $\overline{S^c}$, so that, in particular, $d_{\overline{S^c}}(\xi_t) = t$. Therefore, $\xi_\sigma \in S_{-\sigma}$, and so $d_{S_{-\sigma}}(\xi) \le d_S(\xi) + \sigma$. On the other hand, for all $\xi' \in S_{-\sigma}$ one has obviously $\|\xi' - \xi\| \ge d_S(\xi) + \sigma$, whence

$$d_{S_{-\sigma}}(\xi) = d_S(\xi) + \sigma, \tag{3.11}$$

and the claim follows. By (IS), there exist $B_\xi \in \mathcal{L}, ord(B_\xi) \le k$, and $\zeta_\xi \in \partial^P d_S(\xi)$, such that (3.6) holds. Since $\partial^P d_S(\xi) = \partial^P d_{S_{-\sigma}}(\xi)$ the proof is completed for the case $\xi \in S_\delta \setminus S$. Observe that in this case one can choose $\mu' = \mu$.

Fix now $\xi \in S \setminus S_{-\sigma}$. Let x be the unique projection of ξ onto $\overline{S^c}$. Assume first that $N^P_S(x) \ne \{0\}$ and let ζ be the (unique) unit vector in $N^P_S(x)$. Let $x_n = x + \frac{\zeta}{n}, n \in \mathbb{N}$. Then, since $\nabla d_S(x_n) = \zeta$ for all n large enough, the assumption (IS) yields that there exist $B_n \in \mathcal{L}, ord(B_n) \le k$, such that

$$\langle \zeta, B_n(x_n) \rangle \le -\mu, \quad \forall n \text{ large enough.}$$

Since the order of the B_n's is bounded, up to a subsequence we may assume that $B_n(x) = B(x)$ is independent of n. Therefore, by passing to the limit we obtain

$$\langle \zeta, B(x) \rangle \le -\mu. \tag{3.12}$$

Let now $N^P_S(x) = \{0\}$ and let $\zeta \in N^L_S(x), \|\zeta\| = 1$. By definition of limiting normal, there exist sequences $\{x_n\} \subset S, \{\zeta_n\} \subset \mathbb{R}^n$ such that $\zeta_n \in N^P_S(x_n)$, $\|\zeta_n\| = 1$, $x_n \to x$, and $\zeta_n \to \zeta$ as $n \to \infty$. Then, for every n there exists $B_n \in \mathcal{L}, ord(B_n) \le k$, such that

$$\langle \zeta_n, B_n(x_n) \rangle \le -\mu.$$

By passing to the limit as above, we obtain (3.12).

Now we wish to prove that an inequality of the type (3.12) holds at ξ. Recalling that $x = \pi_{\overline{S^c}}(\xi)$. As before, we assume first that $N^P_S(x) \ne \{0\}$. Then, since S has both an inner and an outer nonvanishing proximal normal at x, we have that $N^P_S(x) = \zeta\mathbb{R}^+ = -N^P_{\overline{S^c}}(x)$ for a suitable unit vector ζ, and

$$d_{S_{-\sigma}}(\xi) = \sigma - d_{\overline{S^c}}(\xi). \tag{3.13}$$

Thus $d_{S_{-\sigma}}$ is differentiable at ξ and moreover

$$\nabla d_{S_{-\sigma}}(\xi) = -\nabla d_{\overline{S^c}}(\xi) = \zeta.$$

By the uniform Lipschitz continuity of Lie brackets of order $\le k$, we obtain from (3.12) that

$$\langle \zeta, B(\xi) \rangle \le \langle \zeta, B(x) \rangle + L_{\mathcal{L}} \|\xi - x\| \le -\mu + L_{\mathcal{L}}\sigma.$$

Therefore, if $\sigma < \bar{\sigma} := \frac{\mu - \mu'}{L_{\mathcal{L}}}$ we obtain

$$\langle \zeta, B(\xi) \rangle \le -\mu',$$

which was to be proved.

Assume now again that $N_S^P(x) = \{0\}$. Recalling Lemma 5 in [22][1], we have that $N_S^C(x) = -N_{S^c}^P(x)$. Therefore, by (3.13), $\zeta' := \nabla d_{S_{-\sigma}}(\xi) = \frac{x-\xi}{\|x-\xi\|} \in N_S^C(x)$. By the assumption (IS), there exist a unit vector $\zeta \in N_S^L(x)$ and a Lie bracket $B \in \mathcal{L}$, with $ord(B) \leq k$, such that $\langle \zeta, B(x)\rangle \leq -\mu$. Recalling (3.10) we obtain $\|\zeta - \zeta'\| < \frac{\mu'}{C_B}$. By putting the above inequalities together, we finally have

$$\langle \zeta', B(\xi)\rangle = \langle \zeta' - \zeta, B(\xi)\rangle + \langle \zeta, B(x)\rangle + \langle \zeta, B(\xi) - B(x)\rangle < \mu' - \mu + L_{\mathcal{L}}\sigma.$$

Therefore, if $\sigma \leq \bar{\sigma} := \frac{\mu - 2\mu'}{L_{\mathcal{L}}}$ we finally reach $\langle \zeta', B(\xi)\rangle < -\mu'$. The proof is concluded. $\square$

The second perturbation result is concerned with the case where the target S has positive reach.

Proposition 3.8. *Let the assumption (ES) of Theorem 3.5 hold and let $0 < \sigma < \delta$. Then the minimum time to reach S_σ from $S_\delta \setminus S_\sigma$ (is finite and) satisfies (3.8), where the constant Λ is independent of σ.*

Proof. It is enough to observe that if $\xi \in S_\delta \setminus S_\sigma$, then $d_{S_\sigma}(\xi) = d_S(\xi) - \sigma$. $\square$

Remark 3.9. Observe that, under the assumptions of Proposition 3.8, the enlargement of S_σ satisfies an internal sphere condition, and so, as far as it is enough to consider an approximation of the target, one can concentrate only on the (IS) case.

4. A higher order scheme for the minimum time function.

This section is devoted to designing a suitable fully discrete scheme for the approximation of T. We follow the well established method based on dynamic programming, which was first designed by Bardi and Falcone [3] (see also [4], [2], and [11] and references therein). We apply to T the Kružkov transform and then, through discrete dynamic programming, we approximate the viscosity solution of a suitable boundary value problem. Since, due to controllability assumptions which are based on higher order Lie brackets, T is not locally Lipschitz, we need to use a scheme which is of a suitably high order in time and of first order in space.

This section is divided into a number of subsections. First we present a higher order one step semidiscrete scheme for our dynamics (3.1), taking controls subject to suitable switchings. Given a step size h, for every initial condition ξ we construct a discrete trajectory which converges as $h \to 0$ to a suitable trajectory of (3.1). Moreover, the time needed to reach the target is bounded by a fractional power of $d_S(\xi)$ (discrete controllability). Next we apply Kružkov transform to T, and relying on a discrete dynamic programming principle and a convergence results due to [3] we prove that a discrete value function v_h converges to the transformation v of T, also providing an error estimate. Finally, we introduce a fully discrete scheme and prove its convergence and a related error estimate.

4.1. Time discretization.

Given the control system (3.1), (3.2), we write

$$\dot{x} = F(x, u), \quad x(0) = \xi, \tag{4.1}$$

where $F(x,u) = f(x) + \sum_{i=1}^M g_i(x)u_i$, $u = (u_1, ..., u_M) \in [-1,1]^M$. Given a fixed step $h > 0$ small enough, we approximate (4.1) by a one step $(q+1)$-th order scheme

[1] The statement of Lemma 5 in [22] actually requires S to be convex, but this is used only to provide wedgedness, which indeed we assume.

which has the form

$$\begin{cases} y_{n+1} &= y_n + h\Phi(y_n, A_n, h) \\ y_0 &= \xi \end{cases} \tag{4.2}$$

where A_n is a $M \times l$ matrix, $A_n = (u_n^1, ..., u_n^l)$ with $u_n^i \in [-1,1]^M$. Here $l > 0$ depends on the specific method. We make the following assumptions on the method:

$$\begin{cases} \lim_{h\to 0} \Phi(\xi, (\bar{u}, ..., \bar{u}), h) = F(\xi, \bar{u}) \quad (l \text{ copies of } \bar{u}), \\ \|\Phi(\xi_1, A, h) - \Phi(\xi_2, A, h)\| \le L_\Phi \|\xi_1 - \xi_2\|. \end{cases} \tag{4.3}$$

In order to prove the discrete controllability, we now consider the following Cauchy problem, instead of (4.1)

$$\dot{x} = F(x, \upsilon) = f(x) + \sum_{i=1}^{M} g_i(x) u_i, \quad x(0) = \xi \tag{4.4}$$

where $\upsilon = (u_1, ..., u_M)$, $u_i \in \{-1, 1\}$, is supposed to be constant in an interval $[0, \tau]$, $0 < \tau \le 1$. Let $0 < h < \tau$ and $k \in \mathbb{N}$, $k \ge 1$, be given. Here k will play the role of the order of a suitable Lie bracket which will be identified later. We consider the one step order scheme (4.2) for (4.4). In this case, the control matrix A is generated by l copies of υ, therefore the conditions (4.3) can be rewritten, by an abuse of notation, in the following way:

$$\begin{cases} \lim_{h\to 0} \Phi(\xi, \upsilon, h) = F(\xi, \upsilon), \\ \|\Phi(\xi_1, \upsilon, h) - \Phi(\xi_2, \upsilon, h)\| \le L_\Phi \|\xi_1 - \xi_2\|. \end{cases}$$

Furthermore, we require a suitably high order of approximation, namely

$$\|x_\upsilon(h, \xi) - (\xi + h\Phi(\xi, \upsilon, h))\| \le C_\Phi h^{q+2}, \tag{4.5}$$

where $q \ge k$ and $x_\upsilon(\cdot, \xi)$ is the exact solution of (4.4). The classical Runge-Kutta method, for example, enjoys the above properties (see [16]).

Set

$$\begin{cases} \xi_0 &= \xi, \\ \xi_{n+1} &= \xi_n + h\Phi(\xi_n, \upsilon, h), \end{cases} \tag{4.6}$$

and, for $N \in \mathbb{N}$, $N \ge 1$,

$$h = \frac{\tau}{N}. \tag{4.7}$$

Then there exists C_Φ such that

$$\|x_\upsilon(\tau, \xi) - \xi_N(\tau, \xi, \upsilon)\| \le C_\Phi h^{q+1}, \tag{4.8}$$

for all h small enough (see, e.g., Theorem 3.6 in [16]). Observe that the point $\xi_N(\tau, \xi, \upsilon)$ defined through (4.6) and (4.7) depends on the M-tuple υ. We denote this point by $y(\upsilon, \tau, h, \xi)$, i.e.,

$$y(\upsilon, \tau, h, \xi) := \xi_N(\tau, \xi, \upsilon).$$

Let $p \in \mathbb{N}, p \ge 1$. We consider now a p-tuple $\underline{u}$ of M-tuples of controls $u_i \in \{-1, 1\}$, namely $\underline{u} = (\upsilon^1, ..., \upsilon^p)$, where $\upsilon^j = (u_1^j, ..., u_M^j) \in \{-1, 1\}^M$ and subsequently apply the process (4.6), with υ^j in place of υ, N times for each j. More precisely, we set

$$\begin{cases} y^1 = y(\upsilon^1, \tau, h, \xi), \\ \vdots \\ y^j = y(\upsilon^j, \tau, h, y^{j-1}), \end{cases} \tag{4.9}$$

where $j = 2, ..., p$. We denote the point y^p constructed above by $y^p(\underline{u}, \tau, h, \xi)$.

Let $x_{\underline{u}}(\tau, \xi)$ be the final point of the exact solution of (4.4) corresponding to the p-tuples of controls $\underline{u}$. More precisely, we set

$$\begin{cases} x^1 = x_{\upsilon^1}(\tau, \xi), \\ \quad \vdots \\ x^m = x_{\upsilon^m}(\tau, x^{m-1}), \qquad m = 2, \ldots, p. \end{cases}$$

By applying (4.8) subsequently on p intervals of length τ, we obtain

$$\|x^p - y^p\| \le C_p h^{q+1}, \tag{4.10}$$

where C_p is a suitable constant depending only on p, Φ.

4.2. **Discrete controllability.** The following result falls in the framework of (approximate) discrete controllability: under assumptions including either (IS) or (ES), given $0 < \eta < \delta$, for all $\xi \in S_\delta \setminus S_\eta$ we construct a finite sequence of points of the types y^p described just above, say $y_1, ..., y_{n(\xi)}$, and of increasing times t_i, $i = 0, ..., n(\xi) - 1$, such that $d_S(y_{n(\xi)}) < \eta$ and the time to reach $y_{n(\xi)}$, namely $\sum_{i=0}^{n(\xi)-1} p_{i+1}(t_{i+1} - t_i)$, is bounded from above by $d_S(\xi)^{1/k}$. The number of discretization steps, namely $[(t_{i+1} - t_i)/h]$ where $h > 0$ is fixed, will be labeled here for simplicity as N.

Theorem 4.1. *Let $S \subset \mathbb{R}^n$ be closed and let $\delta, \rho, \mu > 0$, $k \in \mathbb{N}, k \ge 1$ be given, with $\delta < 1$. Assume that for every $\xi \in S_{2\delta} \setminus S$ there exist $\zeta_\xi \in \partial^P d_S(\xi)$ and a compatible $B_\xi \in \mathcal{L}$, with $ord(B_\xi) \le k$, such that*

$$\langle \zeta_\xi, B_\xi(\xi) \rangle \le -\mu. \tag{4.11}$$

Let $0 < \eta < \delta$ be given and consider a one step $(q+1)$-th order scheme with $q \ge k$. Then for every $\xi \in S_\delta \setminus S_\eta$ there exist a number of steps N, independent of ξ, and finite sequences of natural numbers p_{i+1}, of p_{i+1}-tuples $\{\underline{u}_{i+1}\}$ of M-tuples of ± 1, of points $\{y_i\}$, and of times $\{t_i\}$, $t_{i+1} > t_i$, $i = 0, ..., n(\xi) - 1$, satisfying the properties

$$\begin{cases} t_0 = 0,\ y_0 = \xi, \\ y_{i+1} = y^{p_i}\left(\underline{u}_{i+1}, p_{i+1}(t_{i+1} - t_i), \dfrac{t_{i+1} - t_i}{N}, y_i\right), \\ \quad i = 0, ..., n(\xi) - 1, \end{cases}$$

$$y_{n(\xi)} \in S_\eta,$$

$$\sum_{i=0}^{n(\xi)-1} p_{i+1}(t_{i+1} - t_i) \le C\big(d_S(\xi)\big)^{1/k}, \tag{4.12}$$

for a suitable constant C independent of ξ. Here y^{p_i} is defined according to (4.9).

Proof. Fix $\xi \in S_\delta \setminus S_\eta$. By our assumptions, there exist $\zeta_\xi \in \partial^P d_S(\xi)$ and a Lie bracket $B_\xi \in \mathcal{L}$ with $ord(B_\xi) \le k$ such that (4.11) holds. Now we are going to prove that there exist a time $0 < t_1 \le 1$, a number $p_1 \ge 1$, a (finite) sequence of M-tuples of ± 1, say $\underline{u}_1 = (\upsilon_1^1, ..., \upsilon_1^{p_1})$, $\upsilon_1^j \in \{-1, 1\}^M$, corresponding to B_ξ through Theorem

3.2, such that the trajectory $x_\xi^{B_\xi}(\cdot)$ of (3.1), (3.2) associated to this sequence of controls satisfies the following properties for all $t \in [0, t_1]$:

$$\begin{cases} d_S\Big(x_\xi^{B_\xi}(p_1 t)\Big) > \dfrac{d_S(\xi)}{2}, \\ d_S\Big(x_\xi^{B_\xi}(p_1 t)\Big) \le d_S(\xi) - \mu t^k + K\Big(t^{k+1} + \dfrac{2t^{2k}}{d_S(\xi)}\Big), \end{cases} \tag{4.13}$$

where K is the constant appearing in (3.5). Indeed, in order to obtain the first inequality in (4.13), recalling (3.3), (3.4), and (3.9) it is enough to choose $0 < t_1 \le 1$ such that

$$(C_B + K_{S_\delta}) t_1^k \le \frac{d_S(\xi)}{2}, \tag{4.14}$$

where K_{S_δ} is the constant appearing in (3.4) with S_δ in place of C, while the second one follows from (3.5) in Proposition 3.3 together with (4.11). In particular, we obtain

$$0 < d_S\Big(x_\xi^{B_\xi}(p_1 t_1)\Big) < 2\delta.$$

Observe furthermore that there exists a constant p_k (the maximal power of a Lie bracket of order $\le k$ in $\mathbb{R}^n$) depending only on k, such that

$$p_1 \le p_k.$$

Let $N \in \mathbb{N}, N \ge 1$, and set $h_1 = \frac{t_1}{N}$. We assume N to be so large that the discretization error corresponding to the step size h_1 satisfies (4.8). Let y_1 be the point $y^{p_1}(\underline{u}_1, p_1 t_1, h_1, \xi)$ constructed according to (4.6), (4.9). By (4.10) we have

$$\Big\| x_\xi^{B_\xi}(p_1 t_1) - y_1 \Big\| \le C_{p_k} h_1^{q+1}. \tag{4.15}$$

Remembering that $q \ge k$ and putting together the above inequality and (4.13), we receive

$$d_S(y_1) \le d_S(\xi) - \mu t_1^k + K\Big(t_1^{k+1} + \frac{2t_1^{2k}}{d_S(\xi)}\Big) + C_{p_k}\Big(\frac{t_1}{N}\Big)^{k+1}.$$

We rewrite the above estimate as

$$\begin{aligned} d_S(y_1) &\le d_S(\xi) - \mu t_1^k + \Big(K + \frac{C_{p_k}}{N^{k+1}}\Big) t_1^{k+1} + \frac{2K}{d_S(\xi)} t_1^{2k} \\ &=: d_S(\xi) - \mu t_1^k + K_1 t_1^{k+1} + \frac{K_2}{d_S(\xi)} t_1^{2k}. \end{aligned} \tag{4.16}$$

By imposing the supplementary conditions

$$t_1 K_1 + \frac{K_2}{d_S(\xi)} t_1^k \le \frac{\mu}{2} \qquad \text{and} \qquad C_{p_k} t_1^{k+1} \le \frac{N^{k+1} d_S(\xi)}{4}, \tag{4.17}$$

we obtain from (4.13), (4.15), and (4.16)

$$\frac{d_S(\xi)}{4} \le d_S(y_1) \le d_S(\xi) - \frac{\mu}{2} t_1^k. \tag{4.18}$$

Observe that all conditions previously imposed on t_1 (in particular (4.14) and (4.17)) are satisfied if

$$0 < t_1 = \min\left\{1, \sqrt[k]{\frac{N^{k+1} d_S(\xi)}{4 C_{p_k}}}, \sqrt[k]{\frac{\mu d_S(\xi)}{4K_2}}, \frac{\mu}{4K_1}, \sqrt[k]{\frac{d_S(\xi)}{2(C_B + K_{S_\delta})}}\right\}. \tag{4.19}$$

Assume now that we have constructed recursively times t_i, numbers p_i, controls $\underline{u}_i = (v_i^1, ..., v_i^{p_i})$, $v_i^j \in \{-1,1\}^M$ and points y_i up to $i = \bar{i}$, such that

$$t_{i-1} < t_i,$$
$$t_i - t_{i-1} = \min\left\{t_1, \sqrt[k]{\frac{N^{k+1} d_S(y_{i-1})}{4C_{p_k}}}, \sqrt[k]{\frac{\mu d_S(y_{i-1})}{4K_2}}, \sqrt[k]{\frac{d_S(y_{i-1})}{2(C_B + K_{S_\delta})}}\right\},$$

and

$$\frac{d_S(y_{i-1})}{4} \le d_S(y_i) \le d_S(y_{i-1}) - \frac{\mu}{2}(t_i - t_{i-1})^k. \tag{4.20}$$

We are now going to construct the next step. By the assumptions, there exist a Lie bracket $B_{y_{\bar{i}}}$ and $\zeta_{y_{\bar{i}}} \in \partial^P d_S(y_{\bar{i}})$ such that $\left\langle \zeta_{y_{\bar{i}}}, B_{y_{\bar{i}}}(y_{\bar{i}})\right\rangle \le -\mu$. By applying again Proposition 3.3 and the argument designed for t_1 we find a time $t_{\bar{i}+1}$, a number $p_{\bar{i}+1} \le p_k$, a control $\underline{u}_{\bar{i}+1} \in \{-1,1\}^{M \times p_{\bar{i}+1}}$, and a point $y_{\bar{i}+1}$ satisfying the properties

$$0 < t_{\bar{i}+1} - t_{\bar{i}} < t_1, \tag{4.21}$$

$$t_{\bar{i}+1} := t_{\bar{i}} + \min\left\{\sqrt[k]{\frac{N^{k+1} d_S(y_{\bar{i}})}{4C_{p_k}}}, \sqrt[k]{\frac{\mu d_S(y_{\bar{i}})}{4K_2}}, \sqrt[k]{\frac{d_S(y_{\bar{i}})}{2(C_B + K_{S_\delta})}}\right\}, \tag{4.22}$$

from which, taking into account (4.21) and (4.22), we obtain finally

$$\frac{d_S(y_{\bar{i}})}{4} \le d_S(y_{\bar{i}+1}) \le d_S(y_{\bar{i}}) - \frac{\mu}{2}(t_{\bar{i}+1} - t_{\bar{i}})^k,$$

which concludes our construction.

Now we are going to show that we can reach S_η after finitely many iterations $n(\xi)$ and that (4.12) holds. To this aim, set

$$\alpha = \min\left\{1, \frac{\mu}{4K_1}\right\}, \qquad \beta = \min\left\{\frac{1}{\sqrt[k]{2(C_B + K_{S_\delta})}}, \sqrt[k]{\frac{N^{k+1}}{4C_{p_k}}}, \sqrt[k]{\frac{\mu}{4K_2}}\right\}.$$

Then, for every $i \in \mathbb{N}$, we have $t_{i+1} - t_i = \min\left\{\alpha, \beta \sqrt[k]{d_S(y_i)}\right\}$. Observe that (4.20) implies that the sequence $\{d_S(y_i)\}$ is strictly decreasing. Therefore, there exists an index $\bar{i}$ such that for all $i \ge \bar{i}$ we have

$$t_{i+1} - t_i = \beta \sqrt[k]{d_S(y_i)}. \tag{4.23}$$

Let $d = \lim_{i\to\infty} d_S(y_i)$. From (4.20) and (4.23) we obtain, for all $i \ge \bar{i}$,

$$d_S(y_{i+1}) - d_S(y_i) \le -\frac{\mu\beta^k}{2} d_S(y_i),$$

from which necessarily $d = 0$. Therefore, there exists some index $n(\xi)$ such that $d_S(y_{n(\xi)}) \le \eta$. Finally, we deal with (4.12). Owing again to (4.20) and (4.23), we have for all $i \le n(\xi) - 1$

$$d_S(y_{i+1}) - d_S(y_i) \le -\frac{\mu}{2}(t_{i+1} - t_i)^k = -\frac{\mu}{2}\min\left\{\alpha^{k-1}, \beta^{k-1} d_S(y_i)^{\frac{k-1}{k}}\right\}(t_{i+1} - t_i).$$

Thus,

$$t_{i+1} - t_i \le \frac{2}{\mu}\left(\frac{d_S(y_i) - d_S(y_{i+1})}{\beta^{k-1} d_S(y_i)^{\frac{k-1}{k}}} + \frac{d_S(y_i) - d_S(y_{i+1})}{\alpha^{k-1}}\right).$$

By summing the above inequalities and recalling that $p_i \le p_k$ for each i, we obtain

$$\begin{aligned}\sum_{i=0}^{n(\xi)-1} p_{i+1}(t_{i+1}-t_i) &\le \frac{2p_k}{\mu}\sum_{i=0}^{n(\xi)-1}\left(\frac{d_S(y_i)-d_S(y_{i+1})}{\beta^{k-1}\, d_S(y_i)^{\frac{k-1}{k}}}+\frac{d_S(y_i)-d_S(y_{i+1})}{\alpha^{k-1}}\right)\\ &\le \frac{2p_k}{\mu}\int_0^{d_S(\xi)}\left(\frac{1}{\beta^{k-1}r^{\frac{k-1}{k}}}+\frac{1}{\alpha^{k-1}}\right)dr\\ &\le \frac{2p_k}{\mu}\left(\frac{k}{\beta^{k-1}}\sqrt[k]{d_S(\xi)}+\frac{1}{\alpha^{k-1}}d_S(\xi)\right),\end{aligned}$$

which, recalling that $d_S(\xi) < \delta < 1$, implies (4.12) and concludes our proof. $\square$

The second result of this subsection requires some regularity on the target S. Under suitable conditions we prove that the discretized trajectory reaches the target (*not* a neighborhood) after finitely many steps of constant length, and establish an estimate of the type (4.12). Such upper bound will be used in the proof of the convergence of a suitable discretized value function to the viscosity solution of a Hamilton-Jacobi equation.

To this aim, we define the discrete minimum time as follows. Given a step size $h > 0$ and a sequence of control matrices $\{A_i\} \subset [-1,1]^{Ml}$, we recall the discrete dynamics defined in the previous subsection for the control system (4.1)

$$\begin{cases} y_{n+1} &= y_n + h\Phi(y_n, A_n, h)\\ y_0 &= \xi. \end{cases} \tag{4.24}$$

We define the function

$$n_h(\{A_i\},\xi) = \min\{n\in\mathbb{N} : y_n \in S\} \le +\infty, \tag{4.25}$$

where $n_h = \infty$ if y_n never reaches S. Let $N_h(\xi)$ be the minimum number of steps to reach S, namely,

$$N_h(\xi) = \min_{\{A_i\}\in[-1,1]^{Ml}} \{n_h(\{A_i\},\xi)\}. \tag{4.26}$$

The discrete minimum time function is now defined by setting

$$T_h(\xi) = hN_h(\xi). \tag{4.27}$$

We define also the discrete reachable set $\mathcal{R}_h$ by $\mathcal{R}_h = \{\xi\in\mathbb{R}^n : N_h(\xi) < +\infty\}$.

Theorem 4.2. *Let the assumptions of Theorem 3.7 hold and let the target S, ρ, σ, k be as S, ρ, $\bar\sigma$, k in Theorem 3.7. Consider the discrete dynamics (4.24), generated by a one step scheme Φ which satisfies (4.3) and (4.5) for some $q \ge k$, and the discrete minimum time function (4.27).*

Then there exist $\bar\delta$, $\bar h$, $C > 0$ such that for every $0 < \delta < \bar\delta$, $h \le \bar h$, $\xi \in S_\delta \setminus S$, we have

$$T_h(\xi) \le C\sqrt[k]{d_S(\xi)}. \tag{4.28}$$

Proof. Fix $x_0 \in \partial S$ and consider $\xi \in B_{\sigma/2}(x_0)\setminus S$. Recalling the proof of Theorem 3.7, we obtain that $S_{-\sigma}$ satisfies a $(\rho-\sigma)$-internal sphere condition, and so the distance function to $S_{-\sigma}$, $d_{S_{-\sigma}}(\cdot)$, is semiconcave with constant $(\rho-\sigma)^{-1}$. According to (3.11), $\partial^P d_{S_{-\sigma}}(\xi) = \partial^P d_S(\xi)$. Then, for each $\zeta_\xi \in \partial^P d_S(\xi)$ we obtain

$$d_{S_{-\sigma}}(y) \le d_{S_{-\sigma}}(\xi) + \langle \zeta_\xi, y-\xi\rangle + \frac{1}{\rho-\sigma}\|y-x\|^2$$

for every $y \in B_\sigma(x_0)$.

Recalling (3.3) and (3.4), we can find $p \geq 1$, a sequence of bang-bang controls $\{\pm 1\}$, say $\underline{u} = (\upsilon^1, ..., \upsilon^p) \in \{-1, 1\}^{M \times p}$ and $t \in (0, 1]$, such that the corresponding trajectory $x_\xi^{B_\xi}(\cdot)$ of (3.1), (3.2) has the form (3.3), i.e.,

$$x_\xi^{B_\xi}(pt) = \xi + B_\xi t^k + o(t^k), \tag{4.29}$$

where $\|o(t^k)\| \leq K_{B_\sigma} t^{k+1}$. If furthermore $(C_B + K_{B_\sigma})t^k < \frac{\sigma}{2}$, where we recall that C_B was defined in (3.9), then putting together (4.29), (3.5) and the estimate on the semiconcavity constant of $d_{S_{-\sigma}}$ we have also

$$d_{S_{-\sigma}}\left(x_\xi^{B_\xi}(pt)\right) \leq d_{S_{-\sigma}}(\xi) - \mu t^k + K_{B_\sigma} t^{k+1} + \frac{\left(C_B + K_{B_\sigma}\right)^2}{\rho - \sigma} t^{2k}. \tag{4.30}$$

Set $t = Nh$, $N \in \mathbb{N}$, and let y be the point $y^p(\underline{u}, pNh, h, \xi)$ constructed according to (4.6), (4.9). From (4.10) we receive

$$\left\|x_\xi^B(pNh) - y\right\| \leq C_p h^{q+1}. \tag{4.31}$$

By putting together (4.30) and (4.31), we obtain now

$$d_{S_{-\sigma}}(y) \leq d_{S_{-\sigma}}(\xi) - \mu(Nh)^k + \left(K_{B_\sigma} + C_p\right)(Nh)^{k+1} + \frac{\left(K_{B_\sigma} + C_B\right)^2}{\rho - \sigma}(Nh)^{2k}.$$

Then

$$d_{S_{-\sigma}}(y) \leq d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^k,$$

provided

$$Nh < \min\left\{1, \sqrt[k]{\frac{\sigma}{2(C_B + K_{B_\sigma})}}, \frac{\mu}{4(K_{B_\sigma} + C_p)}, \sqrt[k]{\frac{\mu(\rho - \sigma)}{4\left(K_{B_\sigma} + C_B\right)^2}}\right\} =: \alpha. \tag{4.32}$$

On the other hand, we want to impose the condition $d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^k \leq \sigma$, which yields $d_{S_{-\sigma}}(y) \leq \sigma$ and so $y \in S$. This condition, in view of (3.11), is equivalent to

$$Nh \geq \sqrt[k]{\frac{2d_S(\xi)}{\mu}}. \tag{4.33}$$

Now, in order to make (4.32) and (4.33) compatible, we impose a condition on $d_S(\xi)$, namely

$$2\frac{d_S(\xi)}{\mu} \leq 2\frac{\delta}{\mu} < \alpha^k.$$

Then, to reach S it is enough to choose $N^\star \in \mathbb{N}$ and $h^\star$ so that

$$N^\star h^\star = \sqrt[k]{\frac{2d_S(\xi)}{\mu}}.$$

Due to the compactness of S, we finally obtain

$$T_h(\xi) \leq pN^\star h^\star = p\sqrt[k]{\frac{2d_S(\xi)}{\mu}} \leq C\sqrt[k]{d_S(\xi)},$$

for a suitable constant C, which is the desired estimate. □

Remark 4.3. A result similar to Theorem 4.2 can be proved without restrictions on $\delta > 0$ (except $\delta < 1$).

Indeed, the following statement can be proved.

Under the assumptions of Theorem 4.2, let $0 < \delta < 1$, $k \in \mathbb{N}$ *be such that for every* $\xi \in S_\delta \setminus S$ *there exist* $\zeta_\xi \in \partial^P d_S(\xi)$ *and* $B_\xi \in \mathcal{L}$, *with* $ord(B_\xi) \leq k$, *such that (4.11) holds. Let* $0 < \sigma < \delta$. *Then for every step size* h *small enough, there exists a number* $N^\star$ *such that for every* $\xi \in S_\delta \setminus S$ *we can find controls* $\upsilon_1, ..., \upsilon_{N^\star}$ *for which we can reach* S_δ *by* $N^\star$ *iterations of (4.24).*

The proof is a combination of arguments of the proofs of Theorems 4.1 and (4.2).

4.3. **A further result on discrete controllability.** We consider now the case where the approximation of trajectories of (3.1), (3.2) with Lie brackets contains also lower order terms. This case occurs in general when the drift term f does not vanish or when the system is not necessarily time reversible. Our reference is the second order controllability result proved by [17]. For simplicity we treat only one of the sufficient conditions proved in [17, Proposition 4], but an entirely similar result can be obtained with the other one.

Proposition 4.4 (Proposition 4 in [17])**.** *Consider the controlled system (3.1), (3.2) with* $M = 1$ *and let the target* S *satisfy the* ρ*-internal sphere condition. Let* $\delta, \mu > 0$ *be given and assume that for all* $x \in S_\delta \setminus S$ *there exists a control* $u \in [-1, 1]$, *and* $\zeta_x \in \partial^P d_S(x)$ *such that the following inequalities hold:*
Either
(IS.0) $\langle f(x) + g(x)u, \zeta_x \rangle \leq -\mu$,
or
(IS.1) $\langle f(x), \zeta_x \rangle \leq 0$,
(IS.2) $\langle 2\nabla f(x) f(x) + u[f, g](x), \zeta_x \rangle + \frac{4}{\rho} \|f(x)\|^2 \leq -\mu$.
Then $\mathcal{R}$ *contains* S *in its interior and* T *is Hölder continuous with exponent* $1/2$ *in* $\mathcal{R}$.

Remark 4.5. Robustness of the controllability condition of Proposition 4.4 with respect to a shrinking $S_{-\sigma}$ of the target.

Let $\rho > \sigma > 0$ and S satisfy the same properties as in Theorem 3.7, namely ρ-internal sphere condition and wedgedness. By the same arguments as in the proof of Theorem 3.7, for every $\xi \in S_\delta \setminus S_{-\sigma}$ the inequality (IS.2) still holds with some $\zeta_\xi \in \partial^P d_{S_{-\sigma}}(\xi)$ and a suitable $\mu' \leq \mu$ in place of μ. In order to preserve the discrete controllability under a shrinking of the target, the condition (IS.1), instead, needs to be strengthened as follows:
(IS'.1) *for all* $\xi \in S_\delta \setminus S_{-\sigma}$ *there exists* $\zeta_\xi \in \partial^P d_{S_{-\sigma}}$ *such that* $\langle f(x), \zeta_x \rangle \leq 0$.

The following result contains our second order discrete controllability condition in the case where the drift term cannot be neglected.

Theorem 4.6. *Let the target* S, *and* ρ, δ, σ *be as* $S, \rho, \delta, \bar{\sigma}$ *in Theorem 3.7 and let the assumptions (IS.0), (IS'.1) and (IS.2) hold true for* $x \in S_\delta \setminus S_{-\sigma}$. *Consider the discrete dynamics (4.24) generated by the one step scheme* Φ *which satisfies (4.3) and (4.5) for some* $q \geq 2$, *and the discrete minimum time function (4.27).*

Then there exist $\bar{\delta}$, $\bar{h}$, $C > 0$ *such that for every* $0 < \delta_1 < \bar{\delta}$, $h \leq \bar{h}$, $\xi \in S_{\delta_1} \setminus S$, *we have*

$$T_h(\xi) \leq C\sqrt{d_S(\xi)}.$$

Proof. Fix $x_0 \in \partial S$ and consider $\xi \in B_{\sigma/2}(x_0) \setminus S$. By the same argument as at the beginning of the proof of Theorem 4.2, for each $\zeta_\xi \in \partial^P d_S(\xi)$ we obtain

$$d_{S_{-\sigma}}(y) \leq d_{S_{-\sigma}}(\xi) + \langle \zeta_\xi, y - \xi \rangle + \frac{1}{\rho - \sigma} \|y - x\|^2 \tag{4.34}$$

for every $y \in B_\sigma(x_0)$. Assume first that (IS'.1) and (IS.2) hold at ξ. Recalling Lemma 1 in [17], for every $\bar{u} \in [-1,1]$ and $t \in (0,1]$, if we follow first the flow of $f + \bar{u}g$ and then of $f - \bar{u}g$, each one for a time t, the corresponding trajectory $x_\xi(\cdot)$ of (3.1), (3.2) has the form

$$x_\xi(2t) = \xi + 2tf(\xi) + t^2\big(2Df(\xi)f(\xi) + u[f,g](\xi)\big) + o(t^2), \tag{4.35}$$

where $\|o(t^2)\| \le K_{B_\sigma} t^3$, for a suitable constant K_{B_σ}. Set now

$$\begin{aligned} C_f &:= \max\left\{\|f(x)\| : x \in S_\delta \setminus S_{-\sigma}\right\}, \\ C_{ff} &:= \max\left\{\|Df(x)f(x)\| : x \in S_\delta \setminus S_{-\sigma}\right\}, \\ C_{fg} &:= \max\left\{\|[f,g](x)\| : x \in S_\delta \setminus S_{-\sigma}\right\}, \end{aligned}$$

and $M_1 = (2C_f + 2C_{ff} + C_{fg} + K_{B_\sigma})$ and assume that $M_1 t < \frac{\sigma}{2}$. Then, by putting together (4.35), (4.34), (IS'.1), and (IS.2) we have also

$$d_{S_{-\sigma}}\Big(x_\xi(2t)\Big) \le d_{S_{-\sigma}}(\xi) - \mu t^2 + K_{B_\sigma} t^3 + \frac{M_2^2}{\rho - \sigma} t^4, \tag{4.36}$$

where $M_2 := 2C_{ff} + C_{fg} + K_{B_\sigma}$.

Consider now the one step method (4.2) with $q = 2$ and set $t = Nh$ and $\underline{u} := \{\bar{u}, -\bar{u}\}$. Then let $y := y(\underline{u}, 2Nh, h, \xi)$ be the final point of the discrete dynamical system (4.9) after choosing $\bar{u}$ for the first N iterations and $-\bar{u}$ for other N. From (4.10) we receive

$$\|x_\xi(2Nh) - y\| \le C_2 h^3. \tag{4.37}$$

By putting together (4.36) and (4.37), we obtain now

$$d_{S_{-\sigma}}(y) \le d_{S_{-\sigma}}(\xi) - \mu(Nh)^2 + \big(K_{B_\sigma} + C_2\big)(Nh)^3 + \frac{M_2^2}{\rho - \sigma}(Nh)^4,$$

so that

$$d_{S_{-\sigma}}(y) \le d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^2,$$

provided

$$Nh < \min\left\{1, \frac{\sigma}{2M_1}, \frac{\mu}{4(K_{B_\sigma} + C_2)}, \sqrt{\frac{\mu(\rho - \sigma)}{4M_2^2}}\right\} =: \alpha. \tag{4.38}$$

On the other hand, we want to impose the condition $d_{S_{-\sigma}}(\xi) - \frac{\mu}{2}(Nh)^2 < \sigma$, which yields $d_{S_{-\sigma}}(y) \le \sigma$ and so $y \in S$. This condition, in view of (3.11), is equivalent to

$$Nh \ge \sqrt{\frac{2d_S(\xi)}{\mu}}. \tag{4.39}$$

Now, in order to make (4.38) and (4.39) compatible, we impose

$$2\frac{d_S(\xi)}{\mu} \le 2\frac{\delta}{\mu} < \alpha^2.$$

Then, to reach S it is enough to choose $N^\star \in \mathbb{N}$ and $h^\star$ so that

$$N^\star h^\star = \sqrt{\frac{2d_S(\xi)}{\mu}}.$$

where $h^\star < \bar{h}$, $\delta_1 < \bar{\delta} := \min\left\{1, \sigma, \frac{\mu\alpha^2}{2}\right\}$. Due to the compactness of S, we finally obtain

$$T_h(\xi) \le 2N^\star h^\star = 2\sqrt{\frac{2d_S(\xi)}{\mu}},$$

which is the desired estimate.

Assume now that (IS.0) holds at x. In this case (4.34) yields, for a suitable constant M_3

$$d_{S_{-\sigma}}\Big(x_\xi(t)\Big) \le d_{S_{-\sigma}}(\xi) - \mu t + \frac{M_3}{\rho - \sigma} t^2.$$

Then the argument is analogous and simpler than the previous one. Note that in this case the estimate on the discrete time is

$$T_h(\xi) \le C d_S(\xi).$$

□

4.4. **The discrete dynamic programming approach and convergence.** Following the well established literature on the dynamic programming approach (see [3], [12] and references therein) we consider the Kružkov transformation, namely we define

$$v(x) = 1 - e^{-T(x)}, \tag{4.40}$$

and recall that v is the unique bounded viscosity solution of the boundary value problem

$$\begin{cases} v(x) + \sup_{u\in[-1,1]^M} \{\langle -F(x,u), \nabla v(x)\rangle\} = 1 & \text{in } \mathbb{R}^n \setminus S, \\ v(x) = 0 & \text{on } S \end{cases} \tag{4.41}$$

where $F(x,u) = f(x) + \sum_{i=1}^M g_i(x) u_i$ (see Theorem IV.2.6 and Proposition II.2.5 in [2]).

We define also, for a given step size $h > 0$,

$$v_h(x) = 1 - e^{-T_h(x)}, \tag{4.42}$$

where $T_h(x)$ is the discretized minimum time function which was defined in (4.27). Observe that $v_h(x)$ is the value function of a discrete optimal control problem, namely,

$$v_h(x) = \begin{cases} \min_{\{A_i\}\subset[-1,1]^{Ml}} J_x^h(\{A_i\}) & \text{for } x \in \mathcal{R}_h \\ 1 & \text{for } x \notin \mathcal{R}_h, \end{cases} \tag{4.43}$$

where

$$J_x^h(\{A_i\}) = 1 - e^{-hn_h(\{A_i\},x)} = \Big(\sum_{j=0}^{n_h(\{A_i\},x)-1} e^{-jh}\Big)(1-e^{-h})\chi_{S^c}(x), \tag{4.44}$$

and $\chi_{S^c}(x) = 1$ if $x \notin S$ and 0 otherwise. Following Theorem 2.3 in [3], we observe that v_h is the unique bounded solution of the following problem:

$$\begin{cases} V(x) = A(V(x)) & \forall x \in \mathbb{R}^n \setminus S \\ V(x) = 0 & \forall x \in S \end{cases} \tag{4.45}$$

where $A(V(x)) = \inf_{A\in[-1,1]^{Ml}} \left\{e^{-h} V(x + h\Phi(x,A,h))\right\} + 1 - e^{-h}$.

Furthermore, owing to (4.28) and Remark 4.3, there exists a constant C such that

$$T_h(x) \le C\sqrt[k]{d_S(x)}, \forall x \in \mathcal{R}.$$

Therefore, by Theorem 3.3 in [3], we obtain the following

Theorem 4.7. *Let the assumptions of Theorem 4.2 hold and let v, v_h be defined according to (4.40), (4.42), respectively. Then $v_h \to v$ locally uniformly in $\mathbb{R}^n$ and $hN_h \to T$ locally uniformly in $\mathcal{R}$.*

4.5. **Fully discrete scheme and error estimates.** Let $S \subset \mathbb{R}^n$ be a compact nonempty set and let the assumptions of Theorem 4.2 or of Theorem 4.6 hold on some compact neighborhood of S, S_δ. Before continuing, we observe that it is enough to consider any one step method which has at least $(k+1)$-th order of convergence. To make a slightly more general approach, in the sequel we always consider a method with order higher or equal to $k+1$. We will describe our results only for the case of Theorem 4.2, since the other one requires only small modifications. We recall that error estimates for pursuit evasion differential games, under Hölder continuity assumptions but with a first order time discretization, were obtained in [21].

We recall that, according to Theorem 3.5, under our assumptions the minimum time T is Hölder continuous on S_δ and there exists a constant C such that

$$T(x) \leq C\sqrt[k]{d_S(x)}, \quad \forall x \in S_\delta \setminus S. \tag{4.46}$$

This inequality implies that $v(x) \in C^{0,1/k}(S_\delta)$ (see, e.g., [2, Remark 1.7, p. 230]). Moreover, the discrete minimum time function T_h is finite on S_δ and satisfies

$$T_h(x) \leq C\sqrt[k]{d_S(x)}, \quad \forall x \in S_\delta \setminus S, \tag{4.47}$$

provided $h > 0$ is small enough (see Theorem 4.2).

We consider the dynamical system (4.1) and its corresponding one step $(q+1)$-th order scheme (4.2). We make the following assumptions on the scheme to preserve the order of the method:

(A.1) For any $x \in \mathbb{R}^n$ and any measurable $u\colon [0,h) \to [-1,1]^M$ there exists a $M \times l$ (where l depends on the chosen method) matrix $A \in [-1,1]^{Ml}$ such that

$$\|y(h,x,u) - y_h(h,x,A)\| \leq Ch^{q+2}, \tag{4.48}$$

where C is a constant, $q \geq k$, and $y(h,x,u)$ stands for the exact solution of (4.1) following the control u and $y_h(h,x,A) = x + h\Phi(x,A,h)$.

Conversely,

(A.2) for any matrix $A \in [-1,1]^{Ml}$, there exists a measurable control $u\colon [0,h) \to [-1,1]^M$ such that (4.48) holds.

Such assumptions are used, for example, in [14, 11]. Higher order one step methods satisfying (4.48) for control systems of the type considered here are constructed in [15]. The assumption (A.2) is satisfied by taking u to be piecewise constant (with entries of A) on subsequent intervals of length h/l.

We now deal with space discretization. For convenience we recall that $v_h(x) = 1 - e^{-T_h(x)}$ is the unique bounded solution of the problem

$$\begin{cases} v_h(x) = \inf_{A \in [-1,1]^{Ml}} \left\{ e^{-h} v_h(x + h\Phi(x,A,h)) \right\} + 1 - e^{-h} & \text{on } \mathbb{R}^n \setminus S \\ v_h(x) = 0 & \text{on } S \end{cases} \tag{4.49}$$

provided that $h > 0$ is small enough.

Let $\Gamma = \{x_i\colon i = 1, ..., I\}$ be a space grid for the domain $\Omega \subset \mathcal{R}$, with $\overline{\Omega} = \cup_j S_j$, such that the diameter of each cell S_j corresponding to Γ is less than or equal to

Δx. Let

$$W^k = \{\omega\colon \Omega \to \mathbb{R}\colon \omega(\cdot) \in C(\Omega), D\omega(x) = a_j, \forall x \in S_j, \forall j\}$$

be the class of piecewise linear functions on Ω. We look for an approximate solution of (4.49) belonging to W^k. For any $\phi(\cdot)$ defined on Ω, let $I^1_\Gamma[\phi](x) = \sum_i^I \lambda_i(A)\phi(x_i)$, where $x = \sum_i^I \lambda_i(A)x_i$, $\lambda_i(A) \in [0,1]$, $\sum_{i=1}^I \lambda_i(A) = 1$ for any $A \in [-1,1]^{Ml}$, see [9] for more information.

Now we are going to replace (4.49) with its fully discrete version by substituting $v_h(x_i + h\Phi(x_i, A, h))$ with $I^1_\Gamma[v_h](x_i + h\Phi(x_i, A, h))$. More precisely, in order to construct a fully discretized minimum time function we set $\Gamma^\star := \{x \in \Gamma :$ there exists a control matrix A such that $x + h\Phi(x, A, h) \in \Omega\}$ and consider the problem

$$\begin{cases} v_h^{\Delta x}(x) = \min_{A\in[-1,1]^{Ml}} \left\{e^{-h} I^1_\Gamma[v_h^{\Delta x}](x + h\Phi(x, A, h))\right\} + 1 - e^{-h} & \text{if } x \in \Gamma^\star \setminus S, \\ v_h^{\Delta x}(x) = 0 & \text{if } x \in \Gamma^\star \cap S, \\ v_h^{\Delta x}(x) = 1 & \text{if } x \in \Gamma \setminus \Gamma^\star. \end{cases} \tag{4.50}$$

Let V be a function on the grid Γ and define the operator $A_h^{\Delta x}[V](x)$ by setting, for all $x \in \Gamma \setminus S$,

$$A_x^{\Delta x}[V](x) = \min_{A\in[-1,1]^{Ml}} \left\{e^{-h} I^1_\Gamma[V](x + h\Phi(x, A, h))\right\} + 1 - e^{-h}.$$

By using the same arguments of Section 5.2 in [12], it not difficult to prove that $A_h^{\Delta x}$ is monotone, namely if $V_1(x) \le V_2(x)$ for all $x \in \Gamma$, then

$$A_h^{\Delta x}[V_1](x) \le A_h^{\Delta x}[V_2](x).$$

Moreover, $A_h^{\Delta x}[\cdot]$ considered componentwise is a contraction from $\mathbb{R}^I$ to $\mathbb{R}^I$ with contraction coefficient e^{-h}. Therefore the fixed point problem (4.50) has indeed a unique solution for all $0 < h < 1$ and $\Delta x > 0$, which we label $v_h^{\Delta x}$. Notice that $v_h^{\Delta x}$ is computed only at the grid nodes, but it can be extended by interpolation over the whole of Ω. More precisely, from now on, for every $x \in \Omega$ $v_h^{\Delta x}(x)$ means that

$$\begin{cases} v_h^{\Delta x}(x) \text{ is the solution of (4.50)} & \text{if } x \in \Gamma, \\ v_h^{\Delta x}(x) = I^1_\Gamma[v_h^{\Delta x}](x) & \text{if } x \in \Omega \setminus \Gamma. \end{cases} \tag{4.51}$$

The next results are devoted to error estimates. The first lemmas deal with the (semi)discrete minimum time function. More precisely we will prove that $\|v - v_h\|_{\infty,\Omega} \le Ch^{\frac{q+1}{k}}$. We denote by $\|\cdot\|_{\infty,\Omega}$ the usual supremum norm taken on Ω and recall that the functions $n(\{A_i\}, x)$ and $N_h(x)$ were defined in (4.25) and (4.26), respectively.

Lemma 4.8. *Assume that (4.46) holds in a neighborhood S_δ of the target S (in particular this happens under the assumptions of Theorem 4.2), together with (A.2). Then there exist two positive constants $\bar h$ and C such that*

$$T(x) - hN_h(x) \le Ch^{\frac{q+1}{k}}, \text{ for any } x \in \Omega,\ h \le \bar h.$$

Proof. For any fixed $x \in \Omega$, we choose a sequence of control matrices $\{A_i\} \subset [-1,1]^{Ml}$ such that $n(\{A_i\}, x) = N_h(x)$. According to (A.2) and to the fact that x belongs to the compact set Ω, there exists a measurable control u, with $u_i(t) \in [-1,1]^M$ a.e., such that

$$\|y(hN_h(x), x, u) - y_h(hN_h(x), x, \{A_i\})\| \le C_\Phi h^{q+1}.$$

By choosing $h \le \sqrt[q+1]{\frac{\delta}{C_\Phi}}$, we obtain $y(hN_h(x), x, \{u_i\}) \in S_\delta$. Then due to (4.46) we obtain the inequality

$$T(x) \le hN_h(x) + C(C_\Phi h^{q+1})^{1/k}.$$

Equivalently, $T(x) - hN_h(x) \le Ch^{\frac{q+1}{k}}$, for a suitable constant C. □

The analogous estimate for $hN_h(x) - T(x)$ can be obtained by using (A.1) in place of (A.2).

Lemma 4.9. *Assume that (4.47) and (A.1) hold in a neighborhood S_δ of the target S. Then there exist $\bar{h}$ and $C > 0$ such that*

$$hN_h(x) - T(x) \le Ch^{\frac{q+1}{k}}, \ \textit{for any } x \in \Omega, \ h \le \bar{h}.$$

Proof. Let u be an optimal control steering x to S and fix a discretization step $h > 0$ small enough. By (A.1), there exists a sequence of control matrices $\{A_n\}$, $n = 0, \dots, N < +\infty$, with entries in $[-1, 1]$ such that

$$\|y(T(x), x, u) - y_h(T(x), x, \{A_n\})\| \le C_\Phi h^{q+1}.$$

Then by choosing $h \le \sqrt[q+1]{\frac{\delta}{C_\Phi}}$, we obtain $y_h(hN_h(x), x, \{A_n\}) \in S_\delta$. Thus by (4.47), we receive

$$hN_h(x) \le T(x) + Ch^{\frac{q+1}{k}}$$

and the proof is concluded. □

In the sequel, for the sake of simplicity, we will sometimes use the same letter for different constants. Combining Lemma 4.8 and (4.9), we obtain

$$|hN_h(x) - T(x)| \le Ch^{\frac{q+1}{k}} \tag{4.52}$$

and applying the mean value theorem, from (4.52) we obtain

$$|v(x) - v_h(x)| \le Ch^{\frac{q+1}{k}}. \tag{4.53}$$

Remembering that C may depend on $|x|$, we can choose a global constant C such that (4.53) holds for every $x \in \Omega$. Thus we obtain a uniform estimate for v, namely

$$\|v(x) - v_h(x)\|_{\infty,\Omega} \le Ch^{\frac{q+1}{k}}. \tag{4.54}$$

The following result is devoted to establishing an error estimate for the fully discrete value function, namely an upper bound for $\left\|v(x) - v_h^{\Delta x}(x)\right\|_{\infty,\Omega}$.

Theorem 4.10. *Assume that the assumptions of Lemmas 4.9 and 4.8 hold. Then there exist suitable constants C_1, C_2, $\bar{h}$ such that for every $h \in (0, \bar{h}]$*

$$\left\|v - v_h^{\Delta x}\right\|_{\infty,\Omega} \le C_1 h^{\frac{q+1}{k} - 1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

Proof. Recalling the semidiscrete and the fully discrete dynamic programming principle, for any $x \in \Gamma \setminus S$ we have

$$v_h(x) = \inf_{A \in [-1,1]^{Ml}} \left\{ e^{-h} v_h(x + h\Phi(x, A, h)) \right\} + 1 - e^{-h}, \tag{4.55}$$

$$v_h^{\Delta x}(x) = \inf_{A \in [-1,1]^{Ml}} \left\{ e^{-h} I_\Gamma^1[v_h^{\Delta x}](x + h\Phi(x, A, h)) \right\} + 1 - e^{-h}. \tag{4.56}$$

Let $A^\star$ be an optimal control matrix in (4.55). Then for any $x \in \Gamma$ we obtain

$$\begin{aligned}
v_h^{\Delta x}(x) - v_h(x) &\le e^{-h} I_\Gamma^1[v_h^{\Delta x}](x + h\Phi(x, A^\star, h)) - e^{-h} v_h(x + h\Phi(x, A^\star, h)) \\
&\le e^{-h}\Big(\big| I_\Gamma^1[v_h^{\Delta x}](x + h\Phi(x, A^\star, h)) - I_\Gamma^1[v_h](x + h\Phi(x, A^\star, h)) \big| \\
&\quad + \big| I_\Gamma^1[v_h](x + h\Phi(x, A^\star, h)) - I_\Gamma^1[v](x + h\Phi(x, A^\star, h)) \big| \\
&\quad + \big| I_\Gamma^1[v](x + h\Phi(x, A^\star, h)) - v(x + h\Phi(x, A^\star, h)) \big| \\
&\quad + \big| v(x + h\Phi(x, A^\star, h)) - v_h(x + h\Phi(x, A^\star, h)) \big| \Big) \\
&\le e^{-h} \left\| v_h^{\Delta x} - v_h \right\|_{\infty,\Gamma} + C_2(\Delta x)^{1/k} + C_1 h^{\frac{q+1}{k}}
\end{aligned}$$

where in the last inequality we used the monotonicity of $I_\Gamma^1[\cdot]$, the Hölder continuity of $v(\cdot)$, and (4.54). In an entirely similar way, we also obtain

$$v_h(x) - v_h^{\Delta x}(x) \le e^{-h} \left\| v_h^{\Delta x} - v_h \right\|_{\infty,\Gamma} + C_1 h^{\frac{q+1}{k}} + C_2(\Delta x)^{1/k}.$$

Thus $(1 - e^{-h}) \left\| v_h - v_h^{\Delta x} \right\|_{\infty,\Gamma} \le C_1 h^{\frac{q+1}{k}} + C_2(\Delta x)^{1/k}$. Since $1 - e^{-h} = h + O(h^2)$, by possibly modifying C_1 and C_2 we receive, for all $x \in \Gamma$,

$$\left\| v_h - v_h^{\Delta x} \right\|_{\infty,\Gamma} \le C_1 h^{\frac{q+1}{k} - 1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

Therefore, for every $x \in \Omega$,

$$\begin{aligned}
v_h^{\Delta x}(x) - v_h(x) &\le |I_\Gamma^1[v_h^{\Delta x}](x) - I_\Gamma^1[v_h](x)| + |I_\Gamma^1[v_h](x) - I_\Gamma^1[v](x)| \\
&\quad + |I_\Gamma^1[v](x) - v(x)| + |v(x) - v_h(x)| \\
&\le \left\| v_h^{\Delta x} - v_h \right\|_{\infty,\Gamma} + \| v_h - v \|_{\infty,\Omega} + |I_\Gamma^1[v](x) - v(x)| \\
&\quad + \| v - v_h \|_{\infty,\Omega} \\
&\le C_1 h^{\frac{q+1}{h} - 1} + C_2 \frac{(\Delta x)^{1/k}}{h}.
\end{aligned}$$

Analogously, we receive the same estimate for the reversed direction, i.e.,

$$v_h(x) - v_h^{\Delta x}(x) \le C_1 h^{\frac{q+1}{h} - 1} + C_2 \frac{(\Delta x)^{1/k}}{h}.$$

Thus, we have

$$\left\| v_h^{\Delta x} - v_h \right\|_{\infty,\Omega} \le C_1 h^{\frac{q+1}{k} - 1} + C_2 \frac{(\Delta x)^{1/k}}{h}, \tag{4.57}$$

for every $x \in \Omega$. Putting together (4.54) and (4.57), we obtain the error estimate of the fully discrete value function

$$\left\| v(x) - v_h^{\Delta}(x) \right\|_{\infty,\Omega} \le C_1 h^{\frac{q+1}{h} - 1} + C_2 \frac{(\Delta x)^{1/k}}{h}$$

The proof is complete. □

5. **Design of approximate feedback controls and error estimates for the cost function.** This section is devoted to constructing (approximate) suboptimal feedback controls, together with obtaining an error estimate for the related cost function.

Recall that the semidiscrete dynamic programming principle (SDDPP) was stated in (4.49). The *semidiscrete feedback* is defined, for a given time discretization step h, by picking any control matrix $A_h(x)$ such that

$$A_h(x) \in \operatorname{argmin}_{A\in[-1,1]^{Ml}} \left\{e^{-h} v_h(x + h\Phi(x, A, h))\right\}.$$

We define also a sequence of control matrices $A_h(y_m)$, where y_m is the solution of the discrete dynamical system

$$\begin{cases} y_{m+1} &= y_m + h\Phi(y_m, A_h(y_m), h) \\ y_0 &= x. \end{cases}$$

According to (A.2), there exists a measurable control $u_h(y_m)$, corresponding to each $A_h(y_m)$, such that

$$\|y(h, y_m, u_h(y_m)) - y_h(h, y_m, A_h(y_m))\| \le Ch^{q+2}.$$

Let $S_{-\sigma}$ be a shrinking of the target S. Consider the (SDDPP) for $S_{-\sigma}$, namely

$$v_{h,\sigma}(x) = \inf_{A\in[-1,1]^{Ml}} \left\{e^{-h} v_{h,\sigma}(x + h\Phi(x, A, h))\right\} + 1 - e^{-h}, \quad v_{h,\sigma}(x) = 0 \text{ on } S_{-\sigma}. \tag{5.1}$$

Let $A_{h,\sigma}(y_m)$, $u_{h,\sigma}(y_m)$ be defined as $A_h(y_m)$, $u_h(y_m)$ above and set

$$A^{\star,m}_{h,\sigma} := A_{h,\sigma}(y_m) \quad \text{and} \quad u^\star_{h,\sigma}(s) := u_{h,\sigma}(y_m), \tag{5.2}$$

for $s \in [mh, (m+1)h)$, $m = 1, \ldots$. Set also

$$J(u, x) = 1 - e^{-t(u,x)}, \qquad J_{h,\sigma}(\{A_i\}, x) = 1 - e^{-hn_{h,\sigma}(\{A_i\},x)},$$

where $t(u, x)$ was defined in (2.4) and $n_{h,\sigma}$ is the smallest integer n (if any) such that $y_h(nh, x, \{A_i\})$ belongs to $S_{-\sigma}$. The first result of this section is concerned with an error estimate for the cost function $J(u^\star_{h,\sigma}(\cdot), x)$ compared with $\inf_{u(\cdot)\in U} J(u(\cdot), x)$, under suitable assumptions.

Proposition 5.1. *Assume that there exists $\bar h > 0$ such that, for $0 < h < \bar h$, (A.1), (A.2), and the assumptions of Theorem 4.2 hold, where σ is chosen sufficiently small. Then $J(u^\star_{h,\sigma}(\cdot), x) \le \inf_{u(\cdot)\in U} J(u(\cdot), x) + \epsilon(\sigma, h)$ for every $x \in \Omega$, where $\epsilon(\sigma, h) \to 0$, as $\sigma, h \to 0$.*

Proof. Recall that, according to Theorem 4.2, for all $\bar x \in S \setminus S_{-\sigma}$ we have

$$T_{h,\sigma}(\bar x) \le C \sqrt[k]{\sigma} =: \omega(\sigma), \tag{5.3}$$

where k is the maximal order of Lie brackets appearing in Theorem 4.2. Let $x \in \Omega \setminus S$ and assume there exists $N \in \mathbb{N}$ and a sequence of control matrices $\{A^{\star,m}_{h,\sigma}\}$, $m = 1, \ldots, N$, constructed according to (5.2) such that $y_h(Nh, x, \{A^{\star,m}_{h,\sigma}\}) \in S_{-\sigma}$. By the assumption (A.2), there exists a corresponding control $u^\star_{h,\sigma}(\cdot) \in U$ such that $y(Nh, x, u^\star_{h,\sigma}) \in S$ with $0 < h < \bar h$, whence we obtain

$$J(u^\star_{h,\sigma}(\cdot), x) \le J_{h,\sigma}(\{A^{\star,m}_{h,\sigma}\}, x) = v_{h,\sigma}(x).$$

Thus

$$\begin{aligned} J(u^\star_{h,\sigma}(\cdot), x) - \inf_{u(\cdot)\in U} J(u(\cdot), x) &\le J_{h,\sigma}(\{A^{\star,m}_{h,\sigma}\}, x) - v(x) \\ &= v_{h,\sigma}(x) - v_h(x) + v_h(x) - v(x) \le \epsilon(\sigma, h), \end{aligned}$$

where we used the mean value theorem and (5.3), together with (4.54). Note that the desired estimate is trivial for any $x \in \Omega$ where there does not exist any sequence of control matrices $\{A^{\star,m}_{h,\sigma}\}$ which steers x to $S_{-\sigma}$ or for any $x \in S \cap \Omega$. The proof is complete. $\square$

We consider now the fully discrete version of (5.1) and use it to define our approximate feedback.

Definition 5.2. Let the space mesh Γ, with cell diameter Δx, for the domain Ω and $\sigma > 0$ and $h > 0$ be fixed. For each $x \in \Omega \setminus S_{-\sigma}$ we define the *approximate (fully discrete) feedback* $A^\sigma_{\Delta x,h}(x)$, relative to Δx, h and $S_{-\sigma}$, by picking any

$$A^\sigma_{\Delta x,h}(x) \in \operatorname{argmin}_{A\in[-1,1]^{Ml}} \left\{e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A, h))\right\}. \tag{5.4}$$

As we did for the semidiscrete case, we consider the sequence of control matrices $A^\sigma_{\Delta x,h}(y_m)$, where y_m is computed by

$$\begin{cases} y_{m+1} &= y_m + h\Phi(y_m, A^\sigma_{\Delta x,h}(y_m), h) \\ y_0 &= x. \end{cases}$$

Again, according to (A.2), there exists a measurable control $u^\sigma_{\Delta x,h}(y_m)$ corresponding to $A^\sigma_{\Delta x,h}(y_m)$ such that

$$\left\|y(h, y_m, u^\sigma_{\Delta x,h}(y_m)) - y_h(h, y_m, A^\sigma_{\Delta x,h}(y_m))\right\| \le Ch^{q+2}. \tag{5.5}$$

Let

$$u^{\star,\sigma}_{\Delta x,h}(s) := u^\sigma_{\Delta x,h}(y_m), \qquad A^{\star,\sigma,m}_{\Delta x,h} := A^\sigma_{\Delta x,h}(y_m) \tag{5.6}$$

for $s \in [mh, (m+1)h)$, $m = 1, \dots, N$.

We are interested in estimating the difference between the cost $J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x)$, resp. $J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$, and the value function $v(x)$. We prove first a preliminary lemma, similar to Theorem 1.7 in [9].

Lemma 5.3. *Let $v^{\Delta x}_{h,\sigma}(\cdot)$ and $A^{\star,\sigma,m}_{\Delta x,h}$ be defined, respectively, by (4.51) and (5.6). Then, for every $x \in \Omega$, $J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \le v^{\Delta x}_{h,\sigma}(x) + \frac{\varepsilon(h,\Delta x)}{1-e^{-h}}$, where $\varepsilon(h, \Delta x) := C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{(\Delta x)^{1/k}}{h}$.*

Proof. Recall that for all $x \in \Gamma \setminus S$ the equality

$$v^{\Delta x}_{h,\sigma}(x) = e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + 1 - e^{-h}$$

holds. We are now interested in estimating the difference between $v^{\Delta x}_{h,\sigma}(x)$ and $e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}]\big(x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)\big) + 1 - e^{-h}$ for every $x \in \Omega \setminus S$. Recalling that the dynamic programming principle for $S_{-\sigma}$ reads as

$$v_\sigma(x) = \min_{u\in U} \left\{e^{-h} v_\sigma(y(h, x, u)) + 1 - e^{-h}\right\},$$

let

$$u^\star_\sigma(x) \in \operatorname{argmin}_{u\in U} \left\{e^{-h} v_\sigma(y(h, x, u)) + 1 - e^{-h}\right\}$$

and $A^\star_\sigma \in [-1, 1]^{Ml}$ be such that

$$\|y(h, x, u^\star_\sigma(x)) - y_h(h, x, A^\star_\sigma(x))\| \le Ch^{q+2}.$$

Then we have

$$\begin{aligned}
&e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + 1 - e^{-h} - v^{\Delta x}_{h,\sigma}(x) \\
&\le e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + 1 - e^{-h} - v_\sigma(x) + |v_\sigma(x) - v^{\Delta x}_{h,\sigma}(x)| \\
&\le e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\star_\sigma(x), h)) + 1 - e^{-h} - (e^{-h} v_\sigma(y(h, u^\star_\sigma, x)) + 1 - e^{-h})
\end{aligned}$$

$$
\begin{aligned}
&\quad + |v_\sigma(x) - v^{\Delta x}_{h,\sigma}(x)| \\
&\le e^{-h}|I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\star_\sigma(x), h)) - I^1_\Gamma[v_\sigma](x + h\Phi(x, A^\star_\sigma(x), h))| \\
&\quad + |I^1_\Gamma[v_\sigma](x + h\Phi(x, A^\star_\sigma(x), h)) - v_\sigma(x + h\Phi(x, A^\star_\sigma(x), h))| \\
&\quad + |v_\sigma(x + h\Phi(x, A^\star_\sigma(x), h)) - v_\sigma(y(h, u^\star_\sigma, x))| + |v_\sigma(x) - v^{\Delta x}_{h,\sigma}(x)| \\
&\le C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{\Delta x^{1/k}}{h}.
\end{aligned}
$$

Therefore, for every $x \in \Omega \setminus S$ we obtain

$$
e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + 1 - e^{-h} - v^{\Delta x}_{h,\sigma}(x) \le C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{\Delta x^{1/k}}{h},
$$

or equivalently

$$
1 - e^{-h} \le v^{\Delta x}_{h,\sigma}(x) - e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](x + h\Phi(x, A^\sigma_{\Delta x,h}(x), h)) + C_1 h^{\frac{q+1}{k}-1} + C_2 \frac{\Delta x^{1/k}}{h}. \tag{5.7}
$$

By multiplying both sides of (5.7) by e^{-mh} and taking $x = y_m$, we obtain

$$
e^{-mh}(1 - e^{-h}) \le e^{-mh}\Big(v^{\Delta x}_{h,\sigma}(y_m) - e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](y_m + h\Phi(y_m, A^{\star,\sigma,m}_{\Delta x,h})\Big) + e^{-mh}\varepsilon(h, \Delta x).
$$

Let N be the minimum number of steps to reach $S_{-\sigma(h)}$ by $\{y_m\}$. Then, by summing over m, we obtain

$$
\begin{aligned}
\sum_{m=0}^{N-1} e^{-mh}(1 - e^{-h}) \le \sum_{m=0}^{N-1} e^{-mh}\Big(v^{\Delta x}_{h,\sigma}(y_m) - e^{-h} I^1_\Gamma[v^{\Delta x}_{h,\sigma}](y_m + h\Phi(y_m, A^{\star,\sigma,m}_{\Delta x,h}, h))\Big) \\
+ \varepsilon(h, \Delta x) \sum_{m=0}^{N-1} e^{-mh}.
\end{aligned}
$$

After simplifying, the proof is complete. □

Now we are ready to state and prove the main result of this section. It shows that the feedback defined by (5.4) through numerical approximation is suboptimal. For the sake of clarity, we choose $\Delta x = h^{q+1}$ and set $\gamma := \frac{q+1}{k} - 1\,(> 0)$.

Theorem 5.4. *Let the assumptions of Proposition 5.1 hold. Then, for every $x \in \Omega$,*

$$
J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) \le \inf_{u(\cdot)\in U} J(u(\cdot), x) + R(\sigma, h),
$$

moreover,

$$
J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \le \inf_{u(\cdot)\in U} J(u(\cdot), x) + R(\sigma, h),
$$

where $R(\sigma, h) = C\Big(\frac{h^\gamma}{1-e^{-h}} + \omega(\sigma)\Big)$, C being a suitable constant, and $u^{\star,\sigma}_{\Delta x,h}(\cdot)$, $\{A^{\star,\sigma,m}_{\Delta x,h}\}$ and $\omega(\sigma)$ are defined according to (5.6), (5.3), respectively.

Proof. From (4.57) we obtain

$$
\left\|v^{\Delta x}_{h,\sigma} - v_{h,\sigma}\right\|_{\infty,\Omega} \le C' h^\gamma,
$$

whence, recalling Lemma 5.3 we have

$$
J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) - v_{h,\sigma}(x) \le v^{\Delta x}_{h,\sigma}(x) + \frac{\varepsilon(h, \Delta x)}{1 - e^{-h}} - v_{h,\sigma}(x) \le \frac{Ch^\gamma}{1 - e^{-h}}. \tag{5.8}
$$

If $\{A^{\star,\sigma,m}_{\Delta x,h}\}$ does not steer x to $S_{-\sigma}$ through $\{y_m\}$, then $J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) = 1$. Thus $J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) \le J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$. Otherwise, let $N^\star$ be the minimum number of steps to reach $S_{-\sigma}$ by $\{y_m\}$. By the assumption (A.2), there exists a control $u^{\star,\sigma}_{\Delta x,h}(\cdot) \in U$ such that $y(hN^\star, x, u^{\star,\sigma}_{\Delta x,h}) \in S$ for $0 < h < \bar{h}$ small enough, and so $J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) \leq J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$. Therefore

$$
\begin{aligned}
J(u^{\star,\sigma}_{\Delta x,h}(\cdot), x) - \inf_{u(\cdot)\in U} J(u(\cdot), x) &\leq J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) - v(x) \\
&= J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) - v_{h,\sigma}(x) + v_{h,\sigma}(x) - v_h(x) \\
&\quad + v_h(x) - v(x) \\
&\leq R(\sigma, h),
\end{aligned}
\tag{5.9}
$$

where the last inequality is due to (5.8) and the mean value theorem, together with (5.3), and (4.54) as in the proof of Proposition 5.1. To prove $J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \leq \inf_{u(\cdot)\in U} J(u(\cdot), x) + R(\sigma, h)$, we just remark that $J_h(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x) \leq J_{h,\sigma}(\{A^{\star,\sigma,m}_{\Delta x,h}\}, x)$, then by following the same procedure as (5.9) the proof is concluded. □

6. **Examples.** This section is devoted to showing the output of an implementation of our scheme to two examples where the minimum time function is not Lipschitz. The papers [17, 18] contain several cases – including the two ones we are going to describe – where the assumptions of Theorem 3.5 are satisfied. Since in our examples the target is smooth, the assumptions of Theorem 3.7, on which all results of the Sections 4.5 and 5 devoted to algorithms are based, are satisfied as well.

The simplest example where our method applies is the well known double integrator, $\ddot{x} = u$, $|u| \leq 1$. It is well known that the minimum time to reach the origin subject to this dynamics is Hölder continuous with exponent $1/2$ on the whole of $\mathbb{R}^2$. Our method applies when the target satisfies the assumptions of Theorem 4.6. In the second example contained in [17, Section 6], it is shown that such assumptions are satisfied (with $k = 2$) if the target is a ball centered at the origin with any radius r small enough. The following figures show the discrete trajectories obtained via the numerical feedback (Figure 1) and the graph of the value function (after Kružkov transformation, Figure 2). The computed trajectories agree with the theoretical computations which can be made through Pontryagin's Maximum Principle. In particular, the two optimal trajectories which reach the target tangentially are correct.

The second example is bilinear and is taken from [18, Example 5.19], up to the factor $1/8$ in place of 10^{-3}. The dynamics is

$$
\begin{cases}
\dot{x}_1 &= -\frac{x_2}{8} - x_2 u \\
\dot{x}_2 &= \frac{x_1}{8} + 2x_1 u,
\end{cases}
\tag{6.1}
$$

where $|u| \leq 1$, and the target is the unit ball. In [18] the authors prove that the assumptions of Theorem 4.6 are satisfied with $k = 2$. To be more precise, it is not difficult to prove that the first order condition (IS.0) is satisfied in the complement of the union of two strips centered at the axes, while in the two strips the second order conditions (IS'.1) and (IS.2) hold. The following figures are the analogues of Figures 1 and 2 for the dynamics (6.1). Observe that if the initial point is close enough to the target, the estimated optimal trajectory is a "bang" one, while otherwise the estimated trajectory has some switchings. The plot of the value function reveals that the time to reach the target is rapidly decreasing. This is not surprising, since the minimum time function, due to the second order controllability condition, is majorized only by the square root of the distance to the target. Equivalently,

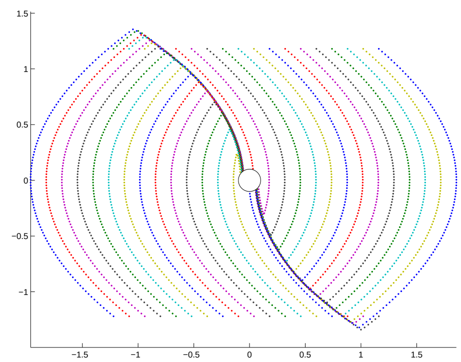

FIGURE 1. Double integrator: computed optimal trajectories (radius of the target $r = 0.1$, $h = 0.025$, $\Delta x = 0.02$, 3rd order Runge-Kutta scheme. Only trajectories issuing from the two horizontal segments are shown.

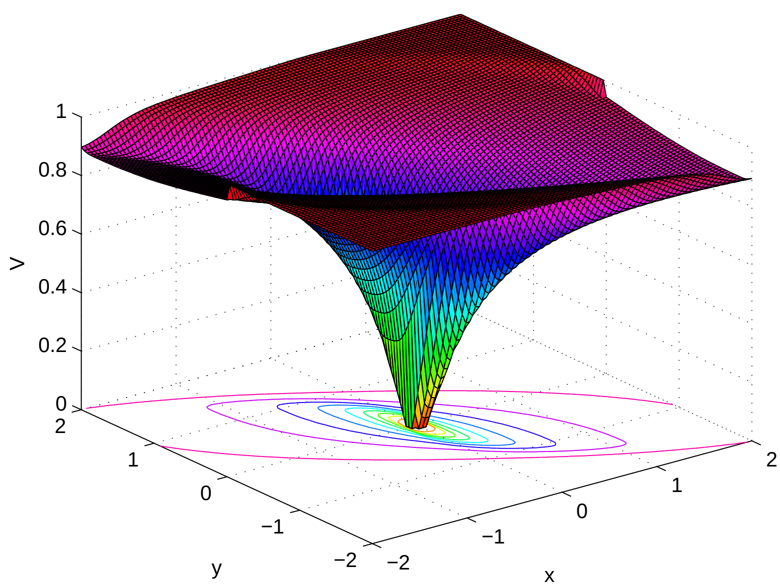

FIGURE 2. Double integrator: graph of the value function, same parameters as in Figure 1.

approaching to the target is very slow (like a sailor which has to beat to windward and therefore proceeds slowly in the desired direction).

Acknowledgments. The authors are indebted with M. Falcone for *really many* useful discussions.

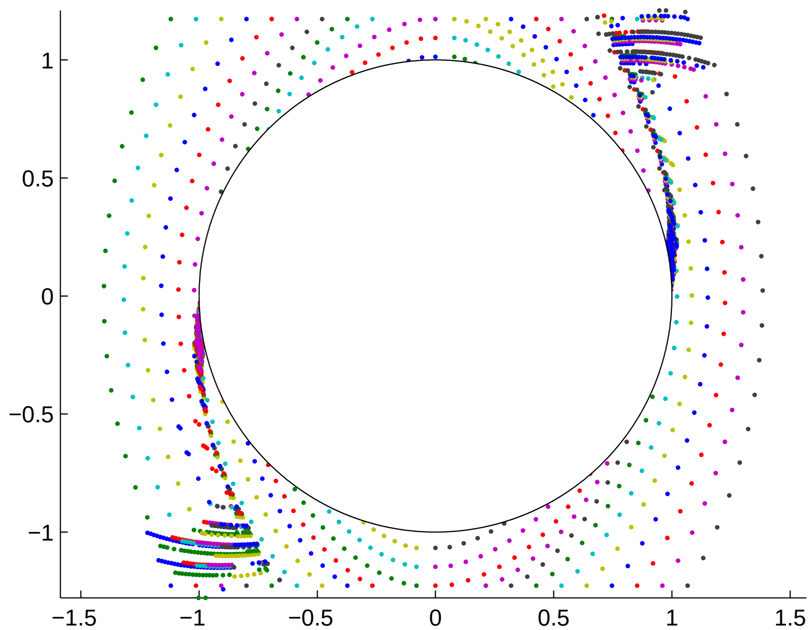

FIGURE 3. Dynamics (6.1): computed optimal trajectories, $h = 0.05$, $\Delta x = 0.027$, 3rd order Runge-Kutta scheme. Only trajectories issuing from the two horizontal segments are shown.

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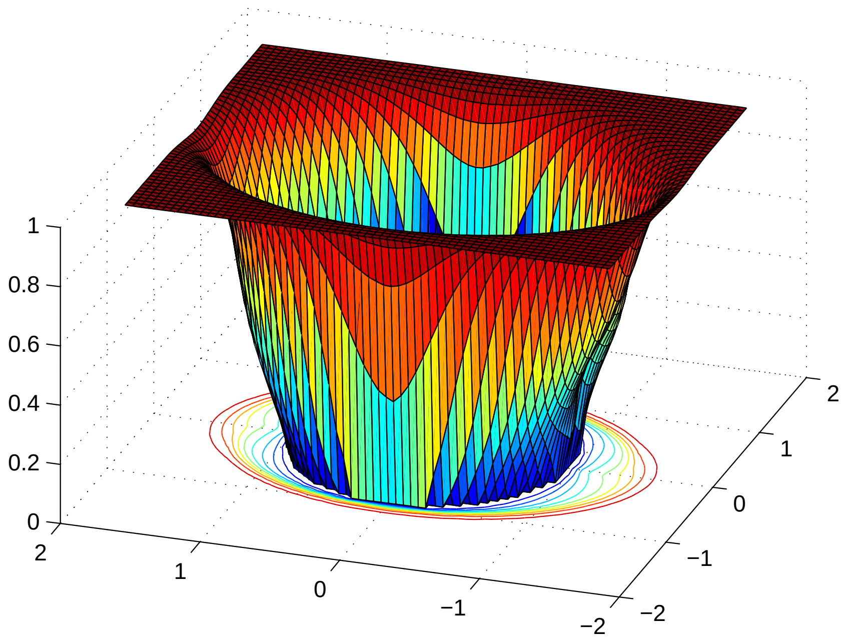

FIGURE 4. Dynamics (6.1): graph of the value function, same parameters as in Figure 3.

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Received April 2014; revised October 2014.

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