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# On a $p$-adic invariant cycles theorem

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**Abstract.** For a proper semistable curve $X$ over a DVR of mixed characteristics and perfect residue field we prove the "invariant cycles theorem" with trivial coefficients, i.e. that the group of elements of the first de Rham cohomology group of the generic fiber of $X$ annihilated by the monodromy operator coincides with the first rigid cohomology group of its special fiber. This is done using an explicit description of the monodromy operator on the de Rham cohomology of the generic fiber of $X$ with coefficients a convergent $F$-isocrystal (see [11]) and the proof exhibits an interesting interplay between this cohomology group and the combinatorics of the graph of the reduction of $X$. The result was proved in a different way in the case the DVR has finite residue field in [4]. We also study the case where the coefficients are unipotent convergent $F$-isocrystals on the special fiber of $X$ (without log-structure): we show that the invariant cycles theorem does not hold in general in this setting and give a sufficient condition for non-exactness.

## 1. Introduction

Let $\mathcal{V}$ be a complete discrete valuation ring of mixed characteristics, $K$ its fraction field and $k$ the residue field, which we assume to be perfect. Let $W := W(k)$ denote the ring of Witt-vectors with coefficients in $k$ seen as a subring of $\mathcal{V}$ and let $K_0$ denote its fraction field.

For a proper variety $X$ over $\mathcal{V}$ with semistable reduction and special fiber $X_k$, via the theory of log schemes and the work of Hyodo–Kato one defines a monodromy operator on the de Rham cohomology groups of its generic fiber $X_K$. It has been known for some time now that associated to this operator there is an analogue of the classical invariant cycles sequence [4]

$$H^i_{\mathrm{rig}}(X_k) \otimes_{K_0} K \to H^i_{\mathrm{dR}}(X_K) \to H^i_{\mathrm{dR}}(X_K).$$

The exactness of such a sequence is implied by the weight-monodromy conjecture [4] if the residue field $k$ is finite. Hence the above invariant cycles sequence is exact if $X$ is a curve or

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a surface (which are the cases in which the weight-monodromy conjecture is known) and in this case the first map is even injective if $i = 1$, i.e. the following sequence is exact:

$$0 \to H^1_{\mathrm{rig}}(X_k) \otimes_{K_0} K \to H^1_{\mathrm{dR}}(X_K) \to H^1_{\mathrm{dR}}(X_K). \tag{1.1}$$

In these cases (i.e. in the cases in which the sequence (1.1) is exact) we obtain an interpretation of the part of the de Rham cohomology which is annihilated by the monodromy operator: it is the rigid cohomology group of the special fiber. On the other hand the same exact sequence gives us an interpretation à la Fontaine of the first rigid cohomology group, in fact we can translate the exactness as follows: since

$$D_{\mathrm{st}}(H^1_{\text{ét}}(X_K \times \overline{K}), \mathbb{Q}_p) = H^1_{\log-\mathrm{crys}}(X_k) \otimes K,$$
$$D_{\mathrm{st}}^{N=0} = D_{\mathrm{crys}},$$

it follows that

$$H^1_{\mathrm{rig}}(X_k) = D_{\mathrm{crys}}(H^1_{\text{ét}}(X_K \times \overline{K}), \mathbb{Q}_p).$$

In [9] and [10] another definition of a monodromy operator was given in the case $X$ is a curve with semistable reduction using the combinatorics of the curve together with the use of the analytic spaces associated to the generic fiber. The authors in [11] also considered the case of cohomology with coefficients and generalized the definition of the monodromy operator on the de Rham cohomology with coefficients non-trivial log-$F$-isocrystals and they showed that it coincides with the previous definition given by Faltings [13]. Using this definition of the monodromy operator we are able (see Section 5) to re-prove the exactness of the invariant cycles sequence (1.1) without any hypothesis on the finiteness of the residue field. It is then natural to ask if such an invariant cycles sequence (1.1) is still exact when the log-$F$-isocrystals are induced from convergent $F$-isocrystals on the special fiber. This is one of the motivations of the present article. As a matter of fact, the invariant cycles sequence (1.1) involves the trivial convergent $F$-isocrystal on the special fiber of $X$ and its rigid cohomology. Hence we start with coefficients which a priori do not have singularities being convergent on the special fiber without any log structure. But, even for the simplest non-trivial coefficients on a curve (i.e. the unipotent ones) the sequence fails sometimes to be exact and we give a sufficient condition (see Theorem 10). Underlying our work, of course, is the aim of giving a cohomological interpretation for the part of the cohomology on which the monodromy operator acts as zero.

Of course the invariant cycles theorem can be studied also in the $\ell$-adic and respectively the complex settings, where it is known for semi-simple perverse sheaves or $\mathcal{D}$-modules of geometric origin and it follows from the decomposition theorem ([1, Corollaire 6.2.8], [18, Theorem 19.47], [20, Theorem 1], [12, Theorem, Section 1.7]). Our $p$-adic setting deals with unipotent, non-trivial coefficients, which are therefore not semi-simple. We did not find any evidence of a similar result for reducible coefficients in the $\ell$-adic or complex settings, although we believe that such results should hold.

Here it is the plan of our article. In Section 2 we introduce notation and recall results on rigid spaces which will be used in the article, in the third section we recall some properties of the monodromy operator on the de Rham cohomology with coefficients on a curve as introduced by Coleman and Iovita and of the associated invariant cycles sequence. In Section 4 we give some properties of such a monodromy operator: in particular for general convergent $F$-isocrystals we prove that the rigid cohomology of the convergent $F$-isocrystal injects on the

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part of the de Rham cohomology of the associated log-$F$ isocrystal where the monodromy acts as zero. In Section 5, we then re-prove ([4]) the invariant cycles theorem for trivial coefficients in a combinatorial way along the lines of the work in [11]. In Section 6 we study the invariant cycles sequence for unipotent convergent $F$-isocrystals and we prove a sufficient conditions for the non-exactness of the sequence. Finally we give an explicit example of this on a Tate curve.

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## 2. Notation and settings. A Mayer–Vietoris exact sequence

We assume the notations in Section 1. Let $X$ be a proper curve over $\mathcal{V}$ (of mixed characteristic) that is semistable, which means that locally for the Zariski topology there is an étale map to $\mathrm{Spec}(\mathcal{V}[x, y]/xy - \pi)$ and we suppose that the special fiber has at least two irreducible components and all components are smooth. We denote by $X_k$ the special fiber of $X$ which we suppose connected, by $X_K$ its generic fiber and by $X_K^{\mathrm{rig}}$ its rigid analytic generic fiber. We suppose that the intersection points of the components of $X_k$ are $k$-rational. As $X$ is a proper, regular curve over $\mathcal{V}$, Theorem 2.8 of [17] implies that $X$ is a projective $\mathcal{V}$-scheme.

Following [10] we associate to $X_k$ a multigraph $\mathrm{Gr}(X_k)$ whose definition we now recall. To every irreducible component $C_v$ of $X_k$ we associate a vertex $v$ and if $v, w$ are vertices, an oriented edge $e$ with origin $v$ and end $w$ corresponds to an intersection point $C_e$ of the components $C_v$ and $C_w$. Let $v$ and $w$ be two vertices which correspond to the components $C_v$ and $C_w$, let us suppose that they intersect in $n \geq 1$ points $P_1, \dots, P_n$, then they correspond to $n$ edges in $\mathrm{Gr}(X_k)$ with origin in $v$ and end $w$ denoted by $[v, w]_1, \dots, [v, w]_n$. To simplify the notation if $e$ is an oriented edge with origin $v$ and end $w$, we write $e = [v, w]$. We denote by $\mathscr{V}$ the set of vertices and by $\mathscr{E}$ the set of oriented edges.

We have the specialization map

$$\mathrm{sp} : X_K^{\mathrm{rig}} \to X_k$$

defined in [2]. For every $v \in \mathscr{V}$ we define

$$X_v := \mathrm{sp}^{-1}(C_v)$$

and for every $e \in \mathscr{E}$

$$X_e := \mathrm{sp}^{-1}(C_e).$$

The set $X_e$ is an open annulus in $X_K^{\mathrm{rig}}$ and $X_v$ is what is called a wide open subspace in [8, Proposition 3.3], that means an open of $X_K^{\mathrm{rig}}$ isomorphic to the complement of a finite number of closed disks, each contained in a residue class, in a smooth proper curve over $K$ with good reduction. If $C_v$ and $C_w$ intersect in $C_e$, then $X_v \cap X_w = X_e$.

One can prove that $\{X_v\}_{v\in\mathscr{V}}$ is an admissible covering of $X_K^{\mathrm{rig}}$ (see [8]) and that wide opens are Stein spaces so that we can use the covering $\{X_v\}_{v\in\mathscr{V}}$ to calculate the de Rham cohomology of $X_K^{\mathrm{rig}}$ using a Čech complex. Moreover one can prove that the first de Rham

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cohomology of a wide open is finite ([8, Theorem 4.2]) proving a comparison theorem with the de Rham cohomology of an algebraic curve minus a finite set of points.

Let $(\mathcal{E}, \nabla)$ be a module with integrable connection on $X_K^{\mathrm{rig}}$. Given the admissible covering $\{X_v\}_{v \in \mathcal{V}}$ which has the property that the intersection of any collection of three distinct $X_v$ is void, we can write the Mayer–Vietoris sequence:

$$\text{(2.1)} \quad \bigoplus_{v \in \mathcal{V}} H^0_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)) \xrightarrow{\alpha} \bigoplus_{e \in \mathcal{E}} H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)) \longrightarrow H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)) \longrightarrow \bigoplus_{v \in \mathcal{V}} H^1_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)) \xrightarrow{\beta} \bigoplus_{e \in \mathcal{E}} H^1_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)).$$

Let us remark that every cohomology group that appears in the long exact sequence except for $H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))$ can be calculated as the cohomology of the global sections of the de Rham complex, due to the fact that every wide open is Stein.

From equation (2.1) we can deduce the short exact sequence

$$\text{(2.2)} \qquad 0 \to H^1(\mathrm{Gr}(X_k), \mathcal{E}) \xrightarrow{\gamma} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla))) \to \mathrm{Ker}(\beta) \to 0,$$

where $H^1(\mathrm{Gr}(X_k), \mathcal{E}) := \mathrm{Coker}(\alpha)$.

## 3. The monodromy operator and the rigid cohomology

We consider again a proper and semistable curve $X$, its generic fiber $X_K$ and its associated rigid space $X_K^{\mathrm{rig}}$. We recall the construction of the monodromy operator in [11, Section 2.2].

By our assumptions there is a proper scheme $P$ over $W$, smooth around $X_k$ and such that we have a global embedding $X \hookrightarrow P \times_{\mathrm{Spec(W)}} \mathrm{Spec}(\mathcal{V}) = P_{\mathcal{V}}$. Let us denote by $P_k$ its special fiber and by $P_{K_0}^{\mathrm{rig}}$ and $P_K^{\mathrm{rig}}$ the rigid analytic spaces associated to $P$ and $P_{\mathcal{V}}$; then one has the following diagram:

$$\begin{array}{ccccc} & & & & P_{K_0}^{\mathrm{rig}} \\ & & & \swarrow^{\mathrm{sp}_P} & \downarrow \\ X_k & \longrightarrow & P_k & \longrightarrow & P, \end{array}$$

where the map between $P_{K_0}^{\mathrm{rig}}$ and $P_k$ is the specialization map that we denote by $\mathrm{sp}_P$. We also have a specialization map $\mathrm{sp}_{P_{\mathcal{V}}} : P_K^{\mathrm{rig}} \to P_k$. One can consider the tubes $\mathrm{sp}_P^{-1}(X_k) := ]X_k[_P$ and $Y_K := \mathrm{sp}_{P_{\mathcal{V}}}^{-1}(X_k) = ]X_k[_{P_{\mathcal{V}}}$. Let now $E$ be a convergent $F$-isocrystal on $X_k$. It has a realization on $]X_k[_P, (\mathcal{E}, \nabla)$, and we denote by $(\mathcal{E}, \nabla)_K$ its base change to $K$. It is a module with connection on $Y_K$. We will denote by the same symbol its restriction to $X_K^{\mathrm{rig}}$. We may then define the first rigid cohomology group with coefficients in $E$ as

$$H^1_{\mathrm{rig}}(X_k, E) := H^1_{\mathrm{dR}}(]X_k[_P, (\mathcal{E}, \nabla)),$$

which is a finite dimensional $K_0$-vector space. We also consider

$$H^1_{\mathrm{rig}}(X_k, E)_K := H^1_{\mathrm{dR}}(]X_k[_{P_{\mathcal{V}}}, (\mathcal{E}, \nabla)_K) = H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K).$$

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On the other hand we can proceed as before and consider $X_K^{\mathrm{rig}}$, the rigid analytic space associated to $X_K$. We then have a morphism of rigid spaces

$$\varphi : X_K^{\mathrm{rig}} \to Y_K$$

given by the immersion of $X$ into $P_{\mathcal{V}}$, which induces the pull-back map in cohomology

$$\varphi^* : H^1_{\mathrm{rig}}(X_k, E)_K := H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K) \to H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K). \tag{3.1}$$

In the notations above we define following [11] a $K$-linear map

$$N : H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K) \to H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K).$$

Due to the fact that wide opens are Stein spaces, every element $[\omega]$ in $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ can be described as a hypercocycle $((\omega_v)_{v\in\mathscr{V}}, (f_e)_{e\in\mathscr{E}})$, with $(\omega_v)$ in $\Omega^1_{X_v} \otimes \mathcal{E}_{X_v}$ and $f_e$ in $\mathcal{E}_{X_e}$ that verifies that $\omega_{v|X_e} - \omega_{w|X_e} = \nabla(f_e)$ if $e = [v, w]$.

Let us remember that every $X_e$ is an ordered open annulus; we can define a residue map

$$\mathrm{Res} : H^1_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) \to H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K)$$

as follows. The module with connection $(\mathcal{E}, \nabla)_K$ has a basis of horizontal sections $e_1, \dots, e_n$ on $X_e$ because $X_e$ is a residue class ([11, Lemma 2.2]). Hence if $z$ is an ordered uniformizer of the ordered annulus $X_e$, every differential form $\mu_e \in H^1_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K)$ can be written as $\mu_e = \sum_{i=1}^n (e_i \otimes \sum_j a_{i,j} z^j \, dz)$ with $a_{i,j} \in K$. Then $\mathrm{Res}(\mu_e) = \sum_{i=1}^n a_{i,-1} e_i$, and it is an isomorphism of vector spaces.

For a cohomology class $[\omega]$ represented as before by $((\omega_v)_{v\in\mathscr{V}}, (f_e)_{e\in\mathscr{E}})$ we define $N$ as the composition of the following maps:

$$\begin{aligned} \tilde{N} : H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K) &\to \bigoplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K)), \\ [\omega] &\mapsto (\mathrm{Res}(\omega_{v|X_e})_{e=[v,w]}) \end{aligned}$$

and the map

$$\begin{aligned} i : \bigoplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) \to \bigoplus_{e\in\mathscr{E}} & H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) / \bigoplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)_K) \\ &\xrightarrow{\gamma} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K), \\ (f_e)_{e\in\mathscr{E}} &= (0, f_e/\mathrm{Im}(\alpha))_{v\in\mathscr{V}, e\in\mathscr{E}}, \end{aligned}$$

and $\alpha$ as in (2.1) and $\gamma$ as in (2.2).

Hence $N$ is defined as $N = i \circ \tilde{N}$. Note that $N^2 = 0$.

In order to give an interpretation of the monodromy operator on the de Rham cohomology defined above we will introduce the log formalism. The curve $X$ can be equipped with a log structure, associated to the special fiber $X_k$ which is a divisor with normal crossing and $\mathrm{Spec}(W)$ with the log structure given by the closed point. Pulling them back to $X_k$ and to $\mathrm{Spec}(k)$ respectively, we may consider $X_k$ and $\mathrm{Spec}(k)$ as log schemes, and when we want to treat them as log schemes, we denote them by $X_k^\times$ and $\mathrm{Spec}(k)^\times$. The log structure

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on Spec$(W)$ induces a log structure on Spf$(W)$, and again when we want to treat it as a log formal scheme, we denote it by Spf$(W)^\times$. We note that in the case of the trivial isocrystal by [16] the de Rham cohomology groups of the generic fiber coincide with the log-crystalline ones of $X_k^\times$, base-changed to $K$. This result holds also in our case with coefficients. In fact if we start with a convergent $F$-isocrystal on $X_k$, then one can associate to it a log-convergent $F$-isocrystal on $X_k^\times$ and then a log(-crystalline) $F$-isocrystal on $X_k^\times$ ([21, Theorem 5.3.1]): we again denote it by $E$.

**Proposition 1.** *In the previous hypothesis and notations if we start with a convergent $F$-isocrystal $E$ on $X_k$ and we denote by $(\mathcal{E}, \nabla)$ its realization on $]X_k[_P$, then the cohomology of the restriction $H^i_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K)$ coincides with the log-crystalline cohomology of the associated log-$F$-isocrystal on $X_k^\times$, $H^i_{\mathrm{log-crys}}(X_k^\times, E) \otimes_{K_0} K$. The monodromy operators coincide as well.*

*Proof.* We are in the case of [13]. The Frobenius structure will imply that the relative log cohomology arising from the deformation gives a locally free module, but it will guarantee also that the exponents of the associated Gauss–Manin differential system are non-Liouville numbers: hence we may trivialize the system by the Transfer Theorem [7]. For the coincidence of the monodromy operators we refer to [11]. □

Using $\varphi^*$ of (3.1) and the monodromy operator $N$, we can form the following sequence:

$$H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K) \xrightarrow{\varphi^*} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K) \xrightarrow{N} H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}}, (\mathcal{E}, \nabla)_K). \tag{3.2}$$

We quote the following theorem from [4], where $k$ is finite, $X_K$ is a variety of dimensions 1 or 2 and $X_k$ is projective (see also [19]).

**Theorem 2.** *In the sequence* (3.2) *if $E$ is the trivial isocrystal, then the map $\varphi^*$ is injective and* $\mathrm{Im}(\varphi^*) = \mathrm{Ker}(N)$.

In the next section we will prove that if $E$ is not necessarily the trivial isocrystal, then in the sequence (3.2) the map $\varphi^*$ is injective and $\mathrm{Im}(\varphi^*) \subset \mathrm{Ker}(N)$. Moreover if $E$ is the trivial isocrystal, we will give a new proof of Theorem 2 using the explicit description of the monodromy operator as introduced before.

**Remark 3.** According to [11] for the definition of the monodromy operator on the de Rham cohomology we did not need either the Frobenius structure or an isocrystal: we just needed a connection on the generic fiber with the property that its restriction to any residue class is trivial. In general we do not know the interpretation of such an operator in terms of the integral structures.

## 4. The behavior of the monodromy operator

We would like to study the properties of the monodromy operator as defined in the previous section and, in particular, the exactness of the sequence (3.2).

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As in Section 2 let us consider the multigraph $\mathrm{Gr}(X_k)$ associated to $X_k$, with vertices in $\mathscr{V}$ and edges in $\mathscr{E}$. For $v \in \mathscr{V}$ we denote by $X_v := \mathrm{sp}_X^{-1}(C_v)$ and $Y_v := \mathrm{sp}_{P_{\mathcal{V}}}^{-1}(C_v)$; because the definition of $\varphi$, we have that $\varphi(X_v) \subset Y_v$. In the same way we denote by $X_e := \mathrm{sp}_X^{-1}(C_e)$ and $Y_e := \mathrm{sp}_{P_{\mathcal{V}}}^{-1}(C_e)$; because of the definition of $\varphi$, we have that $\varphi(X_e) \subset Y_e$.

Let us note that $Y_e$ is a polydisk because $P_{\mathcal{V}}$ is smooth. We choose the admissible covering of $X_K^{\mathrm{rig}}$ given by $\{X_v\}_{v\in\mathscr{V}}$ to calculate the de Rham cohomology using Čech complexes.

As before let $E$ be an $F$-convergent isocrystal on $X_k$, we can also use the Mayer–Vietoris spectral sequence for rigid cohomology with coefficients in $E$ ([23, Theorem 7.1.2]). We use the finite closed covering of $X_k$ given by $\{C_v\}$. Since the intersection of every collection of three distinct components is empty, the spectral sequence degenerates to a Mayer–Vietoris long exact sequence ([14, Theorem 4.6.1])

$$\bigoplus_{v\in\mathscr{V}} H^0_{\mathrm{rig}}(C_v,E) \xrightarrow{\alpha} \bigoplus_{e\in\mathscr{E}} H^0_{\mathrm{rig}}(C_e,E) \longrightarrow H^1_{\mathrm{rig}}(X_k,E) \longrightarrow \bigoplus_{v\in\mathscr{V}} H^1_{\mathrm{rig}}(C_v,E) \xrightarrow{\sigma} \bigoplus_{e\in\mathscr{E}} H^1_{\mathrm{rig}}(C_e,E), \tag{4.1}$$

whose base-change to $K$ can be described in terms of the de Rham cohomology of $Y_K$ as

$$\begin{array}{ccc} \bigoplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K) & \xrightarrow{\alpha} & \bigoplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(Y_e,(\mathcal{E},\nabla)_K) \\ & & \downarrow \\ \bigoplus_{v\in\mathscr{V}} H^1_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K) & \longleftarrow & H^1_{\mathrm{dR}}(Y_K,(\mathcal{E},\nabla)_K) \\ \downarrow \sigma & & \\ \bigoplus_{e\in\mathscr{E}} H^1_{\mathrm{dR}}(Y_e,(\mathcal{E},\nabla)_K). & & \end{array} \tag{4.2}$$

Now we study the exactness property of the sequence (3.2).

**Lemma 4.** *If $E$ is a convergent isocrystal and $(\mathcal{E},\nabla)$ is the coherent module with integrable connection induced by it, then the map $\varphi^*$ in the sequence* (3.2) *is injective.*

*Proof.* We fix an irreducible component $C_v$ of $X_k$ and we would like to prove that the following sequence is exact:

$$0 \to H^1_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K) \to H^1_{\mathrm{dR}}(X_v,(\mathcal{E},\nabla)_K) \to \bigoplus_{e\in\mathscr{E}_v} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K), \tag{4.3}$$

where the last map is the residue map and $\mathscr{E}_v := \{e : \text{there exists a vertex } w \text{ with } e=[v,w]\}$. As $C_v$ is proper and smooth, the above sequence will be isomorphic to the following sequence:

$$0 \to H^1_{\mathrm{crys}}(C_v,E)\otimes K \to H^1_{\mathrm{log-crys}}(C_v^{\times\times},E)\otimes K \to \bigoplus_{e\in\mathscr{E}_v} H^0_{\mathrm{dR}}(X_e,(\mathcal{E},\nabla)_K), \tag{4.4}$$

where $C_v^{\times\times}$ is the log scheme given by the component $C_v$ with the log structure induced by the divisor given by the intersection points of $C_v$ with the other components. The two sequences are isomorphic as

$$H^1_{\mathrm{crys}}(C_v,E)\otimes K \cong H^1_{\mathrm{dR}}(Y_v,(\mathcal{E},\nabla)_K)$$

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since $C_v$ is proper and smooth,

$$H^1_{\log-\mathrm{crys}}(C_v^{\times\times}, E) \otimes K \cong H^1_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)_K)$$

by [11, Lemma 5.2]. Moreover the second one is exact because it is the Gysin sequence for rigid cohomology.

For the convenience of the reader we recall that the Gysin sequence for rigid cohomology, in our setting is the following ([5, Proposition 2.1.4]):

$$0 \to H^1_{\mathrm{rig}}(C_v, E) \otimes K \to H^1_{\mathrm{rig}}(U_v, E) \otimes K \to \bigoplus_{e \in \mathscr{E}_v} H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K), \tag{4.5}$$

where $U_v$ is the complement in $C_v$ of all the points of $C_v$ that intersect the other components of $X_k$.

The isomorphism $H^1_{\mathrm{rig}}(U_v, E) \cong H^1_{\text{log-crys}}(C_v^{\times\times}, E)$ follows from [22, Section 2.4] and [22, Theorem 3.1.1]. Moreover

$$H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) \cong H^0_{\mathrm{dR}}(Y_e, (\mathcal{E}, \nabla)_K)$$

because $Y_e$ and $X_e$ are residue classes and $E$ has a basis of horizontal sections on each residue class, which means that both $H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K)$ and $H^0_{\mathrm{dR}}(Y_e, (\mathcal{E}, \nabla)_K)$ are isomorphic to $K^d$ where $d$ is the rank of $\mathcal{E}$ as $\mathcal{O}_{X_K}$-module. Moreover by the Gysin isomorphism in degree zero ([5, Proposition 2.1.4]), with the same notations as before, we have

$$H^0_{\mathrm{rig}}(C_v, E) \cong H^0_{\mathrm{rig}}(U_v, E),$$

which implies that

$$H^0_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)_K) \cong H^0_{\mathrm{dR}}(Y_v, (\mathcal{E}, \nabla)_K),$$

using the same techniques as before.

Using the Mayer–Vietoris long exact sequence for rigid cohomology (4.2), we can pass to the following short exact sequence:

$$0 \to H^1(\mathrm{Gr}(X_k), \mathcal{E}_K) \xrightarrow{\delta} H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K)) \to \mathrm{Ker}(\sigma) \to 0, \tag{4.6}$$

where $H^1(\mathrm{Gr}(X_k), \mathcal{E}_K) := \mathrm{Coker}(\alpha)$.

Putting together Mayer–Vietoris sequences for the coverings $\{X_v\}$ and $\{Y_v\}$ respectively we obtain the following diagram:

$$\begin{array}{ccccccc}
 & & \bigoplus_e H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) & \to & \bigoplus_e H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) & & \\
 & \swarrow^{\theta} & \uparrow{\scriptstyle \mathrm{Res}} & & \uparrow & & \\
0 \to H^1(\mathrm{Gr}(X_k), \mathcal{E}_K) & \xrightarrow{\gamma} & H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K) & \xrightarrow{\pi_X} & \bigoplus_v H^1_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)_K) & & \\
\uparrow & & \uparrow{\scriptstyle \varphi^*} & & \uparrow{\scriptstyle \varphi^*} & & \\
0 \to H^1(\mathrm{Gr}(X_k), \mathcal{E}_K) & \xrightarrow{\delta} & H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K) & \xrightarrow{\pi_Y} & \bigoplus_v H^1_{\mathrm{dR}}(Y_v, (\mathcal{E}, \nabla)_K) & \to 0 & \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array} \tag{4.7}$$

and by the Snake Lemma one can conclude that $\varphi^* : H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K) \to H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ is injective. □

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**Remark 5.** Let us note that in (4.7) the monodromy operator on $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ acts as $N = \gamma \circ \theta \circ \mathrm{Res}$.

**Lemma 6.** *If $E$ is a convergent $F$-isocrystal and $(\mathcal{E}, \nabla)$ is the coherent module with integrable connection induced by it, then in the sequence* (3.2)

$$N \circ \varphi^* = 0.$$

*Proof.* Let us consider an element $[\omega] \in H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K)$. Then $\varphi^*[\omega]$, which is an element of $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$, can be represented by a hypercocycle $((\alpha_v)_{v \in \mathscr{V}}, (g_e)_{e \in \mathscr{E}})$ where $\alpha_v \in \Omega^1_{X_v} \otimes \mathcal{E}_{X_v}$ and $g_e$ in $\mathcal{E}_{X_e}$ and they verify that $\alpha_{v|X_e} - \alpha_{w|X_e} = \nabla(g_e)$ if $e = [v, w]$. We want to calculate $N(\varphi^*([\omega]))$. We now look at the diagram (4.7). By the definition of $N$ one can see that

$$N(\varphi^*([\omega])) = \gamma \circ \theta \circ \mathrm{Res}(\varphi^*([\omega])) = \gamma \circ \theta \circ \mathrm{Res}_{|X_e}(\pi_X(\varphi^*[\omega])).$$

By the commutativity of the diagram (4.7),

$$\mathrm{Res}_{|X_e}(\pi_X(\varphi^*[\omega])) = \mathrm{Res}_{|X_e}(\varphi^*(\pi_Y([\omega]))).$$

If we denote by $\omega_v = \pi_Y([\omega])$, then we have to compute $\mathrm{Res}_{|X_e}(\varphi^*(\omega_v))$:

$$\mathrm{Res}_{|X_e}(\varphi^*(\omega_v)) = \mathrm{Res}(\varphi^*(\omega_v)_{|X_e}) = \mathrm{Res}(\varphi^*(\gamma_e))$$

where $\gamma_e \in \mathcal{E}_{Y_e} \otimes \Omega^1_{Y_e}$, but as $Y_e$ is an open polydisk we have that $H^1_{\mathrm{dR}}(Y_e, (\mathcal{E}, \nabla)_K) = 0$ and so $\mathrm{Res}(\phi^*(\gamma_e)) = 0$ as claimed. □

From the above lemma we can conclude that $\mathrm{Im}(\varphi^*) \subset \mathrm{Ker}(N)$. Now we would like to characterize in terms of residues the elements of $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ which are in the image of the map $\varphi^*$.

Let us take an element $[\omega] \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$. As before we can choose a representative $\omega = ((\omega_v)_{v \in \mathscr{V}}, (f_e)_{e \in \mathscr{E}})$, with $(\omega_v)$ in $\mathcal{E}_{X_e} \otimes \Omega^1_{X_v}$ and $f_e$ in $\mathcal{E}_{X_e}$, which verifies that $\omega_{v|X_e} - \omega_{w|X_e} = \nabla(f_e)$ if $e = [v, w]$.

In the next lemma we prove a necessary and sufficient condition for an element of the cohomology group $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ to be in the image of the map $\varphi^*$.

**Lemma 7.** *Let us take an element $[\omega] \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ and take a representative $\omega = ((\omega_v)_{v \in \mathscr{V}}, (f_e)_{e \in \mathscr{E}})$ as above. Then $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$ for every $e \in \mathscr{E}$ if and only if $[\omega] \in \mathrm{Im}(\varphi^*)$.*

*Proof.* Let us see first that if $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$ for every $e \in \mathscr{E}$, then $[\omega] \in \mathrm{Im}(\varphi^*)$. If $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$, thanks to the exact sequence (4.3) there exists a $\gamma_v \in H^1_{\mathrm{dR}}(Y_v, (\mathcal{E}, \nabla)_K)$ such that $\varphi^*(\gamma_v) = \omega_v$ for every $v \in \mathscr{V}$. As the map $\pi_Y$ in (4.7) is surjective, there exists an $\alpha \in H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K)$ such that $\pi_Y(\alpha) = (\omega_v)_{v \in \mathscr{V}}$.

Now $\pi_X([\omega] - \varphi^*(\alpha)) = 0$, hence, looking again at the diagram (4.7), there exists an element $c \in H^1(\mathrm{Gr}(X_k), \mathcal{E}_K)$ such that $[\omega] - \varphi^*(\alpha) = \gamma(c)$. By the commutativity of the diagram (4.7) there exists an element $\mu \in H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K)$ such that $\varphi^*(\mu) = \gamma(c)$. (One can choose $\mu = \delta(c)$.)

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Conversely if $[\omega] = \varphi^*(\alpha)$ for $\alpha \in H^1_{\mathrm{dR}}(Y_K, (\mathcal{E}, \nabla)_K)$, then

$$(\omega_v)_{v\in\mathscr{V}} = \varphi^*(\pi_Y(\alpha)) := \varphi^*(\alpha_v)_{v\in\mathscr{V}}.$$

Hence $\mathrm{Res}_{|X_e}(\omega_v) = \mathrm{Res}_{|X_e}(\varphi^*(\alpha_v))$ for every $v \in \mathscr{V}$. But as in the proof of Lemma 6 one can prove that from this it follows that $\mathrm{Res}_{|X_e}(\omega_v) = 0$ for every $v \in \mathscr{V}$. □

## 5. The constant coefficients case

In this section we show that if $E$ is the trivial convergent $F$-isocrystal, then the condition in Lemma 7 is fulfilled. This will imply that the sequence in (3.2) is exact and it will give a new proof of Theorem 2, i.e. the exactness of the invariant cycles sequence under the assumption that $k$ is perfect instead of finite. The realization of $E$ on $X_K^{\mathrm{rig}}$ is the structure sheaf with trivial connection $(\mathcal{O}_{X_K}, d)$.

We would like to prove that if $[\omega] \in H^1_{\mathrm{dR}}(X_K^{\mathrm{rig}})$ is such that $N([\omega]) = 0$, then one can find a hypercocycle $(\omega_v, f_e)$ representing it such that $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$: hence we may apply Lemma 7 and conclude.

Let $(\omega_v, f_e)$ be a hypercocycle representing $[\omega]$ and consider $\mathrm{Res}_{X_e}(\omega_{v|X_e})$; if $[\omega]$ is in $\mathrm{Ker}(N)$, then $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e = 0$ in $H^1(\mathrm{Gr}(X_k), \mathcal{O}_K)$, that means that

$$(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e \in \mathrm{Coker}\Big(\bigoplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(X_v) \to \bigoplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e)\Big).$$

On the other hand, thanks to the Residue Theorem on Wide Opens ([8, Proposition 4.3]), for every irreducible component $C_v$ in $X_k$, the family $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e$ verifies that

$$\sum_{e\in\mathscr{E}_v} \mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0, \tag{5.1}$$

where the notation $\mathscr{E}_v$ refers, as before, to the set $\{e : \text{there exists a vertex } w \text{ with } e = [v, w]\}$.

Hence to prove that $\mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$ we are left to prove that if $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e$ is an element of $\mathrm{CoKer}(\bigoplus_{v\in\mathscr{V}} H^0_{\mathrm{dR}}(X_v) \to \bigoplus_{e\in\mathscr{E}} H^0_{\mathrm{dR}}(X_e))$ and for every $v$ it verifies that $\sum_{e\in\mathscr{E}_v} \mathrm{Res}_{X_e}(\omega_{v|X_e}) = 0$, then $(\mathrm{Res}_{X_e}(\omega_{v|X_e}))_e = 0$ for all $e$. So we are reduced to a linear algebra and graph theory problem, which we can translate as follows.

Let $\mathbb{F}$ be a field of characteristic 0. Let $G$ be a connected multigraph with $n$ vertices and $m$ edges. Let us denote by $\mathscr{V}$ the set of all vertices and by $\mathscr{E}$ the set of all oriented edges. We associate to $G$ a vector space $V = \bigoplus_{e\in\mathcal{E}} \mathbb{F}$ modulo the relations $a_e = -a_{\bar{e}}$, where if $e = [v, w]$, then $\bar{e} = [w, v]$. Then there is a map

$$\phi : \bigoplus_{v\in\mathscr{V}} \mathbb{F} \to \bigoplus_{e\in\mathscr{E}} \mathbb{F},$$
$$(a_v)_{v\in\mathscr{V}} \mapsto (a_e)_{e\in\mathscr{E}},$$

where $a_e = a_v - a_w$ if $e = [v, w]$. We consider two vector subspaces $W$ and $T$ of $\bigoplus_{e\in\mathscr{E}} \mathbb{F}$ where

$$W = \{(a_e)_{e\in\mathscr{E}} : (a_e)_{e\in\mathscr{E}} \in \mathrm{Im}(\phi)\},$$
$$T = \Big\{(a_e)_{e\in\mathscr{E}} : \sum_{e\in\mathscr{E}_v} a_e = 0 \text{ for all } v \in \mathscr{V}\Big\}.$$

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**Proposition 8.** *With notations as before we have $W \cap T = 0$.*

*Proof.* An element $(a_e)_{e\in\mathscr{E}}$ which belongs to $W$ and to $T$ is described by the following equations:

$$a_e = a_v - a_w$$

and

$$\sum_{e\in\mathscr{E}_v} a_e = 0 \quad \text{for all } v \in \mathscr{V}.$$

We can rewrite the equations as follows:

$$\deg(v)a_v = a_{w_1} + \cdots + a_{w_{s_v}} \quad \text{for all } v \in \mathscr{V}, \tag{5.2}$$

where $w_1, \ldots, w_{s_v}$ are the vertices connected to $v$ by an edge and by $\deg(v) := s_v$ we denote the cardinality of the set of the vertices connected to $v$. Requiring that $W \cap T = 0$ is equivalent to requiring that the linear system in (5.2) has a 1-dimensional space of solutions, generated by the vector $(1, \ldots, 1)$. This is equivalent to requiring that the matrix associated to the system in (5.2) has rank $n-1$, i.e. that there exists at least one minor of rank $n-1$ whose determinant is non-zero.

This last condition is independent of the field $\mathbb{F}$, hence to prove that $W \cap T = 0$ it is enough to prove that the equations in (5.2) imply that $a_v = a_{w_i}$ for all $w_i$ and for all $v$ assuming that $\mathbb{F}$ is a totally ordered field. We assume in what follows that $\mathbb{F}$ is a totally ordered field of characteristic 0. Let us suppose by absurd that the equations in (5.2) do not imply that $a_v = a_w$ for all $w$. Let us call

$$a_{v_0} = \min_{v\in\mathscr{V}} a_v,$$

which exists because our assumption that our field $\mathbb{F}$ is totally ordered; then $a_{v_0} \le a_v$ for all $v \in \mathscr{V}$. If $a_{v_0} = a_v$ for all $v \in \mathscr{V}$, we are done; if not, there exists a $v_1$ such that $a_{v_0} < a_{v_1}$. Moreover we can suppose that $v_1$ is connected to $v_0$ by an edge because if not, then this means that $a_{v_0} = a_v$ for all $v$ connected to $v_0$ by an edge. Then if we now fix a $v \neq v_0$ that is connected to $v_0$, we can consider all the $w$ that are connected to it by an edge; if $a_v = a_w$ for all these $w$, we can go on as before. In the end we will find that all the $a_v$ are equal for all $v \in \mathscr{V}$ which proves the claim.

Hence we suppose that there exists a $v_1$ such that $a_{v_0} < a_{v_1}$ for $v_1$ connected to $v_0$ by an edge. We consider equation (5.2) for $v = v_0$ and we get the contradiction

$$\deg(v_0)a_{v_0} < a_{w_1} + \cdots + a_{w_{s_v}}. \qquad \square$$

With this proposition we end the proof of the exactness of the invariant cycles sequence for trivial coefficients.

**Remark 9.** We would like now to give another proof of Proposition 8 more in the spirit of graph theory: it uses [3, Proposition 4.3, Proposition 4.8] and [15, Lemma 13.1.1].

*Proof.* The matrix associated to the linear system in (5.2) is an $n \times n$ matrix $A = (a_{i,j})$, where for $i \neq j$, $a_{i,j} = -1h_{i,j}$ if there are $h_{i,j}$ edges between the vertex $v_i$ and $v_j$ and 0 otherwise, and $a_{i,i} = \deg(v_i)$. We will prove that the rank of the matrix $A$ is $n-1$.

The matrix $A$ is called the Laplacian matrix associated to the multigraph $G$; we will see that $A = DD^t$ and that $D$ is an $n \times m$ matrix with rank $n-1$.

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The following are equivalent:

(i) there exists an $(n-1) \times (n-1)$ minor of $A$ with determinant different from zero,

(ii) the rank of $A$ is $(n-1)$,

(iii) the dimension of the kernel of $A$ is 1,

(iv) $\mathrm{Ker}(D^t) = \mathrm{Ker}(A)$.

Assertion (i) is independent of the field $\mathbb{F}$, so we may suppose that $\mathbb{F}$ is the field of real numbers $\mathbb{R}$.

We will prove assertion (iv). Let us suppose that $z$ is a vector in $\mathbb{R}^n$ that is in $\mathrm{Ker}(A)$. We want to prove that $z \in \mathrm{Ker}(D^t)$. As $z \in \mathrm{Ker}(A)$, we have

$$\begin{aligned} Az &= 0, \\ DD^t z &= 0, \\ z^t DD^t z &= 0. \end{aligned}$$

But the last equality implies that the vector $D^t z$ has inner product with itself in $\mathbb{R}^n$ equal to zero, that means that $D^t z$ is the zero vector, i.e. $z \in \mathrm{Ker}(D^t)$, as we wanted.

We are left to prove that $A = DD^t$ and that $D$ is an $n \times m$ matrix with rank $n-1$. We consider the matrix $D$ associated to the multigraph $G$ defined as follows: $D$ is an $n \times m$ matrix such that $(D)_{i,j} = 1$ if the vertex $v_i$ is such that $e_j = [v_i, -]$, $(D)_{i,j} = -1$ if the vertex $v_i$ is such that $e_j = [-, v_i]$, and $(D)_{i,j} = 0$ otherwise.

Now if we consider $(DD^t)_{i,j}$, this is the inner product of the rows $\mathbf{d}_i$ and $\mathbf{d}_j$. They have a non-zero entry in the same column if and only if there is an edge between $v_i$ and $v_j$, and these entries are one $-1$ and one $+1$, hence $(DD^t)_{i,j}$ is given by $-1$ times the number of edges between $v_i$ and $v_j$. Moreover $(DD^t)_{i,i}$ is the number of entries in $\mathbf{d}_i$ different from zero, which means the degree of $v_i$. This proves that $A = DD^t$.

Let us see now that $D$ has rank $n-1$. On every column there is a $+1$ and a $-1$, hence the sum of all the elements on the columns are zero, hence the rank of $D$ is less or equal to $n-1$. Let us suppose to have a linear relation

$$\sum_i a_i \mathbf{d}_i = 0, \tag{5.3}$$

where as before $\mathbf{d}_i$ is the row corresponding to the vertex $v_i$ and suppose that not all the $a_i$ are zero. Choose a row $\mathbf{d}_k$ for which $a_k \neq 0$. This row has non-zero entries in the columns corresponding to the edges that intersect $v_i$. For every such column there is only one other row $\mathbf{d}_l$ with a non-zero entry in that column. Hence we should have that $a_l = a_k$, hence $a_l = a_k$ for all vertices $v_l$ adjacent to $v_k$. Hence all the $a_k$ are equal as the multigraph $G$ is connected, and the equation in (5.3) is a multiple of $\sum_i \mathbf{d}_i = 0$. But $(a_1, \dots, a_n)$ that verifies (5.3) is in $\mathrm{Ker}(D^t)$, hence we have proven that $\mathrm{Ker}(D^t)$ is 1-dimensional generated by $(1, \dots, 1)$, the rank of $D^t$ is $(n-1)$ as well as the rank of $D$. □

## 6. Unipotent coefficients

In this section we study the sequence (3.2) when the coefficients are unipotent $F$-isocrystals. In particular we prove that, unlike the case of constant coefficients, the sequence in (3.2) is not necessarily exact. We give a sufficient condition for non-exactness.

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Let $E$ be a unipotent convergent $F$-isocrystal for which the sequence in (3.2) is exact and let us consider the following extension in the category of convergent $F$-isocrystals:

$$0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} \mathcal{O} \to 0, \tag{6.1}$$

where $\mathcal{O}$ is the trivial $F$-isocrystal. Let us also consider the element $x \in H^1_{\mathrm{rig}}(X_k, E)$ corresponding to the class of this extension ($x$ is then fixed by the Frobenius operator; see [5, Propositions 1.3.1 and 3.2.1] and [6]). Let us suppose that $x \neq 0$. In the sequel we use sequence (3.2) for the isocrystals $E$, $F$ and $\mathcal{O}$; to avoid confusion we denote the first maps by $\phi^*_{\mathcal{E}}$, $\phi^*_{\mathcal{F}}$ and $\phi^*_{\mathcal{O}}$ respectively and the monodromy operators by $N_{\mathcal{E}}$, $N_{\mathcal{F}}$ and $N_{\mathcal{O}}$ respectively.

Our assumptions imply that $H^1_{\mathrm{rig}}(X_k, E) \otimes K$ is isomorphic via $\varphi^*_{\mathcal{E}}$ to $\mathrm{Ker}(N_{\mathcal{E}})$, and this last group contains the image of $N_{\mathcal{E}}$, as this operator has square zero.

**Theorem 10.** *If $\varphi^*_{\mathcal{E}}(x \otimes 1) = N_{\mathcal{E}}(y)$ for $y \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$, then if we denote by*

$$\alpha_{\mathrm{dR}} : H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K) \to H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$$

*the map induced in cohomology by $\alpha$ of the sequence* (6.1)*, the following holds:*

$$\mathrm{Ker}(N_{\mathcal{F}}) = (H^1_{\mathrm{rig}}(X_k, F) \otimes K) \oplus (\alpha_{\mathrm{dR}}(y)K).$$

*Proof.* Let us consider the following commutative diagram:

$$\begin{array}{ccccc}
H^0_{\mathrm{rig}}(X_k) \otimes K & \xrightarrow{i^0_{\mathcal{O}}} & H^0_{\mathrm{dR}}(X_K) & \xrightarrow{N^0_{\mathcal{O}}} & H^0_{\mathrm{dR}}(X_K) \\
\downarrow{\scriptstyle \delta^0_{\mathrm{rig}}} & & \downarrow{\scriptstyle \delta^0_{\mathrm{log-crys}}} & & \downarrow{\scriptstyle \delta^0_{\mathrm{dR}}} \\
H^1_{\mathrm{rig}}(X_k, E) \otimes K & \xrightarrow{\varphi^*_{\mathcal{E}}} & H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K) & \xrightarrow{N_{\mathcal{E}}} & H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K) \\
\downarrow{\scriptstyle \alpha_{\mathrm{rig}}} & & \downarrow{\scriptstyle \alpha_{\mathrm{dR}}} & & \downarrow{\scriptstyle \alpha_{\mathrm{dR}}} \\
H^1_{\mathrm{rig}}(X_k, F) \otimes K & \xrightarrow{\varphi^*_{\mathcal{F}}} & H^1_{\mathrm{dR}}(X_K, (\mathcal{F}, \nabla)_K) & \xrightarrow{N_{\mathcal{F}}} & H^1_{\mathrm{dR}}(X_K, (\mathcal{F}, \nabla)_K) \\
\downarrow{\scriptstyle \beta_{\mathrm{rig}}} & & \downarrow{\scriptstyle \beta_{\mathrm{dR}}} & & \downarrow{\scriptstyle \beta_{\mathrm{dR}}} \\
H^1_{\mathrm{rig}}(X_k) \otimes K & \xrightarrow{\varphi^*_{\mathcal{O}}} & H^1_{\mathrm{dR}}(X_K) & \xrightarrow{N_{\mathcal{O}}} & H^1_{\mathrm{dR}}(X_K) \otimes K \\
\downarrow{\scriptstyle \gamma_{\mathrm{rig}}} & & \downarrow{\scriptstyle \gamma_{\mathrm{dR}}} & & \downarrow{\scriptstyle \gamma_{\mathrm{dR}}} \\
H^2_{\mathrm{rig}}(X_k, E) \otimes K & \xrightarrow{i^2_{\mathcal{E}}} & H^2_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K) & \xrightarrow{N^2_{\mathcal{E}}} & H^2_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{\mathrm{rig}}(X_k, F) \otimes K & \longrightarrow & H^2_{\mathrm{dR}}(X_K, (\mathcal{F}, \nabla)_K) & \longrightarrow & H^2_{\mathrm{dR}}(X_K, (\mathcal{F}, \nabla)_K) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{\mathrm{rig}}(X_k) \otimes K & \longrightarrow & H^2_{\mathrm{dR}}(X_K) & \longrightarrow & H^2_{\mathrm{dR}}(X_K).
\end{array} \tag{6.2}$$

Let us consider $\varphi^*_{\mathcal{E}}(x \otimes 1) \in \varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K) = \mathrm{Ker}(N_{\mathcal{E}})$, with $\varphi^*_{\mathcal{E}}(x \otimes 1) = N_{\mathcal{E}}(y)$ and $y \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$. We remark that the class of 1 in $H^0_{\mathrm{rig}}(X_k) \otimes K = K$ is sent to the class $x \otimes 1$ in $H^1_{\mathrm{rig}}(X_k, E) \otimes K$ by the map $\delta^0_{\mathrm{rig}}$.

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Let us prove first that $N_{\mathcal{F}}(\alpha_{\mathrm{dR}}(y)) = 0$. By the commutativity of the diagram (6.2) we have that

$$N_{\mathcal{F}}(\alpha_{\mathrm{dR}}(y)) = \alpha_{\mathrm{dR}}(N_{\mathcal{E}}(y)) = \alpha_{\mathrm{dR}}(\varphi_{\mathcal{E}}^{*}(x \otimes 1)) = \alpha_{\mathrm{dR}}(\delta_{\mathrm{dR}}^{0}(1)) = 0,$$

hence $\alpha_{\mathrm{dR}}(y) \in \mathrm{Ker}(N_{\mathcal{F}})$. We claim that $z = \alpha_{\mathrm{dR}}(y) \notin \varphi_{\mathcal{F}}^{*}(H_{\mathrm{rig}}^{1}(X_k, F) \otimes K)$. Let us suppose that $z = \alpha_{\mathrm{dR}}(y) = \varphi_{\mathcal{F}}^{*}(b)$, with $b \in H_{\mathrm{rig}}^{1}(X_k, F) \otimes K$. Then

$$\varphi_{\mathcal{O}}^{*}(\beta_{\mathrm{rig}}(b)) = \beta_{\mathrm{dR}}(\varphi_{\mathcal{F}}^{*}(b)) = \beta_{\mathrm{dR}}(z) = \beta_{\mathrm{dR}}(\alpha_{\mathrm{dR}}(y)) = 0.$$

As $\varphi_{\mathcal{O}}^{*}$ is injective, we have $\beta_{\mathrm{rig}}(b) = 0$, hence $b \in \mathrm{Ker}(\beta_{\mathrm{rig}}) = \mathrm{Im}(\alpha_{\mathrm{rig}})$, i.e. there exists an element $a \in H_{\mathrm{rig}}^{1}(X_k, E) \otimes K$ such that $\alpha_{\mathrm{rig}}(a) = b$. So

$$z = \alpha_{\mathrm{dR}}(y) = \varphi_{\mathcal{F}}^{*}(b) = \varphi_{\mathcal{F}}^{*}(\alpha_{\mathrm{rig}}(a)) = \alpha_{\mathrm{dR}}(\varphi_{\mathcal{E}}^{*}(a)),$$

from which it follows that

$$y - \varphi_{\mathcal{E}}^{*}(a) \in \mathrm{Ker}(\alpha_{\mathrm{dR}}) = \mathrm{Im}(\delta_{\mathrm{dR}}^{0}).$$

But the image of $\delta_{\mathrm{dR}}^{0}$ is generated by $\varphi_{\mathcal{E}}^{*}(x \otimes 1)$, as vector space, hence $y - \varphi_{\mathcal{E}}^{*}(a) = m\varphi_{\mathcal{E}}^{*}(x \otimes 1)$ for some $m \in K$. Now

$$N_{\mathcal{E}}(y) - N_{\mathcal{E}}(\varphi_{\mathcal{E}}^{*}(a)) = N_{\mathcal{E}}(m\varphi_{\mathcal{E}}^{*}(x \otimes 1)) = 0,$$

hence

$$N_{\mathcal{E}}(y) = N_{\mathcal{E}}(\varphi_{\mathcal{E}}^{*}(a)) = 0,$$

but

$$N_{\mathcal{E}}(y) = \varphi_{\mathcal{E}}^{*}(x \otimes 1) = 0,$$

which is absurd.

We are left to prove that for all $\alpha \in \mathrm{Ker}\, N_{\mathcal{F}}$ there exist elements $\beta \in H_{\mathrm{rig}}^{1}(X_k, F) \otimes K$ and $t \in K$ such that $\alpha = \varphi_{\mathcal{F}}^{*}(\beta) + t\alpha_{\mathrm{dR}}(y)$. Let us calculate

$$N_{\mathcal{O}}(\beta_{\mathrm{dR}}(\alpha)) = \beta_{\mathrm{dR}}(N_{\mathcal{F}}(\alpha))) = 0,$$

hence

$$\beta_{\mathrm{dR}}(\alpha) \in \mathrm{Ker}(N_{\mathcal{O}}) = \mathrm{Im}(\varphi_{\mathcal{O}}^{*}),$$

so that there exists an element $\gamma \in H_{\mathrm{rig}}^{1}(X_k) \otimes K$ such that $\varphi_{\mathcal{O}}^{*}(\gamma) = \beta_{\mathrm{dR}}(\alpha)$. By Lemma 11 we have $\gamma_{\mathrm{rig}}(\gamma) = 0$. Hence there exists an element $\beta \in H_{\mathrm{rig}}^{1}(X_k, F) \otimes K$ such that $\beta_{\mathrm{rig}}(\beta) = \gamma$. Let us consider now the element $\alpha - \varphi_{\mathcal{F}}^{*}(\beta)$; it is in $\mathrm{Ker}(\beta_{\mathrm{dR}})$ because

$$\beta_{\mathrm{dR}}(\alpha - \varphi_{\mathcal{F}}^{*}(\beta)) = \beta_{\mathrm{dR}}(\alpha) - \varphi_{\mathcal{O}}^{*}(\beta_{\mathrm{rig}}(b)) = \beta_{\mathrm{dR}}(\alpha) - \varphi_{\mathcal{O}}^{*}(\gamma) = 0.$$

Hence there exists an element $u \in H_{\mathrm{dR}}^{1}(X_K, (\mathcal{E}, \nabla_{\mathcal{E}})_K)$ such that $\alpha_{\mathrm{dR}}(u) = \alpha - \varphi_{\mathcal{F}}^{*}(\beta)$. Now

$$\alpha_{\mathrm{dR}}(N_{\mathcal{E}}(u)) = N_{\mathcal{F}}(\alpha_{\mathrm{dR}}(u)) = N_{\mathcal{F}}(\alpha - \varphi_{\mathcal{F}}^{*}(\beta)) = 0$$

because $\alpha \in \mathrm{Ker}(N_{\mathcal{F}})$ and $N_{\mathcal{F}}(\varphi_{\mathcal{F}}^{*}(\beta)) = 0$ by Lemma 6. Then $N_{\mathcal{E}}(u) \in \mathrm{Ker}(\alpha_{\mathrm{dR}}) = \mathrm{Im}\, \delta_{\mathrm{dR}}^{0}$, i.e

$$N_{\mathcal{E}}(u) = t\varphi_{\mathcal{E}}^{*}(x \otimes 1) = tN_{\mathcal{E}}(y),$$

for some $t \in K$ and $u - ty \in \mathrm{Ker}(N_{\mathcal{E}}) = \varphi_{\mathcal{E}}^{*}(H_{\mathrm{rig}}^{1}(X_k, E) \otimes K)$. Hence there exists an ele-

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ment $\beta' \in H^1_{\mathrm{rig}}(X_k, E) \otimes K$ such that $u = ty + \varphi^*_{\mathscr{E}}(\beta')$. So

$$\alpha - \varphi^*_{\mathscr{F}}(\beta) = \alpha_{\mathrm{dR}}(u) = \alpha_{\mathrm{dR}}(ty + \varphi^*_{\mathscr{E}}(\beta')) = t\alpha_{\mathrm{dR}}(y) + \alpha_{\mathrm{dR}}(\varphi^*_{\mathscr{E}}(\beta')),$$

which means that

$$\alpha = \varphi^*_{\mathscr{F}}(\beta) + t\alpha_{\mathrm{dR}}(y) + \alpha_{\mathrm{dR}}(\varphi^*_{\mathscr{E}}(\beta')),$$

but $\varphi^*_{\mathscr{F}}(\beta) + \alpha_{\mathrm{dR}}(\varphi^*_{\mathscr{E}}(\beta')) = \varphi^*_{\mathscr{F}}(\beta) + \varphi^*_{\mathscr{F}}(\alpha_{\mathrm{rig}}(\beta'))$, hence we are done. □

**Lemma 11.** *With the same hypothesis and notations as in the previous theorem, the co-boundary map $\gamma_{\mathrm{rig}} : H^1_{\mathrm{rig}}(X_k) \otimes K \to H^2_{\mathrm{rig}}(X_k, E) \otimes K$ induced by the exact sequence* (6.1) *is the zero map.*

*Proof.* Clearly, the vanishing of the co-boundary map $\gamma_{\mathrm{rig}}$ is equivalent to the fact that the map $j : H^2_{\mathrm{rig}}(X_k, E) \otimes K \to H^2_{\mathrm{rig}}(X_k, F) \otimes K$ is *injective*.

Let us first make more explicit the group $H^2_{\mathrm{rig}}(X_k, G) \otimes K$, where $G$ is any one of the isocrystals $E, F, \mathcal{O}$ and $(\mathscr{G}, \nabla)$ is the module with integrable connection induced by $G$. Let us recall the notations of Section 3: we consider the diagram

$$X_k \hookrightarrow P_k \xleftarrow{\mathrm{sp}_{P_{\mathcal{V}}}} P_K$$

with $P_k$ smooth and let $Y_K := \mathrm{sp}^{-1}_{P_{\mathcal{V}}}(X_k)$. Then $H^i_{\mathrm{rig}}(X_k, G) \otimes K = H^i_{\mathrm{dR}}(Y_K, (\mathscr{G}, \nabla)_K)$.

The relevant part of the Mayer–Vietoris exact sequence for the admissible covering $\{Y_v\}_v$ of $Y_K$ then reads

$$\bigoplus_e H^1_{\mathrm{dR}}(Y_e, (\mathscr{G}, \nabla)_K) \to H^2_{\mathrm{dR}}(Y_K, (\mathscr{G}, \nabla)_K)$$
$$\to \bigoplus_v H^2_{\mathrm{dR}}(Y_v, (\mathscr{G}, \nabla)_K) \to \bigoplus_e H^2_{\mathrm{dR}}(Y_e, (\mathscr{G}, \nabla)_K).$$

As $Y_e$ is a wide open polydisk, $H^i_{\mathrm{dR}}(Y_e, (\mathscr{G}, \nabla)_K) = 0$ for $i \geq 1$, therefore we have a natural isomorphism $H^2_{\mathrm{dR}}(Y_K, (\mathscr{G}, \nabla)_K) \cong \bigoplus_v H^2_{\mathrm{dR}}(Y_v, (\mathscr{G}, \nabla)_K)$.

Moreover, as $C_v$ which is the irreducible component of $X_k$ corresponding to $v$ was supposed smooth, it follows that we have canonical isomorphisms

$$H^i_{\mathrm{dR}}(Y_v, (\mathscr{G}, \nabla)_K) \cong H^i_{\mathrm{crys}}(C_v, G) \otimes K.$$

In particular, if we denote by $Z_v$ a smooth proper curve over $K$ whose reduction is $C_v$ and which contains the wide open $X_v$, then the isocrystal $G$ can be evaluated on $Z_v$ to give a sheaf with connection which we will denote again by $(\mathscr{G}, \nabla)$. Then

$$H^i_{\mathrm{dR}}(Y_v, (\mathscr{G}, \nabla)_K) \cong H^i_{\mathrm{dR}}(Z_v, (\mathscr{G}, \nabla)_K) \quad \text{for all } i \geq 0.$$

Therefore we have a natural isomorphism

$$H^2_{\mathrm{rig}}(X_k, G) \otimes K \cong \bigoplus_v H^2_{\mathrm{dR}}(Z_v, (\mathscr{G}, \nabla)_K).$$

For every vertex $v$ we denote as before $\mathscr{E}_v = \{e : \text{there exists a vertex } w \text{ with } e = [v, w]\}$. For every vertex $v$ and $e \in \mathscr{E}_v$ we denote by $D_e$ the residue disk of the point in $C_v$ corresponding to $e$ in $Z_v$. Let us then remark that the family $\{X_v, D_e\}_{e \in \mathscr{E}_v}$ is an admissible covering of $Z_v$ and $X_v \cap D_e = X_e$ for every $e \in \mathscr{E}_v$. We will represent classes in $H^2_{\mathrm{dR}}(Z_v, (\mathscr{G}, \nabla)_K)$ by hypercocycles for the above covering.

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We now prove the injectivity of the map $j : H^2_{\mathrm{rig}}(X_k, E) \otimes K \to H^2_{\mathrm{rig}}(X_k, F) \otimes K$. Let $z \in H^2_{\mathrm{rig}}(X_k, E) \otimes K = \bigoplus_v H^2_{\mathrm{dR}}(Z_v, (\mathcal{E}, \nabla)_K)$ such that $j(z) = 0$. Let $z_v \in H^2_{\mathrm{dR}}(Z_v, (\mathcal{E}, \nabla)_K)$ be the $v$-component of $z$ and $j_v : H^2_{\mathrm{dR}}(Z_v, (\mathcal{E}, \nabla)_K) \to H^2_{\mathrm{dR}}(Z_v, (\mathcal{F}, \nabla)_K)$ be the $v$ component of $j$. Obviously $j_v(z_v) = 0$ and it would be enough to show that this implies $z_v = 0$ for every $v$.

Let $(\omega_e)_{e \in \mathcal{E}_v}$ be a 2-hyper cocycle representing $z_v$, where $\omega_e \in H^0(X_e, \mathcal{E}_K \otimes \Omega^1_{Z_v})$ for all $e$. Then $j_v(z_v)$ will be represented by the 2-hyper cocycle $(\alpha(\omega_e))_{e \in \mathscr{E}_v}$, where $\alpha$ is defined by the exact sequence of isocrystals on $X_k$ below:

$$0 \to E \xrightarrow{\alpha} F \xrightarrow{\beta} \mathcal{O} \to 0.$$

As extension on $X_K$ this is given by the class $\varphi^*_{\mathcal{E}}(x \otimes 1) = N_{\mathcal{E}}(y) \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ and therefore, for every $v$, the sequence

$$0 \to H^0(X_v, \mathcal{E}_K) \xrightarrow{\alpha} H^0(X_v, \mathcal{F}_K) \xrightarrow{\beta} H^0(X_v, \mathcal{O}_{X_K}) \to 0$$

is exact because $X_v$ are wide opens and moreover, it is naturally split as an exact sequence of $\mathcal{O}_{X_v}$-modules with connections because $\varphi^*_{\mathcal{E}}(x \otimes 1) = N_{\mathcal{E}}(y)$ can be represented by $(0_v, f_e)$ with $f_e \in H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K)$. Let $s : H^0(X_v, \mathcal{O}_{X_K}) \to H^0(X_v, \mathcal{F}_K)$ be such a section of $\beta$. We remark that it is determined by $s(1)$, which is an element of $H^0_{\mathrm{dR}}(X_v, (\mathcal{F}, \nabla)_K)$ such that $\beta(s(1)) = 1$.

Therefore, $s$ determines, for every $e \in \mathscr{E}_v$, a splitting of the exact sequence

$$0 \longrightarrow H^0(X_e, \mathcal{E}_K) \xrightarrow{\alpha_e} H^0(X_e, \mathcal{F}_K) \xrightarrow{\beta_e} H^0(X_e, \mathcal{O}_{X_K}) \longrightarrow 0$$

which will also be called $s_e$ (it is determined by the element $s_e(1) = s(1)|_{X_e}$).

Now the sequence

$$(6.3) \quad 0 \longrightarrow H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K) \xrightarrow{\alpha_e} H^0_{\mathrm{dR}}(X_e, (\mathcal{F}, \nabla)_K) \xrightarrow{\beta_e} H^0_{\mathrm{dR}}(X_e, (\mathcal{O}_{X_K}, d)) \longrightarrow 0$$

is exact and $s_e$ induces a natural splitting of it.

The isocrystal $G$ (which is any one of $E, F, \mathcal{O}$ regarded as a sheaf with connection on $Z_v$) has a basis of horizontal sections on $D_e$, for every $e \in \mathscr{E}_v$. Therefore the natural restriction map $H^0_{\mathrm{dR}}(D_e, (\mathcal{G}, \nabla)_K) \to H^0_{\mathrm{dR}}(X_e, (\mathcal{G}, \nabla)_K)$ is an isomorphism. Thus the exact sequence (6.3) implies that the sequence

$$(6.4) \quad 0 \longrightarrow H^0_{\mathrm{dR}}(D_e, (\mathcal{E}, \nabla)_K) \xrightarrow{\alpha_e} H^0_{\mathrm{dR}}(D_e, (\mathcal{F}, \nabla)_K) \xrightarrow{\beta_e} H^0_{\mathrm{dR}}(D_e, (\mathcal{O}_{X_K}, d)) \longrightarrow 0$$

is exact and naturally split, where we denote the splitting by $s_e$. By tensoring (6.4) with $\Omega^1_{D_e}$ we obtain that the sequence

$$0 \longrightarrow H^0(D_e, \mathcal{E}_K \otimes \Omega^1_{D_e}) \xrightarrow{\alpha_e} H^0(D_e, \mathcal{F}_K \otimes \Omega^1_{D_e}) \xrightarrow{\beta_e} H^0(D_e, \Omega^1_{D_e}) \longrightarrow 0$$

is exact, naturally split as sequence of $\mathcal{O}_{D_e}$-modules with connection and everything is compatible with restriction to $X_e$.

Using these splittings, we write

$$H^0(X_v, \mathcal{F}_K \otimes \Omega^1_{X_v}) = H^0(X_v, \mathcal{E}_K \otimes \Omega^1_{X_v}) \oplus H^0(X_v, \Omega^1_{X_v})$$

and similarly for sections over $X_e$ and $D_e$.

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Now we go back to proving that $j_v$ is injective for all $v$. Suppose that $j_v(z_v) = 0$, i.e. for every $e \in \mathscr{E}_v$,

$$\alpha_e(\omega_e) = \eta_v|_{X_e} - \rho_e|_{X_e} - \nabla(f_e),$$

where $\eta_v \in H^0(X_v, \mathcal{F}_K \otimes \Omega^1_{X_v})$, $\rho_e \in H^0(D_e, \mathcal{F}_K \otimes \Omega^1_{D_e})$, $f_e \in H^0(X_e, \mathcal{F}_K)$.

Using the decompositions above, we write (uniquely)

$$\eta_v = \eta_{v,E} + \eta_{v,\mathcal{O}}, \quad \rho_e = \rho_{e,E} + \rho_{e,\mathcal{O}}, \quad f_e = f_{e,E} + f_{e,\mathcal{O}},$$

with $\eta_{v,E} \in H^0(X_v, \mathcal{E}_K \otimes \Omega^1_{X_v})$, $\rho_{e,E} \in H^0(D_e, \mathcal{E}_K \otimes \Omega^1_{D_e})$ etc.

Using the fact that the decompositions respect the connections and the restrictions to $X_e$, we obtain

$$\omega_e - \big(\eta_{v,E}|_{X_e} - \rho_{e,E}|_{X_e} - \nabla(f_{e,E})\big) = \eta_{v,\mathcal{O}_X}|_{X_e} - \rho_{e,\mathcal{O}_X}|_{X_e} - d_X(f_{e,\mathcal{O}}).$$

As the decomposition is a direct sum decomposition, the left hand side and the right hand side are zero.

Therefore $\omega_e = \eta_{v,E}|_{X_e} - \rho_{e,E}|_{X_e} - \nabla(f_{e,E})$ for every $e \in \mathscr{E}_v$ and we have $z_v = 0$. □

## A. Appendix: An example for a Tate curve

In this section we use explicit calculations to confirm Theorem 10, i.e. that the sequence (3.2) is not exact for a certain non-trivial unipotent $F$-isocrystal $E$ on a specific Tate curve.

Let $X$ be a Tate elliptic curve over $K$ with invariant $q$, where $q \in m_{\mathcal{V}} = (\pi)$. We consider $x \in H^1_{\mathrm{rig}}(X_k)$. By Lemma 6, $\varphi^*_{\mathcal{O}}(x \otimes 1)$ in $H^1_{\mathrm{dR}}(X_K)$ is such that $N(\varphi^*_{\mathcal{O}}(x \otimes 1)) = 0$; since $H^1_{\mathrm{dR}}(X_K)$ is a 2-dimensional $K$-vector space, we have $\mathrm{Im}(N) = \mathrm{Ker}(N)$, hence we get $\varphi^*_{\mathcal{O}}(x \otimes 1) \in \mathrm{Im}(N)$. In particular in this case the hypothesis of Theorem 10 are satisfied.

Every element in $H^1_{\mathrm{rig}}(X_k)$ corresponds to an extension of the trivial $F$-isocrystal by itself ([5, Proposition 1.3.1]), hence the element $x$ corresponds to the following exact sequence:

$$0 \to \mathcal{O} \to E \to \mathcal{O} \to 0.$$

As before we consider $\varphi^*_{\mathcal{O}}(x \otimes 1) \in H^1_{\mathrm{dR}}(X_K)$ and the exact sequence of modules with connections induced by the one above:

$$0 \to (\mathcal{O}_{X_K}, d) \to (\mathcal{E}, \nabla)_K \to (\mathcal{O}_{X_K}, d) \to 0.$$

We suppose from now on that $\mathrm{ord}_\pi q = 3$. Then the graph associate to $X$ is a triangle with vertices $I, II, III$ and edges $[I, II]$, $[II, III]$, $[I, III]$.

The element $\varphi^*_{\mathcal{O}}(x \otimes 1)$, as hypercocycle, can be written as $(0_v, g_e)$ with $g_e \in H^0(X_e)$; in particular $d(g_e) = 0$, so $g_e \in K$. Moreover since $E$ is an $F$-isocrystal, the class $x$ is fixed by the Frobenius of $H^1_{\mathrm{rig}}(X_k)$ ([5, Proposition 3.2.1]), in particular we can take $g_e \in \mathbb{Q}_p$ for every $e$.

The $\mathcal{O}_{X_K}$-module $\mathcal{E}_K$ is locally free: on $X_v$ it has a basis given by $e_{1,v}, e_{2,v}$ and on $X_w$ it has a basis given by $e_{1,w}, e_{2,w}$. If on $X_e$ we choose the basis $e_{1,v}, e_{2,v}$, then the matrix relating this basis to $e_{1,w}, e_{2,w}$ is given by $\left(\begin{smallmatrix} 1 & g_e \\ 0 & 1 \end{smallmatrix}\right)$ and the connection on $X_e$ is given by the direct sum of the two trivial connections.

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Now we consider $(\omega_v, f_e) \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$. Then

$$\begin{aligned} \omega_v &= h_{1,v} e_{1,v} + h_{2,v} e_{2,v}, \\ \omega_w &= h_{1,w} e_{1,w} + h_{2,w} e_{2,w}, \\ \omega_{w|X_e} &= (h_{1,w} + g_e h_{2,w}) e_{1,v} + h_{2,w} e_{2,w}, \end{aligned}$$

with $h_{1,v}$ and $h_{2,v}$ elements of $\Omega^1_{X_v}$ and $h_{1,w}$ and $h_{2,w}$ elements of $\Omega^1_{X_w}$. Let us suppose now that $(\omega_v, f_e) \in \mathrm{Ker}(N_{\mathcal{E}})$, which means that

$$N_{\mathcal{E}}(\omega_v, f_e) = (0, \mathrm{Res}_{|X_e} \omega_v) = 0 \quad \text{in } H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K),$$

but as the map from $H^1(\mathrm{Gr}, \mathcal{E}_K)$ to $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ is injective, we have that $\mathrm{Res}_{|X_e} \omega_v$ is zero as element of $H^1(\mathrm{Gr}, \mathcal{E}_K)$.

Let us write the system which tells us that an element

$$a_e = (a^1_e, a^2_e) \in H^1(\mathrm{Gr}, \mathcal{E}_K) = \frac{\bigoplus_e H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K)}{\bigoplus_v H^0_{\mathrm{dR}}(X_v, (\mathcal{E}, \nabla)_K)},$$

written in coordinates with respect to the basis $e_{v,1}, e_{v,2}$, is zero:

$$\begin{cases} a^1_{[I,II]} = a^1_I - a^1_{II} - g_{[I,II]} a^2_{II}, \\ a^2_{[I,II]} = a^2_I - a^2_{II}, \end{cases} \qquad \begin{cases} a^1_{[II,III]} = a^1_{II} - a^1_{III} - g_{[II,III]} a^2_{III}, \\ a^2_{[II,III]} = a^2_{II} - a^2_{III}, \end{cases}$$

$$\begin{cases} a^1_{[I,III]} = a^1_I - a^1_{III} - g_{[I,III]} a^2_{III}, \\ a^2_{[I,III]} = a^2_I - a^2_{III}. \end{cases}$$

Moreover from the Gysin sequence ([5, Proposition 2.1.4]), applied to every component $C_v$ of $X_k$ (on every wide open $X_v$, $(\mathcal{E}, \nabla)_K$ is the direct sum of two copies of $(\mathcal{O}_{X_K}, d)$), we can derive the following equations:

$$\begin{cases} a^1_{[I,II]} + a^1_{[I,III]} = 0, \\ a^1_{[II,III]} + a^1_{[II,I]} = 0, \\ a^1_{[III,I]} + a^1_{[III,II]} = 0, \end{cases} \qquad \begin{cases} a^2_{[I,II]} + a^2_{[I,III]} = 0, \\ a^2_{[II,III]} + a^2_{[II,I]} = 0, \\ a^2_{[III,I]} + a^2_{[III,II]} = 0. \end{cases}$$

Putting together the previous equations and writing a linear system in terms of the elements $a_v$, we find the following matrix:

$$A = \begin{pmatrix} 2 & 0 & -1 & -g_{[I,II]} & -1 & -g_{[I,III]} \\ 0 & 2 & 0 & -1 & 0 & -1 \\ -1 & -g_{[II,I]} & 2 & 0 & -1 & -g_{[II,III]} \\ 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & -g_{[III,I]} & -1 & -g_{[III,II]} & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \end{pmatrix},$$

where $g_{[I,II]} = -g_{[II,I]}$, $g_{[II,III]} = -g_{[III,II]}$ and $g_{[I,III]} = -g_{[III,I]}$. The matrix $A$ has determinant equal to zero and dimension of the rank equal to 4. Two generators of the kernel are the

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following vectors:

$$K_1 = (1,0,1,0,1,0),$$
$$K_2 = \left(\frac{1}{3}g_{[I,II]} + \frac{2}{3}g_{[I,III]} + \frac{1}{3}g_{[II,III]}, 1, -\frac{1}{3}g_{[I,II]} + \frac{1}{3}g_{[I,III]} + \frac{2}{3}g_{[II,III]}, 1, 0, 1\right).$$

If we now write $K_1, K_2$ as elements of $H^1(\mathrm{Gr}, \mathcal{E}_K)$, i.e. as elements of $\bigoplus_e H^0_{\mathrm{dR}}(X_e, (\mathcal{E}, \nabla)_K)$, we find the following vectors:

$$H_1 = (0,0,0,0,0,0),$$
$$H_2 = \left(-\frac{1}{3}g_{[I,II]} - \frac{1}{3}g_{[II,III]} + \frac{1}{3}g_{[I,III]}, 0, -\frac{1}{3}g_{[I,II]} - \frac{1}{3}g_{[II,III]} + \frac{1}{3}g_{[I,III]}, 0, \frac{1}{3}g_{[I,II]} + \frac{1}{3}g_{[II,III]} - \frac{1}{3}g_{[I,III]}, 0\right).$$

These computations show that the kernel of $N_{\mathcal{E}}$ consists of $(\omega_v, f_e) \in H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ such that $\mathrm{Res}_{|X_e}\omega_v$ equals $H_1$ or $H_2$. The elements $(\omega_v, f_e)$ of $H^1_{\mathrm{dR}}(X_K, (\mathcal{E}, \nabla)_K)$ which are such that $\mathrm{Res}_{|X_e}\omega_v = H_1$ are the elements that come from $H^1_{\mathrm{rig}}(X_k, E) \otimes K$.

Let us consider now the subvector space

$$V = \{(\omega_v, f_e) : \mathrm{Res}_{|X_e}\omega_v = tH_2 \text{ with } t \in K\}$$

Clearly the elements of $\varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K)$ are contained in the vector space $V$ and one can see that $V/\varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K)$ is a 1-dimensional vector space, in fact two elements in $V$ are multiples one of the other modulo an element of $\varphi^*_{\mathcal{E}}(H^1_{\mathrm{rig}}(X_k, E) \otimes K)$.

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