ON KOLMOGOROV ENTROPY COMPACTNESS ESTIMATES FOR SCALAR CONSERVATION LAWS WITHOUT UNIFORM CONVEXITY*

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Abstract. In the case of scalar conservation laws $u_t + f(u)_x = 0, t \geq 0, x \in \mathbb{R}$, with uniformly strictly convex flux f, quantitative compactness estimates—in terms of Kolmogorov entropy in $\mathbf{L}^1_{loc}$— were established in [C. De Lellis and F. Golse, *Comm. Pure Appl. Math.*, 58 (2005), pp. 989–998; F. Ancona, O. Glass, and K. T. Nguyen, *Comm. Pure Appl. Math.*, 65 (2012), pp. 1303–1329] for sets of entropy weak solutions evaluated at a fixed time $t > 0$, whose initial data have a uniformly bounded support and vary in a bounded subset of $\mathbf{L}^\infty$. These estimates reflect the irreversibility features of entropy weak discontinuous solutions of these nonlinear equations. We provide here an extension of such estimates to the case of scalar conservation laws with a smooth flux function f that either is strictly (but not necessarily uniformly) convex or has a single inflection point with a polynomial degeneracy.

Key words. Kolmogorov entropy, scalar conservation laws, compactness estimates

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1. Introduction. Consider a scalar conservation law in one space dimension

$$(1.1) \qquad\qquad u_t + f(u)_x = 0,$$

where $u = u(t,x)$ is the state variable, and $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable map. Without loss of generality, we will suppose

$$(1.2) \qquad\qquad f'(0) = 0,$$

since one may always reduce the general case to this one by performing the space-variable and flux transformations $x \to x + tf'(0)$ and $f(u) \to f(u) - uf'(0)$. It is well known that, no matter how smooth the initial data are, solutions of the Cauchy problem for (1.1) generally stay smooth only up to a critical time beyond which discontinuities (shocks) develop. Hence, it is natural to consider weak solutions in the sense of distributions that, for the sake of uniqueness, satisfy an entropy admissibility criterion [12, 16] equivalent to the celebrated Oleinik E-condition [22] which generalizes the classical stability conditions introduced by Lax [18]:

Oleinik E-condition. A shock discontinuity located at x and connecting a left state $u^L \doteq u(t,x-)$ with a right state $u^R \doteq u(t,x+)$ is entropy admissible if and only

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if there holds

$$(1.3) \qquad \frac{f(u^L) - f(u)}{u^L - u} \ \geq \ \frac{f(u^R) - f(u)}{u^R - u}$$

for every u between u^L and u^R, where $u(t, x\pm)$ denote the one-sided limits of $u(t, \cdot)$ at x.

Equation (1.1) generates an $\mathbf{L}^1$-contractive semigroup of solutions $(S_t)_{t \geq 0}$ that associates, to given initial datum $u_0 \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$, the unique entropy admissible weak solution $S_t u_0 \doteq u(t, \cdot)$ of the corresponding Cauchy problem (cf. [12, 16]). This yields the existence of a continuous semigroup $(S_t)_{t \geq 0}$ acting on the whole space $\mathbf{L}^1(\mathbb{R})$. Under the assumption that the flux function f is uniformly strictly convex, it was shown by Lax [17] that such a semigroup S_t is compact as a mapping from $\mathbf{L}^1(\mathbb{R})$ to $\mathbf{L}^1_{loc}(\mathbb{R})$, for every $t > 0$. Indeed, in this case entropy admissible weak solutions satisfy the one-side Oleinik inequality [22] which yields uniform bounded variation (BV) bounds on the solutions at any fixed time $t > 0$ which in turn, applying Helly's compactness theorem, imply the compactness of the mapping S_t. This property reflects the irreversibility features of entropy weak (discontinuous) solutions of these equations. De Lellis and Golse [13], following a suggestion by Lax [19, 20], used the concept of Kolmogorov ε-entropy, recalled below, to provide a quantitative estimate of this compactness effect.

DEFINITION 1.1. *Let (X, d) be a metric space and K a totally bounded subset of X. For $\varepsilon > 0$, let $\mathcal{N}_\varepsilon(K)$ be the minimal number of sets in a cover of K by subsets of X having diameter no larger than 2ε. Then the ε-entropy of K is defined as*

$$\mathcal{H}_\varepsilon(K \mid X) \ \doteq \ \log_2 \mathcal{N}_\varepsilon(K).$$

Throughout the paper, we will call an ε-*cover*, a cover of K by subsets of X having diameter no larger than 2ε.

In the case of uniformly strictly convex conservation laws, De Lellis and Golse established in [13] an upper bound on the Kolmogorov ε-*entropy* of the image set $S_t(\mathcal{C})$ for bounded subsets $\mathcal{C}$ of $\mathbf{L}^1$ of order $1/\varepsilon$. In [3], we have supplemented the upper estimate established in [13] with a lower bound on the ε-entropy of $S_t(\mathcal{C})$ of the same order $1/\varepsilon$, thus showing that the estimate of De Lellis and Golse was optimal. Entropy numbers play a central role in various areas of information theory and statistics as well as of learning theory. In the present setting, this concept could provide a measure of the order of "resolution" and of the "complexity" of a numerical method for (1.1), as suggested in [19, 20].

The aim of this paper is to extend this type of quantitative estimate on the compactness property of the mapping S_t, $t > 0$, to the case of conservation laws (1.1) with a flux function that either is strictly (but not necessarily uniformly) convex or has a single inflection point with polynomial degeneracy. Notice that, when one removes the assumption of uniform convexity of the flux function, entropy weak solutions do not satisfy anymore the classical Oleinik inequality and they may have unbounded variation (see [7]). However, it was shown in [8, 14] that for such equations the positive variation of the derivative of the flux composed with a bounded solution is uniformly bounded at any positive time, hence it belongs to the BV space. Hence, in the case of a conservation law with a single inflection point, given a bounded subsets $\mathcal{C}$ of $\mathbf{L}^1$, we shall

- exploit the BV bounds on $f'(u)$ and invoke [6, Theorem 1] to construct an ε'-covering $\mathcal{U}'$ of the set $\mathcal{L} \doteq \{f' \circ u \mid u \in S_t(\mathcal{C})\}$ with cardinality $Card\,(\mathcal{U}') \approx 2^{a/\varepsilon'}$ for some constant $a > 0$;
- associate with $\mathcal{U}'$ an ε-covering $\mathcal{U}$ of the set $S_t(\mathcal{C})$, with cardinality $Card\,(\mathcal{U}) \approx Card\,(\mathcal{U}') \cdot 2^{b/f'(\varepsilon)}$ for some other constant $b > 0$;
- take $\varepsilon' = f'(\varepsilon)$ to produce an ε-covering $\mathcal{U}$ of the set $S_t(\mathcal{C})$ with cardinality $Card\,(\mathcal{U}) \approx 2^{(a+b)/f'(\varepsilon)}$.

As a consequence, we find that the ε-entropy of $S_t(\mathcal{C})$ has an upper bound of order $1/f'(\varepsilon)$. We also show that this estimate is optimal providing a lower bound of the same order $1/f'(\varepsilon)$ for the ε-entropy of a subset of $S_t(\mathcal{C})$, and, hence, for the ε-entropy of $S_t(\mathcal{C})$. Namely, performing a similar analysis as in [3], we establish such a lower bound for the ε-entropy of $S_t(\mathcal{C}^+ \cup \mathcal{C}^-)$, where $\mathcal{C}^+$, $\mathcal{C}^-$, denote the classes of initial data in $\mathcal{C}$ which assume only nonnegative and nonpositive values, respectively. Notice that, for the particular class of fluxes $f(u) = u^{m+1}/(m+1)$, m even, we find that the Kolmogorov ε-entropy of $S_t(\mathcal{C})$ is of order $1/\varepsilon^m$, which shows how accurate this concept is in reflecting the nonlinearity of the flux. We finally prove that even in the case of strictly, but not uniformly, convex flux there hold the same upper and lower bounds of order $1/f'(\varepsilon)$ for the Kolmogorov ε-entropy of $S_t(\mathcal{C})$.

Specifically, we shall assume that the flux function satisfies one of the *standing assumptions*:

(C) $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable, strictly convex function.

(NC) $f : \mathbb{R} \to \mathbb{R}$ is a smooth, nonconvex function with a single inflection point at zero having polynomial degeneracy, i.e., such that

$$f^{(j)}(0) \;=\; 0 \quad \text{for all} \quad j = 2, \ldots, m, \qquad f^{(m+1)}(0) \;\neq\; 0\,,$$

(1.4)

$$f''(u) \cdot u \cdot \mathrm{sign}\big(f^{(m+1)}(0)\big) \;>\; 0 \qquad \text{for all} \;\; u \in \mathbb{R} \setminus \{0\}$$

for some even integer $m \geq 2$.

Notice that, generically, smooth fluxes satisfy one of the assumptions (C), (NC), since a generic property of smooth maps $f : \mathbb{R} \to \mathbb{R}$ is that $f^{(3)}(x) \neq 0$ whenever $f''(x) = 0$.

In connection with a flux $f : \mathbb{R} \to \mathbb{R}$ and any constant $M > 0$, we introduce a map $\Delta_{f,M} : (0, +\infty) \to \mathbb{R}$ measuring the oscillation of f', defined by setting

$$(1.5) \qquad \Delta_{f,M}(s) \;\doteq\; s \cdot \inf_{\substack{|u|,|v| \leq M \;\; u \cdot v \,\geq\, 0 \\ |v-u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \qquad \text{for all} \;\; s > 0\,.$$

Notice that since in (1.5) we are taking the infimum in a compact subset of $\mathbb{R}^2$, if f satisfies either of the assumptions (C) or (NC), it follows that $\Delta_{f,M}(s) > 0$ for all $s > 0$.

We then consider sets of bounded, compactly supported initial data of the form

$$(1.6) \qquad \mathcal{C}_{[L,M]} \;\doteq\; \Big\{ u_0 \in \mathbf{L}^\infty(\mathbb{R}) \;\big|\; \mathrm{Supp}\,(u_0) \subset [-L, L]\,, \; \|u_0\|_{\mathbf{L}^\infty} \leq M \Big\}.$$

The main results of the paper show that the Kolmogorov ε-entropy of $S_t(\mathcal{C}_{[L,M]})$ with respect to the $\mathbf{L}^1$-topology is of order $\approx \varepsilon^{-m}$ for fluxes satisfying the assumption (NC), and has an upper bound of order $\approx (\Delta_{f,M}(\varepsilon))^{-1}$ for fluxes satisfying the assumption (C). Precisely, we prove the following upper and lower bounds for the Kolmogorov ε-entropy of $S_t(\mathcal{C}_{[L,M]})$.

THEOREM 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying* (1.2) *and the assumption* (C), *and let $\{S_t\}_{t \geq 0}$ be the semigroup of entropy weak solutions generated by* (1.1) *on*

the domain $\mathbf{L}^1(\mathbb{R})$. Then, given $L, M, T > 0$, for every $\varepsilon > 0$ sufficiently small the following estimates hold:

$$(1.7) \quad \mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\Big) \leq \ \Gamma_1^+ \cdot \frac{1}{\Delta_{f,M}\big(\frac{\varepsilon}{\gamma_1^+}\big)}\,,$$

$$(1.8) \quad \mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\Big) \geq \ \Gamma_1^- \cdot \frac{1}{\varepsilon \cdot \min\Big\{\max\limits_{z\in\left[0,\,\frac{6\varepsilon}{L}\right]} f''(z),\ \max\limits_{z\in\left[-\frac{6\varepsilon}{L},\,0\right]} f''(z)\Big\}}\,,$$

where

$$(1.9) \qquad \Gamma_1^+ \ = \ c_1\Big(L + T + \frac{L^2}{T}\Big), \qquad \gamma_1^+ \ = \ c_1\Big(1 + L + T\Big),$$

$$(1.10) \qquad \Gamma_1^- \ = \ \frac{c_1}{T}$$

for some constant $c_1 > 0$ depending only on f and M.

Remark 1.3. In the case where the derivative f' of a strictly convex flux f is a convex function on $[0, +\infty)$ and a concave function on $(-\infty, 0]$, and we assume that (1.2) holds, by definition (1.5) it follows that

$$\Delta_{f,M}(s) \ = \ \min\big\{|f'(-s)|, |f'(s)|\big\} \qquad \text{for all } \ s > 0$$

for every $M > 0$, while

$$\min\Big\{\max_{z\in[0,s]} f''(z),\ \max_{z\in[-s,0]} f''(z)\Big\} \ = \ \min\big\{f''(-s),\, f''(s)\big\} \qquad \text{for all } \ s > 0.$$

Therefore, in this case, by (1.7)–(1.8) the $\mathbf{L}^1$-Kolmogorov ε-entropy of $S_t(C_{[L,M]})$ turns out to be of order $\approx 1/|f'(\pm\varepsilon)| \approx 1/\big(\varepsilon \cdot f''(\pm\varepsilon)\big)$. Instead, if we assume that $f''(u) \geq c > 0$ for all $u \in \mathbb{R}$, applying the mean-value theorem to f' it follows that

$$\Delta_{f,M}(s) \ \geq \ c \cdot s \qquad \text{for all } \ s > 0.$$

On the other hand, for every fixed $M > 0$, there exists some constant $c_M > 0$ such that

$$\min\Big\{\max_{z\in[0,s]} f''(z),\ \max_{z\in[-s,0]} f''(z)\Big\} \ \leq \ c_M \qquad \text{for all } \ s \in (0, M].$$

Thus, in this second case we recover the estimate $\mathcal{H}_\varepsilon(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})) \approx 1/\varepsilon$ established in [3, 13] for uniformly strictly convex fluxes.

Remark 1.4. If we consider a smooth, strictly convex flux f with a polynomial degeneracy at zero, i.e., such that

$$(1.11) \qquad \begin{aligned} f^{(j)}(0) \ &= \ 0 \quad \text{for all} \quad j = 1, \ldots, m, \qquad f^{(m+1)}(0) \ \neq \ 0, \\[4pt] f''(u) \ &> \ 0 \qquad \text{for all} \ \ u \in \mathbb{R} \setminus \{0\} \end{aligned}$$

for some odd integer $m \in \mathcal{N}$, one can show that there exist some constant $\alpha_M > 0$ depending on f, M, and $\overline{\alpha} > 0$ depending only on f, such that

$$(1.12) \qquad \Delta_{f,M}(s) \ \geq \ \frac{s^m}{\alpha_M}, \qquad \min\Big\{\max_{z\in[0,s]} f''(z),\ \max_{z\in[-s,0]} f''(z)\Big\} \ \leq \ \overline{\alpha} \cdot s^{m-1}$$

for all $s > 0$ sufficiently small (see Remark 3.6 and Lemma 4.4). Hence, for fluxes satisfying the assumption (1.11), by (1.7)–(1.8) the $\mathbf{L}^1$-Kolmogorov ε-entropy of $S_t(C_{[L,M]})$ turns out to be of order $\approx 1/\varepsilon^m$.

THEOREM 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying* (1.2) *and the assumption* (NC). *Then, in the same setting as Theorem* 1.2, *for any given $L, M, T > 0$, and for every $\varepsilon > 0$ sufficiently small, the following estimates hold:*

$$(1.13) \qquad \mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\Big) \leq \Gamma_2^+ \cdot \frac{1}{\varepsilon^m} \,,$$

$$(1.14) \qquad \mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\Big) \geq \Gamma_2^- \cdot \frac{1}{\varepsilon^m} \,,$$

where

$$(1.15) \qquad \begin{aligned} \Gamma_2^+ &= c_2\left(1 + L + T + \frac{L^2}{T}\right)^{m+1}, \\[2mm] \Gamma_2^- &= c_2 \cdot \frac{L^{m+1}}{T} \end{aligned}$$

for some constant $c_2 > 0$ depending only on f and M.

Remark 1.6. If a flux f satisfies the assumption (NC) and (1.2), one can show that, for every fixed $M > 0$, there exists some constant $\beta_M > 0$ such that

$$(1.16) \qquad \frac{s^m}{\beta_M} \leq \Delta_{f,M}(s) \leq \beta_M \cdot s^m$$

for all $s > 0$ sufficiently small (see Lemma 3.5). Hence, the upper estimates on the Kolmogorov ε-entropy of $S_t(C_{[L,M]})$ provided by Theorem 1.5 are of the same order as the ones stated in Theorem 1.2.

We observe that, for fluxes having one inflection point where all derivatives vanishes the composition of the derivative of the flux with the solution of (1.1) fails in general to belong to the BV space (see [21] and Remark 2.4 here). However, for *weakly genuinely nonlinear* fluxes, that is to say, for fluxes with no affine parts, it is shown in [23, Theorem 26] that equibounded sets of entropy solutions of (1.1) are still relatively compact in $\mathbf{L}^1$ (see also [21]). Therefore, for fluxes of such classes that do not fulfill the assumption (NC), it remains an open problem to provide quantitative compactness estimates on the solutions set of (1.1). In this case, a different approach from the one developed in the present paper must be pursued to obtain upper bounds on the Kolmogorov ε-entropy of the solution set, perhaps exploiting the BV^Φ-bounds obtained in [21, Theorem 1], Φ being a convex function linked to the degeneracy of the flux.

The paper is organized as follows. In section 2 we collect notations and preliminary results concerning the theory of scalar conservation laws and the estimates of the Kolmogorov ε-entropy for sets of functions with uniformly BV. In section 3 we establish the upper bounds on the ε-entropy of the solution set stated in Theorems 1.2–1.5, while the proof of the lower bounds is carried out in section 4.

2. Notations and preliminaries. Throughout the paper we shall denote by
- $\mathbf{L}^1(\mathbb{R})$, the Lebesgue space of all (equivalence classes of) summable functions on $\mathbb{R}$, equipped with the usual norm $\|\cdot\|_{\mathbf{L}^1}$;

- $\mathbf{L}^\infty(\mathbb{R})$, the space of all essentially bounded functions on $\mathbb{R}$, equipped with the usual norm $\|\cdot\|_{\mathbf{L}^\infty}$;
- $Supp(u)$, the essential support of a function $u \in \mathbf{L}^\infty(\mathbb{R})$;
- $TV\{u \,|\, D\}$, the total variation of u on the interval $D \subset \mathbb{R}$; in the case where $D = \mathbb{R}$ we just write $TV\{u\}$;
- $BV(D)$, the set of functions with bounded total variation on D;
- $\lfloor x \rfloor \doteq \max\big\{z \in \mathbb{Z} \,|\, z \leq x\big\}$, the integer part of x.

Remark 2.1. We recall [12, 16] that a scalar conservation law (1.1) generates a unique $\mathbf{L}^1$-contractive semigroup $\big\{S_t : \mathbf{L}^1(\mathbb{R}) \to \mathbf{L}^1(\mathbb{R})\big\}_{t \geq 0}$ that associates with any $u_0 \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^\infty(\mathbb{R})$ the unique entropy solution

$$u(t,x) \;\doteq\; S_t u_0(x)\,, \qquad\qquad x \in \mathbb{R}, t > 0\,,$$

of (1.1) with initial datum $u(0,x) = u_0$. Notice that, if the flux function f satisfies either of the assumptions (C) or (NC) stated in the introduction, although $S_t u_0$ may well have unbounded variation, it is still true that $S_t u_0$ admits one-sided limits $S_t u_0(\pm x)$ at every point $x \in \mathbb{R}$. This is the consequence of the Lax–Oleinik representation formula [18] in the (C) case, and of the $\mathrm{BV}^{\frac{1}{p}}$ regularity (see [21, Theorem 3]) in the (NC) case.

For any $L, M > 0$, consider the class of functions in (1.6) and set

$$(2.1) \qquad\qquad f'_M \;\doteq\; \sup_{|v| \leq M}\, |f'(v)|\,.$$

The next classical result provides an upper bound on the $\mathbf{L}^\infty$-norm and on the support of $S_T u_0$ for every $u_0 \in \mathcal{C}_{[L,M]}$.

LEMMA 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable map. For any $L, M, T > 0$ and $u_0 \in \mathcal{C}_{[L,M]}$, there holds*

$$(2.2) \qquad \big\|S_T u_0\big\|_{\mathbf{L}^\infty(\mathbb{R})} \;\leq\; M \qquad \text{and} \qquad Supp(S_T u_0) \;\subseteq\; \big[-l_{[L,M,T]},\, l_{[L,M,T]}\big]$$

with

$$(2.3) \qquad\qquad l_{[L,M,T]} \;\doteq\; L + T \cdot f'_M\,.$$

Moreover, if $u_0 \in \mathcal{C}_{[L,M]} \cap BV(\mathbb{R})$, then one has $S_T u_0 \in BV(\mathbb{R})$.

Proof. The monotonicity of the solution operator S_t yields [12, 16]:

$$(2.4) \qquad\qquad \big\|S_t u_0\big\|_{\mathbf{L}^\infty(\mathbb{R})} \;\leq\; \big\|u_0\big\|_{\mathbf{L}^\infty(\mathbb{R})} \;\leq\; M \qquad \text{for all} \;\; t \geq 0\,.$$

Next observe that, for any $u_0 \in \mathcal{C}_{[L,M]}$, we can find a sequence $\{u^\nu\}_\nu$, $u^\nu \in BV(\mathbb{R})$, with $Supp(u^\nu) \subset Supp(u_0)$, such that $u^\nu \to u_0$ in $\mathbf{L}^1$. This, in turn, implies that

$$(2.5) \qquad\qquad S_t u^\nu \;\xrightarrow{\;\mathbf{L}^1\;}\; S_t u_0 \qquad \text{for all} \;\; \nu \;\text{for all}\;\; t > 0\,.$$

Moreover, recalling that $S_t u^\nu$ can be obtained as the limit of piecewise constant front tracking approximations [9, Chapter 6], we deduce that

$$(2.6)$$
$$Supp(S_t u^\nu) \;\subseteq\; \big[-l_{[L,M,t]},\, l_{[L,M,t]}\big] \;\; \text{with} \;\; l_{[L,M,t]} \;\doteq\; L + t \cdot f'_M \;\; \text{for all} \;\; \nu \;\text{for all}\;\; t > 0$$

with f'_M as in (2.1). Thus, (2.4)–(2.6) together yield (2.2). The a priori bounds on the total variation of the solution guarantee also that $S_T u_0 \in BV(\mathbb{R})$ whenever $u_0 \in BV(\mathbb{R})$ (see [9, Theorem 6.1]). $\qquad\qquad\square$

We next collect the uniform upper bounds on the total variation of the flux of an entropy weak solution established in [8, Theorem 3.4, Theorem 4.9] (see also [10], [12, section 11.2], [21, Theorem 2]).

LEMMA 2.3. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying either of the* (C) *or* (NC) *conditions. Then, for any $L, M, T > 0$ and for every $u_0 \in \mathcal{C}_{[L,M]}$, there holds*

$$(2.7) \qquad TV\{f' \circ S_T u_0 \mid \mathbb{R}\} \leq C_1 \left(1 + \frac{L}{T}\right)$$

for some constant $C_1 > 0$ depending only on f and M.

Proof. For convenience of the reader we provide a sketch of the proof since the constants in the right-hand side of (2.7) slightly differs form the ones in the cited references.

1. Assume that f satisfies the (C) condition. Observe first that, because of the nonintersection property of minimal and maximal backward characteristics [10], one deduces a one-sided Lipschitz condition on the derivative of the flux [12, section 11.2]:

$$(2.8) \qquad f'\big(S_T u_0(y)\big) - f'\big(S_T u_0(x)\big) \leq \frac{y - x}{T} \qquad \text{for all} \ \ x < y.$$

On the other hand, by Lemma 2.2 we have $Supp(S_T(u_0)) \subseteq [-l_{[L,M,T]}, l_{[L,M,T]}]$. Thus, since (2.8) implies that $x \to f'\big(S_T u_0(x)\big) - \frac{x}{T}$ is a nonincreasing map, we find

$$
\begin{aligned}
TV\{f' \circ S_T u_0 \mid \mathbb{R}\} &= \lim_{\varepsilon \to 0} TV\{f' \circ S_T u_0 \mid (-l_{[L,M,T]} - \varepsilon, l_{[L,M,T]} + \varepsilon)\} \\
&\leq \lim_{\varepsilon \to 0} \left[TV\Big\{f' \circ S_T u_0 - \frac{\cdot}{T} \mid (-l_{[L,M,T]} - \varepsilon, l_{[L,M,T]} + \varepsilon)\Big\} \right. \\
&\qquad \left. + TV\Big\{\frac{\cdot}{T} \mid (-l_{[L,M,T]} - \varepsilon, l_{[L,M,T]} + \varepsilon)\Big\} \right] \leq \frac{4 l_{[L,M,T]}}{T},
\end{aligned}
$$

which, by definition (2.3), yields (2.7).

2. Assume that f satisfies the (NC) condition. Since by Lemma 2.2 we have $\|S_T u_0\|_{\mathbf{L}^\infty(\mathbb{R})} \leq M$, $Supp(S_T(u_0)) \subseteq [-l_{[L,M,T]}, l_{[L,M,T]}]$, invoking [8, Theorem 4.9] (see also [21, Theorem 2]) we deduce that, for every $\varepsilon > 0$, there holds

$$(2.9) \quad TV\{f' \circ S_T u_0 \mid (-l_{[L,M,T]} - \varepsilon, l_{[L,M,T]} + \varepsilon)\} \leq \frac{C_M \cdot 2(l_{[L,M,T]} + \varepsilon)}{T} + \widetilde{C}_M,$$

where $C_M, \widetilde{C}_M > 0$ are constants depending only on the flux f and on M. Hence, relying on (2.3), (2.9) we derive

$$
\begin{aligned}
TV\{f' \circ S_T(u_0) \mid (-\infty, +\infty)\} &= \lim_{\varepsilon \to 0} TV\{f' \circ S_T(u_0) \mid (-l_{[L,M,T]} - \varepsilon, l_{[L,M,T]} + \varepsilon)\} \\
&\leq \frac{8 C_1^{f,M} L}{T} + 2 \cdot \Big(C_M \cdot f'_M + \widetilde{C}_M\Big),
\end{aligned}
$$

which yields (2.7). $\qquad \square$

Remark 2.4. In the nonconvex case a bound as in (2.7) in general does not hold without the assumption of polynomial degeneracy in (1.4). In fact, in [21, section 8.1] an example of a flux $f(u)$ was exhibited having one inflection point at zero, with $f^j(0) = 0$ for all $j \in \mathcal{N}$, $j \geq 2$, and of an initial datum $u_0 \in \mathbf{L}^\infty(\mathbb{R})$ with compact support, such that $f' \circ S_t(u_0) \notin BV(\mathbb{R})$ for almost every t in an interval of $(0, \infty)$.

To complete this section, we recall now two results that provide an upper bound on the ε-entropy for sets of functions with uniformly bounded total variation and a lower bound for the ε-entropy of sets of functions having uniformly bounded one-sided derivative.

LEMMA 2.5 (see [6, Theorem 1]).   *Given $L, V > 0$, consider the set*

$$(2.10) \qquad \mathcal{F}_{[L,V]} \ \doteq \ \Big\{g : \mathbb{R} \to [-V,V] \ \big| \ Supp(g) \subseteq [-L,L], \ TV\{g\} \ \leq \ 2V\Big\}.$$

Then, for all $\varepsilon \leq \frac{VL}{3}$, there holds

$$\mathcal{H}_\varepsilon\big(\mathcal{F}_{[L,V]} \mid \mathbf{L}^1(\mathbb{R})\big) \ \leq \ 48VL \cdot \frac{1}{\varepsilon}\,.$$

Moreover, there exists a set of piecewise constant functions $\{g_1, \ldots, g_p\} \subset \mathcal{F}_{[L,V]}$ with

$$p \ \leq \ \left\lfloor 2^{\left(\frac{48VL}{\varepsilon}\right)} \right\rfloor + 1\,,$$

that satisfy

$$g_i(x) \ = \ g_i\Big(-L + \frac{2L}{N} \cdot \nu\Big)$$

for all $x \in \Big[-L + \frac{2L}{N} \cdot \nu, \ -L + \frac{2L}{N} \cdot (\nu+1)\Big), \quad \nu \in \{0, 1, \ldots, N-1\}$ with

$$N \ \geq \ \left\lfloor \frac{8LV}{\varepsilon} \right\rfloor,$$

and such that

$$(2.11) \qquad\qquad\qquad \mathcal{F}_{[L,V]} \ \subset \ \bigcup_{i=1}^{p} B\big(g_i, \ \varepsilon\big)\,,$$

where $B\big(g_i, \ \varepsilon\big)$ denotes the $\mathbf{L}^1(\mathbb{R})$-ball centered at g_i of radius ε.

LEMMA 2.6 (see [3, Proposition 2.2]).   *Given $L, h, b > 0$, consider the sets*

$$\mathcal{B}_{[L,h,\leq b]} \ \doteq \ \Big\{v \in BV(\mathbb{R}) \ \big| \ Supp(v) \subseteq [-L,L], \ \|v\|_{\mathbf{L}^\infty(\mathbb{R})} \leq h, \ Dv \leq b\Big\},$$
$$(2.12)$$
$$\mathcal{B}_{[L,h,\geq -b]} \ \doteq \ \Big\{v \in BV(\mathbb{R}) \ \big| \ Supp(v) \subseteq [-L,L], \ \|v\|_{\mathbf{L}^\infty(\mathbb{R})} \leq h, \ Dv \geq -b\Big\},$$

where the inequalities $Dv \leq b$, $Dv \geq -b$ must be understood in the sense of measures, i.e., the Radon measure Dv satisfies $Dv(J) \leq b \cdot |J|$, $Dv(J) \geq -b \cdot |J|$, respectively, for every Borel set $J \subset \mathbb{R}$, $|J|$ being the Lebesgue measure of J. Then, for any $0 < \varepsilon \leq \frac{Lh}{6}$, there holds

$$\mathcal{H}_\varepsilon\big(\mathcal{B}_{[L,h,\geq b]} \mid \mathbf{L}^1(\mathbb{R})\big) \ \geq \ \frac{2bL^2}{27\ln 2} \cdot \frac{1}{\varepsilon}\,,$$

$$\mathcal{H}_\varepsilon\big(\mathcal{B}_{[L,h,\leq -b]} \mid \mathbf{L}^1(\mathbb{R})\big) \ \geq \ \frac{2bL^2}{27\ln 2} \cdot \frac{1}{\varepsilon}\,.$$

3. Upper compactness estimates. We derive in this section upper bounds on the ε-entropy in $\mathbf{L}^1$ of $S_T(\mathcal{C}_{[L,M]})$ for the class of initial data $\mathcal{C}_{[L,M]}$ in (1.6), when the flux function f satisfies either of the assumptions (C) or (NC) stated in the introduction.

Towards a proof of (1.7), (1.13), we first establish an upper bound on the ε-entropy in $\mathbf{L}^1$ of the set

$$(3.1) \qquad \mathcal{L}_{[L,M,T]} \;\doteq\; \Big\{ f' \circ u \mid u \in S_T(\mathcal{C}_{[L,M]}) \Big\}.$$

LEMMA 3.1. *In the same setting as Theorem* 1.2 *or* 1.5, *assume that* $f : \mathbb{R} \to \mathbb{R}$ *is a function satisfying either of the* (C) *or* (NC) *conditions and that* (1.2) *holds. Then, given any* $L, M, T > 0$, *for all* $\varepsilon \le \frac{\Gamma_1^+}{288}$, *there holds*

$$(3.2) \qquad \mathcal{H}_\varepsilon\Big(\mathcal{L}_{[L,M,T]} \mid \mathbf{L}^1(\mathbb{R})\Big) \;\le\; \frac{\Gamma_1^+}{2} \cdot \frac{1}{\varepsilon}$$

with Γ_1^+ *as in* (1.9). *Moreover, there exists a set of piecewise constant functions* $\{g_1, \dots, g_p\}$ *with*

$$p \;\le\; \left\lfloor 2^{\left(\frac{\Gamma_1^+}{2\,\varepsilon}\right)} \right\rfloor + 1 \,,$$

that enjoy the following properties:
 (i) *For any* $i = 1, \dots, p$, *one has*

$$Supp(g_i) \;\subseteq\; \big[- l_{[L,M,T]},\, l_{[L,M,T]} \big],$$

$$Im(g_i) \subseteq \big[-f'_M, f'_M \big] \;\; \text{if (C) holds}, \quad Im(g_i) \subseteq \big[0, f'_M \big] \;\; \text{if (NC) holds},$$

and

$$g_i(x) \;=\; g_i(x_\nu) \qquad \text{for all } \; x \in [x_\nu,\, x_{\nu+1}), \qquad \nu \in \{0, 1, \dots, N-1\},$$

with

$$x_\nu \;\doteq\; -l_{[L,M,T]} + \frac{2\, l_{[L,M,T]}}{N} \cdot \nu\,, \;\; \nu \in \{0, 1, \dots, N\}, \;\; N \;\ge\; \left\lfloor \frac{8\, l_{[L,M,T]} V_{[L,M,T]}}{\varepsilon} \right\rfloor,$$

where f'_M, $l_{[L,M,T]}$ *are the constants defined in* (2.1), (2.3), *respectively, and*

$$(3.3) \qquad V_{[L,M,T]} \;\doteq\; \max\left\{ \frac{C_1}{2} \cdot \left(1 + \frac{L}{T}\right),\, f'_M \right\},$$

 C_1 *being the constants defined in* (2.7).
 (ii)

$$\mathcal{L}_{[L,M,T]} \;\subset\; \bigcup_{i=1}^{p} B\big(g_i,\, \varepsilon\big)\,.$$

Proof. Observe first that, relying on Lemmas 2.2–2.3, we have

$$(3.4) \qquad \mathcal{L}_{[L,M,T]} \;\subseteq\; \mathcal{F}_{[l_{[L,M,T]}, V_{[L,M,T]}]}\,,$$

where $\mathcal{F}_{[l_{[L,M,T]}, V_{[L,M,T]}]}$ is a set defined as in (2.10). Therefore, invoking Lemma 2.5, we derive

$$(3.5) \quad \mathcal{H}_\varepsilon \Big(\mathcal{L}_{[L,M,T]} \mid \mathbf{L}^1(\mathbb{R}) \Big) \ \leq \ \mathcal{H}_\varepsilon \Big(\mathcal{F}_{[l_{[L,M,T]}, V_{[L,M,T]}]} \mid \mathbf{L}^1(\mathbb{R}) \Big)$$

$$\leq \ 48 \cdot \max \left\{ \frac{C_1 \big(L + T \cdot f'_M \big)}{2} \cdot \Big(1 + \frac{L}{T} \Big), \ f'_M \big(L + T \cdot f'_M \big) \right\} \cdot \frac{1}{\varepsilon}$$

which yields (3.2), and we deduce the existence of a set of piecewise constant functions $\{g_1, \ldots, g_p\}$ enjoying the properties (i)–(ii). □

3.1. Strictly (not necessarily uniformly) convex fluxes. In this subsection, we will study the case where f is a convex function satisfying the assumption (C) which in particular implies that f' is strictly increasing and hence invertible on $\mathbb{R}$.

In order to establish (1.7), we will use the following technical lemma providing an estimate of the $\mathbf{L}^1$-distance of two elements $u, v \in S_T(\mathcal{C}_{[L,M]})$ in terms of the $\mathbf{L}^1$-distance of $f' \circ u, f' \circ v$. To this end, consider the map

$$(3.6) \qquad \widehat{\Delta}_{f,M}(s) \ \doteq \ s \cdot \inf_{\substack{|u|,|v| \leq M \\ |v-u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \qquad \text{for all} \ \ s > 0 \,,$$

which differs form the map in (1.5) for the fact that the infimum is taken also over pairs u, v of opposite sign. Observe that the maps $s \mapsto \widehat{\Delta}_{f,M}(s)$, $s \mapsto \dfrac{\widehat{\Delta}_{f,M}(s)}{s}$ are strictly increasing and thus invertible. Moreover, one has

$$(3.7) \qquad \widehat{\Delta}_{f,M}\big(|u-v|\big) \ \leq \ |f'(u) - f'(v)| \quad \text{for all} \ \ u, v \in \mathbb{R} \ \ \text{s.t.} \ \ |u|, |v| \leq M \,.$$

LEMMA 3.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a differentiable map. Given any* $L, M > 0$, *for every* $u, v \in \mathbf{L}^\infty(\mathbb{R})$ *with*
(3.8)
$$\|u\|_{\mathbf{L}^\infty} \ \leq \ M, \ \ \|v\|_{\mathbf{L}^\infty} \ \leq \ M, \qquad \text{and} \qquad Supp(u) \subset [-L, L], \ Supp(v) \subset [-L, L],$$

there holds

$$(3.9) \qquad \|u - v\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ (1 + 2L) \cdot \widehat{\Delta}_{f,M}^{-1} \Big(\|f' \circ u - f' \circ v\|_{\mathbf{L}^1(\mathbb{R})} \Big).$$

Proof.
1. We claim that, setting

$$(3.10) \qquad\qquad \rho \ \doteq \ \widehat{\Delta}_{f,M}^{-1} \Big(\|f' \circ u - f' \circ v\|_{\mathbf{L}^1(\mathbb{R})} \Big),$$

one has

$$(3.11) \ \ \big| u(x) - v(x) \big| \ \leq \ \rho \cdot \max \left\{ 1, \ \frac{\big| f'(u(x)) - f'(v(x)) \big|}{\|f' \circ u - f' \circ v\|_{\mathbf{L}^1(\mathbb{R})}} \right\} \qquad \text{for all} \ \ x \in \mathbb{R} \,.$$

Indeed, assume that $|u(x) - v(x)| \geq \rho$. Then, relying on (3.7), (3.10), and on the

monotonicity of $s \to \frac{\widehat{\Delta}_{f,M}(s)}{s}$, we estimate

$$
\begin{aligned}
|u(x) - v(x)| &\leq \frac{|u(x) - v(x)|}{\widehat{\Delta}_{f,M}\big(|u(x) - v(x)|\big)} \cdot \big|f'(u(x)) - f'(v(x))\big| \\
&\leq \frac{\rho}{\widehat{\Delta}_{f,M}(\rho)} \cdot \big|f'(u(x)) - f'(v(x))\big| \\
&= \rho \cdot \frac{\big|f'(u(x)) - f'(v(x))\big|}{\big\|f' \circ u - f' \circ v\big\|_{\mathbf{L}^1(\mathbb{R})}}
\end{aligned}
$$

(3.12)

which yields (3.11).

2. Thanks to (3.10), (3.11), and since by (3.8) one has $u = v = 0$ on $\mathbb{R} \setminus [-L, L]$, we derive

$$
\begin{aligned}
\big\|u - v\big\|_{\mathbf{L}^1(\mathbb{R})} &\leq \int_{-L}^{L} |u(x) - v(x)| dx \\
&\leq \int_{-L}^{L} \rho \cdot \left(1 + \frac{\big|f'(u(x)) - f'(v(x))\big|}{\big\|f' \circ u - f' \circ v\big\|_{\mathbf{L}^1(\mathbb{R})}}\right) dx \\
&\leq 2L\,\rho + \rho\,,
\end{aligned}
$$

(3.13)

which proves (3.9). $\qquad\square$

Remark 3.3. As suggested by one of the referees, one could replace the expression in (3.9) with

(3.14)
$$
\big\|u - v\big\|_{\mathbf{L}^1(\mathbb{R})} \leq 2L\omega\left(\frac{1}{2L}\big\|f'(u) - f'(v)\big\|_{\mathbf{L}^1(\mathbb{R})}\right),
$$

where ω is the concave modulus of continuity of $[f']^{-1}$, as follows from Jenssen's inequality. Following the rest of the proof, this gives another expression for the upper estimates. This gives the same expression in the case of power nonlinearities.

The next lemma shows that $\Delta_{f,M}, \widehat{\Delta}_{f,M}$ are comparable maps.

LEMMA 3.4. *Given a map* $f : \mathbb{R} \to \mathbb{R}$ *satisfying the assumption* (C), *let* $\Delta_{f,M}$, $\widehat{\Delta}_{f,M}$ *be the maps defined in* (1.5), (3.6), *respectively. Then, one has*

(3.15)
$$
\Delta_{f,M}(s/2) \leq \widehat{\Delta}_{f,M}(s) \leq \Delta_{f,M}(s).
$$

Proof. The second inequality in (3.15) is an immediate consequence of the definitions (1.5), (3.6). Towards a proof of the first inequality in (3.15), given $u \leq 0 \leq v$, relying on the monotonicity of f' we find

$$
\begin{aligned}
\left|\frac{f'(v) - f'(u)}{v - u}\right| &= \frac{f'(v) - f'(0)}{v - u} + \frac{f'(0) - f'(u)}{v - u} \\
&= \left(\frac{v}{v - u}\right) \cdot \frac{f'(v) - f'(0)}{v} + \left(\frac{-u}{v - u}\right) \cdot \frac{f'(0) - f'(u)}{-u}\,.
\end{aligned}
$$

(3.16)

Therefore, observing that $v - u \geq s$ implies $\max\{v, -u\} \geq s/2$, we deduce from (3.16)

that for all $-M \leq u \leq 0 \leq v \leq M$, such that $v - u \geq s$, there holds

(3.17)
$$
\left| \frac{f'(v) - f'(u)}{v - u} \right| \geq
\begin{cases}
\dfrac{1}{2} \cdot \dfrac{f'(v) - f'(0)}{v} & \text{if} \quad v = \max\{v, -u\}, \\[2ex]
\dfrac{1}{2} \cdot \dfrac{f'(0) - f'(u)}{-u} & \text{if} \quad -u = \max\{v, -u\},
\end{cases}
$$
$$
\geq \frac{1}{2} \cdot D(s/2),
$$

where

(3.18)
$$
D(s) \;\doteq\; \inf_{\substack{|u|, |v| \leq M, \, u \cdot v \geq 0 \\ |v - u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right|.
$$

Taking the infimum in the left-hand side of (3.17) over all $-M \leq u \leq 0 \leq v \leq M$ with $v - u \geq s$, we thus find

(3.19)
$$
\widehat{D}(s) \;\geq\; \frac{1}{2} \cdot D(s/2) \qquad \text{for all} \;\; s > 0,
$$

where

(3.20)
$$
\widehat{D}(s) \;\doteq\; \inf_{\substack{|u|, |v| \leq M \\ |v - u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right|.
$$

Then, observing that by (1.5), (3.6), we have

(3.21)
$$
\Delta_{f,M}(s) \;=\; s \cdot D(s), \qquad\qquad \widehat{\Delta}_{f,M}(s) \;=\; s \cdot \widehat{D}(s) \qquad\qquad \text{for all} \;\; s > 0,
$$

we recover from (3.19) the first inequality in (3.15). $\qquad\qquad\qquad\qquad\qquad\square$

Proof of upper bound (1.7) *of Theorem* 1.2. By virtue of Lemma 3.1, given any

(3.22)
$$
0 \;<\; \varepsilon < \left(1 + 2 \, l_{[L,M,T]} \right) \cdot \widehat{\Delta}_{f,M}^{-1} \left(\frac{\Gamma_1^+}{124} \right)
$$

with $l_{[L,M,T]}$ as in (2.3), and setting

(3.23)
$$
\varepsilon' \;\doteq\; \widehat{\Delta}_{f,M} \left(\frac{\varepsilon}{1 + 2 \, l_{[L,M,T]}} \right),
$$

there holds

(3.24)
$$
\mathcal{N}_{\varepsilon'} \left(\mathcal{L}_{[L,M,T]} \mid \mathbf{L}^1(\mathbb{R}) \right) \;\leq\; 2^{\left(\frac{\Gamma_1^+}{2 \, \varepsilon'} \right)}.
$$

Therefore, there exists a set of functions

(3.25)
$$
\{g_1, \ldots, g_p\} \;\subset\; \mathcal{L}_{[L,M,T]}
$$

with

(3.26)
$$
p \;\leq\; \left\lfloor 2^{\left(\frac{\Gamma_1^+}{2 \, \varepsilon'} \right)} \right\rfloor + 1,
$$

such that

$$(3.27) \qquad \mathcal{L}_{[L,M,T]} \subseteq \bigcup_{i=1}^{p} B(g_i, \varepsilon'),$$

where $B(g_i, \varepsilon')$ denotes the $\mathbf{L}^1(\mathbb{R})$-ball centered at g_i of radius ε'. Notice that, by Lemma 2.2 and because of (1.2), we have

$$\mathcal{L}_{[L,M,T]} \subseteq \mathcal{C}_{[l_{[L,M,T]},f'_M]}.$$

Hence (3.27) yields

$$(3.28) \qquad \mathcal{L}_{[L,M,T]} \subseteq \mathcal{C}_{[l_{[L,M,T]},f'_M]} \cap \bigcup_{i=1}^{p} B(g_i, \varepsilon').$$

On the other hand, observing that by (1.2) one has

$$(3.29) \qquad g \in \mathcal{C}_{[l_{[L,M,T]},f'_M]} \qquad \Longrightarrow \qquad (f')^{-1} \circ g \in \mathcal{C}_{[l_{[L,M,T]},M]},$$

and because of (3.23), invoking Lemma 3.2 we deduce that for all $i = 1, \ldots, p$, there holds
$$(3.30)$$
$$g \in \mathcal{C}_{[l_{[L,M,T]},f'_M]}, \ \|g - g_i\|_{\mathbf{L}^1(\mathbb{R})} < \varepsilon' \qquad \Longrightarrow \qquad \|(f')^{-1} \circ g - (f')^{-1} \circ g_i\|_{\mathbf{L}^1(\mathbb{R})} < \varepsilon.$$

Hence, we deduce from (3.28), (3.30) that

$$(3.31) \qquad \begin{aligned} S_T(\mathcal{C}_{[L,M]}) &\subseteq \bigcup_{i=1}^{p} \left\{ (f')^{-1} \circ g \mid g \in \mathcal{C}_{[l_{[L,M,T]},f'_M]} \cap B(g_i, \varepsilon') \right\} \\ &\subseteq \bigcup_{i=1}^{p} B((f')^{-1} \circ g_i, \varepsilon). \end{aligned}$$

Thus, for all $\varepsilon > 0$ satisfying (3.22), we have produced an ε-cover of $S_T(\mathcal{C}_{[L,M]})$ in $\mathbf{L}^1$ of cardinality p which, thanks to (3.15), (3.26), is bounded by

$$(3.32) \qquad p \leq 1 + 2^{\left(\frac{\Gamma_1^+}{2\,\varepsilon'}\right)} \leq 2^{\left(\frac{\Gamma_1^+}{\varepsilon'}\right)} = 2^{\left(\frac{\Gamma_1^+}{\hat{\Delta}_{f,M}(2\varepsilon/\gamma_1^+)}\right)} \leq 2^{\left(\frac{\Gamma_1^+}{\Delta_{f,M}(\varepsilon/\gamma_1^+)}\right)}$$

with $\gamma_1^+ \doteq 2(1 + 2\,l_{[L,M,T]})$ as in (1.9) because of (2.3). Taking the base-2 logarithm in (3.32) we then derive the estimate (1.7). $\qquad \square$

3.2. Fluxes with one inflection point having polynomial degeneracy. In this subsection we will assume that f is a nonconvex function satisfying the assumption (NC) and (1.2). To fix the ideas we shall consider the case where $f^{(m+1)}(0) > 0$, the case with $f^{(m+1)}(0) < 0$ being entirely similar. Therefore, throughout this subsection we shall assume that, for some even integer $m \geq 2$, there holds

$$(3.33) \qquad \begin{aligned} f^{(j)}(0) &= 0 \quad \text{for all} \quad j = 1, \ldots, m, \qquad f^{(m+1)}(0) > 0, \\ f''(u) \cdot u &> 0 \qquad \text{for all} \ \ u \in \mathbb{R} \setminus \{0\}. \end{aligned}$$

This implies that the function f' is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$. Moreover, f' is positive on $\mathbb{R} \setminus \{0\}$.

Towards a proof of (1.13) we first establish some technical lemmas concerning the flux f and the function $\Delta_{f,M}$ defined in (1.5), and providing bounds on the $\mathbf{L}^1$-distance of two elements $u, v \in S_T(\mathcal{C}_{[L,M]})$ in terms of the $\mathbf{L}^1$-distance of $f' \circ u, f' \circ v$.

LEMMA 3.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth map satisfying the assumption* (3.33). *For any $M > 0$, there exist constants $\kappa_M \in (0,1)$, $\beta_M, \sigma_M > 0$ depending only on f and M, such that the following hold:*

$$(3.34) \qquad \begin{aligned} \left| f'(u) - f'(u/2) \right| &\geq \kappa_M \cdot \left| f'(u) \right|, \\[4pt] \left| f'(u/2) \right| &\geq \kappa_M \cdot \left| f'(u) \right| \end{aligned} \qquad \text{for all} \ \ u \in [-M, M],$$

$$(3.35) \qquad \sup_{u \in [-M,M] \setminus \{0\}} \left\{ \left| \frac{f(u) - f(0)}{u f'(u)} \right| \right\} \ \leq \ 1 - \frac{\kappa_M}{2} \ < \ 1,$$

$$(3.36) \qquad \frac{s^m}{\beta_M} \ \leq \ \Delta_{f,M}(s) \ \leq \ \beta_M \cdot s^m \qquad \text{for all} \ \ s \in (0, \sigma_M].$$

Proof.

1. Observe first that, by the monotonicity property of f' and since f' is always nonnegative, the inequalities in (3.34) are equivalent to

$$(3.37) \qquad \begin{aligned} f'(u) - f'(u/2) &\geq \kappa_M \cdot f'(u), \\[4pt] f'(u/2) &\geq \kappa_M \cdot f'(u) \end{aligned} \qquad \text{for all} \ \ u \in [-M, M].$$

Next, by writing a Taylor approximation of the derivative of the flux in the origin and relying on (1.2), we find

$$(3.38) \qquad \begin{aligned} \frac{f'(u)}{2} - f'(u/2) &= \frac{f^{(m+1)}(0)}{m!} \left(\frac{u^m}{2} - (u/2)^m \right) + u^m \cdot o(1) \\[6pt] &= u^m \cdot \left(\frac{f^{(m+1)}(0)}{m!} \left((1/2) - (1/2)^m \right) + o(1) \right) \end{aligned}$$

and

$$(3.39) \qquad \begin{aligned} f'(u/2) - \frac{f'(u)}{2^{m+1}} &= \frac{f^{(m+1)}(0)}{m!} \left((u/2)^m - \frac{1}{2}(u/2)^m \right) + u^m \cdot o(1) \\[6pt] &= u^m \cdot \left(\frac{f^{(m+1)}(0)}{m!} \frac{1}{2^{m+1}} + o(1) \right), \end{aligned}$$

where $o(1)$ denotes a function converging to zero when $u \to 0$. Since $f^{m+1}(0) > 0$ and m is even, we deduce from (3.38) that there will be some constant $u_0 > 0$ such that

$$(3.40) \qquad \begin{aligned} f'(u) - f'(u/2) &\geq \frac{1}{2} f'(u), \\[4pt] f'(u/2) &\geq \frac{1}{2^{m+1}} f'(u) \end{aligned} \qquad \text{for all} \ \ u \in [-u_0, u_0].$$

On the other hand, setting
$$(3.41)$$
$$c_0 \doteq \inf_{u_0 \leq |u| \leq M} f'(u) - f'(u/2), \qquad c_0' \doteq \inf_{u_0 \leq |u| \leq M} f'(u/2), \qquad \widehat{c}_0 \doteq \sup_{u_0 \leq |u| \leq M} f'(u),$$

we find

$$f'(u) - f'(u/2) \ \geq \ \frac{c_0}{\widehat{c}_0} \cdot f'(u) \,,$$

(3.42) $$\qquad\qquad\qquad\qquad \text{for all} \ \ u \in [-M, M] \setminus [-u_0, u_0] \,,$$

$$f'(u/2) \ \geq \ \frac{c_0'}{\widehat{c}_0} \cdot f'(u)$$

where $c_0, c_0', \widehat{c}_0$ are positive constants since in (3.41) we are taking the infimum and the supremum of positive continuous functions on a compact subset of $\mathbb{R}$. Hence, (3.40), (3.42) together yield (3.37) taking

$$\kappa_M \ \dot{=} \ \min\left\{ \frac{1}{2^{m+1}}, \ \frac{c_0}{\widehat{c}_0}, \ \frac{c_0'}{\widehat{c}_0} \right\}.$$

2. Notice that condition (3.37) implies

$$f'(u/2) \ \leq \ (1 - \kappa_M) \cdot f'(u) \qquad \text{for all} \ u \ \in [-M, M] \,.$$

Hence, relying on the nonnegativity and monotonicity property of f', for any $u \in [-M, M] \setminus \{0\}$ we derive the estimate:

$$\left| \frac{f(u) - f(0)}{u} \right| \ = \ \left| \frac{\int_0^u f'(s)\,ds}{u} \right| \ \leq \ \left| \frac{\int_0^{u/2} f'(s)\,ds + \int_{u/2}^u f'(s)\,ds}{u} \right|$$

$$\leq \ \frac{1}{2}\Big(f'(u/2) + f'(u) \Big) \ \leq \ \Big(1 - \frac{\kappa_M}{2} \Big) \cdot f'(u) \,,$$

which yields (3.35).

3. In order to establish (3.36), it will be sufficient to show that there exist constants $s_o, k_0 > 0$ such that there holds
(3.43)
$$\frac{1}{k_0} \cdot \frac{\min\left\{ f'(-s/2), \ f'(s/2) \right\}}{s/2} \ \leq \ D(s) \ \leq \ k_0 \cdot \frac{\min\left\{ f'(-s), \ f'(s) \right\}}{s} \qquad \text{for all} \ s \in (0, s_0],$$

with D as in (3.18), since then one recovers (3.36) from (3.43) recalling (3.21) and taking the Taylor expansion of f' at zero.

Towards a proof of (3.43), observe first that by writing the Taylor expansion of $f^{(3)}$ at zero we find

(3.44) $$f^{(3)}(u) \ = \ u^{m-2} \cdot \left(\frac{f^{(m+1)}(0)}{(m-2)!} + o(1) \right),$$

where $o(1)$ denotes a function converging to zero when $u \to 0$. Since $f^{(m+1)}(0) > 0$ and m is even, we deduce from (3.44) that there will be some constant $u_0' \in (0, M)$ such that

$$f^{(3)}(u) \ \geq \ 0 \qquad \text{for all} \ \ u \in [-u_0', u_0'] \,,$$

which in turn implies that f' is a convex map on $[-u_0', u_0']$. Therefore, recalling that $f'(0) = 0$, we deduce that
(3.45)
$$\inf_{\substack{|u|, |v| \leq u_0', \, u \cdot v \geq 0 \\ |v - u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \ = \ \frac{\min\left\{ f'(-s), \ f'(s) \right\}}{s} \qquad \text{for all} \ \ s \in (0, u_0'] \,.$$

Since by definition (3.18) we have

$$
(3.46) \qquad D(s) \;\leq\; \inf_{\substack{|u|,|v| \,\leq\, u_0', \, u\cdot v \,\geq\, 0 \\ |v-u|\geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \qquad \text{for all } s \,,
$$

we obtain from (3.45) the upper bound in (3.43) with $s_0 = u_0'$, $k_0 = 1$.

Concerning the lower bound in (3.43), applying the mean-value theorem to f' we find

$$
(3.47) \qquad \inf_{\substack{u_0' \leq |u|,|v| \leq M \\ u\cdot v \geq 0}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \;\geq\; c_0'' \,,
$$

where

$$
(3.48) \qquad c_0'' \;\doteq\; \inf_{u_0' \leq |u| \leq M} \left| f''(u) \right| .
$$

Here, c_0'' is a positive constant since in (3.48) we are taking the infimum of a continuous function on a compact subset of $\mathbb{R} \setminus \{0\}$, which is positive on $\mathbb{R} \setminus \{0\}$ because of (3.33). On the other hand, observing that by (3.33) we have $\lim_{s\to 0} \frac{f'(|s|)}{s} = f''(0) = 0$, it follows that

$$
(3.49) \qquad \frac{\max\left\{ f'(-s), \, f'(s) \right\}}{s} \;<\; c_0'' \qquad \text{for all } s \in (0, u_0'']
$$

for some constant $u_0'' \in (0, u_0')$. Therefore, by virtue of (3.45), (3.47), (3.49), we derive
(3.50)

$$
D(s) = \min \left\{ \frac{f'(-s)}{s}, \, \frac{f'(s)}{s}, \, \inf_{\substack{|u|\leq u_0' \leq |v| \leq M \\ u\cdot v \geq 0, \; |v-u|\geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \right\} \qquad \text{for all } s \in (0, u_0''] \,.
$$

In order to provide a lower bound for

$$
\inf_{\substack{|u|\leq u_0' \leq |v| \leq M \\ u\cdot v \geq 0, \; |v-u|\geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right|
$$

we shall consider the case where $0 \leq u \leq u_0' \leq v \leq M$. Relying on the monotonicity of f' on $[0, +\infty)$, on convexity of f' on $[-u_0', u_0']$, and on (3.48), we find

$$
(3.51) \qquad
\begin{aligned}
\left| \frac{f'(v) - f'(u)}{v - u} \right| &= \frac{f'(v) - f'(u_0')}{v - u} + \frac{f'(u_0') - f'(u)}{v - u} \\[2mm]
&\geq \left(\frac{v - u_0'}{v - u} \right) \cdot c_0'' + \left(\frac{u_0' - u}{v - u} \right) \cdot \frac{f'(u_0' - u)}{u_0' - u} \,.
\end{aligned}
$$

We now distinguish two cases:

(i) If $u_0' - u > \frac{s}{2}$, then it follows from (3.51) that

$$
(3.52) \qquad \left| \frac{f'(v) - f'(u)}{v - u} \right| \;\geq\; \min \left\{ c_0'', \, \frac{f'(u_0' - u)}{u_0' - u} \right\} \;\geq\; \min \left\{ c_0'', \, \frac{f'(s/2)}{s/2} \right\} .
$$

(ii) If $u_0' - u \leq \frac{s}{2}$, then one has $u_0' - u \leq \frac{v-u}{2}$ which implies $\frac{v-u_0'}{v-u} \geq \frac{1}{2}$. Hence, we deduce from (3.51) that

$$(3.53) \qquad \left| \frac{f'(v) - f'(u)}{v - u} \right| \geq \frac{c_0''}{2}.$$

Therefore, by virtue of (3.49), (3.52), (3.53), and relying again on the convexity of f' on $[-u_0', u_0']$, we find

$$(3.54) \qquad \inf_{\substack{0 \leq u \leq u_0' \leq v \leq M \\ |v-u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \geq \frac{1}{2} \cdot \frac{f'(s/2)}{s/2} \qquad \text{for all } s \in (0, u_0''].$$

The case where $-M \leq v \leq -u_0' \leq u \leq 0$ can be treated in an entirely similar way. Hence, (3.50), (3.54) together yield the lower bound in (3.43) with $s_0 = u_0''$, $k_0 = 2$, thus completing the proof of the lemma. $\qquad\square$

Remark 3.6. If we consider a smooth, convex flux satisfying the assumption (1.11), with the same arguments of the proof of Lemma 3.5 one can show that the same type of lower bound in (3.36) holds. In fact, assume to fix the idea that $f^{(m+1)}(0) > 0$. Then, given $M > 0$, relying on (1.11), (3.44) one deduces that there exist constants $\widetilde{u}_0' > \widetilde{u}_0'' > 0$ such that:

(i) f' is a convex map on $[0, \widetilde{u}_0']$ and a concave map on $[-\widetilde{u}_0', 0]$;

(ii)

$$(3.55) \qquad \inf_{\substack{\widetilde{u}_0' \leq |u|, |v| \leq M \\ u \cdot v \geq 0}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \geq \widetilde{c}_0'' \doteq \inf_{\widetilde{u}_0' \leq |u| \leq M} \left| f''(u) \right| > 0;$$

(iii)

$$(3.56) \qquad \frac{\max\left\{ |f'(-s)|, |f'(s)| \right\}}{s} < \widetilde{c}_0'' \qquad \text{for all } s \in (0, \widetilde{u}_0''].$$

By virtue of (i), (ii), (iii), one then finds that
(3.57)

$$D(s) = \min\left\{ \frac{|f'(-s)|}{s}, \frac{|f'(s)|}{s}, \inf_{\substack{|u| \leq \widetilde{u}_0' \leq |v| \leq M \\ u \cdot v \geq 0, \ |v-u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \right\} \qquad \text{for all } s \in (0, \widetilde{u}_0''],$$

where $D(s)$ is defined as in (3.18). On the other hand, relying on the monotonicity of f' and on (i), (ii), (iii), we derive as in the proof of of Lemma 3.5 that

$$(3.58) \qquad \inf_{\substack{0 \leq u \leq \widetilde{u}_0' \leq v \leq M \\ |v-u| \geq s}} \left| \frac{f'(v) - f'(u)}{v - u} \right| \geq \frac{1}{2} \cdot \frac{|f'(s/2)|}{s/2} \qquad \text{for all } s \in (0, \widetilde{u}_0''].$$

Thus, (3.57), (3.58) together yield the lower bound

$$(3.59) \qquad \Delta_{f,M}(s) \geq \frac{s^m}{\alpha_M} \qquad \text{for all } s \in (0, \widetilde{u}_0'']$$

for some constant $\alpha_M > 0$.

LEMMA 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth map satisfying the assumption* (3.33). *Given any $L, M, T > 0$, for every $u \in S_T(\mathcal{C}_{[L,M]})$, and for any $x < y$ such that*

$$(3.60) \qquad\qquad \mathrm{sign}(u(x)) \ne \mathrm{sign}(u(y)),$$

there holds

$$(3.61) \qquad TV\{f' \circ u \mid [x, y]\} \ge \widetilde{\kappa}_M \cdot \max\{f'(u(x)), f'(u(y))\}$$

for some constant $\widetilde{\kappa}_M \in (0, 1)$ depending only on f and M.

Proof. Recalling that by Lemma 2.2 we have $\|S_T u_0\|_{\mathbf{L}^\infty(\mathbb{R})} \le M$, we shall rely on (3.34), (3.35) to show first that, for any $x < y$ such that (3.60) holds, one has

$$(3.62) \qquad\qquad TV\{f' \circ u \mid [x, y]\} \ge \frac{\kappa_M^2}{2} \cdot f'(u(x)),$$

$\kappa_M \in (0, 1)$ being the constant provided by Lemma 3.5. We will consider only the case where

$$(3.63) \qquad\qquad u(y) < 0 < u(x),$$

the other case with $u(x) < 0 < u(y)$ being entirely similar. We distinguish two subcases:

(i) If there exists $z \in (x, y]$ with $u(z-) \in \left[0, \frac{u(x)}{2}\right]$, by virtue of (3.34) and since f' is increasing on $[0, +\infty)$, we find

$$
\begin{aligned}
(3.64) \qquad TV\{f' \circ u \mid [x, y]\} &\ge \left|f'(u(x)) - f'(u(z-))\right| \\
&= f'(u(x)) - f'(u(z-)) \\
&\ge f'(u(x)) - f'(u(x/2)) \ge \kappa_M \cdot f'(u(x))
\end{aligned}
$$

proving (3.62).

(ii) Otherwise, because of (3.63), $S_T u_0$ must admit an admissible discontinuity located at some point $z \in [x, y]$, such that the left state $u(z-) \in \left[\frac{u(x)}{2}, M\right]$ and the right state $u(z+) < 0$. In the particular cases where $z = x$ or $z = y$, it must be $u(x) = u(z-)$ and $u(y) = u(z+)$, respectively. Thus, one has

$$(3.65) \qquad TV\{f' \circ u \mid [x, y]\} \ge \left|f'(u(z-)) - f'(u(z+))\right|.$$

Notice that the Oleinik E-condition (1.3) implies

$$(3.66) \qquad\qquad f'(u(z-)) \ge f'(u(z+)).$$

Since f' is decreasing on $(-\infty, 0]$, we then obtain

$$
\begin{aligned}
f(u(z-)) - f(0) &= \int_0^{u(z-)} f'(s)\, ds \\
&\ge f'(u(z-)) \cdot u(z-) \ge f'(u(z+)) \cdot u(z-),
\end{aligned}
$$

which yields

$$\frac{f(u(z-)) - f(0)}{u(z-)} \ge f'(u(z+)).$$

Thanks to (3.35), we thus deduce

$$f'(u(z+)) \; \leq \; \frac{f(u(z-)) - f(0)}{u(z-)} \; \leq \; \left(1 - \frac{\kappa_M}{2}\right) \cdot f'(u(z-)),$$

which, relying on (3.34), implies

$$
\begin{aligned}
\left| f'(u(z+)) - f'(u(z-)) \right| &= f'(u(z-)) - f'(u(z+)) \\
&\geq \frac{\kappa_M}{2 - \kappa_M} \cdot f'(u(z-)) \\
&\geq \frac{\kappa_M}{2} \cdot f'(u(x)/2) \\
&\geq \frac{\kappa_M^2}{2} \cdot f'(u(x))
\end{aligned}
$$

(3.67)

since $u(z-) \geq u(x)/2$, and because f' is increasing on $[0, +\infty)$. Hence, (3.65), (3.67) together yield (3.62).

Observing that

$$TV\{f' \circ u \mid [x,y]\} \; \geq \; \left| f'(u(y)) - f'(u(x)) \right|$$

we derive from (3.62) that

(3.68)
$$
\begin{aligned}
\left(1 + \frac{2}{\kappa_M^2}\right) \cdot TV\{f' \circ u \mid [x,y]\} &\geq \left| f'(u(y)) - f'(u(x)) \right| + \left| f'(u(x)) \right| \\
&\geq f'(u(y)).
\end{aligned}
$$

Therefore, (3.68) implies

$$TV\{f' \circ u \mid [x,y]\} \; \geq \; \frac{\kappa_M^2}{\kappa_M^2 + 2} \cdot f'(u(y)),$$

which, together with (3.62), yields (3.61) with

$$\widetilde{\kappa}_M \; \doteq \; \frac{\kappa_M^2}{\kappa_M^2 + 2} \,. \qquad\qquad \square$$

LEMMA 3.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable map. Given any $L, M > 0$, for every $u, v \in \mathbf{L}^\infty(\mathbb{R})$ with*

(3.69)
$$
\begin{aligned}
\|u\|_{\mathbf{L}^\infty} \; \leq \; M, \quad \|v\|_{\mathbf{L}^\infty} \; \leq \; M, \qquad u(x) \cdot v(x) \geq 0 \qquad \text{for all } \; x \in \mathbb{R}, \\
Supp(u) \; \subset \; [-L, L], \; Supp(v) \; \subset \; [-L, L],
\end{aligned}
$$

there holds

(3.70)
$$\left\| u - v \right\|_{\mathbf{L}^1(\mathbb{R})} \; \leq \; (1 + 2L) \cdot \Delta_{f,M}^{-1}\left(\left\| f' \circ u - f' \circ v \right\|_{\mathbf{L}^1(\mathbb{R})} \right),$$

where $\Delta_{f,M}$ is the map defined in (1.5).

Proof. Observe that $s \mapsto \Delta_{f,M}(s)$, $s \mapsto \frac{\Delta_{f,M}(s)}{s}$ are strictly increasing maps and that there holds
(3.71)
$$\Delta_{f,M}\big(|u-v|\big) \; \leq \; |f'(u) - f'(v)| \qquad \text{for all } \; u, v \in \mathbb{R} \quad \text{s.t.} \quad |u|, |v| \leq M, \quad u \cdot v \geq 0 \,.$$

Then, the estimate (3.70) can be obtained with the same arguments as the proof of Lemma 3.2 replacing $\widehat{\Delta}_{f,M}$ with $\Delta_{f,M}$ since, by assumption, $u(x)$ and $v(x)$ have the same sign for all $x \in \mathbb{R}$.   □

The next lemma provides an estimate of the $\mathbf{L}^1$-distance between a given element $u \in S_T(\mathcal{C}_{[L,M]})$ and its projection on the space of piecewise constant functions defined as follows. Fix $N \in \mathcal{N}$, letting $l_{[L,M,T]}$ be the constant in (2.3), set

$$(3.72) \qquad x_\nu \;\dot{=}\; -l_{[L,M,T]} + \frac{2\,l_{[L,M,T]}}{N} \cdot \nu\,, \qquad \nu \in \{0, 1, \ldots, N\}\,,$$

and define (recalling from Remark 2.1 that u admits one-sided limits at each point x)
(3.73)
$$\mathcal{P}^N(u)(x) \;\dot{=}\; \begin{cases} u(x_\nu^+) & \text{for all } x \in [x_\nu, x_{\nu+1}) \quad \text{if} \quad x \in \big[-l_{[L,M,T]},\, l_{[L,M,T]}\big)\,, \\ 0 & \text{otherwise.} \end{cases}$$

We shall express the $\mathbf{L}^1$-distance between $u \in S_T(\mathcal{C}_{[L,M]})$ and $\mathcal{P}^N(u)$ in terms of $TV\{f' \circ u\}$ which, in turn, admits an a priori bound provided by Lemma 2.3.

LEMMA 3.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth map satisfying the assumption (3.33). Given any $L, M, T > 0$, for every $u \in S_T(\mathcal{C}_{[L,M]})$, and for any $N \in \mathcal{N}$, there holds*

$$(3.74) \quad \Big\| f' \circ u - f' \circ \mathcal{P}^N(u) \Big\|_{\mathbf{L}^1(\mathbb{R})} \;\leq\; \frac{2\,l_{[L,M,T]} \cdot TV\{f' \circ u\}}{N}\,,$$

$$(3.75) \qquad \Big\| u - \mathcal{P}^N(u) \Big\|_{\mathbf{L}^1(\mathbb{R})} \leq 4\,l_{[L,M,T]} \cdot \left(2 + \frac{TV\{f' \circ u\}}{\widetilde{\kappa}_M} \right) \cdot \Delta_{f,M}^{-1}\left(\frac{1}{N}\right)\,,$$

where $\widetilde{\kappa}_M$ is the constant provided by Lemma 3.7.

Proof.
1. Observe first that by definition (3.73) there holds
(3.76)
$$\Big| f'\big(u(x)\big) - f'\big(\mathcal{P}^N(u)(x)\big) \Big| \;\leq\; TV\big\{f' \circ u \mid [x_\nu, x_{\nu+1})\big\} \qquad \text{for all } x \in [x_\nu, x_{\nu+1})$$

for all $\nu \in \{0, 1, \ldots, N-1\}$. Hence, since by (2.2), (3.73) one has $u = \mathcal{P}^N(u) = 0$ on $\mathbb{R} \setminus [-l_{[L,M,T]},\, l_{[L,M,T]}]$, we derive

$$\Big\| f' \circ u - f' \circ \mathcal{P}^N(u) \Big\|_{\mathbf{L}^1(\mathbb{R})} = \int_{-l_{[L,M,T]}}^{-l_{[L,M,T]}} \Big| f'\big(u(x)\big) - f'\big(\mathcal{P}^N(u)(x)\big) \Big|\,dx$$

$$(3.77) \hspace{3.5cm} \leq \; \frac{2\,l_{[L,M,T]}}{N} \cdot \sum_{\nu=0}^{N-1} TV\big\{f' \circ u \mid [x_\nu, x_{\nu+1})\big\}$$

$$\hspace{3.5cm} = \; \frac{2\,l_{[L,M,T]}}{N} \cdot TV\{f' \circ u\}\,,$$

proving (3.74).
2. Towards a proof of (3.75), we first show that, setting

$$(3.78) \qquad\qquad\qquad \rho \;\dot{=}\; \Delta_{f,M}^{-1}\left(\frac{1}{N}\right),$$

one has

$$(3.79) \qquad \left| u(x) - \mathcal{P}^N(u)(x) \right| \ \leq \ 2\rho \cdot \max \left\{ 2, \ \frac{N \cdot TV\{f' \circ u \mid [x_\nu, x_{\nu+1})\}}{\widetilde{\kappa}_M} \right\}$$

for all $x \in [x_\nu, x_{\nu+1})$ and $\nu \in \{0, 1, \ldots, N-1\}$.

Indeed, in the case where $u(x)$ and $\mathcal{P}^N(u)(x) = u(x_\nu)$, $x \in [x_\nu, x_{\nu+1})$, have the same sign, relying on (3.71), and recalling that by (2.2), (3.73) we have that $|u(x)|, |\mathcal{P}^N(u)(x)| \leq M$ with the same arguments as the proof of Lemma 3.2, replacing the definition of ρ in (3.10) with (3.78), one obtains the estimate
(3.80)

$$\left| u(x) - \mathcal{P}^N(u)(x) \right| \ \leq \ \rho \cdot \max \left\{ 1, \ N \cdot \left| f'\big(u(x)\big) - f'\big(\mathcal{P}^N(u)(x)\big) \right| \right\} \qquad \text{for all } \ x \in \mathbb{R}.$$

From (3.80) we immediately recover (3.79) because of (3.76) and since $\widetilde{\kappa}_M < 1$.

On the other hand, if $u(x)$ and $\mathcal{P}^N(u)(x)$ have different signs and we assume that

$$(3.81) \qquad \frac{\left| f'(u(x)) - f'(u(x_\nu)) \right|}{\left| u(x) - u(x_\nu) \right|} \ \geq \ \frac{\widetilde{\kappa}_M}{2\,\rho \cdot N}\,,$$

it follows that

$$(3.82) \qquad \begin{aligned} \left| u(x) - u(x_\nu) \right| &\leq \ \frac{2\,\rho \cdot N \left| f'(u(x)) - f'(u(x_\nu)) \right|}{\widetilde{\kappa}_M} \\ &\leq \ \frac{2\,\rho \cdot N \cdot TV\{f' \circ u \mid [x_\nu, x_{\nu+1})\}}{\widetilde{\kappa}_M} \end{aligned}$$

which proves (3.79).

Therefore, it remains to consider the case where $u(x)$ and $\mathcal{P}^N(u)(x) = u(x_\nu)$, $x \in [x_\nu, x_{\nu+1})$, have different signs and there holds

$$(3.83) \qquad \frac{\left| f'(u(x)) - f'(u(x_\nu)) \right|}{\left| u(x) - u(x_\nu) \right|} \ < \ \frac{\widetilde{\kappa}_M}{2\,\rho \cdot N}\,.$$

Since $u(x)$, $u(x_\nu)$ have opposite signs, one has

$$(3.84) \qquad \left| u(x) - \mathcal{P}^N(u)(x) \right| \ = \ \left| u(x) - u(x_\nu) \right| \ = \ \left| u(x) \right| + \left| u(x_\nu) \right|.$$

Moreover, by Lemma 3.7 there holds

$$(3.85) \qquad TV\{f' \circ u \mid [x_\nu, x_{\nu+1})\} \ \geq \ \widetilde{\kappa}_M \cdot \max \left\{ f'(u(x)), f(u(x_\nu)) \right\}.$$

We now denote by $\pi(u)$, $u \in \mathbb{R} \setminus \{0\}$, the unique point in $\mathbb{R}$ such that

$$(3.86) \qquad f'(u) = f'(\pi(u)) \qquad \text{and} \qquad \pi(u) \neq u\,,$$

while we set $\pi(0) \doteq 0$, and we distinguish two subcases:

(i) Assume that

$$(3.87) \quad \max \left\{ |u(x)|, \, |\pi(u(x))| \right\} \ \geq \ \rho, \qquad \max \left\{ |u(x_\nu)|, \, |\pi(u(x_\nu))| \right\} \ \geq \ \rho.$$

Then, recalling definition (1.5) and that $f'(0) = 0$, and relying on the monotonicity of the map $s \to \frac{\Delta_{f,M}(s)}{s}$, we derive

$$
\begin{aligned}
\frac{f'(u(x))}{\max\left\{|u(x)|,\, |\pi(u(x))|\right\}} &\geq \frac{\Delta_{f,M}\big(\max\left\{|u(x)|,\, |\pi(u(x))|\right\}\big)}{\max\left\{|u(x)|,\, |\pi(u(x))|\right\}} \\[2mm]
&\geq \frac{\Delta_{f,M}(\rho)}{\rho}\,, \\[4mm]
\frac{f'(u(x_\nu))}{\max\left\{|u(x_\nu)|,\, |\pi(u(x_\nu))|\right\}} &\geq \frac{\Delta_{f,M}\big(\max\left\{|u(x_\nu)|,\, |\pi(u(x_\nu))|\right\}\big)}{\max\left\{|u(x_\nu)|,\, |\pi(u(x_\nu))|\right\}} \\[2mm]
&\geq \frac{\Delta_{f,M}(\rho)}{\rho}\,.
\end{aligned}
$$
(3.88)

Hence, by virtue of (3.78), (3.85), (3.88), we deduce

$$
|u(x)| \leq \rho \cdot N \cdot f'(u(x)) \leq \frac{\rho \cdot N \cdot TV\left\{f' \circ u \mid [x_\nu,\, x_{\nu+1})\right\}}{\widetilde{\kappa}_M}\,,
$$
(3.89)
$$
|u(x_\nu)| \leq \rho \cdot N \cdot f'(u(x_\nu)) \leq \frac{\rho \cdot N \cdot TV\left\{f' \circ u \mid [x_\nu,\, x_{\nu+1})\right\}}{\widetilde{\kappa}_M}\,,
$$

which, together with (3.84), yield (3.79).

(ii) If (3.87) is not verified and (3.83) holds, we claim that

$$
|u(x)| \leq 2\rho, \qquad |u(x_\nu)| \leq 2\rho,
$$
(3.90)

which, because of (3.84), imply

$$
|u(x) - u(x_\nu)| \leq 4\rho\,,
$$

proving (3.79). In fact, if (3.87), (3.90) are not verified, then it must be

(3.91)
$$
\begin{aligned}
&\min\left\{\max\left\{|u(x)|,\, |\pi(u(x))|\right\},\, \max\left\{|u(x_\nu)|,\, |\pi(u(x_\nu))|\right\}\right\} < \rho\,, \\
&\max\left\{|u(x)|,\, |u(x_\nu)|\right\} > 2\rho\,.
\end{aligned}
$$

Let us assume that

(3.92)
$$
\max\left\{|u(x)|,\, |\pi(u(x))|\right\} < \rho\,, \qquad |u(x_\nu)| > 2\rho
$$

(the other case $\max\left\{|u(x_\nu)|,\, |\pi(u(x_\nu))|\right\} < \rho$, $|u(x)| > 2\rho$ being entirely similar). In this case, by (3.84) and since f' is decreasing on $(-\infty, 0]$ and increasing on $[0, +\infty)$, we have
(3.93)
$$
|u(x) - u(x_\nu)| \leq 2|u(x_\nu)|\,, \qquad f'(u(x_\nu)) > f'\left(\frac{u(x_\nu)}{2}\right) \geq f'(u(x))\,.
$$

Thus, relying on (3.78), (3.83), (3.93), we find

(3.94)
$$
\begin{aligned}
\frac{\Delta_{f,M}(\rho)}{\rho} &= \frac{1}{N \cdot \rho} \\
&> \frac{2 \cdot \big|f'(u(x)) - f'(u(x_\nu))\big|}{\widetilde{\kappa}_M \cdot \big|u(x) - u(x_\nu)\big|} = \frac{2 \cdot \big(f'(u(x_\nu)) - f'(u(x))\big)}{\widetilde{\kappa}_M \cdot \big|u(x) - u(x_\nu)\big|} \\
&\geq \frac{2 \cdot \big(f'(u(x_\nu)) - f'(u(x_\nu)/2)\big)}{\widetilde{\kappa}_M \cdot \big|u(x) - u(x_\nu)\big|} \geq \frac{f'(u(x_\nu))}{|u(x_\nu)|} \\
&\geq \frac{\Delta_{f,M}(|u(x_\nu)|)}{|u(x_\nu)|} \,.
\end{aligned}
$$

The increasing property of $s \to \frac{\Delta_{f,M}(s)}{s}$ together with (3.94) then implies $|u(x_\nu)| \leq \rho$ which yields a contradiction with (3.92). Thus, the bounds in (3.90) hold and the proof of (3.79) is complete.

3. Since by (2.2), (3.73) one has $u = \mathcal{P}^N(u) = 0$ on $\mathbb{R} \setminus [-l_{[L,M,T]}, l_{[L,M,T]}]$, relying on (3.79) we find
(3.95)

$$
\begin{aligned}
\big\| u - \mathcal{P}^N(u) \big\|_{\mathbf{L}^1(\mathbb{R})} &\leq \sum_{\nu=0}^{N-1} \big\| u - \mathcal{P}^N(u) \big\|_{\mathbf{L}^1([x_\nu,\, x_{\nu+1}])} \\
&\leq \frac{2\, l_{[L,M,T]}}{N} \cdot \sum_{\nu=0}^{N-1} \sup_{x \in [x_\nu,\, x_{\nu+1})} \big|u(x) - \mathcal{P}^N(u)(x)\big| \\
&\leq 8\, l_{[L,M,T]} \cdot \rho + \frac{4\, l_{[L,M,T]} \cdot \rho}{\widetilde{\kappa}_M} \cdot \sum_{\nu=0}^{N-1} TV\big\{f' \circ u \mid [x_\nu, x_{\nu+1})\big\},
\end{aligned}
$$

which yields (3.75). $\qquad\square$

We are now ready to provide the following proof.

Proof of upper bound (1.13) *of Theorem* 1.5. By virtue of Lemma 3.1, given any

(3.96)
$$
0 < \varepsilon < \big(2 + 4\, l_{[L,M,T]}\big) \cdot \Delta_{f,M}^{-1}\Big(\frac{\Gamma_1^+}{144}\Big)
$$

with $l_{[L,M,T]}$ as in (2.3), and setting

(3.97)
$$
\varepsilon' \doteq \frac{1}{2} \cdot \Delta_{f,M}\Big(\frac{\varepsilon}{2 + 4\, l_{[L,M,T]}}\Big),
$$

there exists a set of piecewise constant functions

(3.98)
$$
\mathcal{G} \doteq \{g_1, \ldots, g_p\}
$$

with

(3.99)
$$
p \leq \left\lfloor 2^{\left(\frac{\Gamma_1^+}{2\,\varepsilon'}\right)} \right\rfloor + 1,
$$

that enjoy the following properties:

(i) For any $i = 1, \ldots, p$, one has

$$Supp(g_i) \ \subseteq \ \left[- l_{[L,M,T]}, \, l_{[L,M,T]} \right], \qquad Im(g_i) \subseteq \left[0, f'_M \right],$$

and

$$g_i(x) \ = \ g_i(x_\nu) \qquad \text{for all} \ \ x \in [x_\nu, \, x_{\nu+1}), \qquad \nu \in \{0, 1, \ldots, N-1\},$$

with
(3.100)
$$x_\nu \ \doteq \ -l_{[L,M,T]} + \frac{2 \, l_{[L,M,T]}}{N} \cdot \nu, \ \ \nu \in \{0, 1, \ldots, N\}, \ \ N \geq \left\lfloor \frac{8 \, l_{[L,M,T]} \cdot V_{[L,M,T]}}{\varepsilon'} \right\rfloor,$$

where f'_M, $l_{[L,M,T]}$, $V_{[L,M,T]}$ are the constants defined in (2.1), (2.3), (3.3), respectively.

(ii)

(3.101)
$$\mathcal{L}_{[L,M,T]} \ \subset \ \bigcup_{i=1}^{p} B(g_i, \, \varepsilon').$$

For every g_i, $i = 1, \ldots, p$, and in connection with any N-tuple $\iota = (\iota_0, \ldots, \iota_{N-1}) \in \{-1, 1\}^N$, we now define a piecewise constant map $\mathcal{T}_\iota^N(g_i)$ as follows. Let f'_{-1}, f'_1 denote the restrictions of f' to the semilines $(-\infty, 0]$ and $[0, +\infty)$, respectively. Then, set
(3.102)
$$\mathcal{T}_\iota^N(g_i)(x) \ \doteq \ \begin{cases} \left(f'_{\iota_\nu} \right)^{-1} \left(g_i(x_\nu) \right) \text{ for all } \ x \in [x_\nu, x_{\nu+1}) & \text{if } \ x \in \left[-l_{[L,M,T]}, \, l_{[L,M,T]} \right), \\ 0 & \text{otherwise.} \end{cases}$$

Next, given any $u \in S_T(\mathcal{C}_{[L,M]})$, by (3.4) and (3.101) let g_i be a map satisfying property (i) such that

(3.103)
$$\left\| f' \circ u - g_i \right\|_{\mathbf{L}^1(\mathbb{R})} \ < \ \varepsilon'.$$

Observe that, applying Lemmas 2.3 and 3.9, and choosing

(3.104)
$$N \ \geq \ \left\lfloor \frac{2 \, l_{[L,M,T]} \cdot C_1 \left(1 + \frac{L}{T} \right)}{\varepsilon'} \right\rfloor + 1,$$

we find

(3.105)
$$\left\| f' \circ u - f' \circ \mathcal{P}^N(u) \right\|_{\mathbf{L}^1(\mathbb{R})} \ \leq \ \frac{2 \, l_{[L,M,T]} \cdot C_1 \left(1 + \frac{L}{T} \right)}{N} \ \leq \ \varepsilon'.$$

Hence, (3.103), (3.105) imply that, for any

(3.106)
$$N \ \geq \ \max \left\{ \left\lfloor \frac{8 \, l_{[L,M,T]} \cdot V_{[L,M,T]}}{\varepsilon'} \right\rfloor, \ \left\lfloor \frac{4 \, l_{[L,M,T]} \cdot C_1 \left(1 + \frac{L}{T} \right)}{\varepsilon'} \right\rfloor \right\},$$

and $\varepsilon' \leq 2 \, l_{[L,M,T]} \cdot C_1$, one has

(3.107)
$$\left\| f' \circ \mathcal{P}^N(u) - g_i \right\|_{\mathbf{L}^1(\mathbb{R})} \ < \ 2 \, \varepsilon'.$$

Let $\bar{\iota} \in \{-1,1\}^N$ be the N-tuple defined by

(3.108) $$\bar{\iota}_\nu \; = \; \mathrm{sign}\big(u(x_\nu)\big), \qquad \nu \in \{0,1,\ldots,N\} \, .$$

Notice that, by definitions (3.73), (3.102), by Lemma 2.2, and since $f'(0)=0$ and g_i satisfies the property (i), one has

$$\big\|\mathcal{P}^N(u)\big\|_{\mathbf{L}^\infty} \; \leq \; M, \qquad \big\|\mathcal{T}_{\bar{\iota}}^N(g_i)\big\|_{\mathbf{L}^\infty} \; \leq \; M,$$

(3.109) $$\mathrm{sign}\big(\mathcal{P}^N(u)(x)\big) \; = \; \mathrm{sign}\big(\mathcal{T}_{\bar{\iota}}^N(g_i)(x)\big) \qquad \text{for all} \;\; x \in \mathbb{R} \, ,$$

$$Supp\big(\mathcal{P}^N(u)\big), \; Supp\big(\mathcal{T}_{\bar{\iota}}^N(g_i)\big) \; \subset \; \big[-l_{[L,M,T]}, \, l_{[L,M,T]}\big] \, .$$

Therefore, observing that $f' \circ \mathcal{T}_{\bar{\iota}}^N(g_i) = g_i$, applying Lemmas 2.3, 3.8, and 3.9 and relying on (3.97), (3.107), we find that, for all

(3.110) $$N \; \geq \; \max\left\{ \left\lfloor \frac{1}{\Delta_{f,M}\!\left(\frac{\widetilde{\kappa}_M \, \varepsilon}{8\, l_{[L,M,T]} \cdot \left(2\widetilde{\kappa}_M + C_1(1+\frac{L}{T})\right)}\right)} \right\rfloor, \right.$$
$$\left. \left\lfloor \frac{16\, l_{[L,M,T]} \cdot V_{[L,M,T]}}{\Delta_{f,M}\!\left(\frac{\varepsilon}{2+4\, l_{[L,M,T]}}\right)} \right\rfloor, \; \left\lfloor \frac{8\, l_{[L,M,T]} \cdot C_1\big(1+\frac{L}{T}\big)}{\Delta_{f,M}\!\left(\frac{\varepsilon}{2+4\, l_{[L,M,T]}}\right)} \right\rfloor \right\} ,$$

there holds
(3.111)

$$\left\|\mathcal{P}^N(u) - \mathcal{T}_{\bar{\iota}}^N(g_i)\right\|_{\mathbf{L}^1(\mathbb{R})} \; \leq \; \big(1 + 2l_{[L,M,T]}\big) \cdot \Delta_{f,M}^{-1}(2\varepsilon') \; \leq \; \varepsilon/2 \, ,$$

$$\left\|u - \mathcal{P}^N(u)\right\|_{\mathbf{L}^1(\mathbb{R})} \; \leq \; \frac{4\, l_{[L,M,T]}}{\widetilde{\kappa}_M}\left(2\widetilde{\kappa}_M + C_1\Big(1+\frac{L}{T}\Big)\right) \cdot \Delta_{f,M}^{-1}\Big(\frac{1}{N}\Big) \; \leq \; \varepsilon/2 \, .$$

Hence, by (3.111), for any given $u \in S_T(\mathcal{C}_{[L,M]})$ and for every N satisfying (3.110), we can find an element g_i of the set $\mathcal{G}$ in (3.98) and an N-tuple $\bar{\iota} \in \{-1,1\}^N$ such that

$$\left\|u - \mathcal{T}_{\bar{\iota}}^N(g_i)\right\|_{\mathbf{L}^1(\mathbb{R})} \; \leq \; \varepsilon \, ,$$

showing that

(3.112) $$\bigcup_{\iota \in \{-1,1\}^N} \; \bigcup_{i=1}^{p} B\big(\mathcal{T}_{\iota}^N(g_i), \, \varepsilon\big)$$

provides an ε-cover of $S_T(\mathcal{C}_{[L,M]})$ in $\mathbf{L}^1$ of cardnality $p \cdot 2^N$ By virtue of (3.99), (3.110), for $\varepsilon > 0$ sufficiently small one has

(3.113) $$p \cdot 2^N \; \leq \; 2^{\left(\frac{\Gamma^+}{\Delta_{f,M}(\varepsilon/\gamma^+)}\right)}$$

with

$$\Gamma^+ \; \doteq \; 2\Gamma_1^+ + \max\left\{2, \, 32\, l_{[L,M,T]} \cdot V_{[L,M,T]}, \, 16\, l_{[L,M,T]} \cdot C_1\big(1+L/T\big)\right\},$$

(3.114)
$$\gamma^+ \; \doteq \; \max\left\{\frac{8\, l_{[L,M,T]}}{\widetilde{\kappa}_M}\left(2\widetilde{\kappa}_M + C_1\Big(1+\frac{L}{T}\Big)\right), \, 2 + 4\, l_{[L,M,T]}\right\} .$$

Recalling definitions (1.9), (2.3), (3.3) we deduce that there exists some constant $c > 1$ such that

$$(3.115) \qquad \Gamma^+ \leq \eta, \quad \gamma^+ \leq \eta, \qquad \eta \doteq c\left(1 + L + T + \frac{L^2}{T}\right).$$

Thus, relying on (3.36), (3.113), (3.115), it follows that there holds

$$(3.116) \qquad \mathcal{N}_\varepsilon\left(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\right) \leq 2^{\left(\frac{\eta}{\Delta_{f,M}(\varepsilon/\eta)}\right)} \leq 2^{\left(\frac{\Gamma_2^+}{\varepsilon^m}\right)}$$

with

$$(3.117) \qquad \Gamma_2^+ \doteq \beta_M \cdot \eta^{m+1}.$$

Taking the base-2 logarithm in (3.117) we then derive the estimate (1.13). $\qquad \square$

4. Lower compactness estimates. In this section we derive lower bounds on the ε-entropy in $\mathbf{L}^1$ of $S_T(\mathcal{C}_{[L,M]})$ for the class of initial data $\mathcal{C}_{[L,M]}$ in (1.6), when the flux function f satisfies the following assumption:

(A) $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable map such that

$$f'(0) = 0, \qquad f''(x) \neq 0 \qquad \text{for all} \ \ x \in \mathbb{R}\backslash\{0\},$$

which is fulfilled by fluxes satisfying (1.2) and either of the assumptions (C) or (NC) stated in the introduction. Notice that (A) in particular implies that f'' does not change sign on each of the two semilines $(-\infty, 0)$ and $(0, \infty)$.

Following the same approach introduced in [3], we shall derive a proof of (1.8), (1.14) relying on a controllability results for BV functions with one-side bounds on their spatial distributional derivative. Namely, given any $L, h, T > 0$, setting

$$(4.1) \qquad b_h^+ \ \doteq \ \frac{1}{2T \cdot \max\limits_{z \in [0,h]} |f''(z)|}, \qquad b_h^- \ \doteq \ \frac{1}{2T \cdot \max\limits_{z \in [-h,0]} |f''(z)|},$$

consider the sets
(4.2)

$$\mathcal{A}_{[L,h]}^+ \ \doteq \ \begin{cases} \left\{v \in \mathcal{C}_{[L/2,h]} \cap BV(\mathbb{R}) \mid v \geq 0, \ Dv \ \leq b_h^+\right\} & \text{if} \quad f''(h) > 0, \\[2mm] \left\{v \in \mathcal{C}_{[L/2,h]} \cap BV(\mathbb{R}) \mid v \geq 0, \ Dv \ \geq -b_h^+\right\} & \text{if} \quad f''(h) < 0, \end{cases}$$

$$\mathcal{A}_{[L,h]}^- \ \doteq \ \begin{cases} \left\{v \in \mathcal{C}_{[L/2,h]} \cap BV(\mathbb{R}) \mid v \leq 0, \ Dv \ \leq b_h^-\right\} & \text{if} \quad f''(-h) > 0, \\[2mm] \left\{v \in \mathcal{C}_{[L/2,h]} \cap BV(\mathbb{R}) \mid \ \leq 0, \ Dv \ \geq -b_h^-\right\} & \text{if} \quad f''(-h) < 0. \end{cases}$$

Here and throughout the following, the inequalities of the form $Du \geq b$ for a function $u \in BV(\mathbb{R})$, must be understood in the sense of measures, i.e., the Radon measure Du satisfies $Du(J) \geq b \cdot |J|$ for every Borel set $J \subset \mathbb{R}$, $|J|$ being the Lebesgue measure of J. We will show that any element of $\mathcal{A}_{[L,h]}^\pm$ can be obtained as the value at time T of a solution of (1.1) with initial data in the set $\mathcal{C}_{[L,h]}$ in (1.6). To this end, the following lemma provides a priori bounds on the spatial distributional derivative of entropy solutions of (1.1).

LEMMA 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a map satisfying the assumption* (A) *and, given $L, h, T > 0$, let $u_0 \in \mathcal{C}_{[L,h]} \cap BV(\mathbb{R})$ be any function satisfying either of the following conditions:*

$$(4.3) \quad u_0(x) \geq 0 \quad \text{for all } x \in \mathbb{R}, \qquad \text{sign}\big(f''(u_0(h))\big) \cdot Du_0 \geq -b_h^+ ,$$

$$(4.4) \quad u_0(x) \leq 0 \quad \text{for all } x \in \mathbb{R}, \qquad \text{sign}\big(f''(u_0(-h))\big) \cdot Du_0 \geq -b_h^- ,$$

where $b_h^\pm$ are the constants defined in (4.1). *Then, for every $t \in (0, T]$, the entropy solution $u(t, \cdot) \doteq S_t u_0$ is continuous on $\mathbb{R}$ and one has*

$$(4.5) \quad \begin{aligned} \text{sign}\big(f''(u_0(h))\big) \cdot Du(t, \cdot) &\geq -2b_h^+ \qquad \text{if} \qquad (4.3) \quad \text{holds}, \\ \text{sign}\big(f''(u_0(-h))\big) \cdot Du(t, \cdot) &\geq -2b_h^- \qquad \text{if} \qquad (4.4) \quad \text{holds}. \end{aligned}$$

Proof. We shall consider only the case where u_0 satisfies condition (4.3) and $f''(u_0(h)) \geq 0$. The cases where $f''(u_0(h)) \leq 0$ or where condition (4.4) holds can be treated in an entirely similar way.

1. Assume that (4.3) holds and that f' is increasing on $[0, +\infty)$. Observe first that, by Lemma 2.2, we have $u(t, \cdot) \in \mathcal{C}_{[l_{[L,h,t]}, h]} \cap BV(\mathbb{R})$, $u(t, x) \geq 0$, for any $x \in \mathbb{R}$, $t > 0$, and that (4.5) in particular implies

$$(4.6) \qquad u(t, x+) - u(t, x-) = Du(t, \cdot)(\{x\}) \geq 0 \qquad \text{for all } x \in \mathbb{R}.$$

On the other hand, by the Oleinik E-condition [22] we have

$$f'\big(u(t, x-)\big) \geq f'\big(u(t, x+)\big) \qquad \text{for all } x \in \mathbb{R}, \ t > 0,$$

which, in turn, by the monotonicity of f' on $[0, +\infty)$, implies

$$(4.7) \qquad\qquad u(t, x-) \geq u(t, x+) \qquad \text{for all } x \in \mathbb{R}, \ t > 0.$$

Then, (4.6)–(4.7) together yield

$$(4.8) \qquad\qquad u(t, x-) = u(t, x+) \qquad \text{for all } x \in \mathbb{R}, \ t \in (0, T],$$

proving the continuity of $S_t u_0$ at any $x \in \mathbb{R}$ and for any $t \in (0, T]$. Therefore, to complete the proof of the lemma we only have to show that, if the initial datum u_0 satisfies the assumption (4.3), then the corresponding entropy solution satisfies the inequality in (4.5) which, in this case, is equivalent to

$$(4.9) \qquad u(t, x_2+) - u(t, x_1-) \geq -\frac{x_2 - x_1}{T \cdot \max\limits_{z \in [0, h]} |f''(z)|} \qquad \text{for all } x_1 < x_2.$$

Clearly, it will be sufficient to prove that the inequality in (4.9) holds for any pair of continuity points $x_1 < x_2$ of $u(t, \cdot)$ such that

$$(4.10) \qquad\qquad u(t, x_2) - u(t, x_1) < 0.$$

2. Because of (4.3), and since we are assuming that $f''(u_0(h)) \geq 0$, the initial datum u_0 satisfies the inequality

$$(4.11) \qquad u_0(z_2+) - u_0(z_1-) \geq -\frac{z_2 - z_1}{2T \cdot \max\limits_{z \in [0, h]} |f''(z)|} \qquad \text{for all } z_1 < z_2.$$

Notice that, since $u(t, \cdot)$ takes values in the semiline $[0, +\infty)$ for all $t > 0$, we may always view $u(t, x)$ as the entropy solution of a conservation law with convex flux. In fact, if f satisfies the assumption (NC), $u(t, x)$ turns out to be the entropy solution of

$$(4.12) \qquad u_t + \widetilde{f}(u)_x = 0$$

with

$$\widetilde{f}(u) \doteq \begin{cases} f(u) & \text{if} \quad u \geq 0, \\ 2f(0) - f(u) & \text{if} \quad u \leq 0, \end{cases}$$

where $\widetilde{f}$ is a twice continuously differentiable convex map. Therefore, we may employ the theory of generalized characteristics of Dafermos [10, 12] and, for every given point x of continuity of $u(t, \cdot)$, we may trace a unique backward characteristic starting at (t, x) that is a genuine characteristic.

Then, fix $t \in (0, T]$ and consider two continuity points $x_1 < x_2$ of $u(t, \cdot)$ such that (4.10) holds. Let $\xi_i(\cdot)$ be the unique backward characteristics emanating from (t, x_i) for $i = 1, 2$. Since the solution $u(t, \cdot)$ is constant along genuine characteristics, we have

$$(4.13) \qquad x_i = y_i + t \cdot f'(u_0(y_i)) \qquad \text{with} \qquad y_i = \xi_i(0)$$

and

$$(4.14) \qquad u(t, x_i) = u_0(y_i) \qquad \text{for} \quad i = 1, 2.$$

Notice that (4.10), (4.13), (4.14), and the monotonicity of f' on $[0, +\infty)$, together imply

$$(4.15) \qquad \begin{aligned} y_2 - y_1 = x_2 - x_1 - t \cdot \big(f'(u(t, x_2)) - f'(u(t, x_1))\big) &> 0, \\ f'(u_0(y_2)) &< f'(u_0(y_1)). \end{aligned}$$

Thus, relying on (4.11), (4.13), (4.14), (4.15) we find

$$(4.16) \qquad u(t, x_2) - u(t, x_1) = u_0(y_2) - u_0(y_1) \geq -\frac{y_2 - y_1}{2T \cdot \max\limits_{z \in [0, h]} |f''(z)|}$$

and

$$\begin{aligned} x_2 - x_1 &= y_2 - y_1 + t \cdot \big(f'(u_0(y_2)) - f'(u_0(y_1))\big) \\ &\geq y_2 - y_1 + t \cdot \Big(\max\limits_{z \in [0, h]} |f''(z)|\Big) \cdot \big(u_0(y_2) - u_0(y_1)\big) \\ &\geq y_2 - y_1 - t \cdot \Big(\max\limits_{z \in [0, h]} |f''(z)|\Big) \cdot \frac{y_2 - y_1}{2T \cdot \max\limits_{z \in [0, h]} f''(z)} \\ (4.17) \qquad &\geq \frac{y_2 - y_1}{2}. \end{aligned}$$

Combining (4.16), (4.17), we obtain

$$u(t, x_2) - u(t, x_1) \geq -\frac{x_2 - x_1}{T \cdot \max\limits_{z \in [0, h]} |f''(z)|},$$

completing the proof of (4.9) for any pair of continuity points $x_1 < x_2$ of $u(t, \cdot)$ and thus concluding the proof of the lemma. $\qquad\square$

Remark 4.2. An alternative proof, as suggested by one of the referees, would be to regularize u_0 and to rely on classical characteristics to infer (4.9).

Relying on Lemma 4.1, we obtain the following controllability result.

LEMMA 4.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a map satisfying the assumption (A) and, given $L, h, T > 0$, let $\mathcal{C}_{[L,h]}$, $\mathcal{A}^{\pm}_{[L,h]}$ be the sets defined in (1.6), (4.2), respectively. Then, there holds*

$$(4.18) \qquad \mathcal{A}^{+}_{[L, h]} \bigcup \mathcal{A}^{-}_{[L, h]} \ \subseteq \ S_T(\mathcal{C}_{[L,h]})$$

for all $h > 0$ such that

$$(4.19) \qquad f'_h \ \doteq \ \max_{|z| \leq h} \ |f'(z)| \ \leq \ \frac{L}{2T} \,.$$

Proof. We will only show that, for h satisfying (4.19), assuming $f''(h) > 0$ one has

$$(4.20) \qquad \mathcal{A}^{+}_{[L, h]} \ \subseteq \ S_T(\mathcal{C}_{[L,h]}).$$

The proof of (4.20) when $f''(h) < 0$ and the proof of $\mathcal{A}^{-}_{[L, h]} \subseteq S_T(\mathcal{C}_{[L,h]})$ are entirely similar. Then, given an arbitrary function

$$(4.21) \qquad v \ \in \ \mathcal{A}^{+}_{[L, h]},$$

we will determine an element $u_0 \in \mathcal{C}_{[L,h]}$ such that

$$(4.22) \qquad S_T u_0 \ = \ v \,,$$

thus proving (4.20). The function u_0 will be obtained by an entropy admissible solution of (1.1) backward constructed in time, which starts at time T with the value v. Namely, set

$$(4.23) \qquad w_0(x) \ \doteq \ v(-x) \qquad \text{for all} \ \ x \in \mathbb{R} \,,$$

and consider the entropy weak solution $w(t, x) \doteq S_t w_0$ of (1.1) with initial datum w_0. Notice that, letting $l_{[L/2,h,t]}$ be the constant defined in (2.3), because of (4.19) there holds

$$(4.24) \qquad l_{[L/2,h,t]} \ = \ L/2 + t \cdot f'_h \ \leq \ L \qquad \text{for all} \ \ t \in [0, T] \,.$$

Moreover, observe that, by (4.2), (4.21), (4.23), and since we are assuming that $f''(h) > 0$, we have

$$(4.25) \qquad w_0 \ \in \ \mathcal{C}_{[L/2,h]} \cap BV(\mathbb{R}) \,,$$

$$(4.26) \qquad v(x) \ \geq \ 0 \quad \text{for all} \ x \in \mathbb{R}, \qquad Dw_0 \ = - \ Dv \ \geq \ - b^{+}_h \,.$$

Therefore, by virtue of Lemma 2.2 we find

$$(4.27) \quad \big\|w(t, \cdot)\big\|_{\mathbf{L}^\infty(\mathbb{R})} \ \leq \ h \,, \qquad Supp(w(t, \cdot)) \subseteq \big[-L, L\big] \qquad \text{for all} \ \ t \in [0, T] \,,$$

and invoking Lemma 4.1 we deduce that $w(t, \cdot)$ is a continuous map on $\mathbb{R}$ for all $t \in (0, T]$. Next, observe that the map u defined by

$$(4.28) \qquad u(t, x) \doteq w(T - t, -x), \qquad (t, x) \in [0, T] \times \mathbb{R},$$

provides a weak distributional solution of (1.1) which is entropy admissible since it is continuous with respect to the space variable x at any time $t < T$ (see [11, 15]). On the other hand, by (4.23), (4.27), (4.28), we have

$$(4.29) \qquad u_0 \doteq u(0, \cdot) \in \mathcal{C}_{[L,h]}, \qquad S_T u_0 = u(T, \cdot) = v,$$

which completes the proof of the lemma. □

The next lemma shows that, for fluxes with polynomial degeneracy at zero, the constants $b_h^{\pm}$ in (4.1) are of order $\approx \frac{1}{T \cdot s^{m-1}}$.

LEMMA 4.4. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying condition (1.2) and either of (1.4) or (1.11) conditions. Then, there exist constants $\bar{\alpha}, \bar{\sigma} > 0$ such that*

$$(4.30) \quad \max \left\{ \max_{z \in [0,s]} |f''(z)|, \max_{z \in [-s,0]} |f''(z)| \right\} \leq \bar{\alpha} \cdot s^{m-1} \qquad \textit{for all } s \in [0, \bar{\sigma}].$$

Proof. By writing the Taylor expansion of f'' at zero we find

$$(4.31) \qquad f''(u) = u^{m-1} \cdot \left(\frac{f^{(m+1)}(0)}{(m-1)!} + o(1) \right),$$

where $o(1)$ denotes a function converging to zero when $u \to 0$. Since we are assuming that $f^{(m+1)}(0) \neq 0$, the estimate (4.30) immediately follows from (4.31) taking $\bar{\sigma} > 0$ sufficiently small. □

Proof of lower bounds (1.8), (1.14) of Theorems 1.2–1.5. Given any constant $L, h > 0$, recalling definitions (2.12), (4.2), we have

$$\mathcal{A}_{[L,h]}^+ \supseteq \mathcal{B}_{\left[\frac{L}{2}, \frac{h}{2}, \leq b_h^+ \right]} + \frac{h}{2} \cdot \chi_{\left[-\frac{L}{2}, \frac{L}{2} \right]} \qquad \text{if} \qquad f''(h) > 0,$$

$$\mathcal{A}_{[L,h]}^+ \supseteq \mathcal{B}_{\left[\frac{L}{2}, \frac{h}{2}, \geq -b_h^+ \right]} + \frac{h}{2} \cdot \chi_{\left[-\frac{L}{2}, \frac{L}{2} \right]} \qquad \text{if} \qquad f''(h) < 0,$$

$$(4.32)$$

$$\mathcal{A}_{[L,h]}^- \supseteq \mathcal{B}_{\left[\frac{L}{2}, \frac{h}{2}, \leq b_h^+ \right]} - \frac{h}{2} \cdot \chi_{\left[-\frac{L}{2}, \frac{L}{2} \right]} \qquad \text{if} \qquad f''(-h) > 0,$$

$$\mathcal{A}_{[L,h]}^- \supseteq \mathcal{B}_{\left[\frac{L}{2}, \frac{h}{2}, \geq -b_h^+ \right]} - \frac{h}{2} \cdot \chi_{\left[-\frac{L}{2}, \frac{L}{2} \right]} \qquad \text{if} \qquad f''(-h) < 0.$$

To fix the ideas, assume now that

$$f''(h) > 0, \qquad f''(-h) < 0.$$

The cases where $f''(h) > 0$, $f''(-h) > 0$; $f''(h) < 0$, $f''(-h) > 0$; or $f''(h) < 0$, $f''(-h) < 0$, can be treated in an entirely similar way. Then, by virtue of Lemma 4.3

and relying on (4.32), we find
(4.33)

$$\mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\Big) \;\geq\; \max\left\{\mathcal{H}_\varepsilon\Big(\mathcal{A}^+_{[L,\,h]} \mid \mathbf{L}^1(\mathbb{R})\Big),\, \mathcal{H}_\varepsilon\Big(\mathcal{A}^-_{[L,\,h]} \mid \mathbf{L}^1(\mathbb{R})\Big)\right\}$$

$$\geq\; \max\left\{\mathcal{H}_\varepsilon\Big(\mathcal{B}_{\left[\frac{L}{2},\,\frac{h}{2},\,\leq b_h^+\right]} + \frac{h}{2}\cdot\chi_{\left[-\frac{L}{2},\,\frac{L}{2}\right]} \mid \mathbf{L}^1(\mathbb{R})\Big),\right.$$

$$\left.\mathcal{H}_\varepsilon\Big(\mathcal{B}_{\left[\frac{L}{2},\,\frac{h}{2},\,\geq -b_h^+\right]} - \frac{h}{2}\cdot\chi_{\left[-\frac{L}{2},\,\frac{L}{2}\right]} \mid \mathbf{L}^1(\mathbb{R})\Big)\right\}$$

$$=\; \max\left\{\mathcal{H}_\varepsilon\Big(\mathcal{B}_{\left[\frac{L}{2},\,\frac{h}{2},\,\leq b_h^+\right]} \mid \mathbf{L}^1(\mathbb{R})\Big),\, \mathcal{H}_\varepsilon\Big(\mathcal{B}_{\left[\frac{L}{2},\,\frac{h}{2},\,\geq -b_h^+\right]} \mid \mathbf{L}^1(\mathbb{R})\Big)\right\}$$

for all $h > 0$ satisfying (4.19). Hence, invoking Lemma 2.6 and because of (4.1), we derive from (4.33) the estimate
(4.34)

$$\mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\Big) \geq \frac{L^2}{108\,\ln 2 \cdot T} \cdot \frac{1}{\min\left\{\max\limits_{z\in[0,h]}|f''(z)|,\,\max\limits_{z\in[-h,0]}|f''(z)|\right\}} \cdot \frac{1}{\varepsilon}$$

for any $h \geq \frac{6\varepsilon}{L}$ such that (4.19) holds. Choosing $h = \frac{6\varepsilon}{L}$, we recover from (4.34) the estimate (1.8) for all $\varepsilon > 0$ such that

$$(4.35) \qquad\qquad \max_{|z|\leq\frac{6\varepsilon}{L}}\,|f'(z)| \;\leq\; \frac{L}{2T}\,.$$

On the other hand, in the case where f is a nonconvex flux satisfying conditions (1.2), (1.4), applying Lemma 4.4 and taking $h = \frac{6\varepsilon}{L}$, we derive from (4.34) the estimate

$$(4.36) \qquad \mathcal{H}_\varepsilon\Big(S_T(\mathcal{C}_{[L,M]}) \mid \mathbf{L}^1(\mathbb{R})\Big) \;\geq\; \frac{L^{m+1}}{108\,\ln 2 \cdot 6^{m-1}\cdot\overline{\alpha}\cdot T} \cdot \frac{1}{\varepsilon^m}$$

which proves (1.14). □

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