Matrix Based Rational Interpolation for New Coupling Scheme Between MHD and Eddy Current Numerical Models

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In this paper, a matrix-based Padé rational interpolation used to couple a 3D eddy current code with linear Magneto-Hydrodynamic (MHD) solver. This approach is a general methodology viable for multi-physic problems involving the coupling of an electromagnetic model with another non-algebraic physical model, which has to be solved in a certain region of space. The matrix-based Padé interpolation has been applied to a typical plasma physics problem of modelling the plasma response to external perturbation, providing an accurate and reliable mathematical model viable for the feedback stabilization of Resistive Wall Modes (RWMs) in fusion plasmas. The choice of matrix-based Padé interpolation gives a reliable approach for the considered problem, and can be viewed a convenient formalism for the coupling strategy in this class of multi-physic problems.

***Index Terms*—3D quasi-static problems, Magneto-Hydrodynamic, Matrix-based rational interpolation, Matrix function interpolation, Resistive Wall Modes.**

I. INTRODUCTION

THE macroscopic dynamics of magnetically confined fusion plasma can be described through MHD equations [1]. These equations predict, in some cases, the existing of ideal MHD instabilities which cause the deformation the outer part of the plasma surface and leads to a sudden and abrupt loss of the confinement. Such instabilities are the main limitation to the achievable performance. However, the eddy currents induced by these instabilities in the conducting structures that surround the plasma tend to counteract the instability itself, slowing down the growth time from typical timescales of microseconds to millisecond. Such modes are hence called Resistive Wall Modes (RWMs).

Among the many computational tools developed for the study of RWMs, both in order to give additional insight on the RWM physics, and for activities related to its feedback stabilization, only the CarMa code [6] takes rigorously into account the real geometry of conducting structures, including the thickness of the passive conductors. This code is the result of the coupling between MARS [2], a linearized resistive MHD equilibrium code, and CARIDDI [4], a 3D integral eddy current code.

CarMa has been used, with applications to RWM modelling and control system optimization. However, the code suffers of some limitations related to the coupling strategy between CARIDDI and MARS, because it neglects the plasma mass (so-called *massless approximation*). Under this assumption, the arising mathematical model is a linear time-invariant system, which can be written in a state space form for control oriented purposes. However, when the plasma pressure exceeds a certain threshold, the growth rate of the RWM is too high, and the assumption of neglecting the plasma inertia is no longer valid. This makes CarMa unreliable for such operating regimes.

In this paper, a Padé matrix-based rational approximation is used to interpolate the full set of linearized MHD equations in Laplace domain, solved by MARS, in order to obtain an approximated model of RWM behaviour suitable to the coupling with the eddy currents equation solved by CARIDDI. The exploitation of an approximated model of the mode behaviour answer to the need to develop an approach viable for control oriented purposes. This would not be feasible by using the full set of linearized MHD equations. We stress the fact that, although presented for a plasma physics application, this is a general methodology, and can be used for general multi-physic problems which involves the coupling of an electromagnetic model with another non-algebraic physical model. The idea of interpolating nonlinear dynamic equations for feedback oriented purposes has been widely adopted, both in plasma physics [2] and in other fields [3]. This work shows a preliminary assessment of the reliability of Padé interpolation used for this purpose, in order to be used for the development, in a future work, of a model based feedback controller of RWMs.

II. MODEL DESCRIPTION

A. Standard CarMa coupling strategy

The CarMa code is based on the control surface concept to self-consistently couple MARS, a linear MHD solver, and CARIDDI, a 3D code for the computation of eddy currents in the metallic structures surrounding the plasma [6]. The coupling surface, chosen to be axisymmetric and such that it does not intersect neither the plasma boundary nor the conducting structures, is needed to decouple the computational domains related to the MHD problem and the eddy current problem. This approach is useful when different formulations have to be used in each subdomain, and allows the most convenient approach to be used in each case. This approach has been recently generalized for arbitrary shaped coupling surfaces [5].

Introducing the coupling surface, the electromagnetic effects of the plasma, seen from external environment, are described as produced by an equivalent current density $\boldsymbol{j}_{eq}$ flowing on the coupling surface. The most important assumption for the coupling strategy is that the plasma is assumed to be static. The governing equations inside the plasma volume are then single fluid, linearized MHD linear equations solved by the MARS code neglecting plasma mass.

The analysis of the RWM feedback stabilization in tokamaks requires, for a given magnetic field perturbation $\boldsymbol{b}_N$, the separation of the *plasma response* in terms of $\boldsymbol{j}_{eq}$ from the magnetic field $\boldsymbol{b}^{ex}$ produced by the eddy current induced by the perturbation. The relations needed to this purpose are given in [6]:

$$\begin{aligned} \boldsymbol{j}_{eq} &= \boldsymbol{F}\boldsymbol{b}_N \\ \boldsymbol{b}_N &= \boldsymbol{W}\boldsymbol{b}_N^{ex} \end{aligned} \Rightarrow \boldsymbol{j}_{eq} = \boldsymbol{F}\boldsymbol{W}\boldsymbol{b}_N^{ex} = \boldsymbol{P}\boldsymbol{b}_N^{ex} \tag{1}$$

where $\boldsymbol{F}, \boldsymbol{W}$ are static matricial relations needed to separate respectively plasma and external contribution from the total magnetic field perturbation. Such matrices are computed numerically by MARS. In particular, since MARS works with the Fourier harmonics in the poloidal angle for the decomposition of the perturbed quantities on the poloidal plane, these relations has dimension $M \times M$, with M the total number of considered harmonics.

The plasma response matrices are then taken into account in the eddy current equation solved by CARIDDI [4], [6]: the computational problem is treated through an integral formulation which assumes the current density $\boldsymbol{j}$ in conducting structures as primary unknown. With this formulation, only the conducting domain must be discretized, and the regularity conditions at infinity are automatically taken into account. Assuming vanishing initial conditions and introducing Laplace complex variable s, the resulting eddy current equation is:

$$s\boldsymbol{L}\boldsymbol{i} + \boldsymbol{R}\boldsymbol{i} + s\boldsymbol{M}\boldsymbol{j}_{eq} = \boldsymbol{D}\boldsymbol{v} \tag{2}$$

where $\boldsymbol{R}$ and $\boldsymbol{L}$ are resistance and self inductance matrix of 3D conducting structures, $\boldsymbol{M}$ is the mutual inductance between the 3D conducting structures and the equivalent current density $\boldsymbol{j}_{eq}$ on the coupling surface, $\boldsymbol{i}$ is the vector of the induced current in the conducting structures, and $\boldsymbol{v}$ is the vector of voltages applied to externally fed electrodes.

The CarMa eddy current equations is obtained combining, with some manipulations, (1) and (2):

$$s\boldsymbol{L}^{*}\boldsymbol{i} + \boldsymbol{R}\boldsymbol{i} = \boldsymbol{D}\boldsymbol{v} \tag{3}$$

where $\boldsymbol{L}^{*}$ is the perturbed inductance operator that takes into account the RWM contribution into the eddy current problem. This formulation can be both used for stability computations to understand if the RWM is stable/unstable looking for the eigenvalue of the dynamical matrix $\boldsymbol{A} = -(\boldsymbol{L}^{*})^{-1}\boldsymbol{R}$, and for control oriented purposes, because it can be straightforwardly written in a state space form. It is worth noting that CarMa is able, in principle, to deal with conductors of any shape and geometry.

In order to make such methods convenient for large-scale problems, "fast techniques" have been implemented in the CARIDDI code [7]. Such techniques may be extended in principle also to the new coupling scheme here proposed.

B. New coupling strategy based on a frequency dependent plasma response

In the previous section, a static coupling strategy between CARIDDI and MARS has been presented. However, when growth rate of the RWM is high enough to be comparable with the Alfvén time [1], plasma mass plays a crucial role and cannot be neglected. If the mass-less approximation is removed, the static coupling procedure is no longer available, and all the quantities involved in plasma response matrix computation should depend on the frequency of the perturbation field. This means that, working in the Laplace domain, a certain dependance of the matrix $\boldsymbol{P}$ on the complex Laplace variable s is expected, giving rise to a matrix function $\boldsymbol{P}(s)$. This behaviour is fully described by the set of linearized MHD equations in the Laplace domain, solved by MARS, but a direct coupling with CARIDDI to obtain a linear system of equations as eq. (3) is no longer possible. For this reason, an approximated model of the full MHD equations is desirable. This approach is suitable not only for this application, but for all the cases when the coupling of an electromagnetic model with another non-algebraic physical model is required.

The dependence of the plasma response with respect to the perturbation frequency s can be treated as a dynamical linear system: eq. (1), together with eddy current equation (2), is written as:

$$\begin{cases} s\boldsymbol{L}\boldsymbol{i} + \boldsymbol{R}\boldsymbol{i} + s\boldsymbol{M}\boldsymbol{i}_{eq} = \boldsymbol{D}\boldsymbol{v} & (4) \\ \boldsymbol{j}_{eq} = \boldsymbol{P}(s)\boldsymbol{Q}\boldsymbol{i} & (5) \end{cases}$$

where the matrix-based function $\boldsymbol{P}(s)$ is the frequency dependent plasma response matrix and $\boldsymbol{Q}$ is a matrix mapping each discrete current $\boldsymbol{i}$ into the magnetic field component $\boldsymbol{b}_N^{ex}$ normal to the coupling surface.

The behaviour of the matrix function $\boldsymbol{P}(s)$ is modelled by means of Padé matrix based rational approximated of kth order:

$$\boldsymbol{j}_{eq} = \boldsymbol{P}(s)\boldsymbol{Q}\boldsymbol{i} = \frac{\boldsymbol{A}_k s^k + \ \dots\ + \boldsymbol{A}_0}{\boldsymbol{B}_k s^k + \ \dots\ + \boldsymbol{E}}\boldsymbol{Q}\boldsymbol{i} \tag{6}$$

where $\boldsymbol{E}$ is the identity matrix and $\boldsymbol{A}_i, \boldsymbol{B}_i$ are matricial coefficients of dimension $M \times M$.

For a given interpolation order k, the coefficients of (6) can be computed starting from the knowledge of $2k+1$ pairs $(s_i, \boldsymbol{P}(s_i))$, the *basis points*, along the complex plane. Among these points, two particular choices can be:

- $s = 0$, to match the static response
$$\boldsymbol{A}_0 = \boldsymbol{P}_0 \tag{7}$$
with this particular choice, the coefficient $\boldsymbol{A}_0$ is the same matrix for the original CarMa code;
- $|s| \to +\infty$, to match the response at infinity:
$$\lim_{|s| \to +\infty} \boldsymbol{P}(s) = \boldsymbol{B}_k^{-1}\boldsymbol{A}_k = \boldsymbol{P}_\infty \tag{8}$$

Performing $2k-1$ times the response computations with MARS to obtain $\boldsymbol{P}(s_i)$ for different value of s we obtain the required information to compute coefficients $\boldsymbol{A}_i, \boldsymbol{B}_i$.

Using (6) in eq. (5) leads to:

$$\begin{cases} s\boldsymbol{L}\boldsymbol{i} + \boldsymbol{R}\boldsymbol{i} + s\boldsymbol{M}\boldsymbol{j}_{eq} = \boldsymbol{D}\boldsymbol{v} \\ \boldsymbol{j}_{eq} = (\sum_i s^i \boldsymbol{B}_i)^{-1}(\sum_i s^i \boldsymbol{A}_i)\boldsymbol{Q}\boldsymbol{i} \end{cases} \tag{9}$$

if we define $\boldsymbol{x} = [\boldsymbol{i}\ \boldsymbol{j}_{eq}]^T$, $\boldsymbol{u} = [\boldsymbol{D}\boldsymbol{v}\ \boldsymbol{0}]^T$, system (9) can be written as:

$$s^k \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{A}_k\boldsymbol{Q} & \boldsymbol{B}_k \end{bmatrix}}_{\boldsymbol{L}_{ak}} \boldsymbol{x} + ... + s^i \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{A}_i\boldsymbol{Q} & \boldsymbol{B}_i \end{bmatrix}}_{\boldsymbol{L}_{ai}} \boldsymbol{x} + ...$$
$$+ s \underbrace{\begin{bmatrix} \boldsymbol{L} & \boldsymbol{M} \\ -\boldsymbol{A}_1\boldsymbol{Q} & \boldsymbol{B}_1 \end{bmatrix}}_{\boldsymbol{L}_{a1}} \boldsymbol{x} + \underbrace{\begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ -\boldsymbol{P}_0\boldsymbol{Q} & \boldsymbol{E} \end{bmatrix}}_{\boldsymbol{R}_a} \boldsymbol{x} = \boldsymbol{u} \quad (10)$$

with the new definition of the block matrices $\boldsymbol{R}_a, \boldsymbol{L}_{ai}$. With a compact formalism we obtain:

$$\left(\sum_{i=1}^{k} s^i \boldsymbol{L}_{ai}\right)\boldsymbol{x} + \boldsymbol{R}_a\boldsymbol{x} = \boldsymbol{u} \quad (11)$$

It is worth noting that matrices $\boldsymbol{L}_{ai}$, $i = 2, ..., k$, are rank deficient. Therefore the system of equations (11) is a kth order system of Differential Algebraic Equations (DAE). This is due to the fact that the eddy current equation has clearly a first-order dynamic, whereas the plasma response equation has an arbitrary kth order, which depends on the chosen interpolation order. The way how the problem has been addressed will be discussed further on.

The kth order system (11) can be written as a *first order* system of differential equations through the change of variables: $\boldsymbol{x} = \boldsymbol{y}_1, \boldsymbol{y}_1 = \boldsymbol{y}_2, ..., \boldsymbol{y}_{k-1} = \boldsymbol{y}_k$ to obtain:

$$s\boldsymbol{L}^*\boldsymbol{y} + \boldsymbol{R}^*\boldsymbol{y} = \boldsymbol{u}^* \quad (12)$$

where

$$\boldsymbol{L}^* = \begin{bmatrix} \boldsymbol{E} & & & \\ & \boldsymbol{E} & & \\ & & \ddots & \\ & & & \boldsymbol{L}_{ak} \end{bmatrix} \quad \boldsymbol{R}^* = \begin{bmatrix} \boldsymbol{0} & -\boldsymbol{E} & & \\ & \boldsymbol{0} & \ddots & \\ & & \ddots & -\boldsymbol{E} \\ \boldsymbol{R}_a & \boldsymbol{L}_1 & \cdots & \boldsymbol{L}_{ak-1} \end{bmatrix}$$

which is formally equivalent (3), but with a higher number of degrees of freedom to take into account the plasma dynamics. Before exploiting the equation (12) to work out a state space model for control oriented purposes, it is useful if the model is able to accurately describe RWM behaviour taking into account plasma inertia.

It is worth noting that, if the interpolation order is $k > 1$, which is basically always the case, the matrix $\boldsymbol{L}^*$ is rank deficient for the block elements $\boldsymbol{L}_{ai}$, $i \neq 1$. This mean that $\boldsymbol{L}^*$ is not invertible, and the eigenvalues of the RL system $-(\boldsymbol{L}^*)^{-1}\boldsymbol{R}^*$, i.e. the growth/damping rates of the RWM, can not be computed. On the other hand, it can be seen in eq. (10) that the matrix $\boldsymbol{R}^*$ is always invertible, because it is the composition of the identity matrix with $\boldsymbol{R}_a$, combination of static quantities and therefore of full rank. A possible solution is to compute the eigenvalues of the system $-(\boldsymbol{R}^*)^{-1}\boldsymbol{L}^*$, to obtain the growth/damping *times* of the instability rather than the growth/damping rates: the resulting growth rates would be the inverse of the growth times.

The regular structure of block matrices $\boldsymbol{R}^*, \boldsymbol{L}^*$ allows the analytic computation of $-(\boldsymbol{R}^*)^{-1}\boldsymbol{L}^*$:

$$-(\boldsymbol{R}^*)^{-1}\boldsymbol{L}^* = -\begin{bmatrix} \boldsymbol{R}_a^{-1}\boldsymbol{L}_{a1} & \cdots & \boldsymbol{R}_a^{-1}\boldsymbol{L}_{n-1} & \boldsymbol{R}_a^{-1}\boldsymbol{L}_k \\ -\boldsymbol{E} & & & \\ & \ddots & & \\ & & -\boldsymbol{E} & \end{bmatrix} \quad (13)$$

III. Model validation

Although this method is meant to be used for feedback stabilization of RWMs, in this work a preliminary assessment of the reliability of Padé interpolation is shown. For this reason, the new coupling procedure has been used for the stability analysis of a plasma equilibrium configuration with a circular poloidal cross-section. The poloidal spectrum used for MHD computations has $M = 15$ Fourier harmonics. This circular equilibrium present a $n = 1$ unstable RWM, where n is the toroidal mode number. To benchmark the new CarMa code, MARS results are used as reference, requiring an axisymmetric wall, although CarMa is able to deal with conductors of any shape and geometry.

The effectiveness of rational interpolation is now discussed. The function $\boldsymbol{P}(s)$ is a matricial function of s, meaning that every entry $p_{(m,m)}(s)$ is a scalar function of s. On the other hand, for a chosen $s_i = \gamma_i + i\omega_i$, the quantity $\boldsymbol{P}(s_i)$ is a $M \times M$ matrix. The reference $\boldsymbol{P}(s)$ has been computed by MARS for the spectrum $|s| \in (-\infty, +\infty)$. To check the quality of the Padé interpolation, in principle all the M^2 scalar functions $p_{(m,m)}(s)$ should be checked: here only the two major entries of $\boldsymbol{P}(s)$ (i.e. the diagonal elements related to the harmonics $m = [0, 3]$, real and imaginary part) are shown in Figs. 1. In particular, among the several interpolation degrees considered, the $k = 3$ gives the most precise results. It can be seen from Fig. 1 that the Padé interpolation with $k = 3$ is very accurate for the all interval $|s|$, thus the interpolated matrix function is able to reproduce the mode behaviour given by the full set of linearized MHD equations.

The next step in the validation is to prove that the new version of the CarMa code, based on the Padé approximation of the plasma response, is able to deal with the inertia of the plasma. To test this capability, a scan of the resistivity η of the conducting structures is proposed: for the typical value of the wall resistivity, i.e. $\eta/\eta_{ref} = 1$, the wall has a strong

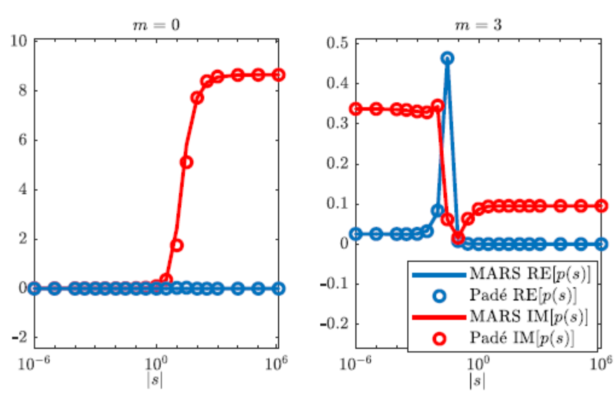

Fig. 1: Reference and interpolation for 2 major entries of $\boldsymbol{P}(s)$.

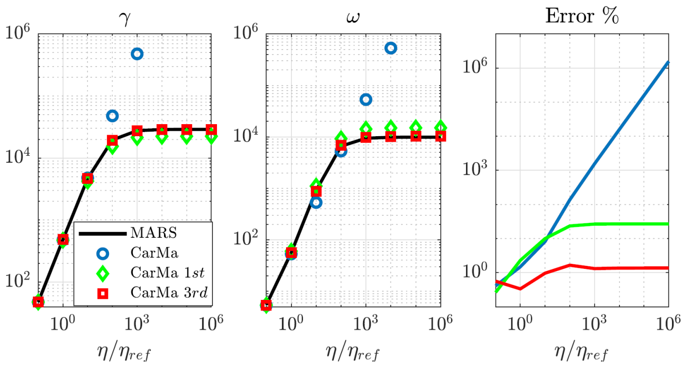

Fig. 2: Unstable eigenvalue (real and imag. part) versus normalized wall resistivity. Logarithmic scale.

stabilizing effect, making the mode to evolve as a RWM. As the resistivity increases, the stabilization effect decreases: in the limit of infinite resistivity, the wall is no longer stabilizing the mode, which evolves as an ideal kink. In this regime the plasma inertia plays a crucial role. Figure 2 compares, in logarithmic scale, the mode unstable eigenvalue (real and imaginary part) as function of normalized wall resistivity η/η_{ref} computed by MARS (reference, black), as well as the percentage error: CarMa with $3rd$ order interpolation, is compared to static CarMa (blue). In addition, also a version with $1st$ order interpolation is shown (green). As can be seen, the new version of CarMa with order $k = 3$ gives a very high accuracy.

An overall view of the axisymmetric mesh, together with the eigenvector corresponding to the unstable mode, can be seen in Fig. 3 for the case with $\eta = \eta_{ref} = 1$. A quantitative comparison of the eigenvector in the poloidal plane is shown in Fig 4.

The sparsity pattern of the matrix $-(\boldsymbol{R}^{*})^{-1}\boldsymbol{L}^{*}$ is reported in Fig. 5. Such matrix is sparse, except for a square dense block on the upper left corner related to the $\boldsymbol{R}_a^{-1}\boldsymbol{L}_1$ in (13), which is exactly the standard CarMa matrix. For this reason, the number of non-zeros is almost due to the original CarMa block, even if the number of unknowns is higher to take into account plasma dynamics. Thus the computational effort to find the eigenvalues remains unchanged if sparse linear algebra subroutines are used.

IV. Conclusion

In this paper, a new matrix-based Padé rational interpolation used for a self consistent coupling scheme between a 3D eddy current code and a linear Magneto-Hydrodynamic (MHD) solver. The goal of this approach is to deal with general multi-physic problems which involves the coupling of an electromagnetic model with another non-algebraic physical model which has to be solved in a given region of space. The matrix-based Padé interpolation has been applied to plasma physics problem of modelling the plasma response to external perturbation, in order to develop a new version of the CarMa code which does not rely on the assumption of neglecting the plasma inertia, as the original version of CarMa does. The arising new version of CarMa, based on this coupling strategy, has shown the capability of model the RWM behaviour with very high accuracy, including in the computation also the three dimensional geometry of the conducting structures. The matrix-based Padé interpolation has shown to be an effective method when a simplified model of a full order model is required, especially for control oriented purposes. Further development would involve the use of state space model related to Eq. (12) for the development of a feedback controller of the RWMs, to be applied to several cases of interest, such as the realistic devices ITER and JT-60SA.

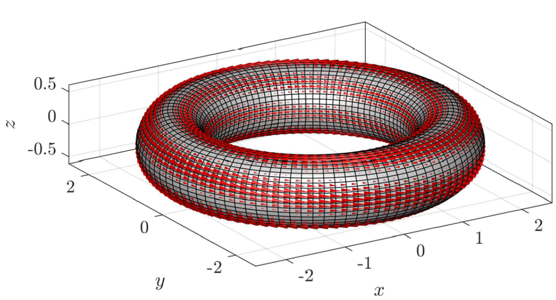

Fig. 3: 3D view of eigenvector corresponding to the unstable mode.

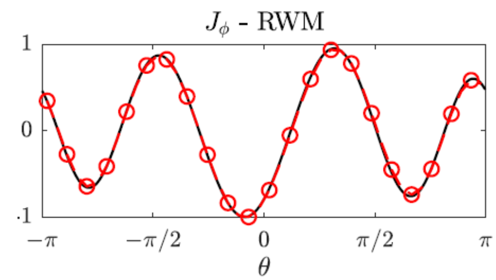

Fig. 4: Unstable eigenvector: MARS (black), CarMa (red, circles).

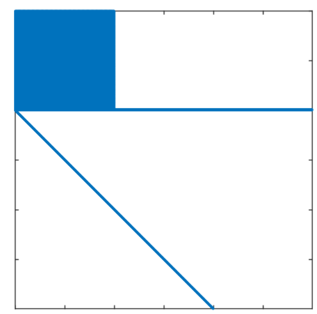

Fig. 5: Sparsity pattern of (13) for $3rd$ order interpolation.

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