Periodic coherent states decomposition and quantum dynamics on the flat torus

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Abstract. We provide a result on the coherent states decomposition for functions in $L^2(\mathbb{T}^n)$ where $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$. We study such a decomposition with respect to the quantum dynamics related to semiclassical elliptic Pseudodifferential operators, and we prove a related invariance result.

1. Introduction

Let us introduce the usual class of semiclassical coherent states on $\mathbb{R}^n$

$$\phi_{(x,\xi)}(y) := \alpha_h\, e^{\frac{i}{h}(x-y)\cdot\xi} e^{-\frac{|x-y|^2}{2h}}, \quad (x,\xi) \in \mathbb{R}^{2n}, \quad y \in \mathbb{R}^n, \quad 0 < h \le 1 \tag{1.1}$$

with the $L^2(\mathbb{R}^n)$ - normalization constant $\alpha_h := 2^{-\frac{n}{2}}(\pi h)^{-3n/4}$, and where h is a 'semiclassical parameter'. For any $\psi \in \mathcal{S}'(\mathbb{R}^n)$ the coherent state decomposition reads, in the distributional sense, as

$$\psi(x_0) = \int_{\mathbb{R}^{2n}} \phi^\star_{(x,\xi)}(x_0)\Big(\int_{\mathbb{R}^n} \phi_{(x,\xi)}(y)\psi(y)dy\Big)dxd\xi \tag{1.2}$$

as shown for example in Prop. 3.1.6 in [11].

We now observe that for the flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$ the well known inclusion $L^2(\mathbb{T}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ implies that distributional equality (1.2) make sense also for functions in $L^2(\mathbb{T}^n)$.

The first aim of our paper is to prove the decomposition of any $\varphi \in L^2(\mathbb{T}^n)$ with respect to the family of periodic coherent states Φ given by the periodization of (1.1). In view of this target, we recall that the periodization operator $\Pi(\phi)(y) := \sum_{k\in\mathbb{Z}^n} \phi(y - 2\pi k)$ maps $\mathcal{S}(\mathbb{R}^n)$ into $C^\infty(\mathbb{T}^n)$, as shown for example Thm. 6.2 in [15]. Thus, we can define for all $0 < h \le 1$

$$\Phi_{(x,\xi)}(y) := \sum_{k\in\mathbb{Z}^n} \phi_{(x,\xi)}(y - 2\pi k) \qquad (x,\xi) \in \mathbb{T}^n \times h\,\mathbb{Z}^n, \qquad y \in \mathbb{T}^n. \tag{1.3}$$

Notice that the family of coherent states in (1.3) is well posed also for $\xi \in \mathbb{R}^n$ and the related phase space is $\mathbb{T}^n \times \mathbb{R}^n$. However, our target is to show that the decomposition of periodic functions can be done with respect to the minimal set of coherent states in (1.3) for $\xi \in h \mathbb{Z}^n \subset \mathbb{R}^n$. Furthermore, we notice that the phase space $\mathbb{T}^n \times h \mathbb{Z}^n$ is necessary in order to deal with a well defined setting of toroidal Weyl operators acting on $L^2(\mathbb{T}^n)$ and more in general with semiclassical toroidal Pseudodifferential operators (see Sect. 2).

The first result of the paper is the following

Theorem 1.1. *Let* $\varphi_h \in C^\infty(\mathbb{T}^n)$ *be such that* $\|\Delta_x \varphi_h\|_{L^2} \le c\, h^{-M}$ *for some* $c > 0, M \in \mathbb{N}$, $\|\varphi_h\|_{L^2} = 1$ *with* $0 < h \le 1$, $h^{-1} \in \mathbb{N}$ *and let* $\Phi_{(x,\xi)}$ *be as in* (1.3). *Then,*

$$\varphi_h = \sum_{\xi \in h\,\mathbb{Z}^n} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, \varphi_h \rangle_{L^2}\, \Phi^\star_{(x,\xi)}\, dx + \mathcal{O}_{L^2}(h^\infty). \qquad (1.4)$$

Moreover, there exists $f(h) > 0$ *depending on* φ_h *such that*

$$\varphi_h = \sum_{\xi \in h\,\mathbb{Z}^n,\ |\xi| \le f(h)} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, \varphi_h \rangle_{L^2}\, \Phi^\star_{(x,\xi)}\, dx + \mathcal{O}_{L^2}(h^\infty). \qquad (1.5)$$

The following inclusion involving the set of frequencies $\xi \in h\,\mathbb{Z}^n \subset \mathbb{R}^n$ allows to consider decomposition (1.4) minimal with respect to (1.2). The above result shows also that the sum over the frequencies can be taken in the bounded region $|\xi| \le f(h)$, i.e. we can consider a finite sum by taking into account an $\mathcal{O}(h^\infty)$ remainder in $L^2(\mathbb{T}^n)$.

An analogous result of (1.4) in the two dimensional setting is shown in prop. 60 of [7] by the use of a different periodization operator. Same construction of coherent states as in [7] for $\mathbb{T}^2$ is used in [4], [8] for the study of quantum cat maps and equipartition of the eigenfunctions of quantized ergodic maps. In the paper [10], covariant integral quantization using coherent states for semi-direct product groups is implemented for the motion of a particle on the circle and in particular the resolution of the identity formula is proved. Another class of coherent states on the torus are defined also in [9], with a related resolution of the identity, in the understanding of the Quantum Hall effect. We also recall [3] where coherent states and Bargmann Transform are studied on $L^2(\mathbb{S}^n)$. The literature on coherent states are quite rich, and thus we address the reader to [1].

We now devote our attention to the periodic coherent states decomposition for eigenfunctions of elliptic semiclassical toroidal Pseudodifferential operators (see Section 2). We will see that the formula (1.4) can be reduced in view of a phase-space localization of eigenfunctions.

This is the content of the second main result of the paper.

Theorem 1.2. *Let* $\mathrm{Op}_h(b)$ *be an elliptic semiclassical* Ψ*do as in* (2.1) *and* $h^{-1} \in \mathbb{N}$. *Let* $E \in \mathbb{R}$, *and let* $\psi_h \in C^\infty(\mathbb{T}^n)$, $\|\psi_h\|_{L^2} = 1$ *s.t.* $\|\Delta_x \psi_h\|_{L^2} \le c\, h^{-M}$ *and which is eigenfunction of the eigenvalue problem on* $\mathbb{T}^n$

$$\mathrm{Op}_h(b)\psi_h = E_h \psi_h$$

where $E_h \leq E$ for any $0 < h \leq 1$. Then, there exists $g(h, E) \in \mathbb{R}_+$ such that

$$\psi_h = \sum_{\xi \in h\,\mathbb{Z}^n, \; |\xi| \leq g(h,E)} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, \psi_h \rangle_{L^2} \, \Phi^\star_{(x,\xi)} \, dx + \mathcal{O}_{L^2}(h^\infty). \qquad (1.6)$$

We notice that for the operators $-h^2 \Delta_x + V(x)$ all the eigenfunctions with eigenvalues $E_h \leq E$ fulfill $\|\Delta_x \psi_h\|_{L^2} \leq c\, h^{-2}$. In particular, we have the asymptotics $g(h, E) \to +\infty$ as $h \to 0^+$. We also underline that the function $g(E, h)$ and the estimate on remainder $\mathcal{O}_{L^2}(h^\infty)$ do not depend on the particular choice of ψ_h. This implies that all these eigenfunctions take the form (1.6) and therefore also any finite linear combination of eigenfunctions of kind $\sum_{1 \leq \alpha \leq N} c_\alpha \psi_{h,\alpha}$ where $|c_\alpha| \leq 1$. We remind that Weyl Law on the number $\mathcal{N}(h)$ of eigenvalues $E_{h,\alpha} \leq E$ (with their multiplicity) for semiclassical elliptic operators (see for example [17]) reads $\mathcal{N}(h) \simeq (2\pi h)^{-n}(\text{vol}(U(E)) + \mathcal{O}(1))$. The proof of the above result is mainly based on a uniform estimate for our toroidal version of the Fourier-Bros-Iagolnitzer (FBI) transform

$$(\mathcal{T}\psi_h)(x, \xi) := \langle \Phi_{(x,\xi)}, \psi_h \rangle_{L^2}$$

on the unbounded region given by all $x \in \mathbb{T}^n$ and $\xi \in h\mathbb{Z}^n$ such that $|\xi| > g(h, E)$. The FBI transform on any compact manifold has already been defined and studied in the literature, see for example [18].

We remind that, in the euclidean setting of $\mathbb{R}^{2n}$, the function $T_h(\psi_h)(x, \xi) := \langle \phi_{(x,\xi)}, \psi_h \rangle_{L^2(\mathbb{R}^n)}$ is the usual version of the FBI transform, which is well posed for any $\psi_h \in \mathcal{S}'(\mathbb{R}^n)$. This is used to study the phase space localization by the Microsupport of ψ_h (see for example [11]), namely $MS(\psi_h)$ the complement of the set of points (x_0, ξ_0) such that $T_h(\psi_h)(x, \xi) \simeq \mathcal{O}(e^{-\delta/h})$ uniformly in a neighborhood of (x_0, ξ_0). In the case of the weaker estimate $T_h(\psi_h)(x, \xi) \simeq \mathcal{O}(h^\infty)$ one can define the semiclassical Wave Front Set $WF(\psi_h)$. Is well known (see [11]) that the Microsupport (or the semiclassical Wave Front Set) of eigenfunctions for elliptic operators is localized in the sublevel sets $U(E) := \{(x, \xi) \in \mathbb{R}^{2n} \mid b(x, \xi) \leq E\}$, i.e. $MS(\psi_h) \subseteq U(E)$. The well posedness of $WF(\psi_h)$ and $MS(\psi_h)$ in the periodic setting can be seen starting from the euclidean setting and thanks to distributional inclusion $L^2(\mathbb{T}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, (see for example section 3.1 of [5]). The semiclassical study in the phase space for eigenfunctions in the periodic setting has also been studied in [20] with respect to weak KAM theory.

In our Theorem 1.2 we are interested to show another kind of semiclassical localization, namely to localize the bounded region $\Omega(E, h) := \{(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n \mid x \in \mathbb{T}^n, |\xi| \leq g(E, h)\}$ which will be bigger than $MS(\psi_h)$, h - dependent and such that the coherent state decomposition of ψ_h can be done up to a remainder $\mathcal{O}_{L^2}(h^\infty)$.

We now focus our attention to the decompositon (1.6) under the time evolution.

Theorem 1.3. *Let $\varphi_h \in C^\infty(\mathbb{T}^n)$, L^2 - normalized such that*

$$\varphi_h = \sum_{1 \leq j \leq J(h)} c_j \, \psi_{h,j}$$

where $\psi_{h,j}$ are given in Thm. 1.2 and $J(h) \leq J_0 h^{-Q}$ for some $J_0, Q > 0$. Let $\mathrm{Op}_h(b)$ be an elliptic semiclassical Ψdo as in (2.1) and $U_h(t) := \exp\{(-i\mathrm{Op}_h(b)t)/h\}$. Then, there exists $\ell(h) > 0$ s.t. for any $t \in \mathbb{R}$

$$U_h(t)\varphi_h = \sum_{\xi \in h\,\mathbb{Z}^n, \ |\xi| \leq \ell(h)} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, U_h(t)\varphi_h \rangle_{L^2} \, \Phi^\star_{(x,\xi)} \, dx + \mathcal{O}_{L^2}(h^\infty). \quad (1.7)$$

The equality (1.7) shows that time evolution under the L^2 - unitary map $U_h(t)$ does not change such a decomposition, since $\ell(h)$ does not depend on time. The function $\ell(h)$ is not necessarily the same as the function $f(h)$ contained in Theorem 1.1 but we have that $\ell(h) \geq f(h)$. In other words, this quantum dynamics preserves the coherent state decomposition (1.5). The same result holds for any eigenfunctions in Thm. 1.2 since in this case $U_h(t)\psi_h = \exp\{(-iE_h t)/h\}\psi_h$. Notice that here we can assume that $Q > n$, namely the linear combination (1.3) can be done with more eigenfunctions than the ones that have eigenvalues $E_h \leq E$ with fixed energy $E > 0$. Notice also that we have $\langle \Phi_{(x,\xi)}, U_h(t)\varphi_h \rangle_{L^2} = \langle U_h(-t)\Phi_{(x,\xi)}, \varphi_h \rangle_{L^2}$ for any $t \in \mathbb{R}$ and that the time evolution of the periodization of coherent states has been used in [19] in the context of optimal transport theory.

2. Semiclassical toroidal Pseudodifferential operators

Let us define the flat torus $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$ and introduce the class of symbols $b \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \leq \delta, \rho \leq 1$, given by functions in $C^\infty(\mathbb{T}^n \times \mathbb{R}^n; \mathbb{R})$ which are 2π-periodic in each variable x_j, $1 \leq j \leq n$ and for which for all $\alpha, \beta \in \mathbb{Z}^n_+$ there exists $C_{\alpha\beta} > 0$ such that $\forall \, (x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n$

$$|\partial^\beta_x \partial^\alpha_\xi b(x,\xi)| \leq C_{\alpha\beta m} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. In particular, the set $S^m_{1,0}(\mathbb{T}^n \times \mathbb{R}^n)$ is denoted by $S^m(\mathbb{T}^n \times \mathbb{R}^n)$.
We introduce the semiclassical toroidal Pseudodifferential Operators by

Definition 2.1. Let $\psi \in C^\infty(\mathbb{T}^n; \mathbb{C})$ and $0 < h \leq 1$,

$$\mathrm{Op}_h(b)\psi(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x - y, \kappa \rangle} b(x, h\kappa)\psi(y)dy.$$

This is the semiclassical version (see [13], [14]) of the quantization by Pseudodifferential Operators on the torus developed in [15] and [16]. See also [2] for the notion of vector valued Pseudodifferential Operators on the torus.
We now notice that we have have a map $\mathrm{Op}_h(b) : C^\infty(\mathbb{T}^n) \longrightarrow \mathcal{D}'(\mathbb{T}^n)$. Indeed, remind that $u \in \mathcal{D}'(\mathbb{T}^n)$ are the linear maps $u : C^\infty(\mathbb{T}^n) \longrightarrow \mathbb{C}$ such that $\exists \, C > 0$ and $k \in \mathbb{N}$, for which $|u(\phi)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha_x \phi\|_\infty \, \forall \phi \in C^\infty(\mathbb{T}^n)$. Given a symbol $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$, the toroidal Weyl quantization reads (see Ref. [13], [14])

$$\mathrm{Op}^w_h(b)\psi(x) := (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x - y, \kappa \rangle} b\left(y, \frac{h}{2}\kappa\right)\psi(2y - x)dy.$$

In particular, it holds

$$\mathrm{Op}_h^w(b)\psi(x) = (\mathrm{Op}_h(\sigma) \circ T_x\,\psi)(x)$$

where $T_x : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$ defined as $(T_x\psi)(y) := \psi(2y - x)$ is linear, invertible and L^2-norm preserving, and σ is a suitable toroidal symbol related to b, i.e. $\sigma \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \triangle_\xi^\alpha D_y^{(\alpha)} b(y, h\xi/2)\big|_{y=x}$, where $\triangle_{\xi_j} f(\xi + e_j) - f(\xi)$ is the difference operator (see Th. 4.2 in Ref. [15]).
The typical example is given by

$$
\begin{aligned}
\mathrm{Op}_h(H) &= \Big(-\frac{1}{2}h^2\Delta_x + V(x) \Big)\psi(x) \\
&= (2\pi)^{-n} \sum_{\kappa \in \mathbb{Z}^n} \int_{\mathbb{T}^n} e^{i\langle x - y, \kappa \rangle} \Big(\frac{1}{2}|h\kappa|^2 + V(x) \Big)\psi(y)dy
\end{aligned}
$$

namely the related symbol is the mechanical type Hamiltonian $H(x,\xi) = \frac{1}{2}|\xi|^2 + V(x)$. Also in the case of the Weyl operators we have

$$-\frac{1}{2}h^2\Delta_x + V(x) = \mathrm{Op}_h^w(H)$$

for the same symbol (see for example [14]).
In our paper we are interested in uniform elliptic operators, namely such that the symbol $b \in S^m(\mathbb{T}^n \times \mathbb{R}^n)$ fulfills for some constants $C, c > 0$ the lower bound

$$|b(x,\xi)| \geq C \langle \xi \rangle^m$$

for any $x \in \mathbb{T}^n$ and $|\xi| \geq c$. This property guarantees bounded sublevels sets for b and discrete spectrum for the operator $\mathrm{Op}_h(b)$ for any fixed $0 < h \leq 1$. As we see in Theorem 1.2, this assumption permits also to prove the semiclassical localization of all the eigenfunctions within these sublevels sets, and this localization can be studied by our semiclassical coherent states (1.3).

3. Main Results

Proof of Theorem 1.1 We remind that $\Phi_{(x,\xi)}(y) := \Pi(\phi_{(x,\xi)})(y)$ and $\Pi(\phi)(y) := \sum_{k \in \mathbb{Z}^n} \phi(y - 2\pi k)$. Thus,

$$
\begin{aligned}
\Phi_{(x+2\pi\beta,\xi)}(y) &= \sum_{k \in \mathbb{Z}^n} \phi_{(x+2\pi\beta,\xi)}(y - 2\pi k) = \sum_{k \in \mathbb{Z}^n} \phi_{(x,\xi)}(y - 2\pi k - 2\pi\beta) \\
&= \Phi_{(x,\xi)}(y).
\end{aligned}
$$

We mainly adapt, in our toroidal setting, the proof of Prop. 3.1.6 shown in [11] written for the euclidean setting. Thus, we define the operator $\mathcal{T}^\star$ on functions $\Psi \in L^2(\mathbb{T}^n \times h\mathbb{Z}^n)$ as

$$(\mathcal{T}^\star\Psi)(y) := \sum_{\xi \in h\,\mathbb{Z}^n} \int_{\mathbb{T}^n} \Psi(x,\xi)\Phi_{(x,\xi)}^\star(y)\,dx.$$

It can be easily seen that $\mathcal{T}^\star$ equals the adjoint of the operator $(\mathcal{T}\psi)(x,\xi) := \langle \Phi_{(x,\xi)}, \psi \rangle_{L^2(\mathbb{T}^n)}$, i.e.

$$\langle \mathcal{T}^\star \Psi, \psi \rangle_{L^2(\mathbb{T}^n)} = \langle \Psi, \mathcal{T}\psi \rangle_{L^2(\mathbb{T}^n \times h\mathbb{Z}^n)}.$$

Thus, $\forall \psi_1, \psi_2 \in C^\infty(\mathbb{T}^n) \subset L^2(\mathbb{T}^n)$ we have

$$\langle \mathcal{T}^\star \circ \mathcal{T}\psi_1, \psi_2 \rangle_{L^2(\mathbb{T}^n)} = \langle \mathcal{T}\psi_1, \mathcal{T}\psi_2 \rangle_{L^2(\mathbb{T}^n \times h\mathbb{Z}^n)}.$$

It remains to prove that

$$\langle \mathcal{T}\psi_1, \mathcal{T}\psi_2 \rangle_{L^2(\mathbb{T}^n \times h\mathbb{Z}^n)} = \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{T}^n)} + \mathcal{O}(h^\infty) \tag{3.1}$$

which implies

$$\mathcal{T}^\star \circ \mathcal{T} = \text{Id} \mod \mathcal{O}(h^\infty) \tag{3.2}$$

on $L^2(\mathbb{T}^n)$, and equality (3.2) is exactly the statement (1.4).

In order to prove (3.1), we recall that the periodization operator Π can be rewritten in the form (see Thm. 6.2 in [15]):

$$\Pi(\phi) = \mathcal{F}_{\mathbb{T}^n}^{-1}\left(\mathcal{F}_{\mathbb{R}^n} \phi \big|_{\mathbb{Z}^n} \right). \tag{3.3}$$

where $\mathcal{F}_{\mathbb{T}^n}^{-1}$ stands for the inverse toroidal Fourier Transform, and $\mathcal{F}_{\mathbb{R}^n}$ is the usual euclidean version. In view of (3.3) it follows

$$\begin{aligned}
(\mathcal{T}\psi)(x,\xi) &:= \langle \Phi_{(x,\xi)}, \psi \rangle_{L^2(\mathbb{T}^n)} = \langle \mathcal{F}_{\mathbb{R}^n} \phi_{x,\xi}|_{\mathbb{Z}^n}, \mathcal{F}_{\mathbb{T}^n} \psi \rangle_{L^2(\mathbb{Z}^n)} \\
&= \sum_{k \in \mathbb{Z}^n} \widehat{\phi}_{x,\xi}(k)^\star \widehat{\psi}(k),
\end{aligned}$$

where $\widehat{\phi}_{x,\xi}(k) := \mathcal{F}_{\mathbb{R}^n} \phi_{x,\xi}(k)$ and $\widehat{\psi}(k) := \mathcal{F}_{\mathbb{T}^n} \psi(k)$. Thus,

$$\begin{aligned}
&\langle \mathcal{T}\psi_1, \mathcal{T}\psi_2 \rangle_{L^2(\mathbb{T}^n \times h\mathbb{Z}^n)} \\
&= \sum_{\xi \in h\,\mathbb{Z}^n} \int_{\mathbb{T}^n} \left(\sum_{k \in \mathbb{Z}^n} \widehat{\phi}_{x,\xi}(k)^\star \widehat{\psi}(k) \right)^\star \left(\sum_{\mu \in \mathbb{Z}^n} \widehat{\phi}_{x,\xi}(\mu)^\star \widehat{\psi}(\mu) \right) dx.
\end{aligned}$$

We can rewrite this equality, in the distributional sense, as

$$\begin{aligned}
&\langle \mathcal{T}\psi_1, \mathcal{T}\psi_2 \rangle_{L^2(\mathbb{T}^n \times h\mathbb{Z}^n)} \\
&= \sum_{k,\mu \in \mathbb{Z}^n} \widehat{\psi}_1(k)^\star \widehat{\psi}_2(\mu) \sum_{\xi \in h\,\mathbb{Z}^n} \int_Q \widehat{\phi}_{x,\xi}(k) \widehat{\phi}_{x,\xi}(\mu)^\star dx
\end{aligned}$$

where $Q := [0, 2\pi]^n$ and $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^n)$. Now let $\xi = h\alpha$ with $\alpha \in \mathbb{Z}^n$, so that

$$\begin{aligned}
&\langle \mathcal{T}\psi_1, \mathcal{T}\psi_2 \rangle_{L^2(\mathbb{T}^n \times h\mathbb{Z}^n)} \\
&= \sum_{k,\mu \in \mathbb{Z}^n} \widehat{\psi}_1(k)^\star \widehat{\psi}_2(\mu) \sum_{\alpha \in \mathbb{Z}^n} \int_Q \widehat{\phi}_{x,h\alpha}(k) \widehat{\phi}_{x,h\alpha}(\mu)^\star dx
\end{aligned}$$

By using the explicit form of $\widehat{\phi}_{x,h\alpha}$ and the condition $h^{-1} \in \mathbb{N}$, a direct computation shows that

$$
\begin{aligned}
&\langle \mathcal{T}\psi_1, \mathcal{T}\psi_2 \rangle_{L^2(\mathbb{T}^n)} \\
&= \sum_{k,\mu \in \mathbb{Z}^n} \widehat{\psi}_1(k)^\star \widehat{\psi}_2(\mu) \Big[\Big(\sum_{\alpha \in \mathbb{Z}^n} e^{i\alpha(k-\mu)} \Big) + \mathcal{O}(h^\infty) \Big]
\end{aligned}
\tag{3.4}
$$

where $\mathcal{O}(h^\infty)$ does not depend on the functions ψ_1, ψ_2. We now use the assumption $\|\Delta_x \psi\|_{L^2} \le c\, h^{-M}$ for some fixed $c, M > 0$ so that Fourier components fulfill $|\widehat{\psi}_k| \le |k|^{-2} (2\pi)^{n/2} c\, h^{-M}$, and $\|\psi\|_{L^2} = 1$ gives $|\widehat{\psi}_0| \le (2\pi)^{n/2}$. Consequently,

$$
\sum_{k \in \mathbb{Z}^n} |\widehat{\psi}_1(k)| \le (2\pi)^{n/2} + (2\pi)^{n/2} c \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^{-2}\, h^{-M},
\tag{3.5}
$$

and

$$
\Big| \sum_{k,\mu \in \mathbb{Z}^n} \widehat{\psi}_1(k)^\star \widehat{\psi}_2(\mu) \Big| \le \sum_{k \in \mathbb{Z}^n} |\widehat{\psi}_1(k)| \sum_{\mu \in \mathbb{Z}^n} |\widehat{\psi}_2(\mu)|.
\tag{3.6}
$$

To conclude, since $\delta(k-\mu) = \sum_{\alpha \in \mathbb{Z}^n} e^{i\alpha(k-\mu)}$, we get

$$
\begin{aligned}
\langle \mathcal{T}\psi_1, \mathcal{T}\psi_2 \rangle_{L^2(\mathbb{T}^n \times h\mathbb{Z}^n)} &= \sum_{k \in \mathbb{Z}^n} \widehat{\psi}_1(k)^\star \widehat{\psi}_2(k) + \mathcal{O}(h^\infty) \\
&= \langle \psi_1, \psi_2 \rangle_{L^2(\mathbb{T}^n)} + \mathcal{O}(h^\infty).
\end{aligned}
\tag{3.7}
$$

The estimates (3.5) - (3.6) together with (3.4) ensure that the remainder in (3.7) has order $\mathcal{O}(h^\infty)$.

In order to prove (1.5), we observe that

$$
\varphi_h = \sum_{\xi \in h\, \mathbb{Z}^n} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, \varphi_h \rangle_{L^2}\, \Phi^\star_{(x,\xi)}\, dx + \mathcal{O}(h^\infty).
$$

is given by an L^2-convergent series. Thus, for any fixed φ_h we can say that there exists $f(h) > 0$ such that

$$
\varphi_h = \sum_{\xi \in h\, \mathbb{Z}^n, |\xi| < f(h)} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, \varphi_h \rangle_{L^2}\, \Phi^\star_{(x,\xi)}\, dx + \mathcal{O}(h^\infty).
$$

$\square$

Proof of Theorem 1.2 We apply the statement of Thm. 1, for a set of linearly independent eigenfunctions $\psi_{h,i}$ generating all the eigenspaces linked to eigenvalues $E_h \le E$ and $f_i(h) > 0$ given by Thm. 1.1.

$$
\psi_{h,i} = \sum_{\xi \in h\, \mathbb{Z}^n, |\xi| < f_i(h)} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, \psi_{h,i} \rangle_{L^2}\, \Phi^\star_{(x,\xi)}\, dx + R_{h,i}
$$

where $\|R_{h,i}\|_{L^2} = \mathcal{O}(h^\infty)$.

Moreover, we recall that the Weyl Law on the number $\mathcal{N}(h)$ of eigenvalues $E_h \le E$ (counted with their multiplicity) for semiclassical elliptic operators

(see for example [17]) reads $\mathcal{N}(E, h) \simeq (2\pi h)^{-n}(\text{vol}(U(E)) + \mathcal{O}(1))$. We define:

$$g(E, h) := \max_{1 \leq i \leq \mathcal{N}(E,h)} f_i(h).$$

Since any eigenfunction ψ_h linked to $E_h \leq E$ will be written as $\psi_h = \sum_i \langle \psi_{h,i}, \psi_h \rangle \psi_{h,i}$ then the linearity of decomposition (1.4) ensures also the decomposition (1.6) for such ψ_h. Namely,

$$\psi_h = \sum_{\xi \in h\,\mathbb{Z}^n,\ |\xi| \leq g(h,E)} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, \psi_h \rangle_{L^2} \, \Phi^{\star}_{(x,\xi)} \, dx + R_h$$

where $R_h = \sum_{1 \leq i \leq \mathcal{N}(E,h)} R_{i,h}$. To conclude:

$$\begin{aligned} \|R_h\|_{L^2} &\leq \sum_{1 \leq i \leq \mathcal{N}(E,h)} \|R_{i,h}\|_{L^2} \leq \mathcal{N}(E, h) \max_{1 \leq i \leq \mathcal{N}(E,h)} \|R_{i,h}\|_{L^2} \\ &= \mathcal{N}(E, h) \cdot \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty). \end{aligned}$$

$\square$

Proof of Theorem 1.3 We assume that $\varphi_h \in C^\infty(\mathbb{T}^n)$ is L^2 - normalized and

$$\varphi_h = \sum_{1 \leq j \leq J(h)} c_j \, \psi_{h,j}$$

where the L^2-normalized eigenfunctions $\psi_{h,j}$ of $\text{Op}_h(b)$ are given in Thm. 1.2 and we assume $J(h) \leq J_0 h^{-Q}$ for some $J_0, Q > 0$ that are independent on $0 < h \leq 1$.
Define

$$\ell(h) := \max_{1 \leq j \leq J(h)} f_j(h)$$

where $f_i(h)$ are associated to the functions $\psi_{h,i}$ and given by Thm 1.1.
We now observe that if $U_h(t) := \exp\{(-i\text{Op}_h(b)t)/h\}$ then

$$U(t)\varphi_h = \sum_{1 \leq j \leq J(h)} c_j \, e^{-\frac{i}{h} E_{j,h}} \psi_{h,j} \tag{3.8}$$

for any $t \in \mathbb{R}$.
We can now apply the decomposition formula (1.4) with the condition on the frequencies $|\xi| \leq \ell(h)$ and for the wave function $U(t)\varphi_h$ and get the expected result, namely

$$U_h(t)\varphi_h = \sum_{\xi \in h\,\mathbb{Z}^n,\ |\xi| \leq \ell(h)} \int_{\mathbb{T}^n} \langle \Phi_{(x,\xi)}, U_h(t)\varphi_h \rangle_{L^2} \, \Phi^{\star}_{(x,\xi)} \, dx + \sum_{1 \leq j \leq J} R_{j,h}$$

for any $t \in \mathbb{R}$. The remainder $R_h := \sum_{1 \leq j \leq J} R_{j,h}$ can be estimated as in the previous Theorem, namely

$$\begin{aligned} \|R_h\|_{L^2} &\leq \sum_{1 \leq i \leq J} \|R_{j,h}\|_{L^2} \leq J_0 \, h^{-Q} \max_{1 \leq j \leq J} \|R_{j,h}\|_{L^2} \\ &= J_0 \, h^{-Q} \cdot \mathcal{O}(h^\infty) = \mathcal{O}(h^\infty). \end{aligned}$$

$\square$

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