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ORIGINAL ARTICLE

Dependence on the observational time intervals and domain of convergence of orbital determination methods

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Abstract In the framework of the orbital determination methods, we study some properties related to the algorithms developed by Gauss, Laplace and Mossotti. In particular, we investigate the dependence of such methods upon the size of the intervals between successive observations, encompassing also the case of two nearby observations performed within the same night. Moreover we study the convergence of Gauss algorithm by computing the maximal eigenvalue of the jacobian matrix associated to the Gauss map. Applications to asteroids and Kuiper belt objects are considered.

Keywords Orbital determination · Gauss method · Laplace method · Mossotti method

1 Introduction

The determination of the orbital motion of a celestial body can be obtained through the celebrated methods of Gauss or Laplace, once a certain number (at least 3) of astronomical observations are available (see Poincaré 1906; Moulton 1914; Plummer 1918; Herrick 1937 for discussions on Laplace and Gauss methods). An alternative technique was developed by Mossotti in the 19th century. The three methods (Gauss, Laplace and Mossotti) have been extensively reviewed and compared in Celletti and Pinzari (2005). In this work, we want to explore the dependence of the three techniques upon the observational time intervals. It is relevant to quote the recent results obtained

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in Milani et al. (2004, 2005), Milani and Knezevic (2005); in these works the authors investigate the problem of the orbit determination using two pairs of data, each of them composed by observations at very close times. As a consequence very few observational data are available to compute the orbit and efficient mathematical methods have been developed to sample the *admissible region* for the undetermined variables.

Let t_1, t_2, t_3 be the times of the three observations; having fixed the intermediate time t_2, we vary the time intervals $t_2 - t_1$ and $t_3 - t_2$, ranging from a few hours (whenever two observations are performed on the same night) to several days. Two sets of data are investigated: the first 10,000 numbered asteroids and 615 Kuiper belt objects. While in the first case Gauss method provides the best results, the orbital determination of Kuiper belt objects seems to privilege Laplace method, being Mossotti's technique intermediate in all cases. For the selected samples of data, the recovery of the orbits of the asteroidal belt improves as the time intervals decrease, while it improves within the Kuiper belt objects whenever the time intervals increase. A statistic of the successful results in terms of the elliptic elements (semi-major axis, eccentricity and inclination) is also performed. In the second part of the paper, we concentrate on Gauss algorithm to investigate the stability domain of such method, by looking at the eigenvalues of the jacobian matrix associated to the Gauss map. We provide a numerical investigation performed on asteroids and Kuiper belt objects. We also develop an analytical estimate of the first-order computation of the largest eigenvalue; we prove a proposition ensuring the convergence of Gauss method, which is related to the contractive character of the Gauss map, at least for small values of the observational times.

2 Implementation of Gauss, Laplace and Mossotti methods

2.1 Basics of the methods

With reference to a heliocentric frame let us denote the unknown elements of the asteroid as follows: a is the semi-major axis, e is the eccentricity, i denotes the inclination, ω is the argument of perihelion, Ω is the longitude of the ascending node and M is the mean anomaly at a fixed epoch T. We assume that the ecliptic geocentric longitudes and latitudes, say λ_i and β_i, $i = 1, \dots, N$, are given through N observations at times t_i referred to the epoch T. Moreover, let $t \to \vec{a}(t)$ denote the Sun–Earth vector, $t \to \vec{r}(t)$ is the Sun–asteroid vector, while $t \to \rho(t)$ is the geocentric distance and $t \to \vec{b}(t)$ with $|\vec{b}(t)| = 1$ denotes the Earth–object direction.

We assume to perform three observations at times t_1, t_2, t_3. The time intervals $t_{ij} = t_j - t_i$, $i, j = 1, 2, 3$, are regarded as small quantities of order ε; for some positive constants γ_{12}, γ_{23}, with $\gamma_{12} + \gamma_{23} = 1$, we set

$$\varepsilon \equiv t_{13} \qquad t_{12} = \gamma_{12}\varepsilon \qquad t_{23} = \gamma_{23}\varepsilon. \tag{1}$$

Let $\vec{k}$ be the unit vector perpendicular to the plane of the orbit; the coplanarity condition of the vectors $\vec{r}_i = \vec{a}_i + \rho_i \vec{b}_i$, $i = 1, 2, 3$, reads as

$$n_{23}\vec{r}_1 - n_{13}\vec{r}_2 + n_{12}\vec{r}_3 = 0,$$

where $n_{ij} = \vec{r}_i \wedge \vec{r}_j \cdot \vec{k}$ is twice the oriented area of the triangle spanned by $\vec{r}_i$ and $\vec{r}_j$. If $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are linearly independent, one can express ρ_i as linear functions (with coefficients of $O(\varepsilon^{-2})$) of the ratios n_{ik}/n_{lk} with $i \neq l \neq k$. The first goal of Gauss method is to find a good approximation of ρ_i, say up to terms of $O(\varepsilon)$. To this end, let

S_{ij} be the areas of the elliptic sectors spanned between t_i and t_j, and let $\eta_{ij} = n_{ij}/S_{ij}$, f_{ij} be half the angle between $\vec{r}_i$, $\vec{r}_j$. Denote by $z = (P, Q)$ a new set of quantities, called Gauss parameters, defined as

$$\begin{aligned} P &= \tfrac{n_{12}}{n_{23}} = \tfrac{\gamma_{12}}{\gamma_{23}} f(\eta_{12}, \eta_{23}), \\ Q &= 2r_2^3\left(\tfrac{n_{12}+n_{23}}{n_{13}} - 1\right) = \gamma_{12}\gamma_{23}\,\varepsilon^2 g\left(\eta_{12}, \eta_{23}, \tfrac{r_1}{r_2}, \tfrac{r_2}{r_3}, f_{12}, f_{23}\right), \end{aligned} \tag{2}$$

where f and g are suitable functions differing from one up to $O(\varepsilon^2), O(\varepsilon)$, respectively (see Celletti and Pinzari 2005). The quantities ρ_i can be expressed in terms of P, Q as

$$\rho_2 = G_2(P, Q, \rho_2), \qquad \rho_1 = G_1(P, Q, \rho_2), \qquad \rho_3 = G_3(P, Q, \rho_2)$$

for suitable functions G_i, $i = 1, 2, 3$ (see Appendix B for explicit expressions of the G_i). In particular $\rho_2 = \rho_2(P, Q)$ is a solution of an implicit equation, from which we derive $\rho_1 = \rho_1(P, Q)$, $\rho_3 = \rho_3(P, Q)$. Finally, setting

$$P_0 = \gamma_{12}/\gamma_{23}, \qquad Q_0 = \gamma_{12}\gamma_{23}\,\varepsilon^2 \tag{3}$$

one finds that $G_i(P, Q, \rho_2) = G_i(P_0, Q_0, \rho_2) + O(\varepsilon)$, namely $\rho_i = \rho_{i,0} + O(\varepsilon)$, where $\rho_{i,0} = \rho_i(P_0, Q_0)$.

Gauss algorithm is inductively based on the following steps:

(1) start from $z_0 = (P_0, Q_0)$;
(2) given $z_n = (P_n, Q_n)$, compute $\rho_{2,n} = \rho_2(P_n, Q_n)$ trying to solve the implicit equation $\rho_{2,n} = G_2(P_n, Q_n, \rho_{2,n})$ and let, for $i = 1, 3$, $\rho_{i,n} = \rho_i(P_n, Q_n)$. The three vectors $\vec{r}_{i,n} = \vec{a}_i + \rho_{i,n}\vec{b}_i$, $i = 1, 2, 3$ are shown to be coplanar;
(3) if the endpoints of $\vec{r}_{1,n}, \vec{r}_{2,n}, \vec{r}_{3,n}$ are not on a straight line, there exists a unique conic $\mathcal{C}_n$ through $\vec{r}_{1,n}, \vec{r}_{2,n}, \vec{r}_{3,n}$; compute the quantities $\eta_{ij,n}$, $f_{ij,n}$ $r_{i,n}$ on $\mathcal{C}_n$;
(4) determine the new parameters $z_{n+1} = (P_{n+1}, Q_{n+1})$ through (2), where the r.h.s. are computed with $\eta_{ij,n}$, $f_{ij,n}$, $r_{i,n}$ replacing η_{ij}, f_{ij}, r_i. Such procedure defines the Gauss map $\mathcal{F}(\mathcal{C}, t_2, \gamma_{12}, \gamma_{23}, \varepsilon) \equiv \mathcal{F}_G$ as $z_{n+1} = \mathcal{F}_G(z_n)$;
(5) look for a fixed point of the Gauss map, motivated by the fact that a conic section $\mathcal{C}$ (on which a Keplerian motion takes place) is a solution of Gauss problem if and only if it corresponds to a fixed point of $\mathcal{F}_G$.

We can finally summarize Gauss method (Gauss 1809; see also Gallavotti 1980) with the following

Theorem 2.1 *Let $\mathcal{C}, t_2, \gamma_{12}, \gamma_{23}, \varepsilon$ be such that $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are linearly independent, and $\partial_\rho G_2(P, Q, \rho)|_{\rho_2} \neq 1$, where $z = (P, Q)$ is the fixed point of $\mathcal{F}_G$, defined in (2). Let D be the domain of definition of $\mathcal{F}_G$, $U \subset D$ a neighbourhood of z, V a neighbourhood of ρ_2, $\rho : z' = (P', Q') \in U \to \rho(P', Q') \in V$ be the smooth solution of $\rho = G_2(P', Q', \rho)$ such that $\rho(P, Q) = \rho_2$. If $z_0 \in U$, the associated conic section $\mathcal{C}_0$ verifies: $\mathcal{C} - \mathcal{C}_0 = O(\varepsilon)$. Finally, if $z_n \in U$, the associated conic section $\mathcal{C}_n$ verifies: $\mathcal{C} - \mathcal{C}_n = O(\varepsilon^{n+1})$.*

A different approach is provided by Laplace method, whose aim is to find an approximation of the position $\vec{r}$ and the velocity $\vec{v}$, so to determine the unknown orbit. Let $r = r(\rho) = |\vec{a} + \rho\vec{b}|$ be the heliocentric distance; using the equations of motion, one gets an implicit equation in the unknowns $\dot{\lambda}, \dot{\beta}, \ddot{\lambda}, \ddot{\beta}$:

$$\rho = \frac{d_1}{d}\left(\frac{1}{r^3} - \frac{1}{a^3}\right) \equiv L(d_1/d, \rho). \tag{4}$$

Moreover, one finds that $\dot\rho = \frac{d_2}{d}\left(\frac{1}{r^3} - \frac{1}{a^3}\right)$, with $d = d(\lambda, \beta, \dot\lambda, \dot\beta, \ddot\lambda, \dot\beta)$, $d_1 = d_1(\lambda, \beta, \dot\lambda, \dot\beta)$, $d_2 = d_2(\lambda, \beta, \ddot\lambda, \ddot\beta)$ (see Celletti and Pinzari 2005, for the explicit expressions of d, d_1, d_2). Given the N observations $(\lambda_1, \beta_1), (\lambda_2, \beta_2), \ldots, (\lambda_N, \beta_N)$, Laplace method (Laplace 1780) consists in replacing $\dot\lambda, \ddot\lambda$ (equivalently $\dot\beta, \ddot\beta$) by the derivatives of some interpolating polynomials of degree $N-1$ obtained through the observed data $(t_1, \lambda_1), (t_2, \lambda_2), \ldots, (t_N, \lambda_N)$ (equivalently $(t_1, \beta_1), (t_2, \beta_2), \ldots, (t_N, \beta_N)$).

An alternative technique was developed by Mossotti (1942) and it is based on the following procedure. Writing the coplanarity condition among $\vec r(t), \vec r_2, \vec v_2$ as

$$\vec r(t) = T(t)\vec r_2 + V(t)\vec v_2 \tag{5}$$

and developing the equation of motion $\ddot{\vec r} = -\vec r/r^3$ in Taylor series with initial data $\vec r(t_2) = \vec r_2, \dot{\vec r}(t_2) = \vec v_2$, one obtains

$$T(t) = 1 - \frac{(t-t_2)^2}{2r_2^3}h(t), \qquad V(t) = (t-t_2)k(t),$$

where $h(t)$ and $k(t)$ are suitable functions; if h_i and k_i denote their values at times t_i, one can show that h_i and k_i differ from one up to $O(\varepsilon)$. Using (2.5) computed at t_1 and t_3, one can express ρ_2 and $\vec v_2$ as

$$\begin{aligned}\rho_2 &= M(h_1, h_3, k_1, k_3, \rho_2) = M(1, 1, 1, 1, \rho_2) + O(\varepsilon),\\ \vec v_2 &= \vec N(h_1, h_3, k_1, k_3, \rho_2) = \vec N(1, 1, 1, 1, \rho_2) + O(\varepsilon)\end{aligned}$$

for suitable (vector) functions $M, \vec N$. In conclusion, it turns out that ρ_2 is a solution of an implicit equation, which can be solved in analogy to Gauss method.

2.2 Iteration of the methods

A major advantage of Gauss method with respect to the others is that it provides an iterative procedure to find better approximations of the solution. On the contrary, the methods of Laplace (implemented over three observations) and Mossotti were originally limited to the first-order approximation. However, an iterative scheme can be implemented along the following lines.

Let us consider first the method of Laplace. Let $R(t)$ denote the remainder function of order 3 of the series expansion of $\lambda(t)$ around t_2, namely $\lambda(t) = P(t) + R(t)$, with $P(t) = \lambda_2 + \dot\lambda(t_2)(t-t_2) + \frac{\ddot\lambda(t_2)}{2}(t-t_2)^2$ (obviously $R(t_2) = 0$). In other words, $\dot\lambda_2 \equiv \dot\lambda(t_2)$, $\ddot\lambda_2 \equiv \ddot\lambda(t_2)$ are the derivatives of the interpolating polynomial $t \to P(t)$ of degree 2 through $\lambda_1 - R_1, \lambda_2, \lambda_3 - R_3$ (here, $R_i = R(t_i)$), at times t_1, t_2, t_3. Similarly for $\dot\beta(t_2), \ddot\beta(t_2)$, where the remainder functions are denoted as S_1, S_3. When $\dot\lambda_2$, $\dot\beta_2$, $\ddot\lambda_2$, $\ddot\beta_2$ are expressed as functions of R_1, R_3, S_1, S_3, Eq. 4, with $t = t_2$, takes the form (without changing the symbol for L) $\rho_2 = L(R_1, R_3, S_1, S_3, \rho_2)$; the first approximation ($N = 3$) of Laplace corresponds to take $R_i = S_i = 0$ ($i = 1, 3$). We are therefore led to define a sequence of remainder functions $R_{i,n}, S_{i,n}$ as follows:

(1) Start with $R_{1,0} = R_{3,0} = 0$ ($S_{1,0} = S_{3,0} = 0$).
(2) Given $R_{1,n}$, $R_{3,n}$ ($S_{1,n}, S_{3,n}$), let $\dot\lambda_n$, $\ddot\lambda_n$ ($\dot\beta_n, \ddot\beta_n$) be defined as the derivatives of the interpolating polynomial $t \to P_n(t)$ ($t \to Q_n(t)$) of degree 2 through $\dot\lambda_1 - R_{1,n}$, λ_2, $\lambda_3 - R_{3,n}$ ($\beta_1 - S_{1,n}$, $\beta_2, \beta_3 - S_{3,n}$) at times t_1, t_2, t_3, respectively. Let $d_n = d(\lambda_2, \beta_2, \dot\lambda_n, \dot\beta_n, \ddot\lambda_n, \ddot\beta_n), d_{1,n} = d_1(\lambda_2, \beta_2, \dot\lambda_n, \dot\beta_n), d_{2,n} = d_2(\lambda_2, \beta_2, \ddot\lambda_n, \ddot\beta_n)$. If $d_n \neq 0$, compute the position $\vec r_{2,n}$ and the velocity $\vec v_{2,n}$. Let $\mathcal{C}_n$, be the conic

describing a Keplerian motion with initial data $\vec{r}_{2,n}, \vec{v}_{2,n}$ (whenever the latter vectors are not parallel), and let $t \to \lambda_n(t), t \to \beta_n(t)$ be the motion of the angles.

(3) Define $R_{i,n+1}$, $S_{i,n+1}$ as the remainder functions of order 3 of the Taylor expansion of $t \to \lambda_n(t), t \to \beta_n(t)$ around $t = t_2$. Introduce the Laplace map $\mathcal{F}_{\rm L}$ as

$$(R_{1,n+1}, R_{3,n+1}, S_{1,n+1}, S_{3,n+1}) = \mathcal{F}_{\rm L}(R_{1,n}, R_{3,n}, S_{1,n}, S_{3,n}).$$

Like for Gauss, all fixed points of $\mathcal{F}_{\rm L}$ provide a solution of the problem, while the nth iteration of $\mathcal{F}_{\rm L}$ gives an approximation of the unknown or bit up to terms of order $O(\varepsilon^n)$, provided that $d \neq 0, \partial_\rho L(R_1, R_3, S_1, S_3, \rho)|_{\rho_2} \neq 1$ and $(R_{1,n}, R_{3,n}, S_{1,n}, S_{3,n})$ belongs to a suitable neighbourhood of (R_1, R_3, S_1, S_3). Let us now present an iterative scheme for the method developed by Mossotti. Define the sequence $h_{i,n}, k_{i,n} (i = 1, 3)$ as follows:

(1) Start with $h_{i,0} = k_{i,0} = 1$.

(2) Given $h_{i,n}$, $k_{i,n}$, let $\vec{r}_{2,n}$, $\vec{v}_{2,n}$ be the vectors obtained replacing h_i, k_i with $h_{i,n}$, $k_{i,n}$. If $\vec{r}_{2,n}, \vec{v}_{2,n}$ are not parallel, let $\mathcal{C}_n$ be the corresponding conic. Finally, let $\vec{r}_{1,n}, \vec{r}_{3,n}$ denote the positions of the same body at times t_1, t_3, respectively.

(3) Define $h_{i,n+1}$, $k_{i,n+1}$ by means of the relations

$$T_{i,n+1} = 1 - \frac{(t_i - t_2)^2}{2r_{2,n}^3} h_{i,n+1}, \quad V_{i,n+1} = (t_i - t_2) k_{i,n+1}, \quad i = 1, 3,$$

where $T_{i,n+1}, V_{i,n+1}$ are the coefficients of the linear relations providing $\vec{r}_{1,n}, \vec{r}_{3,n}$ as a combination of $\vec{r}_{2,n}, \vec{v}_{2,n}$ in analogy to (5). Let the Mossotti map $\mathcal{F}_{\rm M}$ be defined as

$$(h_{1,n+1}, h_{3,n+1}, k_{1,n+1}, k_{3,n+1}) = \mathcal{F}_{\rm M}(h_{1,n}, h_{3,n}, k_{1,n}, k_{3,n}).$$

As for the previous methods, all fixed points of $\mathcal{F}_{\rm M}$ define a solution of the problem, and the nth iteration of $\mathcal{F}_{\rm M}$ provides an approximation of the unknown orbit up to terms of order $O(\varepsilon^n)$, whenever $\partial_\rho M(h_1, h_3, k_1, k_3, \rho)|_{\rho_2} \neq 1, \vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3 \neq 0$ and for $(h_{1,n}, h_{3,n}, k_{1,n}, k_{3,n})$ in a suitable neighbourhood of (h_1, h_3, k_1, k_3).

3 Dependence on the times of observations

In order to study the dependence on the intervals among the times of observations, we consider two samples given by the first 10,000 numbered asteroids and by 615 Kuiper belt objects.[1] We apply Gauss, Mossotti and Laplace methods for different time intervals t_{12} and t_{23}, where the central time t_2 is the real observational time as provided by the astronomical data (see footnote 1). Starting from the elements $(a, e, i, \omega, \Omega, M)$ at the epoch t_2, and given the time intervals t_{12} and t_{23}, we compute the geocentric longitude and latitude at times t_1, t_2, t_3 by means of the coordinates of the object and that of the Earth (see Appendix A). Finally, we apply Gauss, Mossotti and Laplace methods, iterating the procedure as described in the previous section until convergence is reached. In order to be sure that a given method converges in a significant range around the given time t_2 (and not only for the specific time t_2), we

[1] The astronomical data of the asteroids can be found on the web site "Astdys" at http://hamilton.dm.unipi.it/cgi–bin/astdys/astibo; the astronomical data of the Kuiper belt objects can be found at the ephemerides page by D. Jewitt at http://www.ifa.hawaii.edu/faculty/jewitt/kb.html.

proceed as follows. Define $t_{ij}^n \equiv t_{ij} + n/2$, where $n = 0, \pm 1, \pm 2$; if the method converges for the above time lapses t_{12}^n and t_{23}^n ($n = 0, \pm 1, \pm 2$), then we say that the method is successful, otherwise we decide that the method fails.

We consider several choices of the time intervals t_{ij} from 3 to 90 days. Moreover, to cover the case of two observations performed within the same night, we selected t_{12} of the order of some hours and t_{23} ranging from 5 to 30 days. The results are summarized in Table 1, where the first percentage refers to the asteroids, while the second number of each method refers to Kuiper belt objects. Concerning the main belt, one concludes that Gauss method provides the best result, while Mossotti is more successful than Laplace; the opposite conclusion holds for the Kuiper belt objects. For equal time intervals $t_{12} = t_{23}$ (i.e. the first 8 lines of Table 1), the number of successful cases within the asteroidal belt increases as the time interval decreases, while (again) the opposite conclusion can be drawn for the Kuiper belt objects. As discussed in the following section, one might expect that whenever the time interval ε among the observations is sufficiently small (say $\varepsilon < \bar{\varepsilon}$), Gauss method (as well as the other techniques) converges. Of course $\bar{\varepsilon}$ depends on $\mathcal{C}, \gamma_{12}, \gamma_{23}$ (and t_2), implying that smaller is ε, greater is the number of converging orbits for fixed values of γ_{12}, γ_{23}. On the other hand, the dependence of $\bar{\varepsilon}$ on γ_{12}, γ_{23} implies that t_{12}, t_{23} cannot be chosen *too* small, otherwise $\mathcal{C}$ (as well as its approximants $\mathcal{C}_n$) is badly determined. The latter effect is particularly relevant when the semi-major axis is large as it happens for the Kuiper belt (notice that the mean anomalies between two observations differ by $M_{ij} = t_{ij} a^{-3/2}$ and that the difference v_{ij} between the true anomalies, and henceforth between the t_{ij}, goes to zero with M_{ij}).

In order to see the distribution of the previous results as functions of the semi-major axis, eccentricity and inclination, we compute the percentages of successful results of the first 10,000 numbered asteroids by considering four different regions in a, e, i, each one being composed by 2,500 objects. The results are provided in Table 2 for the time intervals $t_{12} = 1^h$ and $t_{23} = 5^d$ and in Table 3 for $t_{12} = t_{23} = 10^d$. In particular, Table 3

Table 1 Percentage of successful results for Gauss, Mossotti and Laplace methods; the first number refers to the asteroids (e.g. 99.86, first line of Gauss method), while the second to Kuiper's objects (e.g. 79.67, same line)

t_{12}	t_{23}	Gauss	Mossotti	Laplace
3^d	3^d	99.86/79.67	99.55/92.03	99.00/93.33
5^d	5^d	99.87/93.33	99.45/93.98	98.90/93.98
10^d	10^d	99.78/93.98	99.23/94.30	98.73/94.63
15^d	15^d	99.58/94.47	99.27/94.47	98.54/94.63
30^d	30^d	99.45/94.63	99.36/94.47	98.17/94.63
60^d	60^d	98.77/94.63	98.41/94.63	96.00/94.63
90^d	90^d	96.80/94.63	96.73/94.63	94.32/94.63
10^d	30^d	99.60/94.63	99.45/94.63	98.01/94.63
5^d	10^d	99.82/94.47	99.56/94.63	98.63/94.63
1^h	5^d	99.77/7.32	99.72/54.79	98.82/93.17
5^h	5^d	99.87/17.40	99.77/78.53	98.86/93.66
1^h	10^d	99.80/17.40	99.66/79.84	98.60/94.31
5^h	10^d	99.81/53.17	99.67/88.62	98.55/94.30
1^h	30^d	99.68/63.25	99.62/90.24	97.59/94.63
5^h	30^d	99.70/83.85	99.64/92.84	97.61/94.63

Table 2 Percentage of successful results for Gauss, Mossotti and Laplace methods in terms of semimajor axis a (in AU), eccentricity e, inclination i (in degrees)

	Gauss	Mossotti	Laplace
$0 \le a < 2.341$	99.56	99.04	97.36
$2.341 \le a < 2.6144$	99.96	99.96	98.48
$2.6144 \le a < 3.0053$	99.80	99.96	99.52
$3.0053 \le a < 100$	99.76	99.92	99.92
$0 \le e < 0.094$	99.68	99.92	99.60
$0.094 \le e < 0.140244$	99.92	99.92	99.56
$0.140244 \le e < 0.187321$	99.84	99.64	98.52
$0.187321 \le e < 1$	99.64	99.40	97.60
$0 \le i < 3.2185$	99.72	99.76	98.68
$3.2185 \le i < 6.0218$	99.84	99.56	98.36
$6.0218 \le i < 10.918$	99.72	99.80	99.08
$10.918 \le i < 360$	99.80	99.76	99.16

Each parameter region is composed by 2,500 objects belonging to the first 10,000 numbered asteroids. The time intervals are $t_{12} = 1^h$ and $t_{23} = 5^d$

Table 3 Percentage of successful results for Gauss, Mossotti and Laplace methods in terms of semimajor axis a (in AU), eccentricity e, inclination i (in degrees)

	Gauss	Mossotti	Laplace
$0 \le a < 2.341$	99.56	97.28	96.88
$2.341 \le a < 2.6144$	99.80	99.68	98.40
$2.6144 \le a < 3.0053$	99.88	99.96	99.72
$3.0053 \le a < 100$	99.88	100	99.92
$0 \le e < 0.094$	99.92	99.80	99.56
$0.094 \le e < 0.140244$	99.84	99.84	99.48
$0.140244 \le e < 0.187321$	99.80	99.40	98.92
$0.187321 \le e < 1$	99.56	97.88	96.96
$0 \le i < 3.2185$	99.84	99.60	99.08
$3.2185 \le i < 6.0218$	99.76	98.68	97.92
$6.0218 \le i < 10.918$	99.76	99.56	98.84
$10.918 \le i < 360$	99.76	99.08	99.08

Each parameter region is composed by 2,500 objects belonging to the first 10,000 numbered asteroids. The time intervals are $t_{12} = 10^d$ and $t_{23} = 10^d$

shows that the success of all methods (slightly) grows if the semi-major axis increases, though a more reliable test should be performed over sample data with equally spaced values of the semi-major axes. On the other hand, all methods seem to be independent on the value of the inclination, while only Laplace method is affected by the value of the eccentricity, performing better for lower eccentricities.

4 Convergence of Gauss algorithm: computation of the eigenvalues of the Jacobian matrix

In the framework of Theorem 2.1, we investigate whether $\mathcal{F}_G : z' = (\zeta_1, \zeta_2) \to \mathcal{F}_G(z') \equiv (\mathcal{F}_G^1(z'), \mathcal{F}_G^2(z'))$ can be indefinitely iterated from the initial point z_0 and, eventually, if the nth iterate $z_n = \mathcal{F}_G^n(z_0)$ tends to its fixed point $z = z(\mathcal{C}, t_2, \gamma_{12}, \gamma_{23}, \varepsilon)$. Let $W \subset U$ be a closed convex neighbourhood of z; by Lagrange's theorem, if $z_1, z_2 \in W$, there exists z_1^*, z_2^* belonging to the interval (z_1, z_2), such that

$\mathcal{F}_{\rm G}(z_1) - \mathcal{F}_{\rm G}(z_2) = \partial\mathcal{F}_{\rm G}(z_1^*, z_2^*)(z_1 - z_2)$, where $\partial\mathcal{F}_{\rm G}(z_1^*, z_2^*)$ has entries $\partial_{\zeta_j}\mathcal{F}^i_{\rm G}(z_i^*)$, for $z' = (\zeta_1, \zeta_2)$. Let us assume that the complex eigenvalues $\lambda_1(x,y), \lambda_2(x,y)$ of $\partial\mathcal{F}_{\rm G}(x,y)$ verify, for $x, y \in W$

$$\lambda_1(x,y) \neq \lambda_2(x,y), \qquad |\lambda_i(x,y)| \leq \theta < 1. \tag{6}$$

For $z_1 \neq z_2 \in W$, let $\vec{v}_i^* \in \mathbf{C}^2$ denote the eigenvector corresponding to $\lambda_i^* \equiv \lambda_i(z_1^*, z_2^*)$; we define $d(z_1, z_2) \equiv |\alpha_1| + |\alpha_2|$, where $\alpha_1, \alpha_2 \in \mathbf{C}$ are such that $z_1 - z_2 = \alpha_1 \vec{v}_1^* + \alpha_2 \vec{v}_2^*$. Otherwise, for $z_1 = z_2$ we set $d(z_1, z_2) = 0$. With this choice of the metric, $\mathcal{F}_{\rm G}$ becomes a contraction on W, being $\mathcal{F}_{\rm G}(z_1) - \mathcal{F}_{\rm G}(z_2) = \lambda_1^*\alpha_1\vec{v}_1^* + \lambda_2^*\alpha_2\vec{v}_2^*$. On the other hand, one can conclude by continuity that setting $x = y = z(\mathcal{C}, t_2, \gamma_{12}, \gamma_{23}, \varepsilon)$, if $\lambda_i(z,z) \equiv \lambda_i(z)$ verify

$$\lambda_1(z) \neq \lambda_2(z) \tag{7}$$

and

$$\mu(\mathcal{C}, t_2, \gamma_{12}, \gamma_{23}, \varepsilon) \equiv \mathrm{Max}_{i=1,2}|\lambda_i(z)| < 1 \tag{8}$$

then, there exists a suitable closed convex set W containing z where (6) holds, namely, $\mathcal{F}_{\rm G}$ is a contraction. As a consequence, its unique fixed point z in W can be obtained as the limit $z = \lim_{n\to\infty} z_n$, starting from *any* $z_0 \in W$. We will see (Proposition 4.1 below) that, under slightly stronger assumptions than in Theorem 2.1 (see (9) and (10) below), condition (8) is always satisfied, provided ε is small enough. The assumptions we make are the following:

(1) The vectors $\vec{b}_2 = \vec{b}(t_2), \dot{\vec{b}}_2 = \dot{\vec{b}}(t_2), \ddot{\vec{b}}_2 = \ddot{\vec{b}}(t_2)$ are linearly independent:

$$\vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \ddot{\vec{b}}_2 \neq 0; \tag{9}$$

(2) setting $\vec{a}_2 = \vec{a}(t_2)$, one has

$$\mathcal{D} \equiv 3\frac{\vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \vec{a}_2}{\vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \ddot{\vec{b}}_2}\,\frac{\rho_2 + \vec{a}_2 \cdot \vec{b}_2}{r_2^5} \neq 1. \tag{10}$$

Remark 4.1 The independence of the $\vec{b}_i$'s is required by Gauss algorithm. Indeed, for $\varepsilon < 1$ let us expand in Taylor series as

$$\vec{b}_1 = \vec{b}_2 - \dot{\vec{b}}_2\,\gamma_{12}\,\varepsilon + \ddot{\vec{b}}_2\frac{\gamma_{12}^2}{2}\varepsilon^2 + o(\varepsilon^3)$$

$$\vec{b}_3 = \vec{b}_2 + \dot{\vec{b}}_2\,\gamma_{23}\varepsilon + \ddot{\vec{b}}_2\frac{\gamma_{23}^2}{2}\varepsilon^2 + o(\varepsilon^3); \tag{11}$$

then, by (9) for ε small one finds that $|\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3| = |\frac{1}{2}\vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \ddot{\vec{b}}_2\,\gamma_{12}\gamma_{23}\,\varepsilon^3 + o(\varepsilon^4)| > 0$. With a similar argument, one finds that condition (10) implies that for ε small $\partial_{\rho_2} G_2(P, Q, \rho_2) \neq 1$ allowing to solve Gauss equation.

Proposition 4.1 *For any* $\mathcal{C}, t_2$ *such that conditions* (9) *and* (10) *are satisfied, one has* $\mu(\mathcal{C}, t_2, \gamma_{12}, \gamma_{23}, \varepsilon) \to 0$ *as* $\varepsilon \to 0$.

The proof is given in Appendix B.

In order to prove the contractive character of $\mathcal{F}_{\rm G}$ for $0 < \varepsilon < \bar{\varepsilon}$ for a suitable $\bar{\varepsilon}$ (and, consequently, the convergence of Gauss algorithm for $0 < \varepsilon < \bar{\varepsilon}$, at least if $\bar{\varepsilon}$ is so small that the initial point $z_0 = (P_0, Q_0)$, defined in (3), belongs to W), we still need the assumption (7). In this context, we provide in Appendix B a sufficient condition (Corollary B.1), based on the computation of $\lambda_1(z), \lambda_2(z)$ at the first order in ε.

4.1 Eigenvalues of $\partial\mathcal{F}_G(z)$

Motivated by the previous discussion and by the fact that the explicit computation of μ is extremely long, we determine numerically the elements of the jacobian matrix $\partial\mathcal{F}_G(z)$, which yield the eigenvalues $|\lambda_1(z)|, |\lambda_2(z)|$. We let t_{12}, t_{23} vary, while t_2 is fixed equal to a given epoch (MJD 53450 for the asteroids, while it changes for Kuiper belt objects according to the astronomical data of footnote 1). More precisely, for each $\mathcal{C}$ (with related set of elements $(a, e, i, \omega, \Omega, M)$ at time t_2) and for each choice of t_{12}, t_{23}, we compute the three vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3$. Together with the three Sun–Earth vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$, we obtain the Earth–object directions $\vec{b}_1, \vec{b}_2, \vec{b}_3$, which provide the Gauss map $\mathcal{F}_G$ and its fixed point z.

The Jacobian $\partial\mathcal{F}_G(z)$ is computed through a polynomial interpolation. Let us consider, for example, the computation of the first element $\partial_P P'(P, Q)$ (for the other derivatives, the computation is quite similar), where $\mathcal{F}_G = (P', Q')$. Having fixed Q, we choose an odd number (say, $2n + 1$) of points $P_i = P + \text{ih}$, $i = -n, \ldots, n$, equally spaced and symmetrically distributed around P with constant step-size h, such that $2nh = 0.1$. Denoting by F_i the value of P' at $z_i = (P_i, Q)$, we approximate $\partial_P P'(P, Q)$ with the quantity $\sum_{|i| \leq n,\, i \neq 0} \frac{(-1)^{i+1}}{ih} \frac{(n!)^2}{(n-i)!(n+i)!} F_i$. The overall number of nodes is such that the difference between the values of the derivatives is smaller than 0.001 as n increases to $n + 1$. The computational details are provided in Appendix C.

4.2 Eigenvalues of asteroids and Kuiper belt objects

We compute the eigenvalues of the jacobian matrix of the Gauss map, following the algorithm outlined in the previous sections. Over a sample of 100 asteroids of the main belt we found 20 objects with at least one eigenvalue with modulus greater than one. Typically the graph of the maximum modulus of the eigenvalues versus the time intervals t_{12} or t_{23} is provided in Fig. 1 (left panel), where t_{12} and t_{23} are taken between

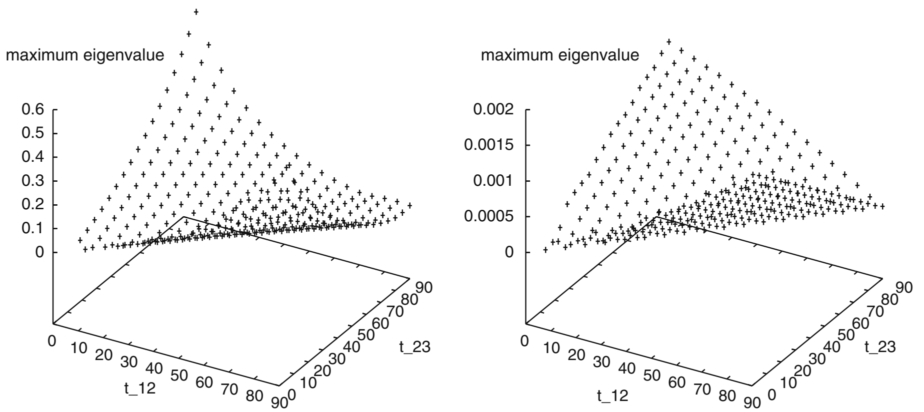

Fig. 1 Maximum modulus of the eigenvalues of the jacobian matrix versus the time intervals t_{12} and t_{23}. Left: asteroid number 8; Right: Kuiper belt object number 12

0 and 90 days with a time-step equal to 5 days. This example refers to the asteroid number 8, whose elements are $a = 0.2012$ AU, $e = 0.1563$, $i = 5.8869°$, $\omega = 284.9649°$, $\Omega = 111.0326°$, $M = 81.1258°$ at epoch MJD 53450. A similar procedure was adopted for the 615 objects of the Kuiper belt; however, contrary to the main belt objects we have not found any sample showing an eigenvalue greater than one. A typical picture of the first eigenvalue of a Kuiper belt object is provided in Fig. 1 (right panel), which corresponds to the Kuiper belt object number 12, whose elements are $a = 42.3035$ AU, $e = 0.2174$, $i = 14.0299°$, $\omega = 236.5808°$, $\Omega = 56.2982°$, $M = 336.4332°$ at epoch MJD 53400.5. We remark that in both cases the graph of $|\lambda_1|$ versus t_{12}, t_{23} is roughly symmetric with respect to the line $t_{12} = t_{23}$, where the eigenvalue approximately attains its minimum; in this situation the contractive character of the Gauss map is stronger. This remarks confirms indeed that Gauss method gives better results whenever equal observational time intervals are considered. In this case Gauss (1809) noticed that at the first iteration his method gives errors of $O(\varepsilon^2)$, instead than $O(\varepsilon)$ as obtained using different time intervals; furthermore, as remarked also by Poincaré (1906), the errors after n iterations become of order $O(\varepsilon^{2n})$ using equal time intervals, while they are $O(\varepsilon^n)$ at different time intervals.

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Appendix

A Computation of the longitude and latitude from the elliptic elements

We derive the ecliptic geocentric longitude and latitude from the elliptic elements, without taking into account topocentric corrections or aberrational effects. We restrict to consider $e < 1$. Let a, e, i, ω, Ω, M be the elliptic elements at a fixed reference epoch $T = 0$; let t_1, t_2, t_3 be the times of observations with $t_{12} = t_2 - t_1$, $t_{23} = t_3 - t_2$. The mean anomaly at time t_2 is given by $M_2 \equiv M(t_2) = M + nt_2$, where $n = ka^{-3/2}$ is the mean motion with $k = 0.985608°$/day. Similarly one has $M_1 = M_2 - nt_{12}, M_3 = M_2 + nt_{23}$. The eccentric anomalies ξ_1, ξ_2, ξ_3 at t_1, t_2, t_3 are obtained solving Kepler's equation $\xi_i - e \sin \xi_i = M_i (i = 1, 2, 3)$. Let $\vec{s} = (x, y, z)$ be the coordinates of the asteroid in the orbital frame with the x-axis coinciding with the perihelion line, i.e. $x = a(\cos \xi - e), y = a(1 - e^2)^{1/2} \sin \xi, z = 0$. Replacing ξ with ξ_1, ξ_2, ξ_3, one obtains the position vectors $\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$, which must be transformed in the ecliptic frame by means of the following three rotations:

(a) a rotation of angle ω around the z-axis;
(b) a rotation of angle i around the x-axis;
(c) a rotation of angle Ω around the z-axis.

Let the resulting vectors in the ecliptic frame be denoted as $\vec{s}_i^{(e)} (i = 1, 2, 3)$; with a similar procedure one obtains the Earth's coordinates $\vec{a}_i^{(e)} (i = 1, 2, 3)$. Defining the generic geocentric vectors as $\vec{R} \equiv \vec{s}^{(e)} - \vec{a}^{(e)} \equiv (X, Y, Z)$, the longitude of $\vec{R}$ is given by the expression $\lambda = \tan^{-1}(Y/X)$ if $X > 0$ and $\lambda = \tan^{-1}(Y/X) + \pi$ if $X < 0$, while the latitude is given by $\beta = \sin^{-1}(Z/(X^2 + Y^2 + Z^2)^{1/2})$.

B Proof of Proposition 4.1

In this appendix, we give a proof of Proposition 4.1 as a byproduct of Proposition B.1 below. Moreover (see Corollary B.1), we provide a sufficient condition to ensure that $\mathcal{F}_{\mathrm{G}}$ is a contraction for ε small. Let $\bar{\mathcal{C}}$ be a conic, and let $\bar{z} = (\bar{P}, \bar{Q})$ be its Gauss parameters.[2] We recall that we keep t_2 fixed, while t_1, t_3 are varied; let ε be the time interval between the first and the third observation and, as in (1), let $t_1 = t_2 - \gamma_{12}\varepsilon,\ t_3 = t_2 + \gamma_{23}\varepsilon$. Denote by $\mathcal{F}_{\mathrm{G}} : z = (P, Q) \to z' = (P', Q')$ the Gauss map, defined in a suitable neighbourhood of $\bar{z}$. We want to compute the eigenvalues of the jacobian matrix of $\mathcal{F}_{\mathrm{G}}$, which we denote as $\mathcal{J} = \mathcal{J}(\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}, \varepsilon) \equiv \{\mathcal{J}_{ij}\}_{i,j=1,2}$.

Proposition B.1 *Fix t_2 and $\bar{\mathcal{C}}$ such that conditions* (9) *and* (10) *are satisfied. Then, there exist $\hat{\mathcal{J}}_{11}, \hat{\mathcal{J}}_{12}, \hat{\mathcal{J}}_{21}, \hat{\mathcal{J}}_{22}$ depending on $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$, such that*

$$\mathcal{J}_{11} = \partial_P\ P'(\bar{P}, \bar{Q}) = \hat{\mathcal{J}}_{11}\varepsilon + o(\varepsilon^2) \qquad \mathcal{J}_{12} = \partial_Q\ P'(\bar{P}, \bar{Q}) = \hat{\mathcal{J}}_{12} + o(\varepsilon),$$
$$\mathcal{J}_{21} = \partial_P\ Q'(\bar{P}, \bar{Q}) = \hat{\mathcal{J}}_{21}\varepsilon^2 + o(\varepsilon^3) \qquad \mathcal{J}_{22} = \partial_Q\ Q'(\bar{P}, \bar{Q}) = \hat{\mathcal{J}}_{22}\varepsilon + o(\varepsilon^2).$$

Remark B.1 The eigenvalues $\lambda_1, \lambda_2 \in \mathbf{C}$ of $\mathcal{J}$ can be written as $\lambda_j = \hat{\iota}_j\varepsilon + o(\varepsilon^2) (j = 1, 2)$ with $\hat{\iota}_j = \tau \pm \sqrt{\tau^2 - \delta}$, where $\tau = (\hat{\mathcal{J}}_{11} + \hat{\mathcal{J}}_{22})/2$ and $\delta = \hat{\mathcal{J}}_{11}\hat{\mathcal{J}}_2 - \hat{\mathcal{J}}_{21}\hat{\mathcal{J}}_{12}$ are the semi–trace and determinant of $\hat{\mathcal{J}} = \{\hat{\mathcal{J}}_{ij}\}_{i,j=1,2}$. Moreover, if $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$ are such that $\Delta \equiv \tau^2 - \delta \neq 0$, then, $\lambda_1(\bar{z}) \neq \lambda_2(\bar{z})$ for $\varepsilon > 0$ sufficiently small.

Corollary B.1 *Let $\bar{\mathcal{C}}, t_2$ verify* (9) *and* (10) *and let γ_{12}, γ_{23} be chosen such that $\Delta \neq 0$. Then, there exists $\bar{\varepsilon} > 0$ such that, if $0 < \varepsilon < \bar{\varepsilon}$, the mapping $\mathcal{F}_{\mathrm{G}} : (W, d) \to \mathbf{R}^2$ is a contraction.*

Let us first recall the definition of the Gauss map, referring to Celletti and Pinzari (2005), for details. Let $\rho_2(P, Q)$ be the solution of Gauss equation:

$$\rho_2 = G_2(P, Q, \rho_2) \equiv -\vec{c}_2 \cdot \vec{a}_2 + \frac{\vec{c}_2 \cdot \vec{a}_1 + \vec{c}_2 \cdot \vec{a}_3 P}{P+1}\left(1 + \frac{Q}{2r_2^3}\right),$$

where $\vec{c}_i = \dfrac{\vec{b}_j \wedge \vec{b}_k}{\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3}\varepsilon_{jki}$, $r_2 = |\vec{a}_2 + \rho_2\vec{b}_2|$ and $\varepsilon_{jki} = 1$ if $\{j, k, i\}$ is an even permutation of $\{1, 2, 3\}$, $\varepsilon_{jki} = -1$ otherwise. Let $\rho_1 \equiv \rho_1(P, Q), \rho_3 \equiv \rho_3(P, Q)$ be defined as

$$\rho_1 = -\vec{c}_1 \cdot \vec{a}_1 + \frac{P+1}{1 + \frac{Q}{2r_2(\rho_2(P,Q))^3}}\vec{c}_1 \cdot \vec{a}_2 - P\vec{c}_1 \cdot \vec{a}_3 \equiv G_1(P, Q, \rho_2(P, Q)),$$
$$\rho_3 = -\frac{1}{P}\vec{c}_3 \cdot \vec{a}_1 + \frac{P+1}{P\left(1 + \frac{Q}{2r_2(\rho_2(P,Q))^3}\right)}\vec{c}_3 \cdot \vec{a}_2 - \vec{c}_3 \cdot \vec{a}_3 \equiv G_3(P, Q, \rho_2(P, Q)).$$

Let $\vec{r}_i(P, Q), i = 1, 2, 3$, be written as

$$\vec{r}_i = \vec{r}_i(P, Q) = \vec{a}_i + \rho_i(P, Q)\vec{b}_i. \tag{12}$$

It can be shown (Celletti and Pinzari 2005) that $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are coplanar and define a unique conic $\mathcal{C} = \mathcal{C}(P, Q)$ with a focus in their common origin. We also recall that the eccentricity $e = e(P, Q)$ of $\mathcal{C}$ and the argument of perihelion $g = g(P, Q)$ are given by

[2] Barred quantities will refer to $\bar{\mathcal{C}}$.

$$e = \frac{\sqrt{A^2+B^2}}{|n_{12}+n_{23}-n_{13}|},$$
$$\cos g = \frac{B}{\sqrt{A^2+B^2}}s, \qquad \sin g = -\frac{A}{\sqrt{A^2+B^2}}s, \tag{13}$$

where $n_{ij} = n_{ij}(P,Q)$ is the oriented area of the triangle formed by $\vec{r}_i, \vec{r}_j$, $s \equiv \mathrm{sgn}(n_{12}+n_{23}-n_{13})$ and, denoting by $v_{ij} \equiv 2f_{ij}$ the angle formed by $\vec{r}_i, \vec{r}_j$, one has

$$A \equiv r_2(r_3-r_1)+r_1(r_2-r_3)\cos v_{12}+r_3(r_1-r_2)\cos v_{23},$$
$$B \equiv -r_1(r_2-r_3)\sin v_{12}+r_3(r_1-r_2)\sin v_{23}. \tag{14}$$

Moreover, let P, Q be expressed by

$$P = \frac{n_{12}}{n_{23}}, \qquad Q = 2r_2^3\left(\frac{n_{12}+n_{23}}{n_{13}}-1\right) \tag{15}$$

and let $\eta_{ij} = \eta_{ij}(P,Q)$ denote the ratio of the area of the triangle formed by $\vec{r}_i$ and $\vec{r}_j$ with the corresponding conic sector. The *Gauss map* $\mathcal{F}_{\mathrm{G}}$ is finally defined by $z' = \mathcal{F}_{\mathrm{G}}(z)$, where $z' = (P', Q')$ takes the form

$$P' = \frac{\gamma_{12}}{\gamma_{23}}\frac{\eta_{12}}{\eta_{23}}, \qquad Q' = \varepsilon^2\gamma_{12}\gamma_{23}\frac{r_2^2}{r_1r_3}\frac{\eta_{12}\eta_{23}}{\cos f_{12}\cos f_{23}\cos f_{13}}. \tag{16}$$

The proof of Proposition B.1 is obtained through some technical lemmas, which provide estimates of the derivatives of $\eta_{ij}, r_i/r_j, f_{ij}$, appearing in (16) (assumptions (9) and (10) are assumed throughout all this appendix).

Lemma B.1 *There exist two constants $\mathcal{R}_P, \mathcal{R}_Q$ depending on $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$, such that for $i = 1,2,3$ one has*

$$\partial_P\varepsilon\rho_i(\bar{P},\bar{Q}) = \mathcal{R}_P + o(\varepsilon), \qquad \partial_Q\varepsilon^2\rho_i(\bar{P},\bar{Q}) = \mathcal{R}_Q + o(\varepsilon). \tag{17}$$

Proof Using (11), (9) and $\vec{a}_3 - \vec{a}_1 = \dot{\vec{a}}_2\varepsilon + o(\varepsilon^2)$, denoting for short $\vec{B} \equiv -2\frac{\vec{b}_2 \wedge \dot{\vec{b}}_2}{\vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \dot{\vec{b}}_2\gamma_{12}\gamma_{23}\varepsilon^2}$ and recalling that $\vec{c}_2 = \frac{\vec{b}_3 \wedge \vec{b}_1}{\vec{b}_1 \wedge \vec{b}_2 \cdot \vec{b}_3}$, one has

$$\vec{c}_2 = \vec{B} + o(\varepsilon^{-1}), \qquad \vec{c}_2 \cdot (\vec{a}_3 - \vec{a}_1) = \vec{B} \cdot \dot{\vec{a}}_2\varepsilon + o(1). \tag{18}$$

The implicit function theorem shows that $\rho_2(P,Q)$ is a smooth function of (P,Q), such that

$$\partial_P\rho_2(\bar{P},\bar{Q}) = \frac{\partial_P G_2(\bar{P},\bar{Q},\bar{\rho}_2)}{1-\partial G_2(\bar{P},\bar{Q},\bar{\rho}_2)} = \frac{\frac{\vec{c}_2\cdot(\vec{a}_3-\vec{a}_1)}{(P+1)^2}}{1+3\vec{c}_2\cdot\vec{a}_3\frac{\bar{Q}}{2\bar{r}_2^5}(\bar{\rho}_2+\vec{a}_2\cdot\vec{b}_2)}$$
$$+3\frac{\frac{[\vec{c}_2\cdot(\vec{a}_3-\vec{a}_1)]^2}{(P+1)^3}\frac{\bar{Q}}{2\bar{r}_2^5}(\bar{\rho}_2+\vec{a}_2\cdot\vec{b}_2)}{\left[1+3\vec{c}_2\cdot\vec{a}_3\frac{\bar{Q}}{2\bar{r}_2^5}(\bar{\rho}_2+\vec{a}_2\cdot\vec{b}_2)\right]^2} + o(\varepsilon). \tag{19}$$

An explicit expression up to $o(1)$ is obtained using (18), (19) and the estimates for $\bar{P}, \bar{Q}$ given by $\bar{P} = \frac{\gamma_{12}}{\gamma_{23}} + o(\varepsilon^2), \bar{Q} = \gamma_{12}\gamma_{23}\varepsilon^2 + o(\varepsilon^3)$:

$$\partial_P \rho_2(\bar{P}, \bar{Q}) = -\frac{2}{1-\mathcal{D}} \frac{\vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \dot{\vec{a}}_2}{\vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \ddot{\vec{b}}_2} \frac{\gamma_{23}}{\gamma_{12}\varepsilon} + o(1) \equiv \frac{\mathcal{R}_P}{\varepsilon} + o(1).$$

Similar computations allow to conclude that $\partial_P \rho_1(\bar{P}, \bar{Q}) \equiv \frac{\mathcal{R}_P}{\varepsilon} + o(1)$ and that $\partial_P \rho_3(\bar{P}, \bar{Q}) \frac{\mathcal{R}_P}{\varepsilon} + o(1)$. Concerning the derivative with respect to Q, one finds that

$$\begin{aligned} \partial_Q \rho_2(\bar{P}, \bar{Q}) &= \frac{\partial_Q G_2(\bar{P}, \bar{Q}, \bar{\rho}_2)}{1 - \partial_{\rho_2} G_2(\bar{P}, \bar{Q}, \bar{\rho}_2)} \\ &= \frac{\left[\vec{c}_2 \cdot \vec{a}_3 - \frac{\vec{c}_2 \cdot (\vec{a}_3 - \vec{a}_1)}{\bar{P}+1}\right] \frac{1}{2\bar{r}_2^3}}{1 + 3\left[\vec{c}_2 \cdot \vec{a}_3 - \frac{\vec{c}_2 \cdot (\vec{a}_3 - \vec{a}_1)}{\bar{P}+1}\right] \frac{\bar{Q}}{2\bar{r}_2^5} (\bar{\rho}_2 + \vec{a}_2 \cdot \vec{b}_2)} \end{aligned}$$

and one easily finds that $\mathcal{R}_Q$ in (17) takes the expression $\mathcal{R}_Q \equiv \frac{1}{2(1-\mathcal{D})} \frac{\varepsilon^2 \vec{B} \cdot \vec{a}_2}{\bar{r}_2^3}$.

As a corollary of the previous lemma we have the following result.

Lemma B.2 *For any $i \neq j$, there exist constants $\mathcal{R}_P^{*i,j}, \mathcal{R}_Q^{*i,j}$ depending on $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$, such that $\partial_P (r_i - r_j) = \mathcal{R}_P^{*i,j} + o(\varepsilon)$, $\partial_Q\, \varepsilon (r_i - r_j) = \mathcal{R}_Q^{*i,j} + o(\varepsilon)$ (a similar expression is valid also for r_i/r_j).*

Next we have the following

Lemma B.3 *There exist two constants $\mathcal{N}_P, \mathcal{N}_Q$ depending on $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$, such that*

$$\partial_P\, n_{13}(\bar{P}, \bar{Q}) = \mathcal{N}_P + o(\varepsilon), \qquad \partial_Q\, \varepsilon n_{13}(\bar{P}, \bar{Q}) = \mathcal{N}_Q + o(\varepsilon). \tag{20}$$

Proof Let $\vec{k}(P, Q) = \dfrac{\vec{r}_1(P, Q) \wedge \vec{r}_3(P, Q)}{|\vec{r}_1(P, Q) \wedge \vec{r}_3(P, Q)|}$ be a unit vector normal to the plane formed by $\vec{r}_1, \vec{r}_2, \vec{r}_3$. Then, $n_{13}(P, Q) = \vec{r}_1 \wedge \vec{r}_3 \cdot \vec{k}$ and

$$\partial_P n_{13}(\bar{P}, \bar{Q}) = (\partial_P \vec{r}_1 \wedge \bar{\vec{r}}_3 \cdot \bar{\vec{k}} + \bar{\vec{r}}_1 \wedge \partial_P \bar{\vec{r}}_3 \cdot \bar{\vec{k}} + \bar{\vec{r}}_1 \wedge \bar{\vec{r}}_3 \cdot \partial_P \vec{k})\,|_{(\bar{P}, \bar{Q})}.$$

Last term is zero, since $\partial_P \vec{k}(\bar{P}, \bar{Q})$ is perpendicular to $\bar{\vec{k}}$ and therefore, it is linearly dependent with $\bar{\vec{r}}_1, \bar{\vec{r}}_3$. For the remaining terms, using (12) we have

$$\begin{aligned} (\partial_P \vec{r}_1 \wedge \bar{\vec{r}}_3 \cdot \bar{\vec{k}})\,|_{(\bar{P}, \bar{Q})} &= (\vec{b}_1 \wedge \vec{a}_3 \cdot \bar{\vec{k}} + \bar{\rho}_3 \vec{b}_1 \wedge \vec{b}_3 \cdot \bar{\vec{k}})\, \partial_P \rho_1(\bar{P}, \bar{Q}), \\ (\bar{\vec{r}}_1 \wedge \partial_P \bar{\vec{r}}_3 \cdot \bar{\vec{k}})\,|_{(\bar{P}, \bar{Q})} &= (\vec{a}_1 \wedge \vec{b}_3 \cdot \bar{\vec{k}} + \bar{\rho}_1 \vec{b}_1 \wedge \vec{b}_3 \cdot \bar{\vec{k}})\, \partial_P \rho_3(\bar{P}, \bar{Q}). \end{aligned}$$

By (11), the two terms in parenthesis are both equalto $\bar{\rho}_2 \vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \bar{\vec{k}} \varepsilon$ up to $o(\varepsilon^2)$, while for the first term we remark that $\vec{b}_1 \wedge \vec{a}_3 + \vec{a}_1 \wedge \vec{b}_3 = (\vec{a}_2 \wedge \dot{\vec{b}}_2 - \dot{\vec{a}}_2 \wedge \vec{b}_2)\varepsilon + o(\varepsilon^2)$. Casting together the previous formulae and using Lemma B.1, we conclude that

$$\partial_P n_{13}(\bar{P}, \bar{Q}) = (\vec{a}_2 \wedge \dot{\vec{b}}_2 \cdot \bar{\vec{k}} - \dot{\vec{a}}_2 \wedge \vec{b}_2 \cdot \bar{\vec{k}} + 2\bar{\rho}_2 \vec{b}_2 \wedge \dot{\vec{b}}_2 \cdot \bar{\vec{k}})\, \mathcal{R}_P + o(\varepsilon),$$

which can be written as $\partial_P n_{13}(\bar{P}, \bar{Q}) \equiv \mathcal{N}_{P+o}(\varepsilon)$ for a suitable constant $\mathcal{N}_P$. In a similar way one obtains the second of (20).

Remark B.2 Similar results hold for $\vec{\partial} n_{12}(\bar{P},\bar{Q}), \bar{\partial} n_{23}(\bar{P},\bar{Q})$. More precisely, for $(i,j) = (1,2),(2,3)$, one has $\partial_P n_{ij} = \mathcal{N}_P\, \gamma_{ij} + o(\varepsilon), \partial_Q \varepsilon n_{ij} = \mathcal{N}_Q \gamma_{ij} + o(\varepsilon)$. As a consequence of Lemmas B.1, B.3 and of the previous remark, a similar estimate holds for v_{ij}, being $\sin v_{ij} = n_{ij}/(r_i r_j)$.

Lemma B.4 *There exist two constants $\mathcal{S}_P, \mathcal{S}_Q$ depending on $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$, such that*

$$\partial_P(\sin v_{12} + \sin v_{23} - \sin v_{13})(\bar{P},\bar{Q}) = \mathcal{S}_P\, \varepsilon^2 + o(\varepsilon^4),$$
$$\partial_Q(\sin v_{12} + \sin v_{23} - \sin v_{13})(\bar{P},\bar{Q}) = \mathcal{S}_Q\, \varepsilon + o(\varepsilon^3).$$

Next step is to evaluate the derivatives of the eccentricity $e(P,Q)$ of $\mathcal{C}(P,Q)$.

Lemma B.5 *There exist two constants $\mathcal{E}_P, \mathcal{E}_Q$ depending on $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$, such that $\partial_P\, \varepsilon e(\bar{P},\bar{Q}) = \mathcal{E}_P + o(\varepsilon),\ \partial_Q\, \varepsilon^2 e(\bar{P},\bar{Q}) = \mathcal{E}_Q + o(\varepsilon)$.*

Proof From (13), we obtain ($\vec{\partial} \equiv (\partial_P, \partial_Q)$):

$$\vec{\partial} e(\bar{P},\bar{Q}) = \frac{\vec{\partial}\sqrt{A^2+B^2}(\bar{P},\bar{Q})}{|\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13}|} - \bar{e}\frac{\vec{\partial}(n_{12}+n_{23}-n_{13})(\bar{P},\bar{Q})}{\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13}};$$

therefore, we can take $\mathcal{E}_P = \mathcal{E}_P^1 - \bar{e}\mathcal{E}_P^2, \mathcal{E}_Q = \mathcal{E}_Q^1 - \bar{e}\mathcal{E}_Q^2$, where $\mathcal{E}_P^i, \mathcal{E}_Q^i$ are such that

$$\varepsilon\frac{\partial_P\sqrt{A^2+B^2}(\bar{P},\bar{Q})}{|\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13}|} = \mathcal{E}_P^1 + o(\varepsilon),$$
$$\varepsilon^2\frac{\partial_Q\sqrt{A^2+B^2}(\bar{P},\bar{Q})}{|\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13}|} = \mathcal{E}_Q^1 + o(\varepsilon) \tag{21}$$

and

$$\varepsilon\frac{\partial_P(n_{12}+n_{23}-n_{13})(\bar{P},\bar{Q})}{(\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13})} = \mathcal{E}_P^2 + o(\varepsilon),$$
$$\varepsilon^2\frac{\partial_Q(n_{12}+n_{23}-n_{13})(\bar{P},\bar{Q})}{(\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13})} = \mathcal{E}_Q^2 + o(\varepsilon). \tag{22}$$

To prove (21) we proceed as follows. From the second of (15) with $(P,Q) = (\bar{P},\bar{Q})$, one has

$$|\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13}| = \bar{n}_{13}\frac{\bar{Q}}{2\bar{r}_2^3} = \sqrt{\bar{p}}\frac{\gamma_{12}\gamma_{23}}{2\bar{r}_2^3}\varepsilon^3 + o(\varepsilon^4), \tag{23}$$

where $\bar{p}$ is the parameter of $\bar{\mathcal{C}}$ and $\bar{n}_{13} = \sqrt{\bar{p}}\ \varepsilon + o(\varepsilon^2)$. Using (13) one has

$$\vec{\partial}\sqrt{A^2+B^2}(\bar{P},\bar{Q}) = \hat{\bar{g}} \cdot \vec{\partial} R^{\perp}, \tag{24}$$

where $R^{\perp} = (sB, -sA)$. Therefore, we need to evaluate $\vec{\partial} A(P,Q), \partial B(P,Q)$. To this end, rewrite (14) as

$$A = -r_1(r_2-r_3)(1-\cos v_{12}) - r_3(r_1-r_2)(1-\cos v_{23})$$
$$B = r_1 r_3(\sin v_{12} + \sin v_{23} - \sin v_{13}) - (n_{12}+n_{23}-n_{13}), \tag{25}$$

where we used $n_{ij} = r_i r_j \sin v_{ij}$. From (25) one has

$$\vec{\partial} A(\bar{P},\bar{Q}) = \sum_{i=1}^{3}\vec{A}_i, \qquad \vec{\partial} B(\bar{P},\bar{Q}) = \sum_{i=1}^{3}\vec{B}_i,$$

where

$$\begin{aligned}
\vec{A}_1 &= -\vec{\partial} r_1(\bar{P},\bar{Q})(\bar{r}_2-\bar{r}_3)(1-\cos\bar{v}_{12}) - \vec{\partial} r_3(\bar{P},\bar{Q})(\bar{r}_1-\bar{r}_2)(1-\cos\bar{v}_{23}),\\
\vec{A}_2 &= -\bar{r}_1\vec{\partial}(r_2-r_3)(\bar{P},\bar{Q})(1-\cos\bar{v}_{12}) - \bar{r}_3\vec{\partial}(r_1-r_2)(\bar{P},\bar{Q})(1-\cos\bar{v}_{23}),\\
\vec{A}_3 &= -\bar{r}_1(\bar{r}_2-\bar{r}_3)\vec{\partial}(1-\cos v_{12})(\bar{P},\bar{Q}) - \bar{r}_3(\bar{r}_1-\bar{r}_2)\vec{\partial}(1-\cos v_{23})(\bar{P},\bar{Q}),\\
\vec{B}_1 &= \vec{\partial} r_1\bar{r}_3(\sin\bar{v}_{12}+\sin\bar{v}_{23}-\sin\bar{v}_{13}) + \bar{r}_1\vec{\partial} r_3(\sin\bar{v}_{12}+\sin\bar{v}_{23}-\sin\bar{v}_{13}),\\
\vec{B}_2 &= \bar{r}_1\bar{r}_3\vec{\partial}(\sin v_{12}+\sin v_{23}-\sin v_{13}),\\
\vec{B}_3 &= -\vec{\partial}(n_{12}+n_{23}-n_{13}).
\end{aligned}$$

Using Taylor formula for $\bar{r}_1,\bar{r}_3,\bar{v}_1,\bar{v}_3$ and recalling Lemmas B.1, B.2, B.4, we find that for suitable constants $\mathcal{A}_P,\mathcal{A}_Q,\mathcal{B}_P,\mathcal{B}_Q$, one has

$$\begin{aligned}
\partial_P\, A(\bar{P},\bar{Q}) &= \mathcal{A}_P\,\varepsilon^2 + o(\varepsilon^3), \quad \partial_Q\, A(\bar{P},\bar{Q}) = \mathcal{A}_Q\,\varepsilon + o(\varepsilon^2),\\
\partial_P\, B(\bar{P},\bar{Q}) &= \mathcal{B}_P\,\varepsilon^2 + o(\varepsilon^3), \quad \partial_Q\, B(\bar{P},\bar{Q}) = \mathcal{B}_Q\,\varepsilon + o(\varepsilon^2).
\end{aligned} \tag{26}$$

The proof of (21) is obtained casting together (26), (24) and (23). The proof of (22) is quite similar: using (15) we have

$$\frac{\vec{\partial}\,[n_{12}+n_{23}-n_{13}]\,(\bar{P},\bar{Q})}{\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13}} = \frac{\vec{\partial} n_{13}(\bar{P},\bar{Q})}{\bar{n}_{13}} + \frac{\vec{\partial} Q(\bar{P},\bar{Q})}{\bar{Q}} - 3\frac{\vec{\partial} r_2(\bar{P},\bar{Q})}{\bar{r}_2}.$$

Therefore, by Lemmas B.1, B.3, we obtain (22).

We remark that (26) allows to evaluate the derivatives of the true anomaly $v_2=-g$; indeed, taking the gradient of $\tan v_2 = A/B$ (see (13)), one has:

$$\begin{aligned}
\vec{\partial} v_2(\bar{P},\bar{Q}) &= \cos^2\bar{v}_2\left[\frac{\vec{\partial} A(\bar{P},\bar{Q})}{\bar{B}} - \frac{\bar{A}}{\bar{B}^2}\vec{\partial} B(\bar{P},\bar{Q})\right]\\
&= \bar{s}\frac{\cos\bar{g}}{\sqrt{\bar{A}^2+\bar{B}^2}}\left[\vec{\partial} A(\bar{P},\bar{Q}) + \tan\bar{g}\,\vec{\partial} B(\bar{P},\bar{Q})\right],
\end{aligned}$$

where $\sqrt{\bar{A}^2+\bar{B}^2} = \bar{e}|\bar{n}_{12}+\bar{n}_{23}-\bar{n}_{13}| = \frac{\bar{e}\sqrt{\bar{p}}\gamma_{12}\gamma_{23}}{2\bar{r}_2^3}\varepsilon^3 + o(\varepsilon^4)$ (see (13), (23)). Therefore, we obtain the following.

Lemma B.6 *There exist two constants $\mathcal{N}_P^2,\mathcal{N}_Q^2$ depending on $\bar{\mathcal{C}},t_2,\gamma_{12},\gamma_{23}$ such that*

$$\varepsilon\,\partial_P v_2(\bar{P},\bar{Q}) = \mathcal{N}_P^2 + o(\varepsilon), \qquad \varepsilon^2\partial_Q v_2(\bar{P},\bar{Q}) = \mathcal{N}_Q^2 + o(\varepsilon).$$

Finally we are able to compute the lowest orders of the quantities $\eta_{ij}=\eta_{ij}(P,Q)=n_{ij}/S_{ij}$ appearing in the definition of $\mathcal{F}_{\mathrm{G}}$ (see (16)). For simplicity we assume to deal with an elliptic trajectory, i.e. $\bar{e}<1$, though the results can be extended to any value of the eccentricity. Let $z=(P,Q)$ vary in a small neighbourhood of $\bar{z}=(\bar{P},\bar{Q})$. If $\xi_i=\xi_i(P,Q)$ denotes the eccentric anomaly and if $M_i=M_i(P,Q)=\xi_i-e\sin\xi_i$ is the mean anomaly, the quantity η_{23} can be expressed as

$$\eta_{23} = \frac{\sin(\xi_3-\xi_2)-e(\sin\xi_3-\sin\xi_2)}{M_{23}} = 1-\frac{\xi_{23}-\sin\xi_{23}}{M_{23}},$$

where $\xi_{ij} = \xi_j - \xi_i$, $M_{ij} = M_j - M_i$. Therefore, we have

$$\begin{aligned}
\vec{\partial}\eta_{23}(\bar{P},\bar{Q}) &= -\frac{\vec{\partial}(\xi_{23} - \sin\xi_{23})}{\bar{M}_{23}} + \frac{(\bar{\xi}_{23} - \sin\bar{\xi}_{23})\vec{\partial}M_{23}}{\bar{M}_{23}^2} \\
&= -\left(\frac{\bar{\xi}_{23}^2}{2\bar{M}_{23}} + o\left(\bar{\xi}_{23}^4/\bar{M}_{23}\right)\right)\vec{\partial}\xi_{23}(\bar{P},\bar{Q}) \\
&\quad + \left(\frac{\bar{\xi}_{23}^3}{6\bar{M}_{23}^2} + o\left(\bar{\xi}_{23}^5/\bar{M}_{23}^2\right)\right)\vec{\partial}M_{23}(\bar{P},\bar{Q}) \\
&= [\varepsilon\mathcal{E}_1 + o(\varepsilon^3)]\vec{\partial}\xi_{23}(\bar{P},\bar{Q}) + [\varepsilon\mathcal{E}_2 + o(\varepsilon^3)]\vec{\partial}M_{23}(\bar{P},\bar{Q}), \qquad (27)
\end{aligned}$$

where we used $\bar{M}_{ij} = \gamma_{ij}\bar{a}^{-3/2}\varepsilon, \bar{\xi}_{ij} = \bar{M}_{ij}/(1 - \bar{e}\cos\bar{\xi}_i) + o(\varepsilon^2)$, with $\mathcal{E}_1, \mathcal{E}_2$ being two suitable constants. We proceed to compute $\vec{\partial}\xi_{23}(\bar{P},\bar{Q}), \vec{\partial}M_{23}(\bar{P},\bar{Q})$. Using the classical relations

$$\xi_i = 2\tan^{-1}\left(f(e)\tan\frac{v_i}{2}\right), \qquad f(e) \equiv \sqrt{\frac{1-e}{1+e}}$$

and recalling Lemmas B.5 and B.6, one finds that

$$\begin{aligned}
\partial_P\xi_{23}(\bar{P},\bar{Q}) &= \mathcal{X}_P^{23} + o(\varepsilon), \qquad \partial_{Q}\varepsilon\xi_{23}(\bar{P},\bar{Q}) = \mathcal{X}_Q^{23} + o(\varepsilon), \\
\partial_P M_{23}(\bar{P},\bar{Q}) &= \mathcal{M}_P^{23} + o(\varepsilon), \qquad \partial_{Q}\varepsilon M_{23}(\bar{P},\bar{Q}) = \mathcal{M}_Q^{23} + o(\varepsilon) \qquad (28)
\end{aligned}$$

for some quantities $\mathcal{X}_Q^{23}, \mathcal{X}_Q^{23}, \mathcal{M}_P^{23}, \mathcal{M}_Q^{23}$ depending only on $\bar{\mathcal{C}}, \gamma_{12}, \gamma_{23}$. Inserting (28) in (27), we obtain the following

Lemma B.7 *Let $i \neq j \in \{1,2,3\}$. There exist two constants $\varepsilon_P^{ij}, \varepsilon_Q^{ij}$ depending on $\bar{\mathcal{C}}, t_2, \gamma_{12}, \gamma_{23}$, such that*

$$\partial_P\eta_{ij}(\bar{P},\bar{Q}) = \mathcal{E}_P^{ij}\varepsilon + o(\varepsilon^2), \qquad \partial_Q\eta_{ij}(\bar{P},\bar{Q}) = \mathcal{E}_Q^{ij} + o(\varepsilon).$$

We are finally ready to complete the
Proof of proposition B.1 From the definition of the Gauss map (16), one has

$$\begin{aligned}
\vec{\partial}P'(\bar{P},\bar{Q}) &= \frac{\bar{\eta}_{12}}{\bar{\eta}_{23}}\left[\frac{\vec{\partial}\eta_{12}(\bar{P},\bar{Q})}{\bar{\eta}_{12}} - \frac{\vec{\partial}\eta_{23}(\bar{P},\bar{Q})}{\bar{\eta}_{23}}\right] \\
\vec{\partial}Q'(\bar{P},\bar{Q}) &= \gamma_{12}\gamma_{23}\bar{q}\left[\varepsilon^2\frac{\vec{\partial}r_2/r_1(\bar{P},\bar{Q})}{\bar{r}_2/\bar{r}_1} + \varepsilon^2\frac{\vec{\partial}r_2/r_3(\bar{P},\bar{Q})}{\bar{r}_2/\bar{r}_3}\right] \\
&\quad + \gamma_{12}\gamma_{23}\bar{q}\left[\varepsilon^2\frac{\vec{\partial}\eta_{12}(\bar{P},\bar{Q})}{\bar{\eta}_{12}} + \varepsilon^2\frac{\vec{\partial}\eta_{23}(\bar{P},\bar{Q})}{\bar{\eta}_{23}}\right] - \gamma_{12}\gamma_{23}\bar{q} \\
&\quad \times\left[\varepsilon^2\tan\bar{f}_{12}\vec{\partial}f_{12}(\bar{P},\bar{Q}) + \varepsilon^2\tan\bar{f}_{23}\vec{\partial}f_{23}(\bar{P},\bar{Q}) + \varepsilon^2\tan\bar{f}_{13}\vec{\partial}f_{13}(\bar{P},\bar{Q})\right],
\end{aligned}$$

where $\bar{q} \equiv \frac{\bar{Q}}{\gamma_{12}\gamma_{23}\varepsilon^2} = \frac{\bar{r}_2^2}{\bar{r}_1\bar{r}_3}\bar{\eta}_{12}\bar{\eta}_{23}\frac{1}{\cos\bar{f}_{12}\cos\bar{f}_{23}\cos\bar{f}_{13}}$. For $i \neq j$, let $\bar{\eta}_{ij} = 1 + o(\varepsilon^2)$, $\bar{r}_i/\bar{r}_j = 1 + o(\varepsilon), \cos\bar{f}_{ij} = 1 + o(\varepsilon^2)$; therefore, $\bar{q} = 1 + o(\varepsilon)$ and using Lemma B.7 to evaluate $\vec{\partial}\eta_{ij}(\bar{P},\bar{Q})$, Lemma B.2 to evaluate $\vec{\partial}[r_i/r_j](\bar{P},\bar{Q})$ and the Remark B.2 to evaluate $\vec{\partial}f_{ij}(\bar{P},\bar{Q}) = \vec{\partial}v_{ij}(\bar{P},\bar{Q})/2$, we find the result of Proposition B.1.

C Computation of the derivatives by polynomial interpolation

Suppose we want to compute the derivative at some point $\bar{x}$ of the function $x \to f(x)$, using a polynomial interpolation. Let $x_i = \bar{x} + ih, i = -n, \ldots, n$ be the nodes around $\bar{x}$ and let $y_i = f(x_i)$; we define the interpolating Laplace polynomial $\mathcal{P}_n$ of degree $2n$ as

$$\mathcal{P}_n(x) = \sum_{i=-n}^{n} \frac{\prod_{j\neq i}(x - x_j)}{\prod_{j\neq i}(x_i - x_j)} y_i.$$

After the change of variable $s = (x - \bar{x})/h$, one obtains

$$\mathcal{P}_n(\bar{x} + sh) = \sum_{i=-n}^{n} \frac{\Pi_{j\neq i}(s - j)}{\Pi_{j\neq i}(i - j)} y_i \equiv \mathcal{Q}_n(s).$$

The derivative $\mathrm{d}f(\bar{x})/\mathrm{d}x$ is approximated by $\mathrm{d}\mathcal{P}_n(\bar{x})/\mathrm{d}x = h^{-1}\mathrm{d}\mathcal{Q}_n(0)/\mathrm{d}s$. Let us consider first the term with $i = 0$:

$$\frac{\Pi_{j\neq 0}(s - j)}{\Pi_{j\neq 0}(-j)} y_0 = \frac{(s-n)(s-n+1)\cdots(s-1)(s+1)\cdots(s+n-1)(s+n)}{(-1)^n(j!)^2} y_0.$$

This term is an even function of s, so that its derivative at $s = 0$ is zero. On the other hand, deriving (through Leibnitz rule) with respect to s the remaining terms of the sum, for any $i \neq 0$ one has:

$$\begin{aligned} &\frac{(s-n)\cdots[s-(i+1)][s-(i-1)]\cdots(s)\cdots(s+n)}{(i-n)\cdots(-1)(1)\cdots(i+n)} y_i \\ &= \frac{(s-n)\cdots[s-(i+1)][s-(i-1)]\cdots(s)\cdots(s+n)}{(-1)^{n-i}(n-i)!(n+i)!} y_i; \end{aligned}$$

computing these terms at $s = 0$, the only one which survives is given by

$$\frac{(-1)^{i+1}}{i} \frac{(n!)^2}{(n-i)!(n+i)!} y_i \quad (i \neq 0).$$

Finally, one concludes that

$$\frac{\mathrm{d}\mathcal{P}_n}{\mathrm{d}x}(\bar{x}) = \frac{1}{h}\frac{\mathrm{d}\mathcal{Q}_n}{\mathrm{d}s}(0) = \sum_{|i|\leq n,\, i\neq 0} \frac{(-1)^{i+1}}{ih} \frac{(n!)^2}{(n-i)!(n+i)!} y_i.$$

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