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A HYSTERETIC HAMMER-STRING INTERACTION MODEL FOR PHYSICAL MODEL SYNTEHESIS

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1 INTRODUCTION

Classical models of piano hammer, based on a mass and a nonlinear spring which gives account of felt force vs. deformation characteristics, have been proposed, for example in [1-4], [6-8]. They are almost always based on the relation $f = k\left(\Delta y\right)^{\alpha} 1(\Delta y)$, where f is the compression force [N], Δy is the felt compression [m], 1(.) is the Heaviside function which implements the so called "contact condition" and k [N/m$^{\alpha}$] and α [adim] are constants depending on felt characteristics. This class of models has been proved efficient and satisfactory by many authors; however, in real time synthesis applications, it has some problems. First problem is that in real pianos k and α vary unpredictably both along the keyboard and in different instruments. This comes from the fact that skillful tuners adjust the felt characteristics in order to get a better sound [4], and this results in a k and α modification. Thus, it would be of great importance to control these parameters in a musically interesting model; unfortunately, it is difficult to implement in a fixed-point DSP architecture. Another problem comes from the hysteretic nature of the hammer felt. Many experiments [6] have proved that the traditional instantaneous law for the compression/force characteristic is inadequate for a real hammer impact simulation.

For these reasons, we first propose a model of piano hammer which offers simple and flexible control over the static characteristic of the felt. The model is stable, accurate and efficient and it gives the possibility of varying the characteristic of the felt through two quite intuitive parameters. As a second step, we extend the static model in order to give account of the hysteresis phenomenon. This is done according to the theory exposed in [6], preserving the properties of stability, accuracy and efficiency of the former model.

2 THE ELASTIC HAMMER

In a perfecly elastic hammer, the compression characteristic of the felt can be obtained applying a known force and measuring the relative compression. The experimental data can then be fitted by a polynomial approximation [7]. However, it has been show that a power law is adequate in characterizing felt properties [4]. For real time synthesis, we need an expression as general as possible, capable of simulating a continuous variation of

α between 2 and 4 (at least) but we also need a polynomial for ease in calculus and, if possible, only one control parameter. For these reasons we have studied a polynomial

model based on linear interpolation of the II and IV order curves through an adimensional coefficient varying from 0 to 1.

$$f(\Delta y, \eta, k) = K\eta\left(\Delta y / y_0\right)^4 + K(1-\eta)\left(\Delta y / y_0\right)^2 \tag{1}$$

In this model K [N] is the force measured at the known y_0 [m] compression and η is the "shape coefficient": when η=0 the felt exhibits a II order nonlinearity, when η=1 it exhibits a IV order nonlinearity; in the intermediate cases we get an intermediate behavior. Note that we are not interested to an *exact* approximation of the power law, but more to a qualitative behavior.

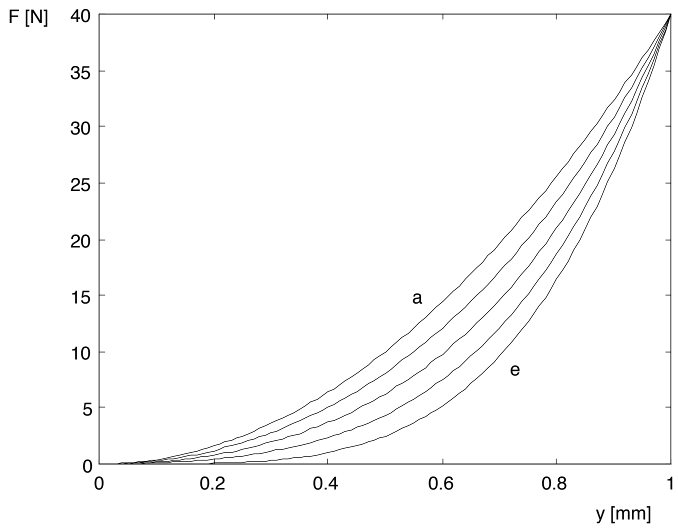

Figure 1. Static force-compression characteristic of the felt. y_0=1 mm, K=40 N, η=a) 0, 0.25, 0.5 0.75 and e) 1 [adim].

3 APPROXIMATED IMPLEMENTATION

In a classical hammer-string interaction model, the equations of the hammer are:

$$\begin{cases} f(t) = k\left(y_h(t) - y(t)\right)^\alpha \\ f(t) = -m_h \dfrac{d^2 y_h}{dt^2} \end{cases} \tag{2}$$

together with initial hammer speed and position. These equation can be solved in a digital environment, together with a suitable model for the string. The discretized form, based on the backward difference approximation of derivatives and a digital waveguides [5] model of the string, is described in [2]:

$$\begin{cases} v(n) = \dfrac{1}{2Z} f(n) + v_i(n) \\ f(n) = k\left(T \displaystyle\sum_{j=0}^{n} v_h(j) - v(j) \right)^{\alpha} \\ v_h(n) = v_h(n-1) - \dfrac{T}{m_h} f(n-1) \end{cases} \tag{3}$$

The equations of the string have been obtainded by applying Kirchhoff's laws to the interaction point (see, for example, [8]). The system (3) is implicit in the variables $f(n)$ and $v(n)$. In the approximated solution method, used for example in [3], the system is not solved but the net velocity at the contact point is extimated by the following formula:

$$v(n) = \frac{1}{2Z} f(n) + v_i(n) \cong \frac{1}{2Z} f(n-1) + v_i(n) \tag{4}$$

Of course, the higher the sampling frequency, the more accurate the approximation of the actual force with the past force. The approximated hammer-string model offers two main advantages: efficiency and flexibility. Efficiency comes from the direct implementation of the equations, with no additional overhead in solving the nonlinear system. Flexibility comes from the fact that almost "every" compression characteristic can be implemented in the model; for example, the linear interpolation method discussed above can be easily tested in this framework. However,the approximated model shows serious drawbacks: being based on an estimation of istantaneous force, it suffers of instability problems. Actually, if sampling frequency is not high enough, if string is short or the hammer is too near to a termination, if the impact velocity of hammer is high or if hammer mass is too small, model behavior exhibits a strong departure from correct values. It may worth note that many hammer models, despite of formal differences in their equations, suffer of these instability problems due to the unit delay inserted for computation purposes (see for example the model in [8]).

4 THE SYSTEM SOLUTION

In order to avoid instability problems, we now derive the digital solution of the hammer-string interaction system in the case of a power law with $\alpha>1$. The method is based on the separation of known terms, both past and istantaneous, from istantaneous and unknown terms; the latter are then rewritten as functions of force variable and the resulting implicit equation is solved. Equations for a hammer-string interaction system have already been presented above: here we slightly change the solution method, by approximating continuous time integrals with discrete sums using the trapezoids method.

$$\begin{cases} v(n) = \dfrac{1}{2Z} f(n) + v_i(n) \\ f(n) = k\left(y_h(n) - y(n) \right)^{\alpha} \\ v_h(n) = v_h(n-1) - \dfrac{T}{2m}\left(f(n) + f(n-1) \right) \end{cases} \tag{5}$$

Here y and y_h are string and hammer position respectively [m] and v_h and v_i [m/s] are the hammer speed and the incoming string velocity at the contact point. By using again digital integration on v_h and v, and substituting the first of (5) into v expression, we find:

$$y_h(n) = y_h(n-1) + \frac{T}{2}\left(v_h(n) + v_h(n-1)\right) \tag{6}$$

$$y(n) = y(n-1) - \frac{T}{2}\left(v_i(n) + \frac{1}{2Z} f(n)\right) - \frac{T}{2}\left(+v_i(n-1) + \frac{1}{2Z} f(n-1)\right) \tag{7}$$

Subtracting eq. (7) from (6), the felt compression becomes:

$$\Delta y(n) \equiv y_h(n) - y(n) = T\left(x(n) - bf(n)\right) \tag{8}$$

where we define $b \equiv \frac{1}{2}\left(\frac{T}{2m} + \frac{1}{2Z}\right)$ and

$$x(n) \equiv \frac{\Delta y(n-1)}{T} + v_h(n-1) - \frac{v_i(n) + v_i(n-1)}{2} - bf(n-1) \tag{9}$$

Finally the force expression is:

$$f(n) = a\left(x(n) - bf(n)\right)^{\alpha} \tag{10}$$

where $a \equiv kT^{\alpha}$. We observe that in f expression, the $x(n)$ definition collects all terms known at time n and the only term which is unknown is $f(n)$ itself. To complete the digital system, we note that substituting eq. (8) in (9) a recursive form for $x(n)$ can be obtained. The system can then be written as:

$$\begin{cases} x(n) = x(n-1) + v_h(n-1) - \dfrac{v_i(n) + v_i(n-1)}{2} - 2bf(n-1) \\ f(n) = a\left(x(n) - bf(n)\right)^{\alpha} \\ v_h(n) = v_h(n-1) - \dfrac{T}{2m}\left(f(n) + f(n-1)\right) \\ v(n) = \dfrac{1}{2Z} f(n) + v_i(n) \end{cases} \tag{11}$$

The second (implicit) equation must now be solved. Of course, it is possible to calculate the analitic solutions for it, and hence for the system, when α=2, 3 and 4. Neverthless, the DSP implementation requires the tabulation of the values of $f=g_{\alpha}(x)$ and so a suitable numerical method can be used to solve the equation above. This is particularly attractive since it is possible to express the implicit equation in adimensional form, allowing the use of a single table for each value of α required, as will be shown in next section.

4.1 Adimensional form of *x* and *f*

Let us consider the implicit expression of $f(x, f)=0$. If we pose:

$$C \equiv (ab)^{1/(\alpha-1)}, \qquad X(n) \equiv Cx(n), \qquad F(n) \equiv Cbf(n) \tag{12}$$

substituting, we find:

$$F(n) = \left(X(n) - F(n)\right)^{\alpha} \tag{13}$$

One of the zeroes of the function gives the values of F(n) as function of X(n), and so allows the calculus of f(n) as a function of x(n). The relevant zero is in 0 when x(n)=0.

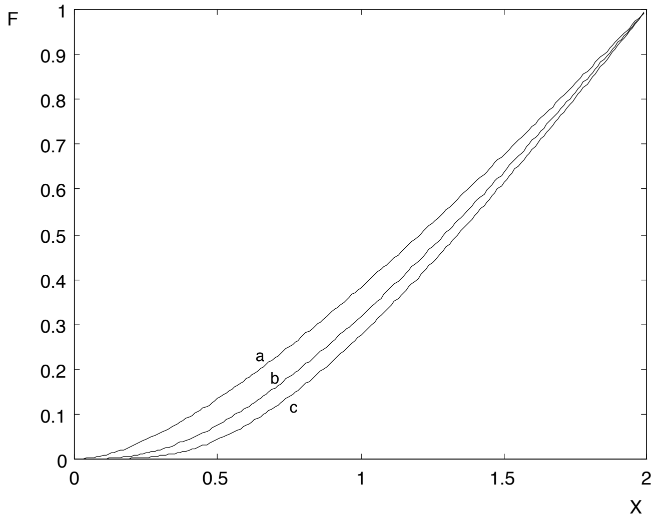

Figure 2. F=Cbf [adim] as a function of X=Cx [adim] for α*= (a) 2.0, (b) 3.0 and (c) 4.0. In these cases the implicit function is a polynomial.*

Note that, if the parameters of the model (Z, m_h, k and α) vary only between interactions, as when a player change a parameter and then listen to the results, C and Cb can be recalculated offline and the calculus can be done possibly by the host processor of the DSP.

4.2 Interpretation of $x(n)$ as a pseudo-compression

As figure 2 suggests, we can also give the hammer-string contact condition as a function of the sign of $x(n)$ or $X(n)$. Actually, as shown above, the expression of the compression of the felt during contact is given by eq. (8): hence, the release condition should be evaluated comparing $x(n)$-$bf(n)$ with zero. However, since:

$$\lim_{x \to 0^+} \frac{f(n)}{x(n)} = 0 \tag{14}$$

the release condition for Δy near zero can be given only as a function of $x(n)$ or $X(n)$. Simmetrically, when there is no contact, f=0 for definition and $\Delta y(n)=Tx(n)$. The union of the two cases exposed allows us to define a contact condition based on the sign of $x(n)$: hence, $x(n)$ can be regarded both as the history of the system and as a "pseudo-compression" variable. In summary, as for Δy, the knowledge of $x(n)$ is sufficient both to evaluate the contact contition and to calculate the value of $f(n)$.

4.3 Force interpolation

It is still possible to interpolate 2nd and 4th degree expressions in order to simulate a smooth variation of α between 2 and 4. This can be done with the following formula:

$$f(n) = \eta g_2(x(n)) + (1-\eta) g_4(x(n)) \tag{15}$$

where $g_2(.)$ and $g_4(.)$ are the second and fourth degree solutions curves. Hence, varying η from 0 to 1 we obtain a linear interpolation of the curves in the (X,F) plane, which corresponds to an interpolation of the curves in the $(\Delta y, f)$ plane.

5 ADDING FELT HYSTERESIS

From Stulov [6], for an hysteretic elastic medium we have in general:

$$\bar{f} = f(\Delta y) - h(t) * f(\Delta y) \tag{16}$$

where the symbol * denotes linear convolution. This can be interpreted as the expression of a material with a time-variying elastic coefficient. In our case we have:

$$f(\Delta y) = k \Delta y^{\alpha} \qquad h(t) = (\varepsilon_0 / \tau_0) \exp(-t/\tau_0) \equiv \varepsilon \exp(-t/\tau_0) \tag{17}$$

where $f(.)$ is the above discussed nonlinear function of the felt compression and $h(.)$ is a low-pass kernel, with ε_0 and τ_0 [s] suitable costants which depend on felt characteristics and $\varepsilon=\varepsilon_0/\tau_0$ an adimensional constant between 0 and 1. The above expression can be rewritten as: $\bar{f} = (\delta(t) - \varepsilon \exp(-t/\tau_0)) * k(\Delta y)^{\alpha}$ and we notice that this kernel is now the impulsive response of a first-order high-pass system. According to Stulov's equation, an hysteretic felt can hence be modeled as the cascade of a nonlinear distortion block and of a linear, high-pass filter.

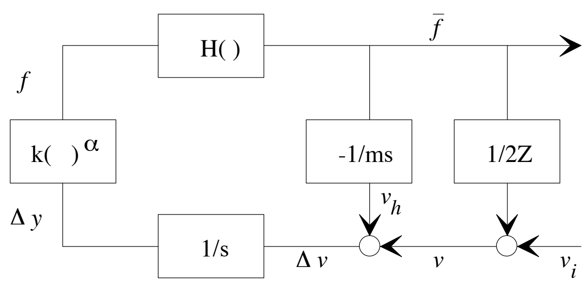

Figure 3. Hysteretic hammer scheme. Note that for H(.)=1 we are in the static case.

In discrete time, we need to implement this system by means of a digital filter. We choose to apply the well-known bilinear trasformation method to the frequency response of the analog filter. We obtain:

$$H(z) = a_1 \frac{1 - a_3 z^{-1}}{1 - a_2 z^{-1}} \tag{18}$$

where:

$$a_1 \equiv \frac{(1-\varepsilon)T + 2\tau_0}{T + 2\tau_0}, \qquad a_2 \equiv \frac{2\tau_0 - T}{2\tau_0 + T}, \qquad a_3 \equiv \frac{2\tau_0 - (1-\varepsilon)T}{2\tau_0 + (1-\varepsilon)T} \tag{19}$$

The compression equation (8) still holds, if we substitute the force $\bar{f}(n)$ at the output of the filter to every occurrence of the static force $f(n)$ previously considered. Hence, using the recursive expression of the filter and defining:

$$\begin{aligned}\bar{x}(n) &\equiv \Delta y(n-1) + \\ &+T\left(v_h(n-1) - b\left(a_2+1\right)\bar{f}(n-1) + ba_1a_3 f(n-1) - \frac{v_i(n)+v_i(n-1)}{2}\right)\end{aligned} \tag{20}$$

we can write the compression law in the usual form:

$$f(n) = a\left(\bar{x}(n) - ba_1 f(n)\right)^{\alpha} \tag{21}$$

Again, applying the method exposed in the static case, it is possibile to give a recursive expression for $\bar{x}(n)$. The complete system becomes:

$$\begin{cases} \bar{x}(n) = \bar{x}(n-1) + v_h(n-1) - \dfrac{v_i(n)+v_i(n-1)}{2} - \\ \qquad -ba_1\left(1-a_3\right)f(n-1) - b\left(a_2+1\right)\bar{f}(n-1) \\ f(n) = a\left(\bar{x}(n) - ba_1 f(n)\right)^{\alpha} = g_{\alpha}\left(\bar{x}(n)\right) \\ \bar{f}(n) = a_2\bar{f}(n-1) + a_1\left(f(n) - a_3 f(n-1)\right) \\ v_h(n) = v_h(n-1) - \dfrac{T}{2m}\left(\bar{f}(n) + \bar{f}(n-1)\right) \\ v(n) = \dfrac{1}{2Z}\bar{f}(n) + v_i(n) \end{cases} \tag{22}$$

Note that the new equation for the output is necessary since the output of the system is now $\bar{f}(n)$ and not $f(n)$. In the following figure we present two simulations of an hammer striking a hard wall; the parameters reported in [6] fig. *3b* and *3c* have been used, and so a comparison can be made with those results. Unfortunately a comparison between hammer-string interaction results is not possible, because of the simplified assumptions made in [6] on the behavior of the struck string.

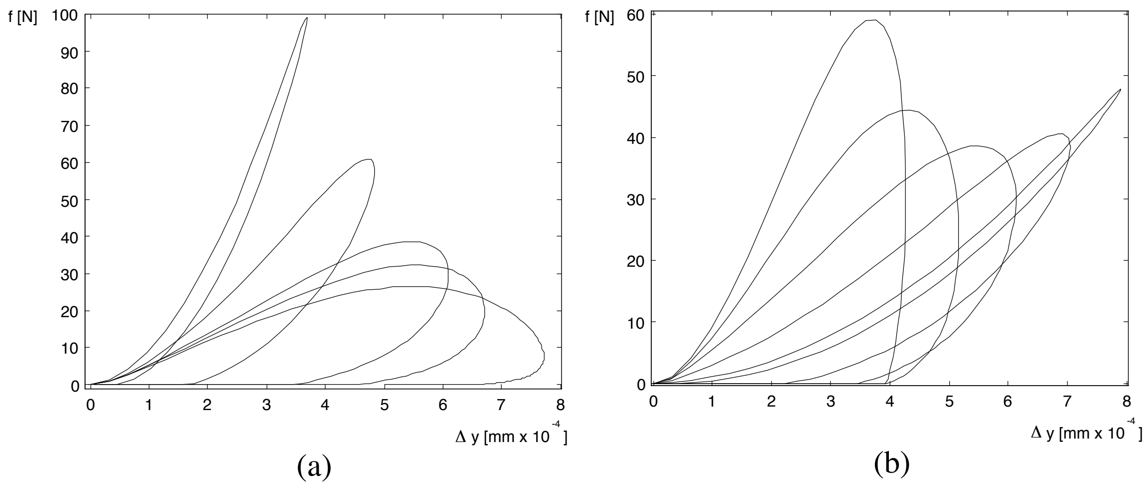

Figure 4. hysteretic hammer; α*=2.0, k=1197 N/mm*^{2}*,* v_h*=1.43 m/s,* m_h*=0.013 Kg. (a):* τ_0*=20* μ*s,* ε*=[0.4, 0.8, 0.936, 0.965, 0.99]; (b)* τ_0*=[80, 40, 20, 8, 1]* μ*s,* ε*=0.936.*

6 CONCLUSIONS

A fast and numerically stable method for hammer-string interaction simulation in physical model synthesis has been proposed. This method was based on the solution of the nonlinear system of a linear ideal string and a nonlinear mass-spring system constituting the hammer. The technique employed allows to precalculate the solutions of hammer nonlinear equation and to put them in a lookup table. An arrangement of these solutions allows to use a single table of adimensional values for each exponent required. Furthermore, the contact condition can be given in a very inexpensive form, which prevents the calculation of the actual felt compression value. Linear interpolation between different nonlinearities allows to simulate in a rough but very efficient way the continuous variation of the nonlinearity exponent. The model has also been extended to take into account the hysteresis phenomenon in real felts. In this case, a filter is inserted in the static model and a simple redefinition of the historical variable is required.

7 ACKNOWLEDGEMENTS

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