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Calculus of Variations — *On a new necessary condition in the Calculus of Variations for Lagrangians that are highly discontinuous in the state and velocity*, by PIERNICOLA BETTIOL and CARLO MARICONDA, communicated on May 10, 2019.[1]

ABSTRACT. — We consider a local minimizer, in the sense of the $W^{1,1}$ norm, of the classical problem of the calculus of variations

$$\text{(P)} \qquad \begin{cases} \text{Minimize} \quad I(x) := \displaystyle\int_a^b \Lambda(t, x(t), x'(t))\, dt + \Psi(x(a), x(b)) \\ \text{subject to:} \quad x \in W^{1,1}([a,b]; \mathbb{R}^n),\ x'(t) \in C \text{ a.e.},\ x(t) \in \Sigma\ \forall t \in [a,b]. \end{cases}$$

where $\Lambda : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is just Borel measurable, C is a cone, Σ is a nonempty subset of $\mathbb{R}^n$ and Ψ is an arbitrary extended valued function: this allows to cover any kind of endpoint constraints. We do not assume further assumptions than Borel measurability and a local Lipschitz condition on Λ with respect to t, allowing $\Lambda(t, x, \xi)$ to be possibly discontinuous, nonconvex in x or ξ. This article reconsiders the results obtained in two recent papers by the authors: we refer to [5, 4] for further details and proofs. Consider a local minimizer x_*, in the sense of the norm of the absolutely continuous functions. We illustrate a new necessary condition: there exists an absolutely continuous function p such that, for almost every t in $[a,b]$,

$$\text{(W)} \qquad \Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \geq p(t)(v-1) \quad \forall v > 0,$$

and moreover, p' belongs to a suitable generalized subdifferential of $s \mapsto \Lambda(s, x_*(t), x_*'(t))$ at $s = t$. The proof of (W) takes full advantage of a classical reparametrization technique, and of recent versions of the maximum principle. The variational inequality turns out to be equivalent to a generalized Erdmann–Du Bois-Reymond (EDBR) type necessary condition, that we are able to express in terms of the classical tools of convex analysis (e.g. convex subdifferentials): in the *autonomous*, real valued case it holds true for *every* Borel Lagrangian. More regularity is required to reformulate the (EDBR) condition in terms of the limiting subdifferential.

From (W) we deduce the Lipschitz regularity of the local minimizers for (P) if the Lagrangian satisfies a growth condition, less restrictive than superlinearity, inspired by those introduced in [8, 17]. In the *autonomous* case the result implies the most general Lipschitz regularity theorem present in the literature, for Lagrangians that are just Borel, and is new in the case of an extended valued Lagrangian.

KEY WORDS: Lipschitz regularity, nonautonomous Lagrangian, Weierstrass, Du Bois-Reymond, maximum principle

MATHEMATICS SUBJECT CLASSIFICATION (primary; secondary): 49N60; 49K05, 90C25

[1] The purpose of this paper is to announce and present results which are to appear (see references [4, 5] in the paper).

1. Introduction

This paper reconsiders the findings of two recent papers [5, 4] by the authors: we describe the most relevant results, the ideas that are involved in the main proofs and fully describe some particular cases of interest.

We consider the classical problem (P) of minimizing a functional

$$I(x) := \int_a^b \Lambda(t, x(t), x'(t))\, dt + \Psi(x(a), x(b))$$

among the absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$ satisfying a given state constraint Σ. $\Lambda : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is referred to as the *Lagrangian*, and generic notation for its three variables is (t, x, ξ). The function Ψ takes its values in $\mathbb{R} \cup \{+\infty\}$ and this allows to cover any kind of endpoint constraints.

The Lagrangian $\Lambda(t, x, \xi)$ is just Borel and satisfies one additional Hypothesis which is unrestrictive with respect to x, ξ. The interest for abstract generality is not the only motivation for investigating such a wide class of Lagrangians: in spite of Tonelli's existence result, some variational problems with a Lagrangian $\Lambda(t, x, \xi)$ that is not convex in the variable ξ may have a minimizer; moreover, discontinuous Lagrangians arise often in real life Engineering problems (e.g., combustion problems in nonhomogeneous media).

We introduce a suitable nonsmooth extension of Cesari's Assumption (S) [9, §2.7 A], described in §3. Once a minimizer x_* for (P) is given, Hypothesis (S_{x_*}) requires a local Lipschitz condition involving only the *real valued* map $s \mapsto \Lambda(s, x_*(t), \xi)$. It is worth noticing that *every* Borelian, *autonomous* Lagrangian $\Lambda(x, \xi)$ satisfies Hypothesis (S_{x_*}).

It turns out that the minimizers for (P), even local in the sense of the $W^{1,1}$ norm, satisfy a new Weierstrass type inequality [5]: there exists an absolutely continuous function p such that, for a.e. $t \in [a, b]$,

$$\text{(W)} \qquad \Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \geq p(t)(v-1) \quad \forall v > 0,$$

where $p'(t)$ is the derivative of $s \mapsto \Lambda(s, x_*(t), x_*'(t))$ at t, in a suitable generalized sense. The above inequality is formally similar to the Weierstrass inequality, which asserts that, for a.e. $t \in [a, b]$,

$$\text{(Weierstrass)} \quad \Lambda(t, x_*(t), \xi) - \Lambda(t, x_*(t), x_*'(t)) \geq v(t) \cdot (\xi - x_*'(t)) \quad \forall \xi \in \mathbb{R}^n,$$

for some absolutely continuous costate arc $v(t)$ satisfying the *Euler equation*

$$v = D_\xi \Lambda(t, x_*, x_*'), \quad v' = D_x \Lambda(t, x_*, x_*')$$

in a suitable generalized sense (here and below, for sake of brevity, we write $\Lambda(t, x_*, x_*')$ instead of $\Lambda(t, x_*(t), x_*'(t))$). However, the assumptions that ensure the validity of (W) and (Weierstrass) are quite different: the most recent versions of the latter require some extra regularity/growth conditions of $(x, \xi) \mapsto$

$\Lambda(t, x, \xi)$, even in the finite valued case (see [10, 16]). Also, whereas *Weierstrass condition* is supported with the Euler equation, our Condition (W) is equivalent to the validity of a nonsmooth extension of the Erdmann–Du Bois-Reymond (EDBR) equation. More precisely, denoting by $\partial_r g(r)_{r=r_0}$ the *convex subgradient* of a function g at r_0, it turns out that, for a.e. $t \in [a, b]$,

$$\text{(EDBR)} \qquad \Lambda(t, x_*, x_*') - p(t) \in \partial_r(0 < r \mapsto \Lambda(t, x_*, r x_*'))_{r=1}.$$

The proof of (W) is based on Clarke's version of the maximum principle applied to an auxiliary optimal control problem. Clearly, when Λ is smooth, (EDBR) gives the classical DuBois-Reymond equation:

$$\Lambda(t, x_*, x_*') - x_*'(t) \cdot D_\xi \Lambda(t, x_*, x_*') = c + \int_a^t D_t \Lambda(s, x_*, x_*')\, ds$$

for a suitable constant c. The (EDBR) is new in such generality, even in the case of an autonomous Lagrangian, in which case its validity holds true if $\Lambda(x, \xi)$ is just Borel.

In the last sections of the paper we introduce a *growth condition* (G_{x_*}), which represents the violation of (EDBR) for those minimizers x_* whose derivative is unbounded. In the smooth case Condition (G_{x_*}) is fulfilled if

$$\text{(1.1)} \qquad \lim_{|\xi| \to +\infty} |\Lambda(t, x_*(t), \xi) - \xi \cdot \Lambda(t, x_*(t), \xi)| = +\infty,$$

uniformly for $t \in [a, b]$. The first nonsmooth analogous of (1.1) was considered by Cellina in [8]. Condition (G_{x_*}) is satisfied by any minimizer whenever $\Lambda(t, x, \xi)$ is superlinear in ξ, and bounded on bounded sets. As a consequence, once Hypothesis (S_{x_*}) is satisfied, the growth condition (G_{x_*}) ensures the Lipschitz continuity of x_*. We recall that an important consequence of such regularity is that it prevents the occurrence of the Lavrentiev phenomenon (see [6, 7, 2, 19]). The study of the Lipschitz continuity of the minimizers in the nonautonomous case was suggested to us by R. Vinter. To the authors, the result seems to be the only one present in the literature that gives back the most general one for Borel, autonomous, superlinear Lagrangians obtained by Dal Maso and Frankowska in [14]. It also extends to the nonautonomous case the regularity results obtained under slower growth assumptions in [8, 17].

In the last section we consider some extensions of the previous results to the case when the Lagrangian Λ is extended valued.

2. Notation

We recall that if X is a subset of $\mathbb{R}^n$, given a function $f : X \to \mathbb{R} \cup \{+\infty\}$ and a point x with $f(x) < +\infty$, the *proximal subdifferential* $\partial^P f(x)$ of f at x is the set of elements an element $\zeta \in \mathbb{R}^n$ such that there exists $\sigma \geq 0$ satisfying

$$f(x') - f(x) + \sigma |x' - x|^2 \geq \zeta \cdot (x' - x)$$

for x' in a neighborhood of x. If f is of class $\mathscr{C}^2$ near x then $\partial^P f(x) = \{\nabla f(x)\}$ (see [11, Proposition 11.1]).

The *convex subdifferential* of f at x is

$$\partial f(x) = \{\zeta \in \mathbb{R}^n : f(x') - f(x) \geq p \cdot (x' - x)\ \forall x' \in \mathbb{R}^n\}.$$

The *limiting subdifferential* of f at $x \in X$ is the set

$$\partial^L f(x) := \{\lim \zeta_i : \zeta_i \in \partial^P f(x_i),\, x_i \to x,\, f(x_i) \to f(x)\}.$$

If, moreover, f is real valued and is locally Lipschitz at x, the *Clarke's generalized gradient* $\partial^C f(x)$ of f at x is the convex hull $\operatorname{co} \partial^L f(x)$ of $\partial^L f(x)$.

If f is of class $\mathscr{C}^1$ near x then $\partial^L f(x) = \partial^C f(x) = \{\nabla f(x)\}$ (see [11, Proposition 11.12]).

- If $x := (x_1, \dots, x_m) \mapsto f(x)$ is a function then $D_{x_i} f$ denotes the classical partial derivative of f with respect to x_i. The symbols

$$\partial_{x_i} f, \partial^P_{x_i} f, \partial^L_{x_i} f, \partial^C_{x_i} f$$

denote, respectively, the convex/proximal/limiting/Clarke subdifferential of $x_i \mapsto f(x_1, \dots, x_i, \dots, x_m)$.
- The L^∞-norm (resp. L^1-norm) on $[a, b]$ is denoted by $\|\cdot\|_\infty$ (resp. $\|\cdot\|_1$).

We refer to [18, 11] for a detailed description of the notions of limiting and generalized gradients: We just recall that the limiting subdifferential of a convex function coincides with the subdifferential in the sense of convex analysis.

The *convex hull* of a subset Y of $\mathbb{R}^n$ is denoted by $\operatorname{co} Y$.

3. Main assumptions

We will consider the following variational problem (P) with *state and derivative constraints.*

Problem (P). We consider a prescribed nonempty subset Σ of $\mathbb{R}^n$, a cone C in $\mathbb{R}^n$, and function $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $\Psi \not\equiv +\infty$.

The *Lagrangian* $\Lambda : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is Borel.

The problem (P) is described as follows:

$$(\mathrm{P}) \quad \begin{cases} \text{Minimize} & I(x) := \displaystyle\int_a^b \Lambda(t, x(t), x'(t))\, dt + \Psi(x(a), x(b)) \\ \text{subject to:} & x \in W^{1,1}([a, b]; \mathbb{R}^n),\ x'(t) \in C \text{ a.e.},\ x(t) \in \Sigma\ \forall t \in [a, b]. \end{cases}$$

We underline the fact that we do not impose any further assumption on the set Σ and the function Ψ.

DEFINITION 3.1. An absolutely continuous arc x is called *admissible* if $x(t) \in \Sigma$ for all t and if $x'(t) \in C$ for a.e. $t \in [a,b]$. A function $x_* \in W^{1,1}([a,b];\mathbb{R}^n)$ is a (global) *minimum* of (P) if $I(x_*)$ is finite and $I(x_*) \le I(x)$ for all admissible x. The function x_* is said to be a $W^{1,1}$ *local minimum* if the above inequality holds just as

$$\|x - x_*\|_{W^{1,1}} := \|x - x_*\|_\infty + \|x' - x_*'\|_1 \le \varepsilon,$$

for some $\varepsilon > 0$.

REMARK 3.2. Clearly a *strong local minimum* i.e., a minimum among the *local* competitors with respect to the L^∞ norm, is a $W^{1,1}$ local minimum.

4. A NEW NECESSARY CONDITION

Given a minimizer x_* for (P), we will suppose the validity of either the following Hypothesis (S_{x_*}) for real valued Lagrangians, or Hypothesis $(\mathrm{S}^\infty_{x_*})$ below that allows the Lagrangian to take the value $+\infty$. We refer to [5] for further details and the proofs of the main results of this section.

Consider an arc x_* in $W^{1,1}([a,b];\mathbb{R}^n)$, Hypothesis (S_{x_*}) is a local Lipschitz condition of $y \mapsto \Lambda(y, x_*(t), \xi)$ for (t,ξ) fixed.

Hypothesis (S_{x_*}). Λ takes values in $\mathbb{R}$. There are $\varepsilon_* > 0$ and a LB-measurable map $S : [a,b] \times \mathbb{R}^n \to [0,+\infty]$ such that

$$S(t, x_*'(t)) \in L^1[a,b]$$

and, for a.e. $t \in [a,b]$, for all $\xi \in \mathbb{R}^n$

$$|\Lambda(t_2, x_*(t), \xi) - \Lambda(t_1, x_*(t), \xi)| \le S(t,\xi)|t_2 - t_1| \tag{4.1}$$

whenever $t_1, t_2 \in [t - \varepsilon_*, t + \varepsilon_*] \cap [a,b]$.

REMARK 4.1. Hypothesis (S_{x_*}) is fulfilled for every $x_* \in W^{1,1}([a,b];\mathbb{R}^n)$ with $\Lambda(t, x_*(t), x_*'(t))$ in $L^1[a,b]$ if the following property holds: for every bounded subset K of $\mathbb{R}^n$, there exist positive ε_K, α_K, A_K and a summable function $\gamma_K \in L^1[a,b]$ satisfying, for all $x \in K$, $\xi \in \mathbb{R}^n$ and for almost every $t \in [a,b]$,

$$|\Lambda(t_2, x, \xi) - \Lambda(t_1, x, \xi)| \le \alpha_K(\Lambda(t,x,\xi) + A_K|\xi| + \gamma_K(t))|t_2 - t_1|, \tag{4.2}$$

for every $t_1, t_2 \in [t - \varepsilon_K, t + \varepsilon_K] \cap [a,b]$.

Condition (4.1) in Hypothesis (S_{x_*}) and Condition (4.2) can be fully characterized in terms of the proximal subgradient $\partial_t^P \Lambda(t,x,\xi)$.

PROPOSITION 4.2 (A proximal characterization of (S_{x_*})). *Assume that the map* $s \mapsto \Lambda(s, x_*(t), \xi)$ *is lower semicontinuous for all* $\xi \in \mathbb{R}^n$ *and a.e.* $t \in [a,b]$. *Then* Λ *satisfies* (4.1) *of Hypothesis* (S_{x_*}) *if and only if, for all* $\xi \in \mathbb{R}^n$,

$$|\partial_t^P \Lambda(t', x_*(t), \xi)| \le S(t,\xi) \quad \forall |t' - t| < \varepsilon_*.$$

Analogously, whenever $t \mapsto \Lambda(t, x, \xi)$ *is lower semicontinuous for all* $x, \xi \in \mathbb{R}^n$, *Condition* (4.2) *holds if and only if, for a.e.* $t \in [a, b]$,

$$|\partial_t^P \Lambda(t', x, \xi)| \leq \alpha_K(\Lambda(t, x, \xi) + A_K|\xi| + \gamma_K(t)) \quad \forall |t' - t| < \varepsilon_*.$$

It is satisfied if, for instance, for some A, $\alpha \geq 0$,

$$|\partial_t^P \Lambda(t, x, \xi)| \leq \alpha(\Lambda(t, x, \xi) + A|\xi| + 1) \quad \text{for a.e. } t \text{ and every } x, \xi \in \mathbb{R}^n.$$

The Weierstrass-type inequality, formulated in [5], may be summarized as follows.

THEOREM 4.3 (The directional Weierstrass type condition (W)). *Let* x_* *be a* $W^{1,1}$ *local minimum of* (P). *Assume that* Λ *is Borel measurable and satisfies either Hypothesis* (S_{x_*}). *There exists* $p \in W^{1,1}[a, b]$ *such that, for almost every* $t \in [a, b]$:

$$\text{(W)} \qquad \Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - \Lambda(t, x_*(t), x_*'(t)) \geq p(t)(v - 1) \quad \forall v > 0.$$

Moreover,

$$\text{(D)} \qquad p'(t) \in \partial_t^C \Lambda(t, x_*(t), x_*'(t)) \quad \textit{almost everywhere in } [a, b].$$

REMARK 4.3. If, for a.e. $t \in [a, b]$, $\Lambda(\cdot, x_*(t), x_*'(t))$ is of class $\mathscr{C}^1$, Condition (D) means that $p'(t) = D_t\Lambda(t, x_*(t), x_*'(t))$ a.e. on $[a, b]$.

Notice that in the case where Λ takes real values, (W) holds with no conditions on Λ with respect to (x, ξ), other than Borel measurability.

Sketch of the proof of Theorem 4.3

We assume here that x_* is a *global minimizer* of (P), and that Λ satisfies Hypothesis (S_{x_*}). The proof of Theorem 4.3 is obtained along the following path:

1. Λ can be extended to $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ in such a manner that (4.1) holds true for almost every $t \in \mathbb{R}$.
2. For $t \in [a, b]$, $y \in \mathbb{R}$ and $v \in \mathbb{R}$ we set

$$\ell(t, y, v) := \begin{cases} \Lambda\Big(y, x_*(t), \dfrac{x_*'(t)}{v}\Big)v & \text{if } v \geq 1/j \text{ and } x_*'(t) \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

We consider the following auxiliary optimal control problem.

DEFINITION 4.5 (An auxiliary control problem).

$$\text{(OC)} \quad \begin{cases} \text{Minimize} & J(y, v) := \displaystyle\int_a^b \ell(t, y(t), v(t))\, dt \\ \text{subject to:} & y \in W^{1,1}([a, b]; \mathbb{R}),\ y' = v \geq 1/j \text{ a.e.},\ y(a) = a,\ y(b) = b. \end{cases}$$

3. A standard reparametrization argument shows that the pair

$$(y_*(t) := t,\ v_*(t) := 1)$$

is optimal for (OC).
4. Hypothesis (S_{x_*}) ensures that problem (OC) satisfies the hypotheses [11, Hypothesis 22.25] of Clarke's maximum principle. The key point here is that the local Lipschitz continuity assumption on the state variable y for $\ell(t, y, v)$ becomes a requirement on the time variable for Λ.
5. Applying the maximum principle [11, Theorem 22.26] provides the existence of an absolutely continuous arc p_j satisfying the conditions of Theorem 4.3 (with p replaced by p_j), with the exception that (W) is satisfied just for $v \geq 1/j$ (instead of $v > 0$).
6. A compactness argument allows to extract a limit function p from the sequence (p_j), that satisfies the required conditions.

5. Erdmann–Du Bois-Reymond necessary conditions

We provide here a convex Erdmann–Du Bois-Reymond (EDBR) necessary condition without convexity assumptions. We first show, for the convenience of the reader, how our Condition (W) yields directly the Du Bois-Reymond equation in the classical setting.

Theorem 5.1 ((EDBR) in the smooth case). *Let x_* be a $W^{1,1}$ local minimizer for* (P) *and suppose the validity of Hypothesis* (S_{x_*}). *Assume, moreover, that, for a.e. $t \in [a,b]$, $(y,\xi) \mapsto \Lambda(y, x_*(t), \xi)$ is of class $\mathscr{C}^1$ in a neighbourhood of $(t, x_*'(t))$. Then*

$$p(t) := \Lambda(t, x_*, x_*') - x_*' \cdot D_\xi \Lambda(t, x_*, x_*')$$

is absolutely continuous and

$$p'(t) = D_t \Lambda(t, x_*(t), x_*'(t)) \quad \text{a.e. } t \in [a,b].$$

Proof. Theorem 4.3 implies the validity of (W). For t and $p(\cdot)$ such that (W) holds, $v = 1$ minimizes

$$\varphi(v) := \Lambda\Big(t, x_*(t), \frac{x_*'(t)}{v}\Big)v - p(t)v \quad \forall v > 0.$$

so that

$$0 = \varphi'(1) = \Lambda(t, x_*, x_*') - x_*' \cdot D_\xi \Lambda(t, x_*, x_*') - p(t).$$

Now, p is absolutely continuous and (D) implies that $p'(t) = D_t \Lambda(t, x_*, x_*')$ a.e. on $[a,b]$. □

We are not aware of formulations of the Erdmann–Du Bois-Reymond for nonautonomous Lagrangians, without assuming that the minimum x_* is Lipschitz or that Λ is somewhat regular in the state or velocity variable. Conditions (W) and (D) can be expressed in terms of the convex directional subdifferential $\partial_r\Lambda(t,x,\xi)$ and yield a natural extension of the (EDBR) equation.

If $(t,x,\xi)\in[a,b]\times\mathbb{R}^n\times\mathbb{R}^n$ is such that $\Lambda(t,x,\xi)<+\infty$, we will denote by $\partial_r\Lambda(t,x,r\xi)_{r=1}$ the *convex subdifferential* of the function $0<r\mapsto\Lambda(t,x,r\xi)$ at $r=1$. Notice that, if $\xi\mapsto\Lambda(t,x,\xi)$ is differentiable, then

$$\partial_r\Lambda(t,x_*(t),r\xi)_{r=1}=\begin{cases}\emptyset,\text{ or}\\ \{\xi\cdot D_\xi\Lambda(t,x_*(t),\xi)\}.\end{cases}$$

THEOREM 5.2 (Erdmann–Du Bois-Reymond inclusion). *Let x_* be a $W^{1,1}$ local minimum of* (P). *Assume that Λ satisfies Hypothesis* (S_{x_*}). *Then, there exist an absolutely continuous function $p\in W^{1,1}[a,b]$ satisfying* (D) *and a measurable function $q(t)$ such that*

$$\begin{cases}\Lambda(t,x_*,x_*')-p(t)=q(t),\\ q(t)\in\partial_r\Lambda(t,x_*,rx_*')_{r=1}\end{cases}\quad a.e.\ t\in[a,b]. \tag{5.1}$$

In particular, if $\Lambda(x,\xi)$ is autonomous *just Borel,* (5.1) *holds and p is constant.*

PROOF. Let $p(t)$ be an absolutely continuous function satisfying the conditions of Theorem (4.3). Let t be such that the directional Weierstrass condition (W) holds. The change of variable $r=\dfrac{1}{v}$ gives

$$\Lambda(t,x_*,rx_*')\frac{1}{r}-\Lambda(t,x_*,x_*')\geq p(t)\Big(\frac{1}{r}-1\Big),\quad\forall r>0. \tag{5.2}$$

By multiplying both terms of (5.2) by r we obtain

$$\Lambda(t,x_*,rx_*')-r\Lambda(t,x_*,x_*')\geq p(t)(1-r)\quad\forall r>0. \tag{5.3}$$

By adding $(r-1)\Lambda(t,x_*,x_*')$ to both terms of (5.3) we get

$$\Lambda(t,x_*,rx_*')-\Lambda(t,x_*,x_*')\geq(\Lambda(t,x_*,x_*')-p(t))(r-1)\quad\forall r>0.$$

It follows that $q(t):=\Lambda(t,x_*,x_*')-p(t)\in\partial_r\Lambda(t,x_*(t),rx_*'(t))_{r=1}$. □

REMARK 5.3. We stress the fact that Theorem 5.2 implies in particular that $0<r\mapsto L(t,r):=\Lambda(t,x_*(t),rx_*'(t))$ is convex at $r=1$ for a.e. $t\in[a,b]$, i.e. $r\mapsto L(t,r)$ has a non empty convex subdifferential at $r=1$. This fact is a well established relaxation result when $(t,x)\mapsto\Lambda(t,x,\xi)$ is continuous (see [15]); notice that continuity of $x\mapsto\Lambda(t,x,\xi)$ is not required here.

Imposing additional regularity assumptions on $\Lambda(t,x,\cdot)$ we obtain versions of the (EDBR) condition in terms of the limiting and of the global convex subdifferential of $\xi \mapsto \Lambda(t,x,\xi)$.

THEOREM 5.4 (Limiting (EDBR) equation). *Let x_* be a $W^{1,1}$ local minimum for* (P). *Assume that Λ satisfies Hypothesis* (S_{x_*}). *Suppose, moreover, that for every t in $[a,b]$, the map $\xi \mapsto \Lambda(t,x_*(t),\xi)$ is locally Lipschitz continuous. There exists a measurable selection v of the multivalued map $t \rightsquigarrow \partial^L_\xi \Lambda(t,x_*(t),x_*'(t))$ such that*

$$p(t) := \Lambda(t,x_*(t),x_*'(t)) - x_*'(t)\cdot v(t) \in W^{1,1}[a,b]$$

and, moreover, p' satisfies (D).

COROLLARY 5.5 ((EDBR) Condition, the case where $\Lambda(t,x,\cdot)$ is convex.). *Let x_* be a $W^{1,1}$ local minimum of* (P). *Assume that Λ satisfies Hypothesis* (S_{x_*}). *Suppose, moreover, that for a.e. t in $[a,b]$, the map $\xi \mapsto \Lambda(t,x_*(t),\xi)$ is convex. There exists a measurable selection v of $t \rightsquigarrow \partial_\xi \Lambda(t,x_*(t),x_*'(t))$ such that*

$$p(t) := \Lambda(t,x_*(t),x_*'(t)) - x_*'(t)\cdot v(t) \in W^{1,1}[a,b]$$

and, moreover, p' satisfies (D).

REMARK 5.6. In the *autonomous* case, the absolutely function p in Theorem 5.2, Theorem 5.4 and Corollary 5.5 is a constant. In this framework, Theorem 5.2 implies [13, Theorems 4.1] and Theorem 5.4 covers [14, Theorem 3.10].

6. GROWTH CONDITIONS

In this section we make use of subdifferentials, in the sense of convex analysis. These may be possibly empty.

Growth assumption (G_{x_*}). Let x_* be a given absolutely continuous arc on $[a,b]$. We say that Λ satisfies (G_{x_*}) if, for every selection $Q(t,\xi)$ of $\partial_r\Lambda(t,x_*,rx_*')_{r=1}$,

$$\lim_{\substack{|\xi|\to+\infty \\ \partial_r\Lambda(t,x_*,r\xi)_{r=1}\neq\emptyset}} |\Lambda(t,x_*(t),\xi) - Q(t,\xi)| = +\infty \quad \text{unif. for a.e. } t \in [a,b].$$

Equivalently:

$$\begin{aligned} &\forall M>0,\ \exists R>0 \quad Q(t,\xi) \in \partial_r\Lambda(t,x_*,r\xi)_{r=1},\ |\xi|>R \\ &\qquad \Rightarrow |\Lambda(t,x_*(t),\xi) - Q(t,\xi)| > M, \quad \text{a.e. } t \end{aligned}$$

REMARK 6.1. We point out that Condition (G_{x_*}) is fulfilled if there exists $R>0$ such that $\partial_r\Lambda(t,x_*(t),r\xi)_{r=1} = \emptyset$ for any $|\xi|>R$ and a.e. $t \in [a,b]$.

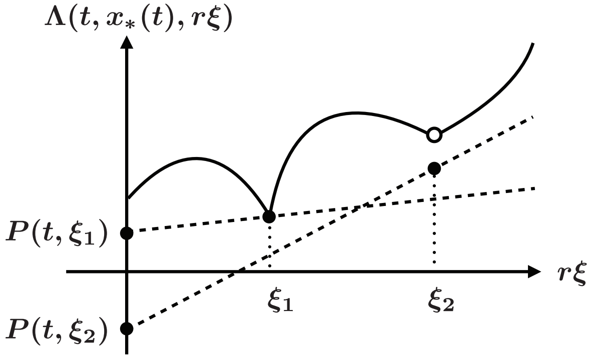

Figure 1. Condition (G_{x_*})

Remark 6.2 (Interpretation of (G_{x_*})). Assume that $\Lambda(t, x_*(t), \xi) < +\infty$. Let $Q(t, \xi) \in \partial_r \Lambda(t, x_*(t), r\xi)_{r=1}$. Then

$$\Lambda(t, x_*(t), r\xi) \geq \phi(r) := \Lambda(t, x_*(t), \xi) + Q(t, \xi)(r-1) \quad \forall r > 0$$

and $\phi(0) = P(t, \xi) := \Lambda(t, x_*(t), \xi) - Q(t, \xi)$ is the intersection of the "tangent" line $z = \phi(r)$ to $0 < r \mapsto \Lambda(t, x_*(t), r\xi)$ at $r = 1$ with the z axis. Condition (G_{x_*}) thus means that the ordinate $P(t, \xi)$ of the above intersection point goes to ∞ as $|\xi|$ goes to ∞, for those points ξ where $0 < r \mapsto \Lambda(t, x_*(t), r\xi)$ has a nonempty convex subdifferential at $r = 1$.

Remark 6.3 (The case when $\Lambda(t, x_*(t), \cdot)$ is smooth). If $\Lambda(t, x_*(t), \cdot)$ is of class C^1 for a.e. $t \in [a, b]$, Assumption (G_{x_*}) is fulfilled if and only if

$$\lim_{\substack{|\xi| \to +\infty \\ \partial_r \Lambda(t, x_*(t), r\xi)_{r=1} \neq \emptyset}} |\Lambda(t, x_*(t), \xi) - \xi \cdot D_\xi \Lambda(t, x_*(t), \xi)| = +\infty, \tag{6.1}$$

uniformly for a.e. $t \in [a, b]$. Indeed, whenever the set $\partial_r \Lambda(t, x_*(t), r\xi)_{r=1}$ is nonempty, it coincides with the singleton $\{\xi \cdot D_\xi \Lambda(t, x_*(t), \xi)\}$. Notice that Condition (6.1) is satisfied if

$$\lim_{|\xi| \to +\infty} |\Lambda(t, x_*(t), \xi) - \xi \cdot D_\xi \Lambda(t, x_*(t), \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$.

Remark 6.4 (The case when $\Lambda(t, x_*(t), \cdot)$ is convex). Assume that $\Lambda(t, x_*(t), \cdot)$ is convex and let $\Lambda(t, x_*(t), \xi) < +\infty$. It turns out easily that Condition (G_{x_*}) is satisfied if, for every selection $v(t, \xi)$ of $\partial_\xi \Lambda(t, x_*(t), \xi)$,

$$\lim_{|\xi| \to +\infty} |\Lambda(t, x_*(t), \xi) - \xi \cdot v(t, \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$.

Superlinearity plays a key role in Tonelli's existence theorem. It has been widely used as a sufficient condition for Lipschitz regularity of minimizers.

Superlinearity. There exist $\Theta : [0, +\infty[\to \mathbb{R}$ such that, for almost every $t \in [a, b]$,

$$(\mathrm{G}_\Theta) \qquad \Lambda(t, x_*(t), \xi) \geq \Theta(|\xi|) \quad \forall \xi \in \mathbb{R}^n, \quad \lim_{r \to +\infty} \frac{\Theta(r)}{r} = +\infty.$$

Boundedness on bounded sets. There exist $0 < \rho$ and $M \geq 0$ such that, for almost every $t \in [a, b]$,

$$(\mathrm{B}_{x_*}) \qquad \Lambda(t, x_*(t), \xi) \leq M \quad \forall \xi \in \mathbb{R}^n, \ |\xi| = \rho.$$

Superlinearity together with the boundedness condition (B_{x_*}) imply the validity of the Growth assumption (G_{x_*}).

PROPOSITION 6.5 (Superlinearity and $(\mathrm{B}_{x_*}) \Rightarrow (\mathrm{G}_{x_*})$). *Let Λ be superlinear and assume that (B_{x_*}) is satisfied. Then Λ satisfies Assumption (G_{x_*}).*

REMARK 6.6 (The autonomous case). If $\Lambda(x, \cdot)$ is convex for all x, the Growth assumption (G_{x_*}) is satisfied if, for every selection $v(t, \xi)$ of the subgradient $\partial_\xi \Lambda(t, x_*(t), \xi)$,

$$(6.2) \qquad \lim_{|\xi| \to +\infty} |\Lambda(x_*(t), \xi) - \xi \cdot v(t, \xi)| = +\infty,$$

uniformly for a.e. $t \in [a, b]$. The growth condition (6.2) was considered by Cellina in [8] where it was pointed that it may be fulfilled by functions that have a linear growth, as in the case of the convex function $\Lambda(\xi) = |\xi| - \sqrt{|\xi|}$.

7. LIPSCHITZ REGULARITY OF THE MINIMIZERS

In the autonomous case the most general Lipschitz regularity result states that if x_* is a minimizer for (P), $\Lambda(x, \xi)$ is Borel and satisfies a growth condition which is a variation of (G_{x_*}), then x_* is Lipschitz (see [17, Theorem 3.2]). This result can be considered as an achievement of many authors: some unnecessary extra assumptions were gradually left aside, starting from [12], where the result was formulated under the assumption that $\Lambda(x, \xi)$ is locally Lipschitz, superlinear and convex in ξ: we mention the fundamental subsequent papers [1], [14], [8]. It seems to the authors, that out of [4], no Lipschitz regularity results have appeared concerning nonautonomous Lagrangians, that give back [17, Theorem 3.2], or even [14, Theorem 2.1] in the autonomous case. All require some extra hypotheses on the regularity of $\Lambda(t, x, \xi)$ with respect to (x, ξ): see for instance [12, Corollary 3.2], [1, Theorem 3.2], [10, Theorem 4.5.2], [10, Theorem 4.5.4]. We believe that this depends on the different approaches that have been used for the two cases: autonomous and nonautonomous. In the autonomous case, the papers that followed the trailbazer paper [12], starting from [1], based the proof on the well known reparametrization technique that we used here at Point 2 of the proof of our necessary condition (Theorem 4.3). In the nonautonomous case, the main argument is often centered in an application of Weierstrass inequality.

The use of Weierstrass inequality

We show here, how the *Weierstrass inequality* (7.1) is used in order to derive regularity of the minimizers. Assume that the Weierstrass inequality holds along a minimizer x_* for (P): for a.e. t in $[a,b]$,

$$\Lambda(t, x_*, \xi) - \Lambda(t, x_*, x_*') \geq v(t) \cdot (\xi - x_*') \quad \forall \xi \in \mathbb{R}^n, \tag{7.1}$$

where v is a suitable absolutely continuous arc. Assume also that there exists $z_0 \in L^\infty[a,b]$ such that $\Lambda(t, x_*, z_0) \in L^\infty[a,b]$. Then, by choosing $\xi = z_0(t)$ in (7.1) we get

$$\Lambda(t, x_*, z_0(t)) \geq \Lambda(t, x_*, x_*') + v(t) \cdot (z_0(t) - x_*').$$

By dividing both terms of the latter inequality by $1 + |x_*'(t)|$ one gets

$$M \geq \frac{\Lambda(t, x_*, z_0(t))}{1 + |x_*'(t)|} \geq \frac{\Lambda(t, x_*, x_*')}{1 + |x_*'(t)|},$$

for a suitable constant M (we used here the fact that v is bounded). Now, if Λ is superlinear in ξ, the limit

$$\lim_{|\xi| \to +\infty} \frac{\Lambda(t, x_*(t), \xi)}{1 + |\xi|} = +\infty,$$

uniformly for $t \in [a,b]$: this forces x_*' to be bounded. In such a way one gets, for instance, the regularity results established in [11, Corollary 18.15] and [1, Theorem 3.2].

A new result

It is well known that if Λ is of class $\mathscr{C}^1$ and satisfies Hypothesis (S_{x_*}), when x_* is an absolutely continuous minimizer of (P), then the condition

$$\lim_{|\xi| \to +\infty} |\Lambda(t, x_*(t), \xi) - \xi \cdot \Lambda(t, x_*(t), \xi)| = +\infty,$$

uniformly for $t \in [a,b]$ implies the Lipschitz continuity of x_*. The proof is based on the fact that (S_{x_*}) implies the validity of the Erdmann–Du Bois-Reymond equation:

$$\Lambda(t, x_*, x_*') - x_*'(t) \cdot D_\xi \Lambda(t, x_*(t), x_*'(t)) = c + \int_a^t \Lambda(s, x_*(s), x_*'(s))\, ds.$$

In particular $\Lambda(t, x_*, \xi) - \xi \cdot D_\xi \Lambda(t, x_*(t), \xi)$ remains bounded as long as ξ takes the values $x_*'(t)$, $t \in [a, b]$, from which the claim follows. The next result is a non smooth extension of the above; it follows directly from the (EDBR) inclusion (Theorem 5.2).

THEOREM 7.1 (Lipschitz regularity under (G_{x_*})). *Let x_* be a $W^{1,1}$ local minimizer for* (P)*, where Λ satisfies Hypothesis* (S_{x_*}). *Suppose in addition that Λ fulfills the Growth assumption* (G_{x_*}). *Then x_* is Lipschitz.*

REMARK 7.2. If Λ is autonomous, Theorem 7.1 gives back [17, Theorem 3.2] and [8, Theorem 4] whenever, in addition, $\xi \mapsto \Lambda(x, \xi)$ is convex.

As an immediate consequence of Theorem 7.1 and of Proposition 6.5 we obtain the following regularity result under the superlinearity assumption.

COROLLARY 7.3 (The superlinear case). *Let x_* be a $W^{1,1}$ local minimum of* (P), *where Λ is Borel and satisfies Hypothesis* (S_{x_*}). *Suppose in addition that Λ is superlinear and satisfies Condition* (B_{x_*}). *Then x_* is Lipschitz.*

REMARK 7.4. Hypothesis (S_{x_*}) is not just technical: the famous counterexample that exhibits a minimizer that does not satisfy the Euler equation introduced in [3] shows that the lack of Hypothesis (S_{x_*}) may even lead to the occurrence of the Lavrentiev phenomenon. The same example violates the validity of the Du Bois-Reymond equation, as shown in [4]. Also, the lack of Condition (B_{x_*}) may lead to non Lipschitz minimizers (see [1]).

REMARK 7.5. If Λ is *autonomous*, Hypothesis (S_{x_*}) is fulfilled. The conclusion of Corollary 7.3 then holds if just Λ is Borel, superlinear and satisfies Condition (B_{x_*}). This formulation appeared in such generality in [14, Theorem 2.1] as a refinement of some results of [12, 1].

8. EXTENDED VALUED LAGRANGIANS

There are few results in the literature that concern the validity of the Du Bois-Reymond equation when the Lagrangian is allowed to take the value $+\infty$. Actually, a suitable version the results reported above is still valid if one replaces Hypothesis (S_{x_*}) with the following, more restrictive, one.

Hypothesis $(\mathrm{S}^\infty_{x_*})$. Given an absolutely continuous arc x_*, the following conditions hold:

(i) The map $(s, \xi) \mapsto \Lambda(s, x_*(t), \xi)$ is lower semicontinuous for each $t \in [a, b]$.
(ii) There exists a non negligible subset E of $[a, b]$ such that for all $t \in E$ there are $0 < \sigma_1 < 1 < \sigma_2$ such that

$$\Lambda(t, x_*(t), \sigma_1 x_*'(t)) < +\infty, \quad \Lambda(t, x_*(t), \sigma_2 x_*'(t)) < +\infty.$$

(iii) There exist $\beta, A \geq 0$ and a positive function $\gamma \in L^1[a,b]$ such that, for a.e. $t \in [a,b]$:

$$\Lambda(\tau, x_*(t), \sigma x_*'(t)) + A\sigma|x_*'(t)| + \gamma(t) \geq 0,$$
$$|\partial_\tau^P \Lambda(\tau, x_*(t), \sigma x_*'(t))| \leq \beta(\Lambda(\tau, x_*(t), \sigma x_*'(t)) + A\sigma|x_*'(t)| + \gamma(t))$$

for all $\tau \in [a,b]$ and $\sigma > 0$ with $\Lambda(\tau, x_*(t), \sigma x_*'(t)) < +\infty$.

More precisely, when Λ has extended values, Theorem 4.3, Theorem 5.2, Theorem 5.4, Corollary 5.5 and Theorem 7.1 do still hold replacing Hypothesis (S_{x_*}) with Hypothesis ($\mathrm{S}_{x_*}^\infty$) and Condition (D) with

$$(\mathrm{D}^\infty) \qquad p'(t) \in \mathrm{co}\left\{\omega : (\omega, p(t)) \in \partial_{(s,v)}^L \left(\Lambda\left(s, x_*(t), \frac{x_*'(t)}{v}\right)v\right)_{\substack{s=t\\ v=1}}\right\}.$$

When $\Lambda(x,\xi)$ is *autonomous* and *convex* in ξ, Corollary 5.5 coincides with [1, Theorem 4.1].

REMARK 8.1. We refer to [5] for a thorough discussion about the validity of Hypotheses (S_{x_*}) and ($\mathrm{S}_{x_*}^\infty$). We just point out the following facts.

1. Point (ii) of Hypothesis ($\mathrm{S}_{x_*}^\infty$) is satisfied if, for a.e. $t \in [a,b]$, $x_*'(t)$ belongs to the interior of the effective domain of $\xi \mapsto \Lambda(t, x_*(t), \xi)$.
2. When Λ is real valued, the validity of Hypothesis ($\mathrm{S}_{x_*}^\infty$) implies that of (S_{x_*}).

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