Chapter 1
Periodic transmission problems for the heat equation

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1.1 Introduction

This paper is devoted to the application of layer potential methods to the solution of some initial-boundary value problems for the heat equation in parabolic cylinders defined as the product of a bounded time interval and unbounded periodic domains.

Layer heat potentials have been systematically exploited in the analysis of boundary value problems for the heat equation. For example, we mention the well-known monographs Ladyženskaja, Solonnikov and Ural'ceva [LaEtAl68] and Friedman [Fr64], where a large variety of parabolic problems are solved. Moreover, by layer potential methods, Fabes and Rivière [FaRi97] have solved the Dirichlet and Neumann problem for the heat equation in C^1 cylinders with data in Lebesgue spaces. Later on, Brown [Br89, Br90] has considered the case of Lipschitz cylinders. Finally, we mention that Costabel [Co90] has obtained the solvability of some boundary value problem for the heat equation in Lipschitz cylinders with data in anisotropic Sobolev spaces.

In this paper, we are interested in developing potential theoretic techniques in order to solve transmission problems in spatially periodic domains. A first step has been done in [Lu18], where space-periodic layer heat potentials have been introduced. Moreover, as a consequence of the results of [LaLu18] for the classical layer heat potentials, regularizing properties for some boundary integral operators related to the space-periodic layer heat potentials have been shown in [Lu18]. Then, the space-periodic versions of the Dirichlet and the Neumann problems for the heat equation have been solved by means of space-periodic layer heat potentials. Here,

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instead, we are interested in exploiting the results of [Lu18] in order to solve space-periodic transmission problems for the heat equation.

Regarding spatially periodic evolution problems, we mention that Rodríguez-Bernal [Ro17] has developed an L^q theory for the space-periodic heat equation.

We now introduce the geometry of our setting. We fix once for all a natural number $n \in \mathbb{N} \setminus \{0,1\}$ and an n-tuple of positive real numbers $(q_{11},\ldots,q_{nn}) \in]0,+\infty[^n$. Then we define the periodicity cell $Q \equiv \prod_{j=1}^n]0,q_{jj}[$ and the diagonal matrix $q \equiv \mathrm{diag}(q_{11},\ldots,q_{nn})$. Clearly, $q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$ is the set of vertices of a periodic subdivision of $\mathbb{R}^n$ corresponding to the cell Q. Then we fix once and for all

$$\alpha \in]0,1[, \qquad m \in \mathbb{N} \setminus \{0\}, \qquad T \in]0,+\infty[,$$

and a bounded open subset Ω of $\mathbb{R}^n$ of class $C^{m,\alpha}$ such that $\mathrm{cl}\,\Omega \subseteq Q$. We denote by $\mathbf{n}_\Omega$ and by $\mathbf{n}_Q$ the outward unit normal fields to $\partial\Omega$ and to ∂Q, respectively. Then, we introduce the following q-periodic sets:

$$\mathbb{S}_q[\Omega] \equiv \bigcup_{z \in \mathbb{Z}^n} (qz + \Omega) = q\mathbb{Z}^n + \Omega, \qquad \mathbb{S}_q[\Omega]^- \equiv \mathbb{R}^n \setminus \mathrm{cl}\,\mathbb{S}_q[\Omega].$$

In the pair of domains $\mathbb{S}_q[\Omega]$ and $\mathbb{S}_q[\Omega]^-$ we consider two transmission problems for the heat equation: problem (1.4), known as non-ideal transmission problem, and problem (1.8), which is called ideal transmission problem. The aim of this paper is to solve the two problems in parabolic Schauder spaces of space-periodic functions: more precisely, after some preliminaries (Section 1.2), in Section 1.3 we solve the non-ideal problem (1.4), while in Section 1.4 we consider the ideal problem (1.8).

1.2 Preliminaries and notation

If $\mathbb{D} \subseteq \mathbb{R}^n$, then we set $\mathbb{D}_T \equiv]-\infty,T] \times \mathbb{D}$, and $\partial_T\mathbb{D} \equiv (\partial\mathbb{D})_T =]-\infty,T] \times \partial\mathbb{D}$. We have $(\mathrm{cl}\,\mathbb{D})_T = \mathrm{cl}\,\mathbb{D}_T$. For the definition of the parabolic Schauder spaces $C^{\frac{m+\alpha}{2};m+\alpha}$ we refer to Ladyženskaja, Solonnikov and Ural'ceva [LaEtAl68, Chapter 1] (see also [Lu18]). We now introduce two subspaces useful for initial-boundary value problems with zero initial condition. If $\tilde{\Omega}$ is a subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$, we set

$$C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\tilde{\Omega}_T) \equiv \left\{ u \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\tilde{\Omega}_T) : u(t,x) = 0 \ \forall t \in]-\infty,0], x \in \mathrm{cl}\,\tilde{\Omega} \right\},$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\tilde{\Omega}_T)$. Moreover, we set

$$C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\tilde{\Omega}) \equiv \left\{ u \in C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\tilde{\Omega}) : u(t,x) = 0 \ \forall t \in]-\infty,0], x \in \partial\tilde{\Omega} \right\},$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\tilde{\Omega})$. Now let $\mathbb{D}$ be a subset of $\mathbb{R}^n$ such that $x + qe_i \in \mathbb{D}$ for all $x \in \mathbb{D}$ and for all $i \in \{1,\ldots,n\}$, where $\{e_1,\ldots,e_n\}$ denotes the canonical basis of $\mathbb{R}^n$. We say that a function u from $\mathbb{D}_T$ to $\mathbb{C}$ is q-periodic in space, or simply q-periodic, if $u(t,x) = u(t,x+qe_i)$ for all $(t,x) \in \mathbb{D}_T$, and for all $i \in \{1,\dots,n\}$. Since we will consider space-periodic problems, we introduce the following subspaces of parabolic Schauder spaces.

$$C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T) \equiv \left\{ u \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T) : u \text{ is } q\text{-periodic in space} \right\}, \tag{1.1}$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$, and

$$C_q^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-) \equiv \left\{ u \in C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-) : u \text{ is } q\text{-periodic in space} \right\}, \tag{1.2}$$

which we regard as a Banach subspace of $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$. Then we can define $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ replacing $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and $C^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ in the right hand side of (1.1) and (1.2), by the spaces $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ and $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$, respectively.

Next, in order to build space-periodic layer heat potentials, we plan to replace the fundamental solution of the heat equation by a periodic analog. Therefore, we introduce the function $\Phi_{q,n}$ from $(\mathbb{R}\times\mathbb{R}^n) \setminus (\{0\}\times q\mathbb{Z}^n)$ to $\mathbb{R}$ defined by

$$\Phi_{q,n}(t,x) \equiv \begin{cases} \sum_{z\in\mathbb{Z}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x+qz|^2}{4t}} & \text{if } (t,x) \in]0,+\infty[\times\mathbb{R}^n, \\ 0 & \text{if } (t,x) \in (]-\infty,0]\times\mathbb{R}^n) \setminus (\{0\}\times q\mathbb{Z}^n) \end{cases}$$

(see [Lu18]). As it is known, $\Phi_{q,n}$ is a q-periodic analog of the (classical) fundamental solution of the heat equation. We are now ready to introduce the q-periodic analog of the single layer heat potential. Let $\mu \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. Then, we set

$$v_q[\partial_T\Omega,\mu](t,x) \equiv \int_0^t \int_{\partial\Omega} (\Phi_{q,n}(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau,$$

for all $(t,x) \in (\mathbb{R}^n)_T$, where $d\sigma$ denotes the area element of a manifold embedded in $\mathbb{R}^n$. Moreover, we set

$$w_{q,*}[\partial_T\Omega,\mu](t,x) \equiv \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial \mathbf{n}_\Omega(x)} \Phi_{q,n}(t-\tau,x-y)\mu(\tau,y)\,d\sigma_y d\tau,$$

for all $(t,x) \in \partial_T\Omega$. The function $v_q[\partial_T\Omega,\mu]$ is the q-periodic in space single layer heat potential with density μ. If $\mu \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, then $v_q[\partial_T\Omega,\mu]$ is continuous in $(\mathbb{R}^n)_T$, is q-periodic in space and $v_q[\partial_T\Omega,\mu] \in C^\infty((\mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega])_T)$. Moreover $v_q[\partial_T\Omega,\mu]$ solves the heat equation in $(\mathbb{R}^n \setminus \partial\mathbb{S}_q[\Omega])_T$. We denote by $v_q^+[\partial\Omega_T,\mu]$ and $v_q^-[\partial\Omega_T,\mu]$ the restriction of $v_q[\partial\Omega_T,\mu]$ to $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ and $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$, respectively. We have,

$$\frac{\partial}{\partial \mathbf{n}_\Omega(x)} v_q^\pm[\partial_T\Omega,\mu](t,x) = \pm\frac{1}{2}\mu(t,x) + w_{q,*}[\partial_T\Omega,\mu](t,x), \tag{1.3}$$

for all $(t,x) \in \partial_T \Omega$ (see [Lu18]). We now recall some basic properties of the space-periodic single layer heat potential. For a proof we refer to [Lu18].

Theorem 1. *The following statements hold.*

(i) The operator from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ *to* $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ *which takes* μ *to* $v_q^+[\partial_T\Omega,\mu]$ *is linear and continuous. The same statement holds with* $v_q^+[\partial_T\Omega,\mu]$ *and* $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T$ *replaced by* $v_q^-[\partial_T\Omega,\mu]$ *and* $\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-$*, respectively.*

(ii) The operator $w_{*,q}[\partial_T\Omega,\cdot]$ *is compact from* $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ *to itself.*

Now we prove the following result on the invertibility of $v_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$.

Theorem 2. *The operator* $v_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ *is a linear homeomorphism from the space* $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ *to* $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$.

Proof. By Theorem 1 (i) and by the continuity of the trace operator, $v_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is linear and continuous from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. Accordingly, by the Open Mapping Theorem, it suffices to show that $v_q[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ is a bijection. We first prove the injectivity. Let $\mu \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ be such that $v_q[\partial_T\Omega,\mu]_{|\partial_T\Omega}=0$. The continuity of the single layer potential implies that both $v_q^+[\partial_T\Omega,\mu]_{|\partial_T\Omega}=v_q^-[\partial_T\Omega,\mu]_{|\partial_T\Omega}=0$. Thus $v_q^+[\partial_T\Omega,\mu]$ solves a Dirichlet problem for the heat equation in $[0,T]\times\Omega$ with zero initial condition and with zero Dirichlet boundary condition. Accordingly, the uniqueness of the solution for the classical Dirichlet problem implies that $v_q^+[\partial_T\Omega,\mu]=0$ in $[0,T]\times\mathrm{cl}\,\Omega$. Moreover, since $v_q^-[\partial_T\Omega,\mu]_{|\partial_T\Omega}=0$, the function $v_q^-[\partial_T\Omega,\mu]=0$ solves the periodic Dirichlet problem

$$\begin{cases} \partial_t u-\Delta u=0 & \text{in }]0,T]\times\mathbb{S}_q[\Omega]^-, \\ u(t,x+qe_i)=u(t,x) & \forall\,(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-,\ \forall\, i\in\{1,\ldots,n\}, \\ u=0 & \text{on } [0,T]\times\partial\Omega, \\ u(0,\cdot)=0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases}$$

Hence, by the maximum principle for the periodic heat equation we have that $v_q^-[\partial_T\Omega,\mu]=0$ in $[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-$ (see [Lu18]). Finally, the jump formula (1.3) implies that

$$\mu=\frac{\partial}{\partial\mathbf{n}_\Omega}v_q^+[\partial_T\Omega,\mu]-\frac{\partial}{\partial\mathbf{n}_\Omega}v_q^-[\partial_T\Omega,\mu]=0 \quad \text{on } [0,T]\times\partial\Omega.$$

Next we prove the surjectivity. Let $\phi\in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$. By the results of [Lu18, Section 5] there exists a unique function $u_\phi^-\in C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$ which solves

$$\begin{cases} \partial_t u-\Delta u=0 & \text{in }]0,T]\times\mathbb{S}_q[\Omega]^-, \\ u(t,x+qe_i)=u(t,x) & \forall\,(t,x)\in[0,T]\times\mathrm{cl}\,\mathbb{S}_q[\Omega]^-,\ \forall\, i\in\{1,\ldots,n\}, \\ u=\phi & \text{on } [0,T]\times\partial\Omega, \\ u(0,\cdot)=0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases}$$

Since $\frac{\partial}{\partial \mathbf{n}_\Omega} u_\phi^- \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$, the results of [Lu18, Section 5] implies that there exists a unique $\mu \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$ such that $v_q^-[\partial_T \Omega, \mu]$ solves the problem

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in }]0,T] \times \mathbb{S}_q[\Omega]^-, \\ u(t,x+qe_i) = u(t,x) & \forall (t,x) \in [0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-, \forall i \in \{1,\dots,n\}, \\ \frac{\partial}{\partial \mathbf{n}_\Omega} u = \frac{\partial}{\partial \mathbf{n}_\Omega} u_\phi^- & \text{on } [0,T] \times \partial\Omega, \\ u(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases}$$

By the uniqueness of the solution for the periodic Neumann problem for the heat equation (see [Lu18]), we have that $v_q^-[\partial_T \Omega, \mu] = u_\phi^-$. In particular,

$$v_q[\partial_T \Omega, \mu]_{|\partial_T \Omega} = v_q^-[\partial_T \Omega, \mu]_{|\partial_T \Omega} = \phi \quad \text{on } [0,T] \times \partial\Omega,$$

and accordingly the statement follows. □

1.3 A periodic non-ideal transmission problem

In this section we consider a periodic transmission problem which models the heat diffusion in a two-phase composite material with thermal resistance at the interface. We fix once and for all $\lambda^+, \lambda^-, \gamma \in]0,+\infty[$. Then we take $f, g \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$ and we consider the following non-ideal transmission problem.

$$\begin{cases} \partial_t u^+ - \Delta u^+ = 0 & \text{in }]0,T] \times \mathbb{S}_q[\Omega], \\ \partial_t u^- - \Delta u^- = 0 & \text{in }]0,T] \times \mathbb{S}_q[\Omega]^-, \\ u^+(t,x+qe_i) = u^+(t,x) & \forall (t,x) \in [0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega], \forall i \in \{1,\dots,n\}, \\ u^-(t,x+qe_i) = u^-(t,x) & \forall (t,x) \in [0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-, \forall i \in \{1,\dots,n\}, \\ \lambda^+ \frac{\partial}{\partial \mathbf{n}_\Omega} u^+ + \gamma(u^+ - u^-) = f & \text{on } [0,T] \times \partial\Omega, \\ \lambda^- \frac{\partial}{\partial \mathbf{n}_\Omega} u^- - \lambda^+ \frac{\partial}{\partial \mathbf{n}_\Omega} u^+ = g & \text{on } [0,T] \times \partial\Omega, \\ u^+(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega], \\ u^-(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-. \end{cases} \tag{1.4}$$

The set $\mathrm{cl}\,\mathbb{S}_q[\Omega]^-$ plays the role of a matrix with thermal conductivity λ^- where the periodic array of inclusions $\mathrm{cl}\,\mathbb{S}_q[\Omega]$ with thermal conductivity λ^+ is inserted. The fifth condition of system (1.4) is the non-ideal transmission (or imperfect contact) condition, which models the thermal resistance at the interface. In particular this condition says that the temperature field at the interface displays a jump proportional to the normal heat flux. Concerning parabolic transmission problems, we mention the works of Donato and Jose [DoJo15], for the study of the asymptotic behavior of the approximate control of a parabolic transmission problem. For the stationary case, we mention [DaMu13], where the authors consider a singularly perturbed stationary version of the above transmission problem in order to study the effective conductivity of a periodic composite. Incidentally, we observe that the discontinuity of the temperature field is a well know phenomenon in physics which has been studied since the work of Kapitza in 1941, in which the author has studied for the first time the thermal interface behavior in liquid helium (see, *e.g.*, Swartz and Pohl [SwPo89], Lipton [Li98] and references therein). We begin our analysis with the following uniqueness result for problem (1.4).

Proposition 1. *Let $u^+ \in C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$, $u^- \in C^{\frac{1}{2};1}_{0,q}(\mathrm{cl}\,\mathbb{S}_q[\Omega]^-_T)$ be one time continuously differentiable with respect to the time variable and two times continuously differentiable with respect to the space variables in $]0,T] \times \mathbb{S}_q[\Omega]$ and $]0,T] \times \mathbb{S}_q[\Omega]^-$, respectively. Moreover, let the pair (u^+,u^-) solve the system (1.4) with $f = g = 0$. Then $u^+ = 0$ in $[0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]$ and $u^- = 0$ in $[0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-$.*

Proof. Let e^+, e^- be the functions from $[0,T]$ to $[0,+\infty[$ defined by

$$e^+(t) \equiv \int_\Omega (u^+(t,y))^2\,dy, \qquad e^-(t) \equiv \int_{Q\setminus \mathrm{cl}\,\Omega} (u^-(t,y))^2\,dy, \qquad \forall t \in [0,T]\,.$$

By the Dominated Convergence Theorem, $e^+, e^- \in C^0([0,T])$. In addition, classical differentiation theorems for integrals depending on a parameter and the approximation argument of Verchota [Ve84, Theorem 1.12, p. 581] if $m = 1$ (see [Lu18]) imply that $e^+, e^- \in C^1(]0,T[)$. Also, by the Divergence Theorem, we have that

$$\begin{aligned}\frac{d}{dt}e^+(t) =& 2\int_\Omega u^+(t,y)\partial_t u^+(t,y)\,dy = 2\int_\Omega u^+(t,y)\Delta u(t,y)\,dy\\ &= -2\int_\Omega |Du^+(t,y)|^2\,dy + 2\int_{\partial\Omega} u^+(t,y)\frac{\partial}{\partial \mathbf{n}_\Omega(y)}u^+(t,y)\,d\sigma_y,\end{aligned}$$

for all $t \in]0,T[$. Moreover, in a similar way, exploiting the Divergence Theorem and the q-periodicity of u we have that

$$\begin{aligned}\frac{d}{dt}e^-(t) =& 2\int_{Q\setminus\mathrm{cl}\,\Omega} u^-(t,y)\partial_t u^-(t,y)\,dy = 2\int_{Q\setminus\mathrm{cl}\,\Omega} u^-(t,y)\Delta u(t,y)\,dy\\ &= -2\int_{Q\setminus\mathrm{cl}\,\Omega} |Du^-(t,y)|^2\,dy - 2\int_{\partial\Omega} u^-(t,y)\frac{\partial}{\partial \mathbf{n}_\Omega(y)}u(t,y)\,d\sigma_y,\end{aligned}$$

for all $t \in]0,T[$. Then, if we set $e \equiv \lambda^+ e^+ + \lambda^- e^-$, we have that

$$\begin{aligned}\frac{d}{dt}e(t) =& -2\left(\lambda^+\int_\Omega |Du^+(t,y)|^2\,dy + \lambda^-\int_{Q\setminus\mathrm{cl}\,\Omega} |Du^-(t,y)|^2\,dy\right)\\ &+2\lambda^+\int_{\partial\Omega} u^+(t,y)\frac{\partial}{\partial \mathbf{n}_\Omega(y)}u^+(t,y)\,d\sigma_y - 2\lambda^-\int_{\partial\Omega} u^-(t,y)\frac{\partial}{\partial \mathbf{n}_\Omega(y)}u^-(t,y)\,d\sigma_y\\ &= -2\left(\lambda^+\int_\Omega |Du^+(t,y)|^2\,dy + \lambda^-\int_{Q\setminus\mathrm{cl}\,\Omega} |Du^-(t,y)|^2\,dy\right)\end{aligned}$$

$$+2\lambda^+\int_{\partial\Omega}(u^+(t,y)-u^-(t,x))\frac{\partial}{\partial \mathbf{n}_\Omega(y)}u^+(t,y)\,d\sigma_y$$
$$=-2\left(\lambda^+\int_\Omega |Du^+(t,y)|^2\,dy+\lambda^-\int_{Q\setminus \mathrm{cl}\,\Omega}|Du^-(t,y)|^2\,dy\right)$$
$$-\frac{2}{\gamma}\int_{\partial\Omega}\left(\lambda^+\frac{\partial}{\partial \mathbf{n}_\Omega(y)}u^+(t,y)\right)^2 d\sigma_y \qquad \forall t\in]0,T[.$$

Hence $\frac{d}{dt}e\le 0$ in $]0,T[$. Since $e(0)=0$ and $e\ge 0$ in $[0,T]$, then $e=0$ for all in $[0,T]$. Then $u^+=0$ in $[0,T]\times \mathrm{cl}\,\Omega$, $u^-=0$ in $[0,T]\times \mathrm{cl}\,Q\setminus\Omega$ and therefore the q-periodicity of u^+,u^- implies the validity of the statement. □

Since we plan to solve the problem (1.4) with two space-periodic single layer heat potentials, we need to solve the related boundary integral equations. In order to do so, we show the invertibility of the operators which appear in such integral equations.

Lemma 1. *Let $J\equiv(J_1,J_2)$ be the operator from the space $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ to $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ defined by*

$$J_1[\mu^+,\mu^-]\equiv\lambda^+\left(\frac{1}{2}\mu^+ + w_{q,*}[\partial_T\Omega,\mu^+]\right) \tag{1.5}$$
$$+\gamma(v_q^+[\partial_T\Omega,\mu^+]_{|\partial_T\Omega}-v_q^-[\partial_T\Omega,\mu^-]_{|\partial_T\Omega}),$$
$$J_2[\mu^+,\mu^-]\equiv\lambda^-\left(-\frac{1}{2}\mu^- + w_{q,*}[\partial_T\Omega,\mu^-]\right)-\lambda^+\left(\frac{1}{2}\mu^+ + w_{q,*}[\partial_T\Omega,\mu^+]\right),$$

for all $(\mu^+,\mu^-)\in(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. Then J is a linear homeomorphism.

Proof. Let $J^\#=(J_1^\#,J_2^\#)$ be the linear operator from $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ to $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ defined by

$$J_1^\#[\mu^+,\mu^-]\equiv\frac{\lambda^+}{2}\mu^+,\qquad J_2^\#[\mu^+,\mu^-]\equiv-\frac{\lambda^-}{2}\mu^- -\frac{\lambda^+}{2}\mu^+,$$

for all $(\mu^+,\mu^-)\in(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. Clearly $J^\#$ is a linear homeomorphism. Moreover, let $\bar J=(\bar J_1,\bar J_2)$ be the linear operator from $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ to $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ defined by

$$\bar J_1[\mu^+,\mu^-]\equiv\lambda^+ w_{q,*}[\partial_T\Omega,\mu^+]+\gamma(v_q^+[\partial_T\Omega,\mu^+]_{|\partial_T\Omega}-v_q^-[\partial_T\Omega,\mu^-]_{|\partial_T\Omega}),$$
$$\bar J_2[\mu^+,\mu^-]\equiv\lambda^- w_{q,*}[\partial_T\Omega,\mu^-]-\lambda^+ w_{q,*}[\partial_T\Omega,\mu^+],$$

for all $(\mu^+,\mu^-)\in(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. By Theorem 1 (ii), the map $w_{*,q}[\partial_T\Omega,\cdot]$ is compact in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. By Theorem 1 (i), $v_q^+[\partial_T\Omega,\cdot]$ and $v_q^-[\partial_T\Omega,\cdot]$

are linear and continuous from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\mathbb{S}_q[\Omega]_T)$ and $C_0^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\mathbb{S}_q[\Omega]^-_T)$, respectively. Then, by the continuity of the trace operators from $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\mathbb{S}_q[\Omega]_T)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ and from $C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\mathbb{S}_q[\Omega]^-_T)$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$, and by the compactness of the embedding of $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ into $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$, which is a consequence of the Ascoli-Arzelà Theorem, $v_q^+[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ and $v_q^-[\partial_T\Omega,\cdot]_{|\partial_T\Omega}$ are compact in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. Then the operator $\bar{J}$ is compact in $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$. Since compact perturbations of linear homeomorphisms are Fredholm operators of index 0, we have that $J = J^\# + \bar{J}$ is a Fredholm operator of index 0. Thus, to show that J is a linear homeomorphism, it suffices to show that J is injective. Let $(\mu^+,\mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ be such that $J[\mu^+,\mu^-] = (0,0)$. Then, $v_q^+[\partial_T\Omega,\mu^+]$ and $v_q^-[\partial_T\Omega,\mu^-]$ satisfy the assumptions of Proposition 1 and thus $v_q^+[\partial_T\Omega,\mu^+] = 0$ in $[0,T]\times\mathrm{cl}\mathbb{S}_q[\Omega]$ and $v_q^-[\partial_T\Omega,\mu^-] = 0$ in $[0,T]\times\mathrm{cl}\mathbb{S}_q[\Omega]^-$. In particular, by the continuity of the periodic single layer heat potential, $v_q[\partial_T\Omega,\mu^+]_{|\partial_T\Omega} = v_q^+[\partial_T\Omega,\mu^+]_{|\partial_T\Omega} = 0$ and $v_q[\partial_T\Omega,\mu^-]_{|\partial_T\Omega} = v_q^-[\partial_T\Omega,\mu^-]_{|\partial_T\Omega} = 0$. Accordingly, Theorem 2 implies that $\mu^+ = \mu^- = 0$ on $[0,T]\times\partial\Omega$, and the statement follows. □

Finally we are ready to prove the following solvability result for the non-ideal transmission problem (1.4).

Theorem 3. *Let* $f,g \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$. *Then problem (1.4) has a unique solution* $(u^+,u^-) \in C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\mathbb{S}_q[\Omega]_T)\times C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\mathbb{S}_q[\Omega]^-_T)$. *Moreover,*

$$u^+ = v_q^+[\partial_T\Omega,\mu^+] \quad \text{in } \mathrm{cl}\mathbb{S}_q[\Omega]_T, \quad u^- = v_q^-[\partial_T\Omega,\mu^-] \quad \text{in } \mathrm{cl}\mathbb{S}_q[\Omega]^-_T, \tag{1.6}$$

where (μ^+,μ^-) *is the unique solution in* $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ *of*

$$J[\mu^+,\mu^-] = (f,g) \quad \text{on } \partial_T\Omega. \tag{1.7}$$

Proof. Proposition 1 implies that problem (1.4) has at most one solution. Then we only need to show that the pair (u^+,u^-) defined by (1.6) is a solution of problem (1.4). Lemma 1 implies that there exists a unique solution (μ^+,μ^-) in $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega))^2$ of (1.7). Then by Theorem 1 (i) and by the definition (1.5) of J, the functions u^+,u^- defined by (1.6) are q-periodic functions which solve the heat equation and which satisfy all the transmission conditions in (1.4). Thus (u^+,u^-) is a solution of problem (1.4). □

1.4 A periodic ideal transmission problem

In this section we consider a periodic transmission problem which models the heat diffusion in a two-phase composite material with perfect contact at the interface. We fix once and for all $\lambda^+, \lambda^- \in]0,+\infty[$. Then we take $f \in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T\Omega)$ and $g \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ and we consider the following ideal transmission problem.

$$
\begin{cases}
\partial_t u^+ - \Delta u^+ = 0 & \text{in }]0,T] \times \mathbb{S}_q[\Omega], \\
\partial_t u^- - \Delta u^- = 0 & \text{in }]0,T] \times \mathbb{S}_q[\Omega]^-, \\
u^+(t, x + q e_i) = u^+(t,x) & \forall (t,x) \in [0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega], \forall i \in \{1,\dots,n\}, \\
u^-(t, x + q e_i) = u^-(t,x) & \forall (t,x) \in [0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-, \forall i \in \{1,\dots,n\}, \\
u^+ - u^- = f & \text{on } [0,T] \times \partial\Omega, \\
\lambda^- \frac{\partial}{\partial \mathbf{n}_\Omega} u^- - \lambda^+ \frac{\partial}{\partial \mathbf{n}_\Omega} u^+ = g & \text{on } [0,T] \times \partial\Omega, \\
u^+(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega], \\
u^-(0,\cdot) = 0 & \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]^-.
\end{cases}
\tag{1.8}
$$

The set $\mathrm{cl}\,\mathbb{S}_q[\Omega]^-$ plays the role of a matrix with thermal conductivity λ^- where the periodic array of inclusions $\mathrm{cl}\,\mathbb{S}_q[\Omega]$ with thermal conductivity λ^+ is inserted. The fifth and sixth conditions of system (1.8) are the ideal transmission (or perfect contact) conditions, which say that heat flux and the temperature field are continuous at the interface between the two materials. We mention Hofmann, Lewis, and Mitrea [HoEtAl03] for the study of the non-periodic version of this transmission problem, in case Ω is a Lipschitz domain and the boundary conditions are in suitable Lebesgue spaces. For the study of the stationary version of ideal transmission problems we mention Ammari, Kang, and Touibi [AmEtAl05] for the computation of the effective conductivity of a material with periodic inclusions and Pukhtaievych [Pu18A, Pu18B] for the asymptotic behavior when the diameter of the periodic inclusions tends to zero. We start our analysis of problem (1.8) with the following uniqueness result that can be proved as the one of Proposition 1.

Proposition 2. *Let $u^+ \in C_{0,q}^{\frac{1}{2};1}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T)$, $u^- \in C_{0,q}^{\frac{1}{2};1}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$ be one time continuously differentiable with respect to the time variable and two times continuously differentiable with respect to the space variables in $]0,T] \times \mathbb{S}_q[\Omega]$ and $]0,T] \times \mathbb{S}_q[\Omega]^-$, respectively. Moreover, let the pair (u^+,u^-) solve the system (1.8) with $f = g = 0$. Then $u^+ = 0$ in $[0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]$ and $u^- = 0$ in $[0,T] \times \mathrm{cl}\,\mathbb{S}_q[\Omega]^-$.*

Our aim is to provide a solution of problem (1.8) in terms of space-periodic single layer heat potentials. By exploiting the potential theoretic method, we will convert problem (1.8) into integral equations. Therefore, in order to prove the solvability of the integral equations, we need to perform a preliminary study of an auxiliary integral operator. We do so in the following lemma.

Lemma 2. *Let K be the operator from $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ to $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T\Omega)$ defined by*

$$K[\mu] = -\frac{1}{2}\mu + \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} w_{q,*}[\partial_T \Omega, \mu]$$

for all $\mu \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$. *Then K is a linear homeomorphism.*

Proof. By Theorem 1 (ii), $w_{q,*}[\partial_T \Omega, \cdot]$ is compact in $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$. Accordingly, the Fredholm Alternative implies that it suffices to show that K is injective. Let $\mu \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$ be such that $K[\mu] = 0$. Then

$$\lambda^+ \frac{\partial}{\partial \mathbf{n}_\Omega} v_q^+[\partial_T \Omega, \mu]_{|\partial_T \Omega} - \lambda^- \frac{\partial}{\partial \mathbf{n}_\Omega} v_q^-[\partial_T \Omega, \mu]_{|\partial_T \Omega} = 0.$$

Accordingly, the pair $(v_q^+[\partial_T \Omega, \mu], v_q^-[\partial_T \Omega, \mu])$ satisfies all the assumptions of Proposition 2 and then $(v_q^+[\partial_T \Omega, \mu], v_q^-[\partial_T \Omega, \mu]) = (0,0)$. In particular, by the continuity of the periodic single layer potential we have that $v_q[\partial_T \Omega, \mu]_{|\partial_T \Omega} = 0$ and accordingly Theorem 2 implies the validity of the statement. □

In the following lemma, we prove the next step, which consists in showing the invertibility of an operator which appears in the integral equations associated to the transmission problem (1.8).

Lemma 3. *Let $H \equiv (H_1, H_2)$ be the operator from the space $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega))^2$ to $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T \Omega) \times C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$ defined by*

$$\begin{aligned} H_1[\mu^+, \mu^-] &\equiv v_q^+[\partial_T \Omega, \mu^+]_{|\partial_T \Omega} - v_q^-[\partial_T \Omega, \mu^-]_{|\partial_T \Omega}, \\ H_2[\mu^+, \mu^-] &\equiv \lambda^- \left(-\frac{1}{2}\mu^- + w_{q,*}[\partial_T \Omega, \mu^-]\right) - \lambda^+ \left(\frac{1}{2}\mu^+ + w_{q,*}[\partial_T \Omega, \mu^+]\right), \end{aligned}$$

for all $(\mu^+, \mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega))^2$. Then H is a linear homeomorphism.

Proof. Theorem 1 implies that H is linear and continuous. Accordingly, by the Open Mapping Theorem, it suffices to show that it is a bijection. Let (ϕ, ψ) be in $C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T \Omega) \times C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$. We show that there exists a unique pair $(\mu^+, \mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega))^2$ such that

$$H[\mu^+, \mu^-] = (\phi, \psi). \tag{1.9}$$

We first show the uniqueness. Let $(\mu^+, \mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega))^2$ be such that (1.9) holds. Theorem 2 implies that there exists a unique $\mu^\# \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$ such that $v_q[\partial_T \Omega, \mu^\#]_{|\partial_T \Omega} = \phi$. Accordingly, $H_1[\mu^+, \mu^-] = \phi$ implies that

$$\mu^\# = \mu^+ - \mu^-. \tag{1.10}$$

By substituting the previous equality in the equality $H_2[\mu^+, \mu^-] = \psi$ we get

$$-\frac{1}{2}\mu^+ + \frac{\lambda^- - \lambda^+}{\lambda^- + \lambda^+} w_{q,*}[\partial_T \Omega, \mu^+] = \frac{\lambda^-}{\lambda^- + \lambda^+}\left(-\frac{1}{2}\mu^\# + w_{q,*}[\partial_T \Omega, \mu^\#]\right) + \frac{1}{\lambda^- + \lambda^+}\psi. \quad (1.11)$$

Since the right hand side of the previous equality belongs to $C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$, Lemma 2 implies that there exists a unique $\mu^+ \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$ such that (1.11) holds, and accordingly μ^+ is uniquely determined. Then the equality (1.10) uniquely determines μ^- and thus uniqueness follows. On the other hand, by reading backward the argument above, one deduces the existence of a pair $(\mu^+, \mu^-) \in (C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega))^2$ such that (1.9) holds. □

Finally, by exploiting Proposition 2, Lemma 3 and the properties of the space-periodic single layer heat potential, we can deduce the following result concerning the solvability of the ideal transmission problem (1.8).

Theorem 4. *Let* $f \in C_0^{\frac{m+\alpha}{2};m+\alpha}(\partial_T \Omega)$, $g \in C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega)$. *Then problem (1.8) has a unique solution* $(u^+, u^-) \in C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T) \times C_{0,q}^{\frac{m+\alpha}{2};m+\alpha}(\mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-)$. *Moreover,*

$$u^+ = v_q^+[\partial_T \Omega, \mu^+] \quad \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]_T, \quad u^- = v_q^-[\partial_T \Omega, \mu^-] \quad \text{in } \mathrm{cl}\,\mathbb{S}_q[\Omega]_T^-,$$

where (μ^+, μ^-) *is the unique solution in* $(C_0^{\frac{m-1+\alpha}{2};m-1+\alpha}(\partial_T \Omega))^2$ *of*

$$H[\mu^+, \mu^-] = (f, g) \quad \text{on } \partial_T \Omega.$$

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