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Research Article

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The p-adic variation of the Gross–Kohnen–Zagier theorem

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Abstract: We relate p-adic families of Jacobi forms to big Heegner points constructed by B. Howard, in the spirit of the Gross–Kohnen–Zagier theorem. We view this as a GL(2) instance of a p-adic Kudla program.

Keywords: Gross–Kohnen–Zagier theorem, big Heegner points, Jacobi forms, Hida families, p-adic Kudla program

MSC 2010: 11F30, 11F27, 14G35, 11F50, 11F85

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1 Introduction

In their seminal paper [19], Gross, Kohnen and Zagier showed that Heegner points are the generating series of a Jacobi form arising from a theta lift. Suppose that $E/\mathbb{Q}$ is an elliptic curve of conductor M. We assume that the sign of the functional equation of $L(E/\mathbb{Q}, s)$ at $s = 1$ is -1. Then it is well known [20, 25, 26] that if the value $L'(E/\mathbb{Q}, s)_{s=1}$ of the first derivative of $L(E/\mathbb{Q}, s)$ at $s = 1$ is non-zero, then the $\mathbb{Q}$-vector space $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is one-dimensional. Moreover, one can show that this vector space is generated by a Heegner point P_K attached to a quadratic imaginary field K in which all primes $\ell \mid N$ are split and such that $L'(E/K_D, s)_{s=1} \neq 0$. As the imaginary quadratic field K varies, it is then a natural question to investigate the relative positions of the points P_K on this one-dimensional line. The theorem of Gross–Kohnen–Zagier answers this question. For each discriminant D and each residue class r mod $2N$ subject to the condition $D \equiv r^2$ mod $4N$, one defines a Heegner point $P_{D,r}$ (so the point P_K considered above corresponds to one of these points for a suitable choice of the pair (D, r)). Then [19, Theorem C] shows that the relative positions of the points $P_{D,r}$, at least under the condition that $(M, D) = 1$, are encoded by the (D, r)-th Fourier coefficient of the Jacobi form ϕ_{f_E} coming from the theta lifting of f_E, where f_E is the weight 2 newform of level $\Gamma_0(M)$ associated with E by modularity; one may express briefly this relation by saying that Heegner points are generating series for Jacobi forms, and actually [19] formulates an *ideal statement* in which the above relation is conjecturally extended to all coefficients of ϕ_{f_E}, including therefore those D such that $(D, M) \neq 1$. Several generalizations are available in the literature, especially by Borcherds [8], using singular theta liftings, and Yuan, Zhang and Zhang [38] using a multiplicity one theorem for automorphic representations. In particular, these works complete [19] by essentially proving the ideal statement alluded to above.

The purpose of this paper is to investigate a variant in families of the GKZ theorem, in which all objects are made to vary via p-adic interpolation. Note that the GKZ theorem (and the closely related famous Hirzebruch–Zagier theorem) can be seen as the historical starting point of the conjectural generalization by Kudla involv-

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ing higher-dimensional varieties, called the Kudla program for brief, relating, for example, algebraic cycles on Shimura varieties and Fourier coefficients of modular forms. We hope that our p-adic GKZ theorem will trigger higher-dimensional generalizations, giving rise to a p-adic analogue of the Kudla program.

We begin by briefly explaining some of the questions motivating this work. Given a primitive branch $\mathcal{R}$ of a Hida family, Howard [23] introduced certain cohomology classes in the Selmer group of Hida's big ordinary representation attached to a Hida family, that he called big Heegner points (we refer to [23] for the terminology, which is not explicitly introduced here). Following Hida's strategy to construct big Galois representations, these big Heegner points are constructed as limits of classical Heegner points on modular curves. Their specializations at arithmetic points of $\mathcal{R}$ of weight bigger than 2 are known, thanks to works of Howard [23], Castella [9, 10] and, more recently, Disegni [13], to interpolate classical Heegner cycles [32, 40]. Moreover, under suitable assumptions, the Selmer group of the Hida big Galois representation is a $\mathcal{R}$-module of rank 1, and big Heegner points are non-torsion elements in this Selmer group [22, 23]. Thus, extending scalars to the fraction field $\mathcal{K} = \mathrm{Frac}(\mathcal{R})$ of $\mathcal{R}$, we obtain a $\mathcal{K}$-vector space of dimension 1, and all Heegner points become proportional to a fixed generator. As in the paper [19], one may thus consider the proportionality coefficients which are elements of $\mathcal{K}$.

On the other hand, the theta correspondence can be applied to each arithmetic point in the given Hida family, and in [28], using methods from [36], the existence of a p-adic family of Jacobi forms whose specialization at arithmetic points of the metaplectic covering of $\mathcal{R}$ gives a Jacobi form coming from a theta lift of an arithmetic point in the Hida family is shown. In other words, [28] constructs a p-adic family of Jacobi forms $\mathbb{S}$ interpolating classical theta lifts of arithmetic points in the given Hida family; here the coefficients of this p-adic family of Jacobi forms $\mathbb{S}$ are elements in $\mathcal{K}$. We refer to Section 5.2 below or [28] for details.

It seems then natural to formulate a conjecture which relates the coefficients of the p-adic family of Jacobi forms $\mathbb{S}$ and the proportionality coefficients associated to big Heegner points in a way similar to the original GKZ. We can thus make the following conjecture.

Conjecture 1.1. *Let $\mathcal{R}$ be a primitive branch of a Hida family. There exists a Zariski open subset in* $\mathrm{Spec}(\mathcal{R})$ *where the proportionality coefficients relating big Heegner points are equal to the Jacobi–Fourier coefficients of the theta lift $\mathbb{S}$ of the Hida family.*

A more precise form of this conjecture is presented below, under some technical assumptions, as Conjecture 5.3, to which the reader is referred for details.

In this paper, we do not attack Conjecture 1.1 directly (see Remark 5.4 for a possible strategy to attach this conjecture). Instead, we present evidence in favor of this conjecture of a more local nature, in which both big Heegner points and Jacobi–Fourier coefficients of theta lifts of classical forms of trivial character in the Hida family are related (up to explicit Euler factors) to certain p-adic L-functions constructed in [1]; as a consequence, big Heegner points and Jacobi–Fourier coefficients of Jacobi forms are indeed related, at least locally, as predicted by Conjecture 1.1, and are both seen as manifestation of the arithmetic of p-adic L-functions.

We now state our results in a more precise way. Pick a positive even integer $2k_0$, and an ordinary p-stabilized newform f_{2k_0} of level $\Gamma_0(Np)$, trivial character and weight $2k_0$, where $N \geq 1$ is an odd integer and $p \nmid N$ a prime number bigger or equal to 5. Let $\mathcal{X}$ be the group of continuous endomorphisms of $\mathbb{Z}_p^\times$; we view $\mathbb{Z}$ inside $\mathcal{X}$ via the map $k \mapsto [x \mapsto x^{k-2}]$. Let

$$f_\infty(\kappa) = \sum_{n \geq 1} a_n(\kappa) q^n$$

be the Hida family passing through f_{2k_0}, where $\kappa \mapsto a_n(\kappa)$ are p-adic analytic functions defined in a neighborhood U of $2k_0$ in $\mathcal{X}$; if we denote by $\mathcal{A}_{2k_0}$ the subring of $\bar{\mathbb{Q}}_p[\![X]\!]$ consisting of power series converging in a neighborhood of $2k_0$, then $a_n \in \mathcal{A}_{2k_0}$. For each even integer $2k \in U$, let f_{2k} be the weight $2k$ modular form with trivial character appearing in the Hida family. The form f_{2k} is then an ordinary p-stabilized form of weight $2k$ and level $\Gamma_0(Np)$, and f_{2k_0} is the form we started with above, justifying the slight abuse of notation. If f_{2k} is not a newform (which is always the case if $k > 1$), then we write $f_{2k}^\sharp$ for the ordinary newform of weight $2k$ and level $\Gamma_0(N)$ whose p-stabilization is f_{2k}. For each of the forms $f_{2k}^\sharp$, we have an associated Jacobi

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form

$$\mathcal{S}_{D_0,r_0}(f_{2k}^\sharp) = \sum_{D=r^2-4Nn} c_{f_{2k}^\sharp}(n,r) q^n \zeta^r$$

of weight $k+1$, index N having the same Hecke eigenvalues as $f_{2k}^\sharp$; this form depends on the choice of a pair of integers (D_0, r_0) such that D_0 is a fundamental discriminant which we assume to be prime to p and $D_0 \equiv r_0^2 \bmod 4N$. To be clear, in the above formula, the sum is over all negative discriminants D, and the coefficients $c_{f_{2k}^\sharp}(n,r)$ only depend on the residue class of $r \bmod 2N$. The first observation we make is that these coefficients $c_{f_{2k}^\sharp}(n,r)$ can be interpolated, up to Euler factors, by p-adic analytic functions. We show that a suitable linear combination $\mathcal{L}_{n,r} \in \mathcal{A}_{2k_0}$ of the square root p-adic L-functions attached to genus characters of real quadratic fields, introduced in [1, 2] and studied extensively in [16, 29, 34], interpolates Fourier–Jacobi coefficients $c_{f_{2k}^\sharp}(n,r)$. Our first result, corresponding to Theorem 3.1 below, is the following.

Theorem 1.2. *For all positive even integers $2k$ in a sufficiently small neighborhood of $2k_0$ and all $D = r^2 - 4nN$ such that $p \nmid D$, we have $\mathcal{L}_{n,r}(2k) \doteq c_{f_{2k}^\sharp}(n,r)$ where $\doteq$ means equality up to an explicit Euler factor and a suitable p-adic period, both independent of (n,r) and non-vanishing.*

On the other hand, as recalled above, B. Howard [23] constructed an analogue of the Heegner point P_K in the setting of Hida families. To explain this, recall that, attached to f_∞, we have a big Galois representation $\mathbb{T}^\dagger$ interpolating self-dual twists of the Deligne representation attached to f_{2k} for $2k$ a positive even integer. If we denote (as before) by $\mathcal{R}$ the branch of the Hida–Hecke algebra corresponding to f_∞ (which is a complete Noetherian integral domain, finite over $\Lambda = \mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$), then $\mathbb{T}^\dagger$ is a free $\mathcal{R}$-module of rank 2 equipped with a continuous action of $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and specialization maps $\mathbb{T}^\dagger \mapsto \mathbb{T}_\kappa^\dagger$ for each $\kappa \in \mathcal{X}$ such that, when $\kappa = 2k$ is a positive even integer, $\mathbb{T}_{2k}^\dagger$ is isomorphic to the self-dual twist of the Deligne representation attached to f_{2k}. We assume throughout this paper that the residual representation $\mathbb{T}^\dagger/\mathfrak{m}_\mathcal{R}\mathbb{T}^\dagger$ (where $\mathfrak{m}_\mathcal{R}$ is the maximal ideal of $\mathcal{R}$) is absolutely irreducible and p-distinguished. Let $\mathrm{Sel}(\mathbb{Q}, \mathbb{T}^\dagger) \subseteq H^1(G_\mathbb{Q}, \mathbb{T}^\dagger)$ be the Greenberg Selmer group attached to the Galois representation $\mathbb{T}^\dagger$. We require that the generic sign of the functional equation of the L-functions of f_{2k} is -1 and that the central critical value of their first derivative is generically non-vanishing; cf. Assumptions 4.1 and 4.2 below. Under these conditions, which might be viewed as analogues to those in [19], the $\mathcal{R}$-module $\mathrm{Sel}(\mathbb{Q}, \mathbb{T}^\dagger)$ is finitely generated of rank 1. Howard constructs certain cohomology classes $\mathfrak{Z}_{D,r}^{\mathrm{How}} \in \mathrm{Sel}(\mathbb{Q}, \mathbb{T}^\dagger)$, taking inverse limits of norm-compatible sequences of Heegner points in towers of modular curves. Therefore, it is a natural question to relate the positions of the points $\mathfrak{Z}_{D,r}^{\mathrm{How}}$ in the 1-dimensional $\mathcal{K}$-vector space $\mathrm{Sel}_\mathcal{K}(\mathbb{Q}, \mathbb{T}^\dagger) = \mathrm{Sel}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_\mathcal{R} \mathcal{K}$, where $\mathcal{K} = \mathrm{Frac}(\mathcal{R})$ is the fraction field of $\mathcal{R}$. To describe our second result, we also require that the height pairing between Heegner cycles is positive definite; cf. Assumption 4.4 below. Define

$$\mathcal{Z}_{n,r}(\kappa) = 2u_D \cdot (2D)^{\frac{\kappa-2}{4}} \cdot \mathfrak{Z}_{D,r}^{\mathrm{How}}(\kappa)$$

for $D = r^2 - 4Nn$ and $\kappa \in \mathcal{X}$; here $2u_D$ is the number of units in $\mathbb{Q}(\sqrt{D})$ and, as above, all primes dividing $D = r^2 - 4Nn$ are required to split in K_D. Suppose $2k_0 \equiv 2 \bmod p-1$. Let $\mathcal{M}_{2k_0}$ be the fraction field of $\mathcal{A}_{2k_0}$. There is a canonical map $\mathcal{K} \to \mathcal{M}_{2k_0}$, and we may define $\mathrm{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T}^\dagger) = \mathrm{Sel}_\mathcal{K}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_\mathcal{K} \mathcal{M}_{2k_0}$. Our second result, under Assumptions 4.1, 4.2 and 4.4 discussed above, is the following.

Theorem 1.3. *There exists an element $\Phi^{\text{ét}} \in \mathrm{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T}^\dagger)$ such that, in a sufficiently small connected neighborhood of $2k_0$ in $\mathcal{X}$ and for all $D = r^2 - 4nN$ such that p splits in $\mathbb{Q}(\sqrt{D})$, we have $\mathcal{Z}_{n,r} = \mathcal{L}_{n,r} \cdot \Phi^{\text{ét}}$.*

This corresponds to Theorem 5.1 below. Combining Theorems 1.2 and 1.3, we obtain a partial evidence toward Conjecture 1.1.

Corollary 1.4. *For all positive even integers $2k$ in a sufficiently small connected neighborhood of $2k_0$ and for all $D = r^2 - 4nN$ such that p splits in $\mathbb{Q}(\sqrt{D})$, we have*

$$\mathcal{Z}_{n,r} \doteq c_{f_\kappa^\sharp}(n,r) \cdot \Phi^{\text{ét}}$$

where $\doteq$ means equality up to simply algebraic factors independent of (n,r).

See (5.1) and (5.5) for a more precise version of this result, including all the algebraic factors involved.

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Remark 1.5. As remarked above, we may view Corollary 1.4 as a fragment of Conjecture 1.1. However, it should be noticed that the coefficients we are considering in this work and in [28] are slightly different. In this paper, we consider coefficients of Jacobi forms which are lifts of modular newforms of level prime to p and trivial character, whose ordinary p-stabilizations belong to our Hida family; the discrepancy between newforms and their p-stabilizations is the origin of the Euler factors (denoted $\mathcal{E}_p$ and defined in (3.1) below) relating p-adic L-functions with Heegner points and Jacobi–Fourier coefficients of Jacobi forms in a way similar to [12]; this should clarify the crucial role played in this paper by the p-adic L-functions from [1]. On the other hand, in [28], we consider Jacobi forms lifting the members of the Hida family f_{2k} (or more generally f_κ for an arithmetic point κ, as in [36]). The theta lifts used in the two papers are then different since, in this paper, the level of the forms is prime to p, while, in [28], the level is divisible by p. The Jacobi–Fourier coefficients coming from different theta liftings can be directly related by means of Euler factors (denoted $\mathcal{E}_2$ and defined in (5.3) below). Since these Euler factors are independent of (n,r), the quotients of Jacobi–Fourier coefficients coming from the two theta liftings, when defined, are the same, and then, in this sense, our result can be seen as an evidence of Conjecture 1.1. The details are discussed in Section 5.2.

2 Theta lifts

In this section, we recall the formalism of theta liftings relating elliptic and Jacobi cuspforms, following [14, 19].

Fix an odd integer $N \geq 1$ and k a positive even integer $2k$. Denote $S_{2k}(\Gamma_0(N))$ the complex vector space of cuspforms of weight $2k$ and level $\Gamma_0(N)$ and $J^{\mathrm{cusp}}_{k+1,M}$ the complex vector space of Jacobi cuspforms of weight $k+1$ and index N (see [14, Chapter I, § 1] for a definition). For $f \in S_{2k}(\Gamma_0(N))$ and $\phi \in J^{\mathrm{cusp}}_{k+1,N}$, we write the corresponding q-expansions $f(z) = \sum_{n\geq 1} a_n q^n$ and (q,ζ)-expansions

$$\phi(\tau,z) = \sum_{r^2-4Nn<0} c(n,r) q^n \zeta^r,$$

where $q = e^{2\pi i\tau}$ and $\zeta = e^{2\pi i z}$ and the second sum is over all pairs (n,r) of integers such that $r^2 - 4Nn < 0$.

Definition 2.1. A level N index pair is a pair (D,r) of integers consisting of a negative discriminant D of an integral quadratic form $Q = [a,b,c]$ such that $D \equiv r^2 \bmod 4N$. A level N index pair is said to be fundamental if D is a fundamental discriminant.

The Fourier expansion of forms $\phi \in J^{\mathrm{cusp}}_{k+1,N}$ is enumerated by level N index pairs, explaining the terminology. Since N is fixed throughout the paper, unless otherwise stated, we simply call index pairs or fundamental index pairs the pairs (D,r) in Definition 2.1. If (D,r) is an index pair, then we usually denote n the integer such that $D^2 = r^2 - 4Nn$. The spaces $S_{2k}(\Gamma_0(N))$ and $J^{\mathrm{cusp}}_{k+1,N}$ are equipped with the action of standard Hecke operators $\mathrm{T}(m)$ and $\mathrm{T}_J(m)$, respectively, for integers $m \geq 1$. We recall the formula from [14, Theorem 4.5] for the action of $\mathrm{T}_J(m)$ on Jacobi cuspforms when $(N,m) = 1$. If $D = r^2 - 4Nn$ is a fundamental discriminant, $\phi = \sum_{n,r} c(n,r) q^n \zeta^r$ belongs to $J^{\mathrm{cusp}}_{k+1,N}$ and $\phi \mid \mathrm{T}_J(m) = \sum_{n,r} c^*(n,r) q^n \zeta^r$, then we have

$$c^*(n,r) = \sum_{d\mid m} d^{k-1} \Big(\frac{D}{d}\Big) c\Big(\frac{nm^2}{d^2}, \frac{rm}{d}\Big).$$

For any ring R, let $\mathcal{P}_{k-2}(R)$ denote the R-module of homogeneous polynomials in two variables of degree $k-2$ with coefficients in R, equipped with the right action of the semigroup $\mathrm{M}_2(R)$ defined by the formula

$$(F|\gamma)(X,Y) = F(aX+bY, cX+dY) \tag{2.1}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $\mathcal{V}_{k-2}(R)$ denote the R-linear dual of $\mathcal{P}_{k-2}(R)$, equipped with the left $\mathrm{M}_2(R)$-action induced from that on $\mathcal{P}_{k-2}(R)$.

Let f be a cuspform of integral even weight $2k$ and level $\Gamma_0(N)$. To f, we may associate the modular symbol $\tilde{I}_f \in \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_{2k-2}(\mathbb{C}))$ by the integration formula

$$\tilde{I}_f\{r \to s\}(P) = 2\pi i \int_r^s f(z) P(z,1)\, dz.$$

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Here, for any congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and any $\mathbb{Z}[\mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{M}_2(\mathbb{Z})]$-module M, we denote $\mathrm{Symb}_\Gamma(M)$ the group of Γ-invariant modular symbols with values in M (cf. [18, formula (4.1)]). The matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ normalizes $\Gamma_0(N)$ and hence induces an involution on the space of modular symbols $\mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_{2k-2}(\mathbb{C}))$; for each $\varepsilon \in \{\pm 1\}$, we denote $\tilde{I}_f^\varepsilon$ the ε-eigencomponents of $\tilde{I}_f$ with respect to this involution. It is known that there are complex periods Ω_f^ε such that

$$I_f^\varepsilon = \frac{\tilde{I}_f^\varepsilon}{\Omega_f^\varepsilon}$$

belong to $\mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_{2k-2}(F_f))$, where F_f is the extension of $\mathbb{Q}$ generated by the Fourier coefficients of f. These periods can be chosen so that the Petersson norm $\langle f, f\rangle$ equals the product $\Omega_f^+ \cdot \Omega_f^-$; note that the Ω_f^ε are well-defined only up to multiplication by non-zero factors in $F_f^\times$.

For each integer Δ, let $\mathcal{Q}_\Delta$ be the set of integral quadratic forms

$$Q = [a, b, c] = ax^2 + bxy + cy^2$$

of discriminant Δ and, for any integer ρ, let $\mathcal{Q}_{N,\Delta,\rho}$ denote the subset of $\mathcal{Q}_\Delta$ consisting of integral binary quadratic forms $Q = [a, b, c]$ of discriminant Δ such that $b \equiv \rho \pmod{2N}$ and $a \equiv 0 \bmod N$. Let $\mathcal{Q}^0_{N,\Delta,\rho}$ be the subset of $\mathcal{Q}_{N,\Delta,\rho}$ consisting of forms which are $\Gamma_0(N)$-primitive, i.e., those $Q \in \mathcal{Q}_{N,\Delta,\rho}$ which can be written as $Q = [Na, b, c]$ with $(a, b, c) = 1$. These sets are equipped with the right action of $\mathrm{SL}_2(\mathbb{Z})$ described in (2.1).

Fix a fundamental index pair (D_0, r_0). For any index pair (D, r), let $\mathcal{F}^{(D,r)}_{D_0,r_0}(N)$ be the set of integral binary quadratic forms $Q = [a, b, c]$ modulo the right action of $\Gamma_0(N)$ described in (2.1), such that

- $\delta_Q = b^2 - 4ac = D_0 D$,
- $b \equiv -r_0 r \bmod 2N$,
- $a \equiv 0 \bmod N$.

Let $Q \mapsto \chi_{D_0}(Q)$ be the generalized genus character attached to D_0 defined in [19, Proposition 1]. We recall the definition for $Q \in \mathcal{Q}_{N,\Delta,\rho}$. If $Q = \ell \cdot Q'$ for some form $Q' \in \mathcal{Q}^0_{N,\Delta,\rho}$, then define $\chi_{D_0}(Q) = \left(\frac{D_0}{\ell}\right) \cdot \chi_{D_0}(Q')$, so it is enough to define it on $\Gamma_0(N)$-primitive forms. Fix $Q \in \mathcal{Q}^0_{N,\Delta,\rho}$. If $(a/N, b, c, D_0) = 1$, then pick any factorization $N = m_1 \cdot m_2$ with $m_1 > 0$, $m_2 > 0$ and any integer n coprime with D_0 represented by the quadratic form $[a/m_1, b, cm_2]$; then put $\chi_{D_0}(Q) = \left(\frac{D_0}{n}\right)$. If $(a/N, b, c, D_0) \neq 1$, then put $\chi_{D_0}(Q) = 0$. In the previous notation, if Q is a representative of a class in $\mathcal{F}^{(D,r)}_{D_0,r_0}(N)$, then Q belongs to $\mathcal{Q}_{M,D_0D,-r_0r}$. In particular, we may consider the genus character $Q \mapsto \chi_{D_0}(Q)$ for all classes Q in $\mathcal{F}^{(D,r)}_{D_0,r_0}(N)$.

Fix $Q = [a, b, c] \in \mathcal{Q}_{M,\Delta,\rho}$. Following [19, 36], we define certain geodesic lines in $\mathcal{H}$, with respect to the Poincaré metric on $\mathcal{H}$, as follows. If $\delta_Q = m^2$ ($m > 0$) is a perfect square, let C_Q be the geodesic line from $(-b - m)/2a$ to $(-b + m)/2a$ when $a \neq 0$, while if $a = 0$, let C_Q be the geodesic line from $-c/b$ to $i\infty$ if $b > 0$ and from $i\infty$ to $-c/b$ if $b < 0$. If δ_Q is not a perfect square, we denote by γ_Q the matrix in $\mathrm{SL}_2(\mathbb{Z})$ corresponding to a unit of the quadratic form Q and denote C_Q the geodesic between z_0 and $\gamma_Q(z_0)$, where z_0 is any point in $\mathbb{P}^1(\mathbb{Q})$ (we can take $z_0 = i\infty$ for example). We let r_Q and s_Q denote the source and the target of the geodesic line C_Q. Note that, in any case, s_Q and r_Q belong to $\mathbb{P}^1(\mathbb{Q})$, and therefore the modular symbol $I_f^-\{r_Q \to s_Q\}$ is well-defined.

For $D = r^2 - 4Nn$, define

$$c_f(n, r) = \sum_{Q \in \mathcal{F}^{(D,r)}_{D_0,r_0}(N)} \chi_{D_0}(Q) \cdot I_f^-\{r_Q \to s_Q\}(Q^{k-1}), \tag{2.2}$$

and set

$$\mathcal{S}_{D_0,r_0}(f) = \sum_{r^2 - 4Nn < 0} c_f(n, r) q^n \zeta^r.$$

Let $S^-_{2k}(\Gamma_0(N))$ be the subspace of $S_{2k}(\Gamma_0(N))$ consisting of forms whose L-function admits a functional equation with sign -1. The association $f \mapsto \mathcal{S}_{D_0,r_0}(f)$ gives a $\mathbb{C}$-linear map, called theta lifting,

$$\mathcal{S}_{D_0,r_0} : S^-_{2k}(\Gamma_0(N)) \to J^{\mathrm{cusp}}_{k+1,N},$$

which is equivariant for the action of Hecke operators on both spaces and such that $c(n, r)$ belong to F_f for all n, r; see [14, Chapter II] for details. The restriction of $\mathcal{S}_{D_0,r_0}$ to newforms of $S^-_{2k}(\Gamma_0(N))$ is an isomorphism onto the image and, for different choices of (D_0, r_0), we get multiples of the same Jacobi form.

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Remark 2.2. The definition of the geodesic line C_Q is slightly different in [19] since it is defined to be any geodesic in the upper half plane connecting a point $z_0 \in \mathcal{H}$ to the point $\gamma_Q(z_0)$. However, the value of the integral is independent of the choice of $z_0 \in \mathcal{H}$, and therefore, passing to the limit, one can take z_0 to be any cusp as well. See also [36, § 2.1] for a similar definition, keeping in mind that the convention in loc. cit. and in this paper (which follow more closely [19]) are slightly different.

3 p-adic analytic theta lifts

In this section, we construct a p-adic analytic family of Jacobi forms using the p-adic variation of integrals. This family interpolates Fourier–Jacobi coefficients of newforms of level $\Gamma_0(N)$ whose p-stabilizations belong to a fixed Hida family (in contrast with [28], where Λ-adic families directly interpolate arithmetic specializations in Hida families). The results of this section are, as in [28], consequences of Stevens and Hida's works on half-integral weight modular forms [21, 36] via the formalism of theta liftings and the work of Greenberg and Stevens [18] and Bertolini and Darmon [1] on measure-valued modular symbols. Fix throughout a prime number $p \nmid N$, where $p \geq 5$.

We first set up the notation for Hida families. Let $\Gamma = 1 + p\mathbb{Z}_p$, and let $\Lambda = \mathcal{O}[\![\Gamma]\!]$ be its Iwasawa algebra, where $\mathcal{O}$ is the valuation ring of a finite field extension of $\mathbb{Q}_p$. We fix an isomorphism $\Lambda \simeq \mathcal{O}[\![X]\!]$. Let

$$\mathcal{X} = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

be the group of continuous group $\mathbb{Z}_p^\times$-valued homomorphisms of $\mathbb{Z}_p^\times$. Embed $\mathbb{Z}$ in $\mathcal{X}$ by $k \mapsto [x \mapsto x^{k-2}]$; if we equip $\mathcal{X}$ with the rigid analytic topology, then $\mathbb{Z} \subseteq \mathcal{X}$ is rigid Zariski dense. We see elements $a \in \Lambda$ as functions on $\mathcal{X}$ by $a(\kappa) = \varphi_\kappa(a)$ for all $\kappa \in \mathcal{X}$, where $\varphi_\kappa : \Lambda \to \mathcal{O}$ denotes the $\mathcal{O}$-linear extension of κ.

Recall the fixed p-stabilized form f_0 of trivial nebentype and weight k_0, and $\mathcal{R}$ be the branch of the Hida family passing through f_0. Then $\mathcal{R}$ is an integral Noetherian domain, finite over Λ; we denote $\mathcal{X}(\mathcal{R})$ the set of continuous homomorphisms of $\mathcal{O}$-algebras $\mathcal{R} \to \bar{\mathbb{Q}}_p$ and $\mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ the subset of arithmetic points (see [23, Definition 2.1.1] for its definition). For each positive even integer $2k \in \mathcal{X}$, using that $\mathcal{R}/\Lambda$ is unramified at the point φ_{2k}, we obtain a unique arithmetic point $\tilde{\varphi}_{2k}$ lying over φ_{2k}. Hida theory shows that there exists a formal power series $F_\infty = \sum_{n\geq 1} A_n q^n \in \mathcal{R}[\![q]\!]$ such that, for each $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$, the power series $F_\varphi = \sum_{n \geq 1} \varphi(A_n) q^n$ is the q-expansion of a p-stabilized newform and $F_{\tilde{\varphi}_{2k_0}} = f_0$.

Fix an even positive integer $2k_0$, and let $\mathcal{A}_{2k_0}$ be the ring of power series in $\bar{\mathbb{Q}}_p[\![X]\!]$ which converge in a neighborhood of $2k_0$. If $\mathcal{R}_{2k_0}$ denotes the localization of $\mathcal{R}$ at $\tilde{\varphi}_{2k_0}$, then, using that $\mathcal{A}_{2k_0}$ is Henselian, we obtain a canonical morphism $\psi_{2k_0} : \mathcal{R}_{2k_0} \to \mathcal{A}_{2k_0}$ ([18, formula (2.7)]), and therefore, using the localization map, we have a canonical map $\mathcal{R} \to \mathcal{A}_{2k_0}$. If $\mathcal{K}$ and $\mathcal{M}_{2k_0}$ are the fraction fields of $\mathcal{R}$ and $\mathcal{A}_{2k_0}$, respectively, then we obtain a map, still denoted $\psi_{2k_0} : \mathcal{K} \to \mathcal{M}_{2k_0}$. The domain of convergence about $2k_0$ is the intersections of the discs of convergence of $\psi_{2k_0}(a)$ for $a \in \mathcal{R}$. We denote U_{2k_0} the domain of convergence, which we may assume to be connected. We let

$$f_\infty = \sum_{n\geq 1} a_n q^n$$

be the image of F_∞ via ψ_{2k_0}, so $a_n = \psi_{2k_0}(A_n)$ are rigid analytic functions converging in the domain of convergence about $2k_0$ such that, for each even positive integer $2k \in U_{2k_0}$, the power series $\sum_{n\geq 1} a_n(2k) q^n$ is the Fourier expansion of an ordinary p-stabilized modular form $f_{2k} \in S_k(\Gamma_0(Np))$.

We now describe universal measure-valued modular symbols, following [1, 18]. Let $\mathbb{D}_*$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^2$ which are supported on the set of primitive vectors $(\mathbb{Z}_p^2)'$ of $\mathbb{Z}_p^2$. The $\mathcal{O}$-module $\mathbb{D}_*$ is equipped with the action induced by the action of $\mathrm{GL}_2(\mathbb{Z}_p)$ on $\mathbb{Z}_p^2$ by

$$(x, y) \mapsto (ax + by, cx + dy) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p),$$

and a structure of $\mathcal{O}[\![\mathbb{Z}_p^\times]\!]$-module induced by the scalar action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p^2$ (cf. [36, § 5] and [2, § 2.2]); in particular, $\mathbb{D}_*$ is also equipped with a structure of a Λ-module.

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Let $\Gamma_0(p\mathbb{Z}_p)$ denote the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ consisting of matrices which are upper triangular modulo p. The group $\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{D}_*)$ is equipped with an action of Hecke operators, including the Hecke operator at p, denoted $\mathrm{U}(p)$, and we denote

$$\mathbb{W} = \mathrm{Symb}^{\mathrm{ord}}_{\Gamma_0(N)}(\mathbb{D}_*)$$

the ordinary subspace of $\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{D}_*)$ for the action of $\mathrm{U}(p)$; see [18, formulas (2.2), (2.3)] for details and definitions.

For any integer $k \in \mathcal{X}$, we have a $\Gamma_0(p\mathbb{Z}_p)$-equivariant homomorphism $\rho_\kappa : \mathbb{D}_* \to \mathcal{V}_{k-2}(\mathbb{C}_p)$ defined by the formula

$$\rho_k(\mu)(P) = \int\limits_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} P(x,y)\,d\mu(x,y),$$

which gives rise to an homomorphism, denoted by the same symbol,

$$\rho_k : \mathbb{W} \to \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_{k-2}(\mathbb{C}_p)).$$

We may then define $\mathbb{W}_{\mathcal{A}_{2k_0}} = \mathbb{W} \otimes_\Lambda \mathcal{A}_{2k_0}$ and consider the extension of ρ_k, still denoted by the same symbol,

$$\rho_k : \mathbb{W}_{\mathcal{A}_{2k_0}} \to \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_{k-2}(\mathbb{C}_p)).$$

By [18, Theorem 5.13], there exist a connected neighborhood U_{2k_0} of $2k_0$ in $\mathcal{X}$ and an element $\Phi_{2k_0} \in \mathbb{W}_{\mathcal{A}_{2k_0}}$ such that, for all positive even integers $2k \in U_{2k_0}$ with $k \geq 1$, we have

$$\rho_k(\Phi_{2k_0}) = \lambda(k) \cdot I_{f_{2k}},$$

where $\lambda(k) \in F_{f_{2k}}$ is the field extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of f_{2k}, and $\lambda(k_0) = 1$.

Fix a fundamental index pair (D_0, r_0) so that $p \nmid D_0$, and an index pair (D, r) with $p \nmid D$. Fix a system of representatives $\mathcal{R}^{(D,r)}_{D_0,r_0}(N)$ of $\mathcal{F}^{(D,r)}_{D_0,r_0}(N)$ so that each form $[a,b,c]$ in $\mathcal{R}^{(D,r)}_{D_0,r_0}(N)$ satisfies $p \nmid a$ and $p \nmid c$. Such a system can easily be obtained up to multiplying quadratic forms by matrices of the form $\left(\begin{smallmatrix}1 & i\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 0\\ Ni & 1\end{smallmatrix}\right)$ for suitable integers i. Fix a matrix $Q = [a,b,c]$ in $\mathcal{R}^{(D,r)}_{D_0,r_0}(N)$. Let $\Delta = D_0 D$. If p is inert in K_Δ, we define $X_Q = (\mathbb{Z}_p^2)'$. If p is split in K_Δ, then $Q(X,Y)$ splits into the product $a \cdot q_1(X,Y) \cdot q_2(X,Y)$, where $q_1(X,Y) = X - \alpha Y$ and $q_2(X,Y) = X - \beta Y$ are two linear forms in $\mathbb{Z}_p[X,Y]$. Define the elements $v_1 = (\alpha, 1)$ and $v_2 = (\beta, 1)$, and put $X_Q = \mathbb{Z}_p^\times \cdot v_1 + \mathbb{Z}_p^\times v_2$. Since $p \nmid ac$, then $\mathbb{Z}_p v_1 + \mathbb{Z}_p v_2 = \mathbb{Z}_p^2$. Both in the split and inert cases, it is easy to show that $Q(x,y) \in \mathbb{Z}_p^\times$ for all $(x,y) \in X_Q$.

Define for $\kappa \in U_{2k_0}$ and $Q \in \mathcal{R}^{(D,r)}_{D_0,r_0}(N)$,

$$\mathcal{L}_Q(\kappa) = \int\limits_{X_Q} \omega(Q(x,y))^{k_0-1} \langle Q(x,y)\rangle^{\frac{k-2}{2}}\, d\Phi_{2k_0}\{r_Q \to s_Q\}(x,y).$$

The next result exploits the interpolation formulas of $\mathcal{L}_Q(\kappa)$. Put

$$\mathcal{E}_p = \begin{cases} \left(1 - \frac{p^{k-1}}{a_p(2k)}\right)^2 & \text{if } p \text{ is split in } K_\Delta, \\ 1 - \frac{p^{2k-2}}{a_p^2(2k)} & \text{if } p \text{ is inert in } K_\Delta. \end{cases} \tag{3.1}$$

Then, for all even positive integers $2k \in U_{2k_0}$ with $k > 1$ and all quadratic forms Q in $\mathcal{R}^{(D,r)}_{D_0,r_0}(N)$, we have

$$\mathcal{L}_Q(2k) = \lambda(k) \cdot \mathcal{E}_p \cdot I_{f_{2k}^\sharp}\{r_Q \to s_Q\}. \tag{3.2}$$

In the inert case, (3.2) is [2, Proposition 2.4]. In the split case, a proof of this result can be found in [34, Proposition 3.3.1]; see also [16, Section 5.1], [29, Proposition 4.24], [27, Proposition 3.3].

Suppose that both D_0 and D are fundamental discriminants. With the usual convention that $D = r^2 - 4nN$, put

$$\mathcal{L}_{n,r}(\kappa) = \sum_{Q \in \mathcal{R}^{(D,r)}_{D_0,r_0}(N)} \chi_{D_0}(Q) \cdot \mathcal{L}_Q(\kappa). \tag{3.3}$$

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The function $\mathcal{L}_{n,r}(\kappa)$ is called the square-root p-adic L-function attached to the genus character χ_{D_0} of the real quadratic field K_Δ since its square for $\kappa = 2k$, an even positive integer in U_{2k_0}, interpolates special values $L(f_k^\sharp/K_\Delta, \chi_{D_0}, k)$. More precisely, setting $L_{n,r} = \mathcal{L}_{n,r}^2$, we have for all even positive integers $2k \in U_{2k_0}$,

$$L_{n,r}(2k) = \lambda(k)^2 \cdot \mathcal{E}_p^2 \cdot \Delta^{k-1} \cdot L^{\mathrm{alg}}(f_{2k}^\sharp/K_\Delta, \chi_{D_0}, k),$$

where the algebraic part of the special value of the L-series of $f_{2k}^\sharp$ twisted by χ_{D_0} is defined as

$$L^{\mathrm{alg}}(f_{2k}^\sharp/K_\Delta, \chi_{D_0}, k) = \frac{(k-1)!^2\sqrt{\Delta}}{(2\pi i)^{2k-2} \cdot (\Omega_{f_{2k}^\sharp}^-)^2} \cdot L(f_{2k}^\sharp/K, \chi_{D_0}, k).$$

In the inert case, the above formula is proved in [2, Theorem 3.5], while, in the split case, the reader may consult [34, Theorem A, Remark 3.3.3], [16, Proposition 5.5] or [29, Proposition 4.25].

Theorem 3.1. *Fix a fundamental index pair (D_0, r_0), and let $2k_0$ be a positive even integer. Then, for all index pairs (D, r) such that $p \nmid \Delta = D_0 D$ and for all even positive integers $2k \in U_{2k_0}$, we have*

$$\mathcal{L}_{n,r}(2k) = \lambda(k) \cdot \mathcal{E}_p^2 \cdot c_{f_{2k}^\sharp}(n, r).$$

Proof. This is an immediate combination of (2.2) and (3.3). □

4 Big Heegner points and Jacobi forms

4.1 Big Heegner points

Let $\mathbb{T}$ denote Hida's big Galois representation and $\mathbb{T}^\dagger$ its critical twist, introduced in [23, Definition 2.1.3]. Recall that $\mathbb{T}^\dagger$ is a free $\mathcal{R}$-module of rank 2 equipped with a continuous $G_\mathbb{Q} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ action such that, for each arithmetic point $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$, the $G_\mathbb{Q}$-representation $V_\varphi^\dagger := \mathbb{T}^\dagger \otimes_\mathcal{R} L_\varphi$ is the self-dual twist of the Deligne representation attached to the module form F_φ; here L_φ is a finite field extension of $\mathbb{Q}_p$, and the tensor product is taken with respect to $\varphi\colon \mathcal{R} \to L_\varphi$. Assume that the semi-simple residual $G_\mathbb{Q}$-representation $\mathbb{T}^\dagger/\mathfrak{m}_\mathcal{R}\mathbb{T}^\dagger$, where $\mathfrak{m}_\mathcal{R}$ is the maximal ideal of $\mathcal{R}$, is absolutely irreducible and p-distinguished. The definition of $\mathbb{T}^\dagger$ depends on the choice of a critical character $\theta\colon G_\mathbb{Q} \to \Lambda^\times$ and that there are two possible choices (see [23, Definition 2.1.3 and Remark 2.1.4]), related by the quadratic character $x \mapsto (\frac{p}{x})$ of conductor p; more precisely, if we write the cyclotomic character ϵ_{cyc} as the product $\epsilon_{\mathrm{tame}} \cdot \epsilon_{\mathrm{wild}}$ with ϵ_{tame} taking values in $(\mathbb{Z}/p\mathbb{Z})^\times$ and ϵ_{wild} taking values in $1 + p\mathbb{Z}_p$, then the two possible critical characters are defined to be

$$\theta = \left(\frac{p}{\cdot}\right)^a \cdot \epsilon_{\mathrm{tame}}^{\frac{k_0-2}{2}} \cdot [\epsilon_{\mathrm{wild}}^{1/2}]$$

for $a = 0$ or $a = 1$, where $\epsilon_{\mathrm{wild}}^{1/2}$ is the unique square root of ϵ_{wild} taking values in $1 + p\mathbb{Z}_p$, and $x \mapsto [x]$ is the inclusion of group-like elements $1 + p\mathbb{Z}_p \hookrightarrow \Lambda$; here recall that k_0 is the weight of our fixed form f_0 through which the Hida family f_∞ passes. We fix the choice of θ for $a = 0$ as prescribed in [9, formula (4.1) and Remark 4.1].

For any extension H of $\mathbb{Q}$, denote by $\mathrm{Sel}(H, \mathbb{T}^\dagger)$ the strict Greenberg Selmer group, whose definition by means of the ordinary filtration of $\mathbb{T}$ can be found in [23, Definition 2.4.2].

Let K_D be an imaginary quadratic field of discriminant $D < 0$; denote by $\mathcal{O}_D$ its ring of algebraic integers and by H_D its Hilbert class field. Suppose that D is a square mod $2N$. Fix a residue class r mod $2N$ such that $r^2 \equiv D \bmod 4N$, and consider the integral $\mathcal{O}_D$-ideal $\mathfrak{N} = (N, \frac{r+\sqrt{D}}{2})$; then $\mathcal{O}_D/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$. Also, take a generator ω_D of $\mathcal{O}_K/\mathbb{Z} \simeq \mathbb{Z}$ so that the imaginary part of ω_D (viewed as a complex number via our fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$) is positive. Depending on these choices, B. Howard [23, § 2.2] constructs a point

$$\mathfrak{X}_{D,r}^{\mathrm{How}} \in H^1(H_D, \mathbb{T}^\dagger),$$

the big Heegner point of conductor 1 associated to the quadratic imaginary field K_D (see [23, Definition 2.2.3]); the notation, slightly different from loc. cit., reflects the dependence on D and r. Finally, define

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$\mathfrak{Z}_{D,r}^{\mathrm{How}} \in H^1(K_D, \mathbb{T}^\dagger)$ to be the image of $\mathfrak{X}_{D,r}^{\mathrm{How}}$ via the corestriction map. Recall that all primes dividing N are split in K_D, and then, by [23, Proposition 2.4.5], we have $\mathfrak{Z}_{D,r}^{\mathrm{How}} \in \mathrm{Sel}(K_D, \mathbb{T}^\dagger)$.

Let $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$, and let $\chi \mapsto L_p(F_\varphi, \chi)$ denote the Mazur–Tate–Teitelbaum p-adic L-function of F_φ, viewed as a function on characters $\chi\colon \mathbb{Z}_p^\times \to \bar{\mathbb{Q}}_p^\times$. The functional equation satisfied by this function is

$$L_p(F_\varphi, \chi) = -w\chi^{-1}(-N)\theta_\varphi(-N) \cdot L_p(F_\varphi, \chi^{-1}[\,\cdot\,]_\varphi),$$

where θ_φ is the composition of the chosen critical character θ with φ and $[\,\cdot\,]_\varphi$ is the composition of the tautological character $[\,\cdot\,]\colon \mathbb{Z}_p^\times \hookrightarrow \Lambda \hookrightarrow \mathcal{R}$ with φ; here $w = \pm 1$, and it is independent of φ (but it depends on the choice of θ) and satisfies the equation $L_p(F_\varphi, \theta_\varphi) = -wL_p(F_\varphi, \theta_\varphi)$ (see [23, Proposition 2.3.6]).

In this paper, we will work under the following assumption on the sign of the functional equation of the Mazur–Tate–Teitelbaum p-adic L-function.

Assumption 4.1. $w = 1$.

We now discuss the analytic condition that we will assume in this paper to obtain our results. First let $\theta\colon \mathbb{Z}_p^\times \to \mathcal{R}^\times$ be the character obtained by factoring θ through the p-cyclotomic extension $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ (where μ_{p^∞} is the group of p-power roots of unity) and identifying $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$ with $\mathbb{Z}_p^\times$ via the cyclotomic character χ_{cyc}; thus we have the relation $\theta = \theta \circ \chi_{\mathrm{cyc}}$. Decompose $\mathbb{Z}_p^\times \simeq \Delta \times \Gamma$ with $\Gamma = 1 + p\mathbb{Z}_p$ and $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$ is identified with the group μ_{p-1} of $(p-1)$-roots of unity via the Teichmüller character, which we denote by ω as usual. The character $\theta\colon \mathbb{Z}_p^\times \to \mathcal{R}^\times$ satisfies the condition $\theta_\varphi(\delta\gamma) = \omega^{k_0-1}(\delta) \cdot \psi^{1/2}(\gamma) \cdot \gamma^{k-1}$ for all arithmetic morphisms φ of weight $2k$ and wild character ψ.

Following the terminology in [22, Definition 2], we say that an arithmetic morphism $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ of weight 2 is generic for θ if one of the following conditions holds:
(i) F_φ has non-trivial nebentype;
(ii) F_φ is the p-stabilization of a newform in $S_2(\Gamma_0(N))$, and θ_φ is trivial;
(iii) F_φ is a newform of level Np and $\theta_\varphi = \omega^{(p-1)/2}$.
For any $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$, define the character $\chi_{D,\varphi}\colon \mathbb{A}_{K_D}^\times \to \bar{\mathbb{Q}}_p^\times$ by the formula

$$\chi_{D,\varphi}(x) := \theta_\varphi(\mathrm{art}_{\mathbb{Q}}(\mathrm{N}_{K_D/\mathbb{Q}}(x)));$$

here $\mathrm{art}_{\mathbb{Q}}$ is the arithmetically normalized Artin map of class field theory, and $\mathrm{N}_{K_D/\mathbb{Q}}$ is the norm map. For an arithmetic morphism $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ which is generic for θ, we say that $(F_\varphi, \chi_{D,\varphi})$ has analytic rank 1 if

$$\mathrm{ord}_{s=1} L(F_\varphi, \chi_{D,\varphi}, s) = 1.$$

We will work under the following analytic condition.

Assumption 4.2. There exists a fundamental discriminant D_1 and a weight two prime φ_1 which is generic for θ such that $(F_\varphi, \chi_{D_1,\varphi_1})$ has analytic rank equal to one.

Remark 4.3. It is a conjecture of Greenberg that if φ is generic for θ, then $(F_\varphi, \chi_{D_1,\varphi_1})$ has possible analytic rank equal to 0 or 1 only. Thus if the Greenberg conjecture holds, Assumption 4.2 is implied by Assumption 4.1. See [23, § 3.4] and [22, Section 4] for details.

We fix such a pair (D_1, φ_1). Also fix a residue class r_1 mod $2N$ such that $D_1 \equiv r_1^2$ mod $4N$. As a consequence of this assumption, we see by [22, Corollary 5] that $(F_\varphi, \chi_{D_1,\varphi})$ has analytic rank 1 for all generic arithmetic morphisms φ, except possibly a finite number of them, that $\mathfrak{Z}_{D_1,r_1}^{\mathrm{How}}$ is non-$\mathcal{R}$-torsion, and

$$\mathrm{rank}_{\mathcal{R}}(\mathrm{Sel}(\mathbb{Q}, \mathbb{T}^\dagger)) = 1.$$

Further, if we denote by $\mathfrak{Z}_{D_1,r_1}^{\mathrm{How}}(\varphi)$ the image via localization map $\mathbb{T}^\dagger \to V_\varphi^\dagger$ of $\mathfrak{Z}_{D_1,r_1}^{\mathrm{How}}$ in the Nekovář-extended Bloch–Kato Selmer group $\tilde{H}_f^1(\mathbb{Q}, V_\varphi^\dagger)$ attached to the representation $V_\varphi^\dagger$ (introduced in [33]), we see that $\mathfrak{Z}_{D_1,r_1}^{\mathrm{How}}(\varphi)$ is non-zero for all arithmetic morphisms φ, except possibly a finite number of them; also, it follows from the discussion in [23, § 2.4] and [33, § 9] that, for all except a finite number of arithmetic primes, $\mathfrak{Z}_{D_1,r_1}^{\mathrm{How}}(\varphi)$ belongs to the Bloch–Kato Selmer group $H_f^1(\mathbb{Q}, V_\varphi^\dagger)$, which, again for all arithmetic primes except a finite number of them, is equal to the strict Greenberg Selmer group $\mathrm{Sel}(\mathbb{Q}, V_\varphi^\dagger)$ of $V_\varphi^\dagger$ (see [6, 17] for details on the definitions of these Selmer groups, or [23, § 2.4]).

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4.2 The GKZ theorem for Heegner cycles

The aim of this Section is to review results of Hui Xue [37] extending the Gross–Kohnen–Zagier theorem [19, Theorem C] to modular forms of higher weight.

Fix in this section a newform f of even weight $2k$ and level $\Gamma_0(N)$ with $p \nmid N$. Let K_D be a quadratic imaginary field, of discriminant D, with ring of algebraic integers $\mathcal{O}_{K_D}$, and assume all primes dividing Np are split in K. Finally, assume that $w = 1$, where w is the sign of the functional equation of $L(f, s)$.

Fix a residue class r mod $2N$ such that $r^2 \equiv D$ mod $4N$. Let $z_{D,r}$ be the solution in the upper half plane $\mathcal{H}$ of the equation $az^2 + bz + c = 0$, where $Q = [a, b, c] \in \mathcal{Q}_{N,D,r}$. Then $x_{D,r} = [z_{D,r}]$ is a Heegner point of conductor 1 in $X_0(N)$, namely, it represents an isogeny of elliptic curves with complex multiplication by $\mathcal{O}_{K_D}$ and cyclic kernel of order N. If H_D denotes as above the Hilbert class field of K_D, there are $h_D = [H_D : K_D]$ such points, permuted transitively by $G_D = \mathrm{Gal}(H_D/K_D)$. To $x_{D,r}$, we may attach a codimension k in the Chow group of the $(2k-1)$-dimensional Kuga–Sato variety W_{2k-2}, which is rational over H_D, as described in [31, Section 5]. Briefly, one starts by considering the elliptic curve $E_{x_{D,r}}$ equipped with a cyclic subgroup of order N, corresponding to $x_{D,r}$ via the moduli interpretation, and let Γ be the graph of the multiplication by $\sqrt{D}$ on $E \times E$; we then consider the cycle

$$Z(x_{D,r}) = \Gamma - E \times \{0\} - D(\{0\} \times E)$$

in $E \times E$ and define

$$W(x_{D,r}) = \sum_{g \in \Sigma_{2k-2}} \mathrm{sgn}(g) g\big(Z(x_{D,r})^{k-1}\big),$$

where the action of the symmetric group Σ_{2k-2} on $2k-2$ letters on E^{2k-2} is via permutation. Thanks to the canonical desingularization, this defines a cycle, denoted by $W_{D,r}^{\mathrm{Heeg}}$, of codimension k in the Chow group of the $(2k-1)$-dimensional Kuga–Sato variety W_{2k-2}, which is rational over H_D. Adopting a standard notation for Chow groups, we write $X_{D,r} \in \mathrm{CH}^k(W_{2k-2})_0(H_D)$. We finally define $X_{D,r}^{\mathrm{Heeg}} = \sum_{\sigma \in G_D} W_{D,r}^{\mathrm{Heeg}}$ and, again in a standard notation, we write $X_{D,r}^{\mathrm{Heeg}} \in \mathrm{CH}^k(W_{2k-2})_0(K_D)$. The Heegner cycle considered in [40, § 2.4] and [37, § 2], which we denote by $S_{D,r}^{\mathrm{Heeg}}$, is the multiple of $X_{D,r}^{\mathrm{Heeg}}$ such that the self-intersection of $S_{D,r}^{\mathrm{Heeg}}$ is $(-1)^{k-1}$. Then

$$S_{D,r}^{\mathrm{Heeg}} \in \mathrm{CH}^k(W_{2k-2})_0(K_D) \otimes_{\mathbb{Q}} \mathbb{R}$$

and, since the self-intersection of $Z(x_{D,r})$ is $-2D$ (see, for example, [32, § (3.1)]) in this vector space, we have $S_{D,r}^{\mathrm{Heeg}} = X_{D,r}^{\mathrm{Heeg}} \otimes |2D|^{-\frac{k-1}{2}}$, where we make the choice of square root of $|2D|$ in $\mathbb{R}$ to be the positive one.

Let $x \mapsto \bar{x}$ denote the action of the non-trivial element τ_D of $\mathrm{Gal}(K_D/\mathbb{Q})$ on the Chow group of K_D-rational cycles, and define

$$(S_{D,r}^{\mathrm{Heeg}})^* := S_{D,r}^{\mathrm{Heeg}} + \overline{S_{D,r}^{\mathrm{Heeg}}}.$$

Denote by $\mathrm{Heeg}_k(X_0(N))$ the $\mathbb{Z}$-submodule of $\mathrm{CH}^k(W_{2k-2})_0(K_D) \otimes_{\mathbb{Q}} \mathbb{R}$ generated by the elements $(S_{D,r}^{\mathrm{Heeg}})^*$ as D varies. Let $\mathbb{T}_N$ be the standard Hecke algebra (over $\mathbb{Z}$) of level $\Gamma_0(N)$ and, for any $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}$-module M, let M_f denote its f-isotypical component. Let finally

$$(S_{D,r}^{\mathrm{Heeg}})_f^* \in \mathrm{Heeg}_k(X_0(N))_f$$

be the f-isotypical components of $(S_{D,r}^{\mathrm{Heeg}})^*$. The assumption $w = 1$ implies then that the image $S_{D,r,f}^{\mathrm{Heeg}}$ of $S_{D,r}^{\mathrm{Heeg}}$ in $(\mathrm{CH}^k(W_{2k-2})_0(K_D) \otimes_{\mathbb{Z}} \mathbb{R})_f$ belongs to $(\mathrm{CH}^k(W_{2k-2})_0(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R})_f$, and therefore

$$(S_{D,r}^{\mathrm{Heeg}})_f^* = 2 \cdot S_{D,r,f}^{\mathrm{Heeg}}.$$

Let

$$\phi_f(\tau, z) = \sum_{r^2 \le 4Nn} c_f(n, r) q^n \zeta^r$$

be the Jacobi form corresponding to f under the Skoruppa–Zagier correspondence [35], where as usual $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$.

Let

$$\langle\ ,\ \rangle_{\mathbb{R}} : \mathrm{Heeg}_k(X_0(N)) \otimes_{\mathbb{Z}} \mathbb{R} \times \mathrm{Heeg}_k(X_0(N)) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R} \tag{4.1}$$

be the restriction to $\mathrm{Heeg}_k(X_0(N)) \otimes_{\mathbb{Z}} \mathbb{R}$ of the height pairing defined via arithmetic intersection theory by

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Gillet and Soulé [15]. Choose $s_f^* \in (\mathrm{Heeg}_k(X_0(N)(\mathbb{Q})) \otimes_{\mathbb{Z}} \mathbb{R})_f$ such that

$$\langle s_f^*, s_f^* \rangle = \frac{(2k-2)!N^{k-1}}{2^{2k-1}\pi^k(k-1)!\,\|\phi_f\|^2} L'(f,k)$$

where $\|\phi_f\|$ is the norm of ϕ_f in the space of Jacobi forms equipped with the Petersson scalar product. For the next theorem, we need the following assumption.

Assumption 4.4. The height pairing $\langle\ ,\ \rangle_{\mathbb{R}}$ in (4.1) is positive definite for each even integer $k \geq 2$.

Theorem 4.5 (Xue). *Assume the height pairing $\langle\ ,\ \rangle_{\mathbb{R}}$ in* (4.1) *is positive definite. For all fundamental discriminants D which are coprime with $2N$, we have*

$$(D)^{\frac{k-1}{2}} \cdot (S_{D,r}^{\mathrm{Heeg}})_f^* = c_f\Big(\frac{r^2-D}{4N}, r\Big) \cdot s_f^*.$$

Remark 4.6. The validity of Assumption 4.4 is a consequence of the Bloch–Beilinson conjectures; see [4, 5, 18] and [40, Conjectures 1.1.1 and 1.3.1]. It could be possible to remove this assumption, as suggested in [37], using Borcherds's approach [7, 8] to the GKZ theorem via singular theta liftings. Actually, Zemel [39] proved such an analogue of Borcherd's results for higher weight modular forms, and therefore it seems reasonable to establish Theorem 4.5 unconditionally using [39]. It would be very interesting to obtain such a result, which however does not seem to be an immediate consequence of the methods developed in [39]. Indeed, [39, Theorem 4.6] proves that the generating series of Heegner cycles is a modular form of half-integral weight with values in $\mathrm{Heeg}_k(X_0(N)) \otimes_{\mathbb{Z}} \mathbb{Q}$. Using a suitable version of the Eichler–Zagier isomorphism between vector-valued half-integral weight and Jacobi forms, and projecting to the f-eigencomponent, this result shows that the generating series of Heegner cycles in [39, Theorem 4.6] gives rise to a $\mathrm{Heeg}_k(X_0(N))_f \otimes_{\mathbb{Z}} \mathbb{Q}$-valued Jacobi form. However, to the best knowledge of the authors, the 1-dimensionality of $\mathrm{Heeg}_k(X_0(N))_f \otimes_{\mathbb{Z}} \mathbb{Q}$ does not follow directly from the work of Zemel, and therefore any comparison with Xue's result would probably require some new idea.

4.3 Specialization of big Heegner points

The aim of this section is to review the explicit comparison result between Howard's big Heegner points and Heegner cycles that could be found conditionally in Castella's thesis [9], announced by Castella and Hsieh [11] and proved explicitly in [10] under a number of arithmetic conditions, the most prominent being that primes p split in the imaginary quadratic field K_D.

Let $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ be an arithmetic point of trivial character and weight $2k$, and let $f_{2k}^\sharp$ be the form whose p-stabilization is F_φ. For any field extension $L/\mathbb{Q}$, we may consider the étale Abel–Jacobi map

$$\Phi_{W_{2k-2},L}^{\text{ét}} : \mathrm{CH}^k(W_{2k-2})(L) \to H^1(L, V_\varphi^\dagger).$$

Let $X_{D,r,2k}^{\mathrm{Heeg}}$ be the image of $X_{D,r}^{\mathrm{Heeg}}$ in $(\mathrm{CH}^k(W_{2k-2})_0(K_D))_{f_{2k}^\sharp}$, and define

$$\mathfrak{Z}_{D,r,2k}^{\mathrm{Heeg}} := \Phi_{W_{2k-2},\mathbb{Q}}^{\text{ét}}(X_{D,r,2k}^{\mathrm{Heeg}}).$$

Denote $\mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = \varphi(\mathfrak{Z}_{D,r}^{\mathrm{How}})$. The following result is due to Castella; cf. [10, Theorem 5.5].

Theorem 4.7 (Castella). *Let φ_0 be an arithmetic point in $\mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ with trivial nebentype and even weight $2k_0 \equiv 2 \bmod p-1$. If p is split in K_D, then, for any arithmetic point φ of even integer $2k > 2$ and trivial character with $2k \equiv 2k_0 \pmod{2(p-1)}$, we have*

$$\mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = \frac{(1 - \frac{p^{k-1}}{\varphi(A_p)})^2}{u_D(4|D|)^{\frac{k-1}{2}}} \cdot \mathfrak{Z}_{D,r,2k}^{\mathrm{Heeg}}$$

as elements in $H^1(\mathbb{Q}, V_\kappa^\dagger)$, where $u_D = |\mathcal{O}_{K_D}^\times|/2$.

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Since $S_{D,r}^{\mathrm{Heeg}}$ is equal to $X_{D,r}^{\mathrm{Heeg}} \otimes (2D)^{-\frac{k-1}{2}}$, it follows immediately from Theorem 4.7 that, for all φ as in the above theorem, we have

$$2^{\frac{k-1}{2}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = \left(1 - \frac{p^{k-1}}{a_p(2k)}\right)^2 \cdot \Phi_{W_{2k-2},\mathbb{Q}}^{\mathrm{ét}}(S_{D,r}^{\mathrm{Heeg}})_{f_{2k}^\sharp} \tag{4.2}$$

as elements in the $\bar{\mathbb{Q}}_p$-vector space $H^1(\mathbb{Q}, V_{f_{2k}^\sharp}^\dagger) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p$, where we fix an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$ and, with a slight abuse of notation, we write $\Phi_{W_{2k-2},\mathbb{Q}}^{\mathrm{ét}}$ for the $\bar{\mathbb{Q}}_p$-linear extension of $\Phi_{W_{2k-2},\mathbb{Q}}^{\mathrm{ét}}$; here we use the fact that the element $S_{D,r}^{\mathrm{Heeg}}$ belongs to $\mathrm{CH}^k(W_{2k-2})_0(\mathbb{Q}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$ and not merely in $\mathrm{CH}^k(W_{2k-2})_0(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$.

Remark 4.8. The referee of this paper suggested that the case of primes p which are inert in $\mathbb{Q}(\sqrt{D})$ could be addressed using a recent result of Daniel Disegni [13]. The relation between the specialization of big Heegner points and Heegner cycles is the content of [13, Theorem C (2)], which is proved at the end of § 6.4. The proof of this nice result is based on [13, Propositions 6.3.6 and 3.1.2], which can be seen as a sort of Hida's control theorem. However, since the control theorem in question is a comparison between two 1-dimensional vector spaces, it does not seem enough to prove the relation

$$\mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = \frac{(1 - \frac{p^{2k-2}}{a_p^2(2k)})}{u_D(4|D|)^{\frac{k-1}{2}}} \cdot \mathfrak{Z}_{D,r,2k}^{\mathrm{Heeg}}, \tag{4.3}$$

which we conjecture to be true in the inert setting, in analogy with the split case (on the right-hand side, $\mathfrak{Z}_{D,r,2k}^{\mathrm{Heeg}}$ denotes classical Heegner cycles, see Section 4.2, and u_D denotes half of the units of K_D). It might however be very interesting to see if a straightening of the results and methods of [13] could be used to prove the conjectural formula (4.3).

4.4 The Λ-adic GKZ theorem

Suppose from now on that Assumptions 4.1, 4.2 and 4.4 are satisfied.

Proposition 4.9. *Suppose that Assumptions 4.1, 4.2 and 4.4 are satisfied. Let φ_0 be a fixed arithmetic point with trivial character and weight $2k_0 \equiv 2 \bmod p-1$. Choose a fundamental index N pair (D_0, r_0) such that p is split in K_{D_0}. Then, for any arithmetic point φ of even positive integer $2k \equiv 2k_0 \bmod p-1$ and trivial character, and any index N pair (D, r) with $(D, Np) = 1$ and p split in K_D, we have*

$$(2D)^{\frac{k-1}{2}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = \left(1 - \frac{p^{k-1}}{a_p(2k)}\right)^2 \cdot c_{f_{2k}^\sharp}\left(\frac{r^2 - D}{4N}, r\right) \cdot \Phi_{W_{2k-2},\mathbb{Q}}^{\mathrm{ét}}(s_{f_{2k}^\sharp}^*)$$

where $c_{f_{2k}^\sharp}(n, r)$ stands for the Fourier–Jacobi coefficients of $\mathcal{S}_{D_0,r_0}(f_{2k}^\sharp)$

Proof. As a preliminary observation, note that Assumption 4.1 ensures that the sign of the functional equation of the L-function of $f_{2k}^\sharp$ is -1 (this sign is the same as the sign of the function equation of the L-function of f_{2k}; see, for example, [22, Proposition 4]). If the index pair (D, r) is fundamental, the result is an obvious consequence of Theorem 4.5 and equation (4.2), so suppose that D is not fundamental, but still $(Np, D) = 1$. In this case, we may argue as in [19, p. 558]. We start with the above equation for D fundamental and fix an integer m prime to Np. Multiply both sides by $a_m(2k)$, the coefficient q^m in $f_{2k}^\sharp$. Since $a_m(2k)$ is the eigenvalue of $\mathrm{T}(m)$ acting on $f_{2k}^\sharp$, the left-hand side is $(2D)^{\frac{k-1}{2}} \cdot \mathrm{T}(m) \cdot 2u_D \cdot \mathfrak{Z}_{D,r,k}$, while, in the right-hand side, we substitute the factor $c_{f_{2k}^\sharp}(n, r)$ with the coefficient $c_{f_{2k}^\sharp}^*(n, r)$ of $q^n \zeta^r$ in $(\mathcal{S}_{D_0,r_0}(f_{2k}^\sharp)) \mid \mathrm{T}_J(m)$ (here as usual $D = r^2 - 4Nn$). The formula in [19, p. 508] (top of the page) shows that

$$a_m(2k) \cdot \mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = \mathrm{T}_J(m) \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = \sum_{d \mid m} d^{k-1}\left(\frac{D}{d}\right) 2u_D \cdot \mathfrak{Z}_{\frac{Dm^2}{d^2}, \frac{rm}{d}, k}^{\mathrm{How}},$$

and by the equation for the action of Hecke operators in Section 2, we also have

$$a_m(\kappa) \cdot c_{f_{2k}^\sharp}(n, r) = c_{f_{2k}^\sharp}^*(n, r) = \sum_{d \mid m} d^{k-1}\left(\frac{D}{d}\right) c_{f_{2k}^\sharp}(n, r)\left(\frac{nm^2}{d^2}, \frac{rm}{d}\right).$$

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It follows by induction from the case of fundamental index pair that the equality in the statement remains true if (D, r) is replaced by (Dm^2, rm). Now each integer R satisfying the congruence $R^2 \equiv Dm^2 \bmod 4N$ also satisfies the congruence $R \equiv rm \bmod 2N$ for some integer r with $r^2 \equiv D \bmod 4N$, which implies the result. □

Remark 4.10. Tracing back the relation between big Heegner points and Heegner cycles, one can deduce the main result of [37] for non-fundamental discriminants as well. However, a more direct proof can be also obtained working directly with Heegner cycles and following the same argument using [40, Proposition 2.4.2].

Remark 4.11. Observe that the product on the right-hand side of Proposition 4.9 does not depend on the choice of the fundamental index N pair (D_0, r_0), even if the last two factors of this product depend on it.

5 The p-adic GKZ theorem

5.1 The main result

We suppose in this section that Assumptions 4.1, 4.2 and 4.4 are satisfied. Fix a fundamental index pair (D_0, r_0) such that p is split in K_{D_0}, and write as above

$$\mathcal{S}_{D_0,r_0}(f_{2k}^\sharp) = \sum_{r^2 \leq 4Nn} c_{f_{2k}^\sharp}(n, r) q^n \zeta^r$$

for an even positive integer $2k$. Fix an even positive integer $2k_0 \equiv 2 \bmod p-1$. Recall the domain of convergence U_{2k_0} about $2k_0$ introduced in Section 4.4, and fix a connected neighborhood $U \subseteq U_{2k_0}$ in $\mathcal{X}$ such that $\lambda(k) \neq 0$ for all k in U_{2k_0}. Define for $\kappa \in U_{2k_0}$ and $D = r^2 - 4nN$,

$$\mathcal{Z}_{n,r}(\kappa) = (2D)^{\frac{\kappa-2}{4}} \cdot 2u_D \cdot \tilde{\varphi}_{2k}(\mathfrak{Z}_{D,r}^{\mathrm{How}}).$$

Recall the notation

$$\mathrm{Sel}_{\mathcal{K}}(\mathbb{Q}, \mathbb{T}^\dagger) = \mathrm{Sel}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_{\mathcal{R}} \mathcal{K} \quad \text{and} \quad \mathrm{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T}^\dagger) = \mathrm{Sel}_{\mathcal{K}}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_{\mathcal{K}} \mathcal{M}_{2k_0},$$

where, for the second tensor product, we use the map $\mathcal{K} \to \mathcal{M}_{2k_0}$ whose construction is recalled in Section 3.

Theorem 5.1. *There exists $\Phi^{\text{ét}}$ in $\mathrm{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T})$ such that, in a sufficiently small neighborhood of $2k_0$, we have $\mathcal{Z}_{n,r} = \mathcal{L}_{n,r} \cdot \Phi^{\text{ét}}$.*

Proof. Combining Proposition 4.9 with Theorem 3.1 (in the split case), we obtain

$$\lambda(k) \cdot \mathcal{Z}_{n,r}(2k) = \mathcal{L}_{n,r}(2k) \cdot \Phi^{\text{ét}}_{W_{2k-2},\mathbb{Q}}(s^*_{f^\sharp_{2k}}).$$

The result follows by setting $\Phi^{\text{ét}}(\kappa) = \frac{\mathcal{Z}_{n_0,r_0}(\kappa)}{\mathcal{L}_{n_0,r_0}(\kappa)}$ for any index pair with $\mathcal{L}_{n_0,r_0}(2k_0) \neq 0$ and using the density of the set of even positive integers in $\mathcal{X}$. □

Remark 5.2. The paper [10] shows that big Heegner points are closely related to p-adic L-functions à la Bertolini–Darmon–Prasanna, and Theorem 5.1 shows still another relation of this nature. However, note that the proof of this result makes an essential use of Castella's result.

5.2 Twisted Gross–Kohnen–Zagier

In this section, we discuss the general conjecture suggested by the results obtained so far.

Combining Theorems 3.1 and 5.1 proves the formula

$$(2D)^{\frac{2k-2}{4}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\mathrm{How}} = c_{f_{2k}^\sharp}(n, r) \cdot \frac{(2D_0)^{\frac{2k-2}{4}} \cdot 2u_{D_0} \cdot \mathfrak{Z}_{D_0,r_0,2k}^{\mathrm{How}}}{c_{f_{2k}^\sharp}(n_0, r_0)} \tag{5.1}$$

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for each even positive integer $2k \equiv 2k_0 \equiv 2 \bmod p-1$ and all discriminants D with $(D,N)=1$ and p split in K_D, where we chose an index pair (D_0, r_0) such that $\mathcal{L}_{n_0,r_0}(2k_0) \neq 0$ and p splits in K_{D_0}.

In [28], we construct a p-adic family of Jacobi forms $\mathbb{S}_{D_0,r_0}(\tilde{\varphi}) = \sum_{n,r} c_{n,r}(\tilde{\varphi}) q^n \zeta^r$ defined for $\tilde{\varphi}$ in the metaplectic covering $\tilde{\mathcal{X}}(\mathcal{R})$ of the weight space $\mathcal{X}(\mathcal{R})$ such that the specialization of $\mathbb{S}_{D_0,r_0}(\tilde{\varphi})$ at arithmetic points $\tilde{\varphi}$ lying over arithmetic points $\varphi \in \mathcal{X}^{\mathrm{arith}}(\mathcal{R})$ interpolates certain theta lifts of the classical forms F_φ; see [28, Theorem 5.5] for details. In particular, [28, Theorem 5.8] shows the equation

$$c_{n,r}(\tilde{\varphi}) = \lambda(k) \cdot \varphi(A_p) \cdot \mathcal{E}_2 \cdot c_{f_{2k}^\sharp}(n,r), \tag{5.2}$$

where

$$\mathcal{E}_2 = \left(1 - \frac{2p^{k-1}}{a_p(\kappa)} - \frac{p^{2k-2}}{a_p^2(\kappa)}\right) \tag{5.3}$$

for each arithmetic point $\varphi \in \mathcal{X}(\mathcal{R})$ of trivial character and weight $2k$, where $\tilde{\varphi}$ is the lift of φ to the metaplectic covering $\tilde{\mathcal{X}}(\mathcal{R})$ of $\mathcal{X}(\mathcal{R})$ having trivial character (see [28, Theorem 5.8]).

Combining (5.1) and (5.2), we obtain for each arithmetic morphism $\varphi \in \mathcal{X}(\mathcal{R})$ of trivial character and weight $2k \equiv 2k_0 \equiv 2 \bmod p-1$ with $2k > 2$ (so that $\mathcal{E}_2 \neq 0$) the formula

$$(2D)^{\frac{2k-2}{4}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r}^{\mathrm{How}}(\varphi) = c_{n,r}(\tilde{\varphi}) \cdot (2D_0)^{\frac{2k-2}{4}} \cdot 2u_{D_0} \cdot \mathfrak{Z}_{D_0,r_0}^{\mathrm{How}}(\varphi), \tag{5.4}$$

where we write $\mathfrak{Z}_{D,r}^{\mathrm{How}}(\varphi)$ for the specialization of $\mathfrak{Z}_{D,r}^{\mathrm{How}}$ at φ.

Assume now that k_0 is congruent to 2 modulo 4. Then there are two distinct ways to p-adically interpolate the term $(2D)^{\frac{2k-2}{4}}$ appearing in (5.4), by choosing a square root of the critical character θ used to define big Heegner points. Indeed, we may define the square roots $\theta_a^{1/2} \colon \mathbb{Z}_p^\times \to \Lambda^\times$ for $a \in \{0,1\}$ of the critical character θ by

$$\theta_a^{1/2} = \left(\frac{\cdot}{p}\right)^a \cdot \epsilon_{\mathrm{tame}}^{\frac{2k_0-2}{4}} \cdot [\epsilon_{\mathrm{wild}}^{1/4}],$$

where $\epsilon_{\mathrm{wild}}^{1/4}$ is the unique square root of $\epsilon_{\mathrm{wild}}^{1/2}$ taking values in $1 + p\mathbb{Z}_p$. Since k_0 is congruent to 2 modulo 4, the expression $\epsilon_{\mathrm{tame}}^{(k_0-2)/4}$ is without ambiguity. For each $\varphi \in \mathcal{X}(\mathcal{R})$, we let $\theta_{a,\varphi}^{1/2}$ be the composition of $\theta_a^{1/2}$ with the restriction of φ to $\Lambda^\times$. On the $\mathcal{O}$-module Λ, we define a new structure of Λ-algebra $\sigma\colon \Lambda \to \Lambda$ given by the map $\sigma(t) = t^2$, and we denote this new Λ-algebra by $\tilde{\Lambda}$. For any $\tilde{\varphi} \in \mathcal{X}(\tilde{\mathcal{R}})$, choose $a(\tilde{\varphi}) \in \{0,1\}$ such that the restriction of $\tilde{\varphi}$ to $\tilde{\Lambda}$ is equal to $\theta_{a(\tilde{\varphi}),\varphi}^{1/2}$. Then if φ has trivial character and weight $2k$ and $\tilde{\varphi}$ is the lift of φ with trivial character as above, we have $\theta_{a(\tilde{\varphi}),\varphi}^{1/2}(2D) = (2D)^{\frac{2k-2}{4}}$. It makes then sense to define for any $\tilde{\varphi} \in \mathcal{X}(\tilde{\mathcal{R}})$ the element

$$\mathcal{Z}_{D,r}(\tilde{\varphi}) = \theta_{a(\tilde{\varphi}),\varphi}^{1/2}(2D) \cdot 2u_D \cdot \mathfrak{Z}_{D,r}^{\mathrm{How}}(\varphi),$$

where $\varphi = \pi(\tilde{\varphi})$ and $\pi\colon \tilde{\mathcal{X}}(\mathcal{R}) \twoheadrightarrow \mathcal{X}(\mathcal{R})$ is the covering map (for the choice of $\varphi = \varphi_{2k}$ with trivial character and $\tilde{\varphi}$ the lift of φ with trivial character; this coincides with the element $\mathcal{Z}_{n,r}(2k)$ defined above, so the new notation is consistent with the old one). Then (5.4) reads as

$$\mathcal{Z}_{D,r}(\tilde{\varphi}) = c_{n,r}(\tilde{\varphi}) \cdot \mathcal{Z}_{D_0,r_0}(\tilde{\varphi}). \tag{5.5}$$

We remark that this formula holds under the restrictive conditions that φ has trivial character and weight $2k \equiv 2k_0 \equiv 2 \bmod p-1$, $2k_0 \equiv 2 \bmod 4$, $2k > 2$, p splits in D (and D_0), and $\tilde{\varphi}$ is the lift of φ with trivial character. However, (5.5) makes sense for all arithmetic primes $\tilde{\varphi} \in \tilde{\mathcal{X}}(\mathcal{R})$ and even if p is inert in D (at least under the condition that $2k_0 \equiv 2 \bmod 4$), and it might be viewed as a fuller analogue of the GKZ theorem over a larger portion of the whole weight space. It is then natural to state the following conjecture.

Conjecture 5.3. *Suppose that* $2k_0 \equiv 2 \bmod 4$. *Then, for all* (D,r) *and all* φ *in a Zariski open of* $\mathcal{X}(\mathcal{R})$*, we have*

$$\mathcal{Z}_{D,r}(\tilde{\varphi}) = c_{n,r}(\tilde{\varphi}) \cdot \mathcal{Z}_{D_0,r_0}(\tilde{\varphi}).$$

Remark 5.4. If φ is a weight two arithmetic prime (with possibly non-trivial character), Conjecture 1.1 would imply a twisted Gross–Kohnen–Zagier theorem. More precisely, the specialization of $\mathcal{Z}_{D,r}(\tilde{\varphi})$ a weight 2 prime φ is a linear combination of Heegner points (see [22, Section 3], [9, Section 5.1], [30, Section 3.2]), and therefore Conjecture 1.1 says that the ratio between these specializations is given by coefficients of Jacobi

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forms, which we interpret as a twisted GKZ theorem; recall that twisted Gross–Zagier theorems are already available in the literature (see [24]). Moreover, on the opposite direction, note that if this type of twisted GKZ theorems would be known in the rather general context, similar to that of [24], of modular forms of level $\Gamma_1(M)$, then Conjecture 1.1 would follow from the Zariski density of weight 2 primes in the weight space. It would be very interesting to prove such twisted Gross–Kohnen–Zagier theorems (in parallel to what was been done by [24] for twisted Gross–Zagier theorems), and the authors hope to come back to this problem in the future.

Remark 5.5. Finally, we would like to indicate how our work might shed some light on the main theorem of [3, Theorem 1.4]; we thank F. Castella for pointing this out to us. There it is been shown that the cohomology class $\Phi^{\text{ét}}(\Delta_t)$ of a generalized Heegner cycle Δ_t is equal to $m_{D,t} \cdot \sqrt{D} \cdot \delta(P_D)$, where P_D is a point independent of t and $m_{D,t} \in \mathbb{Z}$ satisfies

$$m_{D,t}^2 = \frac{2t!\,(2\pi\sqrt{-D})^t}{\Omega^{2t+1}} L(\psi^{2t+1}, t+1),$$

where $\Omega(A)$ is some complex period and $L(\psi^{2t+1}, t+1)$ a Hasse–Weil L-series attached to a Hecke character ψ^{2t+1}. This suggests, in line with the philosophy of the original GKZ theorem, that the coefficients $m_{D,t}$ might have some relationship with Fourier coefficients of a Jacobi form lifting a Hecke theta series.

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