

# Simple Lax Description of the ILW Hierarchy

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**Abstract.** In this note we present a simple Lax description of the hierarchy of the intermediate long wave equation (ILW hierarchy). Although the linear inverse scattering problem for the ILW equation itself was well known, here we give an explicit expression for all higher flows and their Hamiltonian structure in terms of a single Lax difference-differential operator.

*Key words:* intermediate long wave hierarchy; ILW; Lax representation; integrable systems; Hamiltonian

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## 1 Introduction

The intermediate long wave (ILW) equation was introduced in [7] to describe the propagation of waves in a two-layer fluid of finite depth. The model represents the natural interpolation between the Benjamin–Ono deep water and Kortweg–de Vries shallow water theories. The ILW equation for the time  $s$  evolution of a function  $w = w(x)$  defined on the real line has the following form:

$$w_s + 2ww_x + T(w_{xx}) = 0, \quad T(f) := \text{p.v.} \int_{-\infty}^{+\infty} \frac{1}{2\delta} \left( \text{sgn}(x - \xi) - \coth \frac{\pi(x - \xi)}{2\delta} \right) f(\xi) d\xi,$$

where  $\delta$  is a parameter and  $\text{p.v.} \int \cdot d\xi$  represents the principal value integral. This equation can be rewritten in the formal loop space formalism (see for instance [3], from which we borrow notations for the rest of the paper) as

$$u_t = uu_x + \sum_{g \geq 1} \mu^{g-1} \varepsilon^{2g} \frac{|B_{2g}|}{(2g)!} u_{2g+1}, \quad (1.1)$$

where  $B_{2g}$  are the Bernoulli numbers. Indeed the action of the operator  $T$  on a function  $f$  can be written in terms of the derivatives of  $f$  as  $T(f) = \sum_{n \geq 1} \delta^{2n-1} 2^{2n} \frac{|B_{2n}|}{(2n)!} \partial_x^{2n-1} f$ , and, by setting

$w = \frac{\sqrt{\mu}}{\varepsilon} u$ ,  $s = -\frac{\varepsilon}{2\sqrt{\mu}} t$  and  $\delta = \frac{\varepsilon\sqrt{\mu}}{2}$ , we see how the above equations are equivalent.

In [10] the integrability of the ILW equation was established, finding an infinite number of conserved densities and giving a corresponding inverse scattering problem. From this inverse scattering problem a Lax representation can of course be deduced. Also, through a simple

Hamiltonian structure, the conserved densities generate an infinite number of commuting flows. However the explicit relation between the Lax representation of the ILW equation, its higher flows and the Hamiltonian structure was never, to the best of our knowledge, clarified in the literature.

In this paper we give an explicit and remarkably simple Lax description of the full hierarchy of commuting flows of the ILW equation (the ILW hierarchy) and their Hamiltonian structure in terms of a single Lax mixed difference-differential operator.

As it turns out, this Lax description corresponds to the equivariant bi-graded Toda hierarchy of [9, Section B.2] in the somewhat degenerate case of bi-degree  $(1, 0)$ , hence a reduction of the 2D-Toda hierarchy. In fact, in [9], because of the geometric origin of their problem, the authors always work with strictly positive bi-degree, but this is an unnecessary restriction. What we do here is prove that the bi-degree  $(1, 0)$  case corresponds to the ILW hierarchy in its Hamiltonian formulation. A similar question is raised in [5, Remark 31], where the author notices that the definition of all but the extended flows of the extended bi-graded Toda hierarchy survives when one of the bi-degrees vanishes. Since in the equivariant case there is no need of extended flows, this problem does not arise in the Lax representation of the equivariant bi-graded Toda hierarchy. Of course this means that equivariant bi-graded Toda hierarchies of any bidegree  $(N, 0)$ , with  $N \geq 1$ , are well defined (see also [1] for their relation with the rational reductions of the 2D-Toda hierarchy). We will study the relation of such hierarchies with the geometry of equivariant orbifold Gromov–Witten theory in an upcoming publication.

## 2 ILW hierarchy

Here and in what follows we will use the formal loop space formalism in the notations of [3]. The Hamiltonian structure of the ILW equation (1.1) is given by the Hamiltonian

$$\bar{h}_1^{\text{ILW}} = \int \left( \frac{u^3}{6} + \sum_{g \geq 1} \mu^{g-1} \varepsilon^{2g} \frac{|B_{2g}|}{2(2g)!} u u_{2g} \right) dx$$

and the Poisson bracket  $\{\cdot, \cdot\}_{\partial_x}$ , associated to the operator  $\partial_x$ . The Hamiltonians  $\bar{h}_d^{\text{ILW}} \in \widehat{\Lambda}_u^{[0]}$ ,  $d \geq 2$ , of the higher flows of the ILW hierarchy are uniquely determined by the properties

$$\bar{h}_d^{\text{ILW}} \Big|_{\varepsilon=0} = \int \frac{u^{d+2}}{(d+2)!} dx, \quad \left\{ \bar{h}_d^{\text{ILW}}, \bar{h}_1^{\text{ILW}} \right\}_{\partial_x} = 0.$$

For example,

$$\bar{h}_2^{\text{ILW}} = \int \left( \frac{u^4}{4!} + \frac{\varepsilon^2}{48} u^2 u_{xx} + \sum_{g \geq 2} \frac{|B_{2g}|}{(2g)!} \varepsilon^{2g} \left( \mu^{g-2} \frac{g+1}{2} u u_{2g} + \mu^{g-1} \frac{1}{4} u^2 u_{2g} \right) \right) dx.$$

We refer the reader to the paper [2, Section 8], which explains a relation between the Hamiltonians  $\bar{h}_d^{\text{ILW}}$  and the conserved quantities of the ILW equation, constructed in [10]. It is convenient to introduce an additional Hamiltonian  $\bar{h}_0^{\text{ILW}} := \int \frac{u^2}{2} dx$ , which generates spatial translations. So the flows of the ILW hierarchy are given by

$$\frac{\partial u}{\partial t_d} = \partial_x \frac{\delta \bar{h}_d^{\text{ILW}}}{\delta u}, \quad d \geq 0,$$

where we identify the times  $t_1$  and  $t$ .

### 3 Lax description of the ILW hierarchy

Our Lax description of the ILW hierarchy is presented in Section 3.2, see Theorem 1. Before that, in Section 3.1, we recall necessary definitions from the theory of shift operators.

#### 3.1 Shift operators

Let  $\Lambda := e^{i\varepsilon\partial_x}$ . We will consider formal series of the form

$$A = \sum_{n \leq m} a_n \Lambda^n, \quad a_n \in \widehat{\mathcal{A}}_u, \quad m \in \mathbb{Z}.$$

Via the operation of composition  $\circ$ , the vector space of such formal operators is endowed with the structure of a non-commutative associative algebra. The positive part  $A_+$ , the negative part  $A_-$  and the residue  $\text{res } A$  of the operator  $A$  are defined by

$$A_+ := \sum_{n=0}^m a_n \Lambda^n, \quad A_- := A - A_+, \quad \text{res } A := a_0.$$

Let  $z$  be a formal variable. The symbol  $\widehat{A}$  of the operator  $A$  is defined by

$$\widehat{A} := \sum_{n \leq m} a_n e^{nz}.$$

For an operator  $L$  of the form

$$L = \Lambda + \sum_{n \geq 0} a_n \Lambda^{-n}, \quad a_n \in \widehat{\mathcal{A}}_u,$$

one can define the dressing operator  $P$ ,

$$P = 1 + \sum_{n \geq 1} p_n \Lambda^{-n},$$

by the identity

$$L = P \circ \Lambda \circ P^{-1}.$$

Note that the coefficients  $p_n$  of the dressing operator do not belong to the ring  $\widehat{\mathcal{A}}_u$ , but to a certain extension of it (see, e.g., [6, Section 2]). The dressing operator  $P$  is defined up to the multiplication from the right by an operator of the form  $1 + \sum_{n \geq 1} \widehat{p}_n \Lambda^{-n}$ , where  $\widehat{p}_n$  are some constants.

The logarithm  $\log L$  is defined by

$$\log L := P \circ i\varepsilon\partial_x \circ P^{-1} = i\varepsilon\partial_x - i\varepsilon P_x \circ P^{-1},$$

where  $P_x = \sum_{n \geq 1} (p_n)_x \Lambda^{-n}$ . The ambiguity in the choice of dressing operator is cancelled in the

definition of  $\log L$  and, moreover, the coefficients of  $\log L$  do belong to  $\widehat{\mathcal{A}}_u$  (see the proof of Theorem 2.1 in [6]). To be more precise, one has the commutation relations

$$[\log L, L^m] = 0, \quad m \geq 1, \tag{3.1}$$

which imply that

$$\text{res } [i\varepsilon P_x \circ P^{-1}, L^m] = i\varepsilon\partial_x \text{res } L^m, \quad m \geq 1.$$

These relations allow to compute recursively all the coefficients of the operator  $i\varepsilon P_x \circ P^{-1}$ . As a result, if we write

$$\log L = i\varepsilon \partial_x + \sum_{n \geq 1} f_n \Lambda^{-n},$$

then the coefficient  $f_n$  can be expressed as a differential polynomial in the coefficients  $a_0, a_1, \dots, a_{n-1}$  of the operator  $L$ ,  $f_n = f_n(a_0, \dots, a_{n-1}) \in \widehat{\mathcal{A}}_{a_0, \dots, a_{n-1}}^{[0]}$ . For example,

$$f_1 = \frac{i\varepsilon \partial_x}{\Lambda - 1} a_0 = \sum_{n \geq 0} \frac{B_n}{n!} (i\varepsilon \partial_x)^n a_0. \quad (3.2)$$

### 3.2 Lax description

Let  $\tau$  be a formal variable and

$$\mathcal{L} := \Lambda + u - \tau i\varepsilon \partial_x.$$

From the discussion of the construction of the logarithm  $\log L$  in the previous section it is easy to see that there exists a unique operator  $L$  of the form

$$L = \Lambda + \sum_{n \geq 0} a_n \Lambda^{-n}, \quad a_n \in \widehat{\mathcal{A}}_u^{[0]}[\tau],$$

satisfying

$$L - \tau \log L = \mathcal{L}. \quad (3.3)$$

Since  $[L^{d+1}, \mathcal{L}] = 0$ ,  $d \geq 0$ , the commutator  $[(L^{d+1})_+, \mathcal{L}]$  doesn't contain terms with non-zero powers of  $\Lambda$ . Consider the following system of PDEs:

$$\frac{\partial u}{\partial T_d} = \frac{\partial \mathcal{L}}{\partial T_d} = \frac{1}{\tau i\varepsilon (d+1)!} [(L^{d+1})_+, \mathcal{L}], \quad d \geq 0. \quad (3.4)$$

The following theorem is the main result of our paper.

#### Theorem 1.

1. The flows  $\frac{\partial}{\partial T_d}$ , given by (3.4), pairwise commute.
2. The system of Lax equations (3.4) possesses a Hamiltonian structure given by the Hamiltonians

$$\overline{h}_d^{\text{Lax}} = \int \left( \frac{\text{res } L^{d+2}}{(d+2)!} - \frac{\tau}{d+1} \frac{\text{res } L^{d+1}}{(d+1)!} \right) dx, \quad d \geq 0,$$

and the Poisson bracket associated to the operator  $\partial_x$ .

3. Let  $y$  be a formal variable and define polynomials  $P_d(y) \in \mathbb{Q}[y, \tau]$ ,  $d \geq 1$ , by

$$P_d(y) := y \prod_{i=1}^{d-1} \left( y + \frac{\tau}{i} \right) = \sum_{j=1}^d P_{d,j} y^j \tau^{d-j}, \quad P_{d,j} \in \mathbb{Q}.$$

The ILW hierarchy is related to the hierarchy (3.4) by the following triangular transformation:

$$\overline{h}_d^{\text{Lax}} = \sum_{j=0}^d P_{d+1, j+1} \tau^{d-j} \overline{h}_j^{\text{ILW}} \Big|_{\substack{\mu = -\tau^{-1} \\ \varepsilon \mapsto \varepsilon \sqrt{-\tau}}}, \quad d \geq 0. \quad (3.5)$$

**Proof.** 1. Hereafter, for simplicity, we will use  $L_+^m$  to denote  $(L^m)_+$ . Let

$$H_d := \frac{1}{\tau i \varepsilon (d+1)!} L_+^{d+1}, \quad d \geq 0.$$

Let us first check that

$$\frac{\partial L}{\partial T_d} = [H_d, L], \quad \frac{\partial \log L}{\partial T_d} = [H_d, \log L]. \quad (3.6)$$

Equations (3.1) and (3.3) imply that

$$\frac{\partial L}{\partial T_d} - \tau \frac{\partial \log L}{\partial T_d} = [H_d, L] - \tau [H_d, \log L], \quad (3.7)$$

$$\text{res} \left[ \frac{\partial \log L}{\partial T_d}, L^m \right] + \sum_{a+b=m-1} \text{res} \left[ \log L, L^a \circ \frac{\partial L}{\partial T_d} \circ L^b \right] = 0, \quad m \geq 1. \quad (3.8)$$

We consider these equations as a system of equations for the pair of operators  $\frac{\partial L}{\partial T_d}, \frac{\partial \log L}{\partial T_d}$ . Similarly to the discussion of the computation of the logarithm  $\log L$  in the previous section, equations (3.7) and (3.8) allow to compute recursively all the coefficients of the operators  $\frac{\partial L}{\partial T_d}$  and  $\frac{\partial \log L}{\partial T_d}$ . Then it remains to note that the operators  $\frac{\partial L}{\partial T_d} = [H_d, L]$  and  $\frac{\partial \log L}{\partial T_d} = [H_d, \log L]$  satisfy system (3.7)–(3.8). This completes the proof of equations (3.6).

When we know formulas (3.6), the commutativity of the flows  $\frac{\partial}{\partial T_d}$  is proved by a standard computation:

$$\begin{aligned} & \tau i \varepsilon (d_1 + 1)! \tau i \varepsilon (d_2 + 1)! \left( \frac{\partial}{\partial T_1} \frac{\partial u}{\partial T_2} - \frac{\partial}{\partial T_2} \frac{\partial u}{\partial T_1} \right) \\ &= [[L_+^{d_1+1}, L^{d_2+1}]_+, \mathcal{L}] + [L_+^{d_2+1}, [L_+^{d_1+1}, \mathcal{L}]] \\ &\quad - [[L_+^{d_2+1}, L^{d_1+1}]_+, \mathcal{L}] - [L_+^{d_1+1}, [L_+^{d_2+1}, \mathcal{L}]] \\ &\stackrel{\text{Jacobi}}{=} [[L_+^{d_1+1}, L^{d_2+1}]_+, \mathcal{L}] - [[L_+^{d_2+1}, L^{d_1+1}]_+, \mathcal{L}] + [[L_+^{d_2+1}, L_+^{d_1+1}], \mathcal{L}] \\ &= - \underbrace{[[L_+^{d_1+1}, L_+^{d_2+1}]_+, \mathcal{L}] - [[L_+^{d_2+1}, L_+^{d_1+1}]_+, \mathcal{L}]}_{=-[[L_+^{d_2+1}, L_+^{d_1+1}]_+, \mathcal{L}]} + [[L_+^{d_2+1}, L_+^{d_1+1}], \mathcal{L}] = 0. \end{aligned}$$

2. Note that the flows  $\frac{\partial}{\partial T_d}$  can be written as

$$\frac{\partial u}{\partial T_d} = \frac{1}{(d+1)!} \partial_x \text{res} L^{d+1}.$$

Let us compute the flow  $\frac{\partial}{\partial T_1}$ . For the coefficients of the operator  $L$ , one can immediately see that  $a_0 = u$  and then, using formula (3.2), we get

$$a_1 = \tau \frac{i \varepsilon \partial_x}{\Lambda - 1} u.$$

This allows to compute

$$\begin{aligned} \frac{\partial u}{\partial T_1} &= \frac{1}{2} \partial_x \text{res} L^2 = \partial_x \left( \frac{u^2}{2} + \frac{\tau i \varepsilon \partial_x}{2} \frac{\Lambda + 1}{\Lambda - 1} u \right) = uu_x + \tau u_x - \tau \sum_{g \geq 1} \frac{|B_{2g}|}{(2g)!} \varepsilon^{2g} u_{2g+1} \\ &= \partial_x \frac{\delta}{\delta u} \left( \frac{u^3}{6} + \tau \frac{u^2}{2} - \tau \sum_{g \geq 1} \frac{|B_{2g}|}{2(2g)!} \varepsilon^{2g} uu_{2g} \right). \end{aligned}$$

The local functionals  $\bar{h}_d^{\text{Lax}}$  are conserved quantities for the flow  $\frac{\partial}{\partial T_1}$ . Indeed,

$$\frac{\partial}{\partial T_1} \int \text{res } L^d dx = \frac{1}{2\tau i \varepsilon} \int \text{res } [L_+^2, L^d] dx,$$

which is zero because

$$\int \text{res } [f \Lambda^m, g \Lambda^n] dx = \delta_{m+n,0} \int (f \cdot \Lambda^m g - g \cdot \Lambda^n f) dx = 0, \quad f, g \in \widehat{\mathcal{A}}_u, \quad m, n \in \mathbb{Z}.$$

Therefore, the local functionals  $\bar{h}_d^{\text{Lax}}$  together with the Poisson bracket  $\{\cdot, \cdot\}_{\partial_x}$  generate the flows which commute with the flow  $\frac{\partial}{\partial T_1}$ . Then these flows are uniquely determined by their dispersionless parts (see [8, Lemma 3.3] or [4, Lemma 4.14]). Hence, it is sufficient to check the equation

$$\partial_x \frac{\delta \bar{h}_d^{\text{Lax}}}{\delta u} = \partial_x \frac{\text{res } L^{d+1}}{(d+1)!}$$

at the dispersionless level.

Denote  $\widehat{L}_0 := \widehat{L}|_{\varepsilon=0}$ . We see that it is sufficient to check that

$$\partial_x \frac{\partial}{\partial u} \left( \frac{\text{res } \widehat{L}_0^{d+2}}{(d+2)!} - \frac{\tau}{d+1} \frac{\text{res } \widehat{L}_0^{d+1}}{(d+1)!} \right) = \partial_x \text{res} \left( \frac{\widehat{L}_0^{d+1}}{(d+1)!} \right), \quad d \geq 0. \quad (3.9)$$

For this we compute

$$\widehat{L}_0 - \tau \log \widehat{L}_0 = e^z + u - \tau z \Rightarrow \frac{\partial \widehat{L}_0}{\partial u} - \tau \frac{\partial \widehat{L}_0}{\widehat{L}_0} = 1 \Rightarrow \frac{\partial \widehat{L}_0}{\partial u} = \frac{1}{1 - \tau \widehat{L}_0^{-1}}.$$

Therefore,

$$\frac{\partial}{\partial u} \left( \frac{\text{res } \widehat{L}_0^{d+1}}{(d+1)!} \right) = \frac{1}{d!} \text{res} \left( \widehat{L}_0^d \frac{\partial \widehat{L}_0}{\partial u} \right) = \frac{1}{d!} \sum_{j=0}^d \tau^j \text{res} (\widehat{L}_0^{d-j}), \quad d \geq 0,$$

which gives

$$\frac{\partial}{\partial u} \left( \frac{\text{res } \widehat{L}_0^{d+1}}{(d+1)!} \right) = \frac{\text{res } \widehat{L}_0^d}{d!} + \frac{\tau}{d} \frac{\partial}{\partial u} \left( \frac{\text{res } \widehat{L}_0^d}{d!} \right), \quad d \geq 1. \quad (3.10)$$

This implies equation (3.9).

3. We see that

$$\frac{\partial u}{\partial T_1} = \frac{\partial u}{\partial t_1} \Big|_{\substack{\mu = -\tau^{-1} \\ \varepsilon \mapsto \varepsilon \sqrt{-\tau}}} + \tau u_x.$$

Using again the result of [8, Lemma 3.3] (see also [4, Lemma 4.14]), we conclude that it is sufficient to prove equation (3.5) at the dispersionless level, namely,

$$\text{res} \left( \frac{\widehat{L}_0^{d+2}}{(d+2)!} - \frac{\tau}{d+1} \frac{\widehat{L}_0^{d+1}}{(d+1)!} \right) = \sum_{j=0}^d P_{d+1, j+1} \tau^{d-j} \frac{u^{j+2}}{(j+2)!}, \quad d \geq 0.$$

Using formula (3.10) and the property  $\widehat{L}_0|_{u=0} = e^z$ , the last equation can be equivalently written as

$$\frac{\text{res } \widehat{L}_0^{d+1}}{(d+1)!} = \sum_{j=0}^d P_{d+1,j+1} \tau^{d-j} \frac{u^{j+1}}{(j+1)!}. \quad (3.11)$$

Recursion (3.10) implies that

$$\frac{\text{res } \widehat{L}_0^{d+1}}{(d+1)!} = \left( \prod_{j=1}^d \left( \partial_u^{-1} + \frac{\tau}{j} \right) \right) u,$$

where we define the action of the operator  $\partial_u^{-1}$  in the polynomial ring  $\mathbb{Q}[u, \tau]$  by  $\partial_u^{-1} u^j := \frac{u^{j+1}}{j+1}$ ,  $j \geq 0$ . Since we obviously have

$$\sum_{j=0}^d P_{d+1,j+1} \tau^{d-j} \frac{u^{j+1}}{(j+1)!} = \left( \prod_{j=1}^d \left( \partial_u^{-1} + \frac{\tau}{j} \right) \right) u,$$

identity (3.11) becomes clear. This completes the proof of the theorem. ■

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