FINITE-DIMENSIONAL POINTED HOPF ALGEBRAS OVER FINITE SIMPLE GROUPS OF LIE TYPE IV. UNIPOTENT CLASSES IN CHEVALLEY AND STEINBERG GROUPS

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ABSTRACT. We show that all unipotent classes in finite simple Chevalley or Steinberg groups, different from $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$, collapse (i.e. are never the support of a finite-dimensional Nichols algebra), with a possible exception on one class of involutions in $\mathbf{PSU}_n(2^m)$.

CONTENTS

1. INTRODUCTION

1.1. **The motivation and the context.** This is the fourth paper of our series on finite-dimensional complex pointed Hopf algebras whose group of group-likes is isomorphic to a finite simple group of Lie type $\boldsymbol{G}$. See Part I [ACGI] for a comprehensive Introduction. For the benefit of the reader

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and prompted by the referee, we sketch the reductions from the original classification problem to the group-theoretical questions dealt with in this series.

Let G be a finite group. There is a braided tensor category ${}^G_G\mathcal{YD}$ whose objects–called Yetter-Drinfeld modules–are $\mathbb{C}G$-modules M with a G-grading $M = \oplus_{g\in G} M_g$ compatible by $h \cdot M_g = M_{hgh^{-1}}$, for all $h, g \in G$. The map $c \in GL(M \otimes M)$ given by

$$c(m \otimes \widetilde{m}) = g \cdot \widetilde{m} \otimes m, \qquad m \in M_g,\ g \in G,\ \widetilde{m} \in M,$$

is a solution of the braid equation (which is equivalent to the quantum Yang-Baxter equation) and is called the braiding of M. The category ${}^G_G\mathcal{YD}$ is semisimple and its simple objects up to isomorphism are parametrized by pairs $(\mathcal{O}, \rho)$ where $\mathcal{O}$ is a conjugacy class of G and ρ is an irreducible representation of the centralizer of a fixed but arbitrary point in $\mathcal{O}$.

Any $M \in {}^G_G\mathcal{YD}$ gives rise to a graded algebra $\mathfrak{B}(M) = \oplus_{n\in\mathbb{N}_0} \mathfrak{B}^n(M) = T(M)/\mathcal{J}(M)$, where $\mathcal{J}(M) = \oplus_{n\geq 2} \mathcal{J}^n(M)$ is a homogeneous ideal. Thus $\mathfrak{B}(M)$ is generated by $\mathfrak{B}^1(M) = M$. Besides, $\mathfrak{B}(M)$ is characterized uniquely by two facts: it is a (graded) Hopf algebra in ${}^G_G\mathcal{YD}$, and all primitive elements are in degree one. The algebra $\mathfrak{B}(M)$ is called the *Nichols algebra* associated with M. Beware that there is no general method to compute $\mathfrak{B}(M)$. See e.g. the survey [A] for more information on Nichols algebras.

Let H be a Hopf algebra with group of grouplike elements $G(H) \simeq G$. Assume that H is pointed, i.e. all its simple subcoalgebras have dimension one. A fundamental invariant of H is a Nichols algebra $\mathfrak{B}(M)$ that satisfies

$$\dim H \geq |G| \dim \mathfrak{B}(M).$$

We omit the details and also the discussion of the methods to recover H from G and $\mathfrak{B}(M)$. We just emphasize that for the initial question of classifying finite-dimensional Hopf algebras with group of group-likes $\simeq G$, we need to address the determination of all $M \in {}^G_G\mathcal{YD}$ such that $\dim \mathfrak{B}(M) < \infty$ and the presentation of the latter. If G is abelian, then these two problems have been solved by Heckenberger and Angiono respectively, see [AA] for an exposition and references. Assume that G is non-abelian. This problem splits in two sub-problems, depending on whether M is simple or not; the second one has been essentially solved in [HV].

We discuss the first sub-problem. So we are in the following setting: G is a finite non-abelian group and $M = M(\mathcal{O}, \rho)$ is the simple Yetter-Drinfeld module corresponding to $(\mathcal{O}, \rho)$. We say that the support of M, or by abuse of terminology of $\mathfrak{B}(M)$, is $\mathcal{O}$. An easy but crucial observation is that the algebra and coalgebra structures of $\mathfrak{B}(M)$ depend just on the braiding c, that is to say on $\mathcal{O}$ (as a rack, i.e. with the operation $h \triangleright g = hgh^{-1}$) and a suitable (rack) cocycle $q : \mathcal{O} \times \mathcal{O} \to GL_n(\mathbb{C})$, where n is the dimension of ρ (actually q is determined up to a coboundary). This has the advantage that the same pair $(\mathcal{O}, q)$ may appear when dealing with different groups. In short, we may write $\mathfrak{B}(M) = \mathfrak{B}(\mathcal{O}, q)$. To sum up, to classify finite-dimensional pointed Hopf algebras with group G we need to determine for every conjugacy class $\mathcal{O}$ and q arising from a representation of the centralizer as mentioned, when $\dim \mathfrak{B}(\mathcal{O}, q) < \infty$. But we could also look at all racks arising from conjugacy classes of finite groups, compute the cohomology groups $H^2(\mathcal{O}, GL_n(\mathbb{C}))$ and then try to compute the Nichols algebras $\mathfrak{B}(\mathcal{O}, q)$; this is more economical, as at the end, we would just need to check which racks appear as conjugacy classes of our G.

Surprisingly there are three group-theoretical criteria allowing to conclude that for a given rack $\mathcal{O}$, $\dim \mathfrak{B}(\mathcal{O}, q) = \infty$ for any cocycle q. Thus, if any of the criteria applies to $\mathcal{O}$, then we are dispensed from computing the various $H^2(\mathcal{O}, GL_n(\mathbb{C}))$. These criteria were developed in [AFGV, ACGI, ACGIII] and are recalled in §2.1. The verification of any of these criteria in a conjugacy class might be difficult. Here is another remarkable property: if $\widetilde{\mathcal{O}} \to \mathcal{O}$ is a surjective morphism of racks and $\mathcal{O}$ satisfies one of the criteria, then $\widetilde{\mathcal{O}}$ also does. This observation leads to consider simple racks $\mathcal{O}$ (i.e. without projections onto a rack with more than one element), whose classification is known. In particular, non-trivial conjugacy classes of finite simple non-abelian groups are simple racks. In this series of papers we deal with finite simple groups of Lie type; see [AFGV, AFGV2] for alternating and sporadic simple groups. We point out that in the series we are not proceeding group-by-group but analyzing different sorts of conjugacy classes–and the groups PSL_n that could be treated by linear algebra. In the present paper we conclude the unipotent classes and in [ACGV], the mixed ones. The semisimple classes, apparently the most difficult ones, will be treated next.

The three criteria are based on the existence of suitable subracks of the rack in question–this is another instance of the flexibility of the notion of rack. Let us say that a rack is *kthulhu* if neither of the three criteria applies. One may wonder whether there exists a fourth criterium that may dispose of (at least some) kthulhu racks. However frequently kthulhu racks possess very few subracks (see §2.1.6 for precise formulations) making it hopeless the search of such a fourth criterium. If a conjugacy class is kthulhu, then there is still another weapon at our disposal, which is to analyze abelian subracks; since they span a so-called braided vector space of diagonal type, then one may decide from the classification of Heckenberger that the dimension of $\mathfrak{B}(M)$ is infinite (this depends on the cocycle). But even this technique might not apply.

1.2. **The main result and its place in the series.** As explained, the primary task is to study Nichols algebras over $\boldsymbol{G}$ with support in a conjugacy class $\mathcal{O}$ of $\boldsymbol{G}$.

Let p be a prime number, $m \in \mathbb{N}$, $q = p^m$, $\mathbb{F}_q$ the field with q elements and $\Bbbk := \overline{\mathbb{F}_q}$. There are three families of finite simple groups of Lie type (according to the shape of the Steinberg endomorphism): Chevalley, Steinberg and Suzuki-Ree groups; see the list in [ACGI, p. 38] and [MT, 22.5] for details. Here are the contents of the previous papers:

- ⋄ In [ACGI] we dealt with unipotent conjugacy classes in $\mathbf{PSL}_n(q)$, and as a consequence with the non-semisimple ones (since the centralizers of semisimple elements are products of groups with root system A_ℓ).
- ⋄ The paper [ACGII] was devoted to unipotent conjugacy classes in $\mathbf{PSp}_{2n}(q)$.
- ⋄ The subject of [ACGIII] was the semisimple conjugacy classes in $\mathbf{PSL}_n(q)$. But we also introduced the criterion of type C, and applied it to some of the classes not reached with previous criteria in [ACGI, ACGII].

In this paper we consider unipotent conjugacy classes in Chevalley and Steinberg groups, different from $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$. Concretely, these are the groups in Table 1. Notice that $\mathbf{PSU}_3(2)$ is not simple but needed for recursive arguments.

Table 1. Finite groups considered in this paper; q odd for $\mathbf{P\Omega}_{2n+1}(q)$; $q \geq 3$ for $G_2(q)$

Chevalley		Steinberg	
$\boldsymbol{G}$	Root system	$\boldsymbol{G}$	Root system
$\mathbf{P\Omega}_{2n+1}(q)$	B_n, $n \geq 3$	$\mathbf{PSU}_n(q)$	A_{n-1}, $n \geq 3$
$\mathbf{P\Omega}^+_{2n}(q)$	D_n, $n \geq 4$	$\mathbf{P\Omega}^-_{2n}(q)$	D_n, $n \geq 4$
$G_2(q)$	G_2	${}^3D_4(q)$	D_4
$F_4(q)$	F_4	${}^2E_6(q)$	E_6
$E_j(q)$	E_6, E_7, E_8		

As in [AFGV, 2.2], we say that a conjugacy class $\mathcal{O}$ of a finite group G *collapses* if the Nichols algebra $\mathfrak{B}(\mathcal{O}, \mathbf{q})$ has infinite dimension for every finite faithful 2-cocycle $\mathbf{q}$. Our main result says:

Main Theorem. *Let $\boldsymbol{G}$ be as in Table 1. Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class in $\boldsymbol{G}$. Then either $\mathcal{O}$ collapses, or else $\boldsymbol{G} = \mathbf{PSU}_n(q)$ with q even and $(2, 1, \dots, 1)$ is the partition corresponding to $\mathcal{O}$.*

In the terminology of §2.1, the classes not collapsing in the Main Theorem are austere, see Lemma 5.2. This means that the group-theoretical criteria do not apply for it; however, we ignore whether these classes collapse by other reasons. The classes in $\mathbf{PSL}_n(q)$ or $\mathbf{PSp}_{2n}(q)$ not collapsing (by these methods) are listed in Table 3.

1.3. **The scheme of the proof and organization of the paper.** Let $\boldsymbol{G}$ be a finite simple group of Lie type. Then there is q as above, a simple simply connected algebraic group $\mathbb{G}_{\mathrm{sc}}$ defined over $\mathbb{F}_q$ and a Steinberg endomorphism F of $\mathbb{G}_{\mathrm{sc}}$ such that $\boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$. We refer to [MT, Chapter 21] for details. Conversely, $\boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$ is a simple group, out of a short list of exceptions, see [MT, Theorem 24.17]. For our inductive arguments, it is convenient to denote by $\boldsymbol{G}$ the quotient $\mathbb{G}_{\mathrm{sc}}^F/Z(\mathbb{G}_{\mathrm{sc}}^F)$ even when it is not simple. Often there is a simple algebraic group $\mathbb{G}$ with a projection $\boldsymbol{\pi}: \mathbb{G}_{\mathrm{sc}} \to \mathbb{G}$ such that F descends to $\mathbb{G}$ and $[\mathbb{G}^F, \mathbb{G}^F]/\boldsymbol{\pi}(Z(\mathbb{G}_{\mathrm{sc}}^F)) \simeq \boldsymbol{G}$.

The proof of the Main Theorem is by application of the criteria of type C, D or F (see §2.1), that hold by a recursive argument on the semisimple rank of $\mathbb{G}_{\mathrm{sc}}$. The first step of the induction is given by the results on unipotent classes of $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$, while the recursive step is a reduction to Levi subgroups. Then we proceed group by group and class by class. The experience suggests that a general argument is not possible. There are some exceptions in low rank for which Levi subgroups are too small and we need the representatives of the classes to apply *ad-hoc* arguments.

Here is the organization of the paper: We recall some notations and facts in §2, where we also state the needed notation for groups of Lie type. In §3 we describe the reduction to Levi subgroups and collect the known results on unipotent classes of $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$.

Let $\mathcal{O}$ be a non-trivial unipotent class in a group $\boldsymbol{G}$ listed in Table 1. The proof that $\mathcal{O}$ collapses (with the exception stated above) is given in §4, respectively §5, when $\boldsymbol{G}$ is a Chevalley, respectively Steinberg, group.

Indeed, if $\boldsymbol{G} = \mathbf{P\Omega}_{2n+1}(q)$, $n \geq 3$, and q odd, the claim is Proposition 4.3. If $\boldsymbol{G} = \mathbf{P\Omega}^+_{2n}(q)$, $n \geq 4$, $E_6(q)$, $E_7(q)$, or $E_8(q)$, then the claim is Proposition 4.2. If $\boldsymbol{G} = F_4(q)$, the result follows from Lemmata 4.4 and 4.5; and if $\boldsymbol{G} = G_2(q)$, $q \geq 3$, the assertion follows from Lemmata 4.6, 4.7, 4.8, 4.10, 4.11 and 4.12.

In turn, $\mathbf{PSU}_n(q)$ is settled in Proposition 5.1; $\mathbf{P\Omega}^-_{2n}(q)$ in Proposition 5.6; ${}^2E_6(q)$ in Proposition 5.8; and ${}^3D_4(q)$ in Proposition 5.9.

In this way, the Theorem is proved.

1.4. **Applications and perspectives.** The results in this paper will be applied to settle the non-semisimple classes in Chevalley and Steinberg groups.

Next we will deal with unipotent and non-semisimple classes in Suzuki-Ree groups. These are too small to apply the recursive arguments introduced in this paper.

The semisimple conjugacy classes in $\boldsymbol{G}$ different from $\mathbf{PSL}_n(q)$ are more challenging. We expect that classes represented by elements in tori different from the Coxeter ones would collapse while those represented only by elements in tori corresponding to Coxeter classes would be kthulhu, as is the case for $\mathbf{PSL}_2(q)$ and $\mathbf{PSL}_3(q)$ (with some exceptions). Both cases require a deeper understanding of the classes, and in addition the irreducible case seems to need an inductive argument on the maximal subgroups.

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2. Preliminaries

If $a \leq b \in \mathbb{N}$, then $\mathbb{I}_{a,b}$ denotes $\{a, a+1, \dots, b\}$; for simplicity we write $\mathbb{I}_a = \mathbb{I}_{1,a}$. For a set Y, the group of permutations of Y is denoted by $\mathbb{S}_Y$.

2.1. **Glossary of racks.** See [ACGIII] for details and more information.

2.1.1. A *rack* is a finite set $X \neq \emptyset$ with a self-distributive operation $\triangleright : X \times X \to X$ such that $x \triangleright \_\_$ is bijective for every $x \in X$. The archetypical example is the conjugacy class $\mathcal{O}_z^G$ of an element z in a group G with the operation $x \triangleright y = xyx^{-1}$, $x, y \in \mathcal{O}_z^G$. A rack X is *abelian* if $x \triangleright y = y$, for all $x, y \in X$. In this paper we are only concerned with racks that are conjugacy classes in a finite group; one advantage of the rack language is that it could be realized as a conjugacy class in many groups.

Recall that a decomposition of a rack Y is an expression $Y = R \coprod S$ where R and S are subracks. Since $x \triangleright \_\_$ is bijective for $x \in R$, it follows that $R \triangleright S = S$, and also $S \triangleright R = R$.

2.1.2. [AFGV, Definition 3.5] A rack X is *of type D* if it has a decomposable subrack $Y = R \coprod S$ with elements $r \in R$, $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$.

If $X = \mathcal{O}$ is a finite conjugacy class in a group G, then this is equivalent to the existence of r, $s \in \mathcal{O}$ such that $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle}$ and $(rs)^2 \neq (sr)^2$.

Lemma 2.1. [ACGI, Lemma 2.10] *Let X and Y be racks, $y_1 \neq y_2 \in Y$, $x_1 \neq x_2 \in X$ such that $x_1 \triangleright (x_2 \triangleright (x_1 \triangleright x_2)) \neq x_2$, $y_1 \triangleright y_2 = y_2$. Then $X \times Y$ is of type D.* □

Remark 2.2. One of the hypothesis of Lemma 2.1 holds in the following setting. Let $\mathcal{O}$ be a real conjugacy class, i.e. $\mathcal{O} = \mathcal{O}^{-1}$, with no involutions. Then $y_1 \neq y_2 = y_1^{-1}$, that obviously commute.

2.1.3. [ACGI, Definition 2.4] A rack X is *of type F* if it has a family of subracks $(R_a)_{a \in \mathbb{I}_4}$ and elements $r_a \in R_a$, $a \in \mathbb{I}_4$, such that $R_a \triangleright R_b = R_b$, for $a, b \in \mathbb{I}_4$, and $R_a \cap R_b = \emptyset$, $r_a \triangleright r_b \neq r_b$ for $a \neq b \in \mathbb{I}_4$.

In case $X = \mathcal{O}$ is a finite conjugacy class in a group G, then this is equivalent to the existence of $r_a \in \mathcal{O}$, $a \in \mathbb{I}_4$, such that $\mathcal{O}_{r_a}^{\langle r_a : a \in \mathbb{I}_4 \rangle} \neq \mathcal{O}_{r_b}^{\langle r_a : a \in \mathbb{I}_4 \rangle}$ and $r_a r_b \neq r_b r_a$, for $a \neq b \in \mathbb{I}_4$.

2.1.4. [ACGIII, Definition 2.3] A rack X is *of type C* when there are a decomposable subrack $Y = R \coprod S$ and elements $r \in R$, $s \in S$ such that $r \triangleright s \neq s$, and

$$R = \mathcal{O}_r^{\operatorname{Inn} Y}, \quad S = \mathcal{O}_s^{\operatorname{Inn} Y}, \quad \min\{|R|, |S|\} > 2 \text{ or } \max\{|R|, |S|\} > 4.$$

Here $\operatorname{Inn} Y$ is the subgroup of $\mathbb{S}_Y$ generated by $y \triangleright \_\_$, $y \in Y$.

The criterium of type C in group-theoretical terms reads as follows, see [ACGIII, Lemma 2.8]: A conjugacy class $\mathcal{O}$ in a finite group G is of type C if and only if there are a subgroup H of G and elements $r, s \in H \cap \mathcal{O}$

such that $rs \neq sr$; $\mathcal{O}_r^H \neq \mathcal{O}_s^H$; $H = \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle$ and $\min\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 2$ or $\max\{|\mathcal{O}_r^H|, |\mathcal{O}_s^H|\} > 4$.

Here is a new formulation suitable for later applications.

Lemma 2.3. *Let $\mathcal{O}$ be a conjugacy class in a group H. If there are $r, s \in \mathcal{O}$ such that $r^2 s \neq s r^2$, $s^2 r \neq r s^2$ and $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle}$, then $\mathcal{O}$ is of type C.*

Proof. We check that the conditions in [ACGIII, Lemma 2.8] hold with $H = \langle r, s \rangle = \langle \mathcal{O}_r^{\langle r,s \rangle}, \mathcal{O}_s^{\langle r,s \rangle} \rangle$. By hypothesis, $rs \neq sr$ and $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle}$. Now $r, s \triangleright r, s^2 \triangleright r$ are all distinct, so $|\mathcal{O}_r^{\langle r,s \rangle}| > 2$, and similarly for $\mathcal{O}_s^{\langle r,s \rangle}$. $\square$

2.1.5. The utility of the above criteria becomes clear in the following theorem.

Theorem 2.4. [AFGV, Theorem 3.6], [ACGI, Theorem 2.8], [ACGIII, Theorem 2.9]. *A rack X of type D, F or C collapses.*

The proof rests on results from [CH, HS, HV].

2.1.6. A rack is

- *kthulhu* if it is neither of type C, D nor F;
- *sober* if every subrack is either abelian or indecomposable;
- *austere* if every subrack generated by two elements is either abelian or indecomposable.

Clearly, sober implies austere and austere implies kthulhu.

The criteria of type C, D, F are very flexible:

Lemma 2.5. [AFGV, ACGI, ACGIII] *Let Y be either a subrack or a quotient rack of a rack X. If Y is not kthulhu, then X is not kthulhu.* $\square$

2.2. **Conjugacy classes.**

2.2.1. Let $q = p^m$ be as above. We fix a simple algebraic group $\mathbb{G}$ defined over $\mathbb{F}_q$, a maximal torus $\mathbb{T}$, with root system denoted by Φ, and a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$. We denote by $\mathbb{U}$ the unipotent radical of $\mathbb{B}$ and by $\Delta \subset \Phi^+$ the corresponding sets of simple and positive roots. Also $\mathbb{U}^-$ is the unipotent radical of the opposite Borel subgroup $\mathbb{B}^-$ corresponding to Φ^-. We shall use the realisation of the associated root system and the numbering of simple roots in [B]. The coroot system of $\mathbb{G}$ is denoted by $\Phi^\vee = \{\beta^\vee \mid \beta \in \Phi\} \subset X_*(\mathbb{T})$, where $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$, for all $\alpha \in \Phi$. Hence

$$\alpha(\beta^\vee(\zeta)) = \zeta^{\frac{2(\alpha,\beta)}{(\beta,\beta)}}, \qquad \alpha, \beta \in \Phi, \zeta \in \mathbb{F}_q^\times.$$

We denote by $\mathbb{G}_{\text{sc}}$ the simply connected group covering $\mathbb{G}$.

For $\Pi \subset \Delta$, we denote by Φ_Π the root subsystem with base Π and $\Psi_\Pi := \Phi^+ - \Phi_\Pi$. For $\alpha \in \Phi$, we write $s_\alpha \in W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ for the reflection with respect to α. Also, $s_i = s_{\alpha_i}$, if α_i is a simple root with the alluded numeration. Also, there is a monomorphism of abelian groups $x_\alpha : \Bbbk \to \mathbb{U}$; the image $\mathbb{U}_\alpha$ of x_α is called a root subgroup. We adopt the normalization of x_α and the notation for the elements in $\mathbb{T}$ from [Sp, 8.1.4]. We recall the commutation rule: $t \triangleright x_\alpha(a) = tx_\alpha(a)t^{-1} = x_\alpha(\alpha(t)a)$, for $t \in \mathbb{T}$ and $\alpha \in \Phi$. In particular, if $t = \beta^\vee(\xi)$ for some $\xi \in \Bbbk^\times$, then $t \triangleright x_\alpha(a) = x_\alpha(\alpha(\beta^\vee(\xi))a) = x_\alpha(\xi^{\frac{2(\alpha,\beta)}{(\beta,\beta)}} a)$.

We denote by $\mathbb{P}$ a standard parabolic subgroup of $\mathbb{G}$, with standard Levi subgroup $\mathbb{L}$ and unipotent radical $\mathbb{V}$. Thus there exists $\Pi \subset \Delta$ such that $\mathbb{L} = \langle \mathbb{T}, \mathbb{U}_{\pm\gamma} \mid \gamma \in \Pi \rangle$.

If $u \in \mathbb{U}$ then for every ordering of Φ^+, there exist unique $c_\alpha \in \Bbbk$ such that $u = \prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$. We define $\operatorname{supp} u = \{\alpha \in \Phi^+ \mid c_\alpha \neq 0\}$. In general the support depends on the chosen ordering of Φ^+. However, if $u \in \mathbb{V}$ as above, then $\operatorname{supp} u \subset \Psi_\Pi$ for every ordering of Φ^+.

2.2.2. In this paper we deal with Chevalley and Steinberg groups. Let F be a Steinberg endomorphism of $\mathbb{G}$; it is the composition of the split endomorphism Fr_q (the q-Frobenius map) with an automorphism induced by a Dynkin diagram automorphism ϑ. So, Chevalley groups correspond to $\vartheta = \mathrm{id}$. We assume that $\mathbb{T}$ and $\mathbb{B}$ are F-stable. Let $W^F = N_{\mathbb{G}^F}(\mathbb{T})/\mathbb{T}^F$. Thus $W^F \simeq W$ for Chevalley groups. For each $w \in W^F$, there is a representative $\dot{w}$ of w in $N_{\mathbb{G}^F}(\mathbb{T})$, cf. [MT, Proposition 23.2]. Notice that $\dot{w} \triangleright (\mathbb{U}_\alpha) = \mathbb{U}_{w(\alpha)}$ for all $\alpha \in \Phi$. Hence, if F is Chevalley and $\alpha, \beta \in \Phi$ have the same length, then $\mathbb{U}_\alpha^F$ and $\mathbb{U}_\beta^F$ are conjugated by an element in $N_{\mathbb{G}^F}(\mathbb{T})$ by [HuLA, Lemma 10.4 C].

2.2.3. We shall often use the Chevalley's commutator formula (2.1), see [St, pp. 22 and 24]. Let $\alpha, \beta \in \Phi$. If $\alpha + \beta$ is not a root, then $\mathbb{U}_\alpha$ and $\mathbb{U}_\beta$ commute. Assume that $\alpha + \beta \in \Phi$. Fix a total order in the set Γ of pairs $(i, j) \in \mathbb{N}^2$ such that $i\alpha + j\beta \in \Phi$. Then there exist $c_{ij}^{\alpha\beta} \in \mathbb{F}_q$ such that

$$x_\alpha(\xi)x_\beta(\eta)x_\alpha(\xi)^{-1}x_\beta(\eta)^{-1} = \prod_{(i,j)\in\Gamma} x_{i\alpha+j\beta}(c_{ij}^{\alpha\beta}\xi^i\eta^j), \qquad \forall \xi, \eta \in \Bbbk. \tag{2.1}$$

Let $\boldsymbol{G} = [\mathbb{G}^F, \mathbb{G}^F]/Z(\mathbb{G}^F)$.

Definition 2.6. [ACGII, Definition 3.3] Let $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta \in \Phi$ but the pair α, β does not appear in Table 2. We fix an ordering of Φ^+. A unipotent conjugacy class $\mathcal{O}$ in $\boldsymbol{G}$ has the *$\alpha\beta$-property* if there exists $u \in \mathcal{O} \cap \mathbb{U}^F$ such that $\alpha, \beta \in \operatorname{supp} u$ and for any expression $\alpha + \beta = \sum_{1 \le i \le r} \gamma_i$, with $r > 1$ and $\gamma_i \in \operatorname{supp} u$, necessarily $r = 2$ and $\{\gamma_1, \gamma_2\} = \{\alpha, \beta\}$.

Let $\alpha, \beta \in \Phi^+$. The scalar $c_{1,1}^{\alpha,\beta} \neq 0$ in (2.1) if $\alpha + \beta \in \Phi$ and the pair does not appear in Table 2.

Table 2

$p = 3$

Φ	α	β
G_2	α_1	$2\alpha_1 + \alpha_2$
	$2\alpha_1 + \alpha_2$	α_1
	$\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2$
	$2\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$

$p = 2$

Φ	α	β
B_n, C_n, F_4	orthogonal	to each other
G_2	α_1	$\alpha_1 + \alpha_2$
	$\alpha_1 + \alpha_2$	α_1

Proposition 2.7. [ACGII, Proposition 3.5] *Let $\boldsymbol{G}$ be a finite simple group of Lie type, with q odd. Assume $\mathcal{O}$ has the $\alpha\beta$-property, for some $\alpha, \beta \in \Phi^+$ such that $q > 3$ when $(\alpha, \beta) = 0$. Then $\mathcal{O}$ is of type D.* □

Remark 2.8. Assume u satisfies the conditions in Definition 2.6. Then it is never an involution. Indeed if q is odd this is never the case. If q is even then the argument in the proof of [ACGII, Proposition 3.5] shows that the coefficient of $x_{\alpha+\beta}$ in the expression of u^2 is nonzero.

2.2.4. Let us choose an ordering of the positive roots and let $w \in W$ and $u \in \mathbb{U}$ be such that $\Sigma := w(\operatorname{supp} u) \subset \Phi^+$. Then, $\dot{w} \triangleright u \in \mathbb{U}$ and there is an ordering of the positive roots for which Σ is the support of $\dot{w} \triangleright u$. If, in addition, $w\Sigma \not\subset \Psi_\Pi$ for some $\Pi \subset \Delta$, then by the discussion in 2.2.1, $\dot{w} \triangleright u \in \mathbb{U} - \mathbb{V}$.

2.2.5. We shall need a fact on root systems. Recall that there is a partial ordering $\preceq$ on the root lattice $\mathbb{Z}\Phi$ given by $\alpha \preceq \beta$ if $\beta - \alpha \in \mathbb{N}_0\Phi^+ = \mathbb{N}_0\Delta$.

Lemma 2.9. *Let $\gamma, \beta \in \Phi^+$ with $\beta \preceq \gamma$. Then there exists a sequence $\alpha_{i_1}, \dots, \alpha_{i_k} \in \Delta$ such that*

(1) $\forall j \in \mathbb{I}_k$ we have $\gamma_j := \beta + \alpha_{i_1} + \dots + \alpha_{i_j} \in \Phi^+$;

(2) $\gamma = \gamma_k$.

If, in addition, Φ is simply-laced, then $\gamma_j = s_{i_j} \cdots s_{i_1}\beta$ for every $j \in \mathbb{I}_k$.

Proof. (1) and (2) are consequences of [So, Lemma 3.2], with $\alpha_1 = \beta$, and the α_j being simple. Assume that Φ is simply-laced. Clearly, it is enough to prove it for a pair of roots. If $\alpha, \delta \in \Phi$ and $\alpha + \delta \in \Phi$, then $\Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\delta)$ is a root system of type A_2, so $s_\alpha(\delta) = \alpha + \delta$. The last claim follows. □

3. Unipotent classes in finite groups of Lie type

3.1. **Reduction to Levi subgroups.** We start by Lemma 3.2, that is behind the inductive step in most proofs below. We consider the following setting and notation, that we will use throughout the paper:

$\mathbb{P}^1, \dots, \mathbb{P}^k$ are standard F-stable parabolic subgroups of $\mathbb{G}$;
$\mathbb{P}^i = \mathbb{L}_i \ltimes \mathbb{V}_i$ are Levi decompositions, with $\mathbb{L}_i$ F-stable;
$U_i := (\mathbb{U} \cap \mathbb{L}_i)^F$, $U_i^- := (\mathbb{U}^- \cap \mathbb{L}_i)^F$, $P^i := (\mathbb{P}^i)^F$, $L_i := \mathbb{L}_i^F$, $V_i := \mathbb{V}_i^F$;
$\pi_i : P^i \to L_i$ is the natural projection;
$M_i = \langle U_i, U_i^- \rangle \leq L_i$; for $i \in \mathbb{I}_k$.

Remark 3.1. Assume that $\mathbb{G} = \mathbb{G}_{sc}$. If $\mathbb{L}_i$ is standard, then $M_i = [\mathbb{L}_i, \mathbb{L}_i]^F$.

Proof. Since $\mathbb{G}$ is simply connected, so is $[\mathbb{L}_i, \mathbb{L}_i]$ (Borel-Tits, see [SpSt, Corollary 5.4]). Then [MT, Theorem 24.15] applies. □

Lemma 3.2. *Let $u \in \mathbb{U}^F$; so in particular $\pi_i(u) \in M_i$ for all $i \in \mathbb{I}_k$.*

(a) *Assume that $\mathcal{O}^{M_i}_{\pi_i(u)}$ is not kthulhu for some $i \in \mathbb{I}_k$. Then neither of*

$$\mathcal{O}^{L_i}_{\pi_i(u)}, \qquad \mathcal{O}^{P^i}_u, \qquad \mathcal{O}^{\mathbb{G}^F}_u$$

is kthulhu.

(b) *Assume that*

$$\text{No non-trivial unipotent class in } M_i \text{ is kthulhu, } \forall i \in \mathbb{I}_k. \tag{3.1}$$

If $u \notin \cap_{i \in \mathbb{I}_k} V_i$, then $\mathcal{O}^{\mathbb{G}^F}_u$ is not kthulhu.

(c) *Assume that (3.1) holds. Let $\mathcal{O}$ be a unipotent conjugacy class in $\mathbb{G}^F$. If $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_{i \in \mathbb{I}_k} V_i$, then $\mathcal{O}$ is not kthulhu, hence collapses.*

Proof. Since $\mathbb{U} \leq \mathbb{P}^i$, it follows that $u = u_1 u_2$ with $u_1 \in \mathbb{L}_i$ and $u_2 \in \mathbb{V}_i \leq \mathbb{U}$. Hence $u_1 \in \mathbb{L}_i \cap \mathbb{U}$. Since clearly u_1 and u_2 are F-invariant, $u_1 = \pi_i(u) \in M_i$. Now (a) follows from Lemma 2.5 and implies (b), since $\pi_j(u) \neq 1$ for some $j \in \mathbb{I}_k$. (c) follows from (b) and Theorem 2.4 because $\mathcal{O} \cap \mathbb{U}^F \neq \emptyset$. □

3.2. **Unipotent classes in $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$.** We recall now results in the previous papers of the series that constitute the basis of the induction argument. We will also need some of the non-simple groups of Lie type of small rank and small characteristic listed in [ACGII, 3.2.1].

The following Theorem collects information from [ACGI, Table 2], [ACGII, Lemma 3.12 & Tables 3, 4, 5] and [ACGIII, Tables 2 & 3].

Theorem 3.3. *Let $\boldsymbol{G}$ be either $\mathbf{PSL}_n(q)$ or $\mathbf{PSp}_{2n}(q)$ and let $\mathcal{O} \neq \{e\}$ be a unipotent conjugacy class in $\boldsymbol{G}$, not listed in Table 3. Then it is not kthulhu.*

We explain the notation of Table 3, see [ACGI, ACGII] for further details:

(i) Unipotent classes in $\mathbf{PSL}_n(\Bbbk)$ are parametrized by partitions of n; i.e. $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$ and $\sum_j \lambda_j = n$. Thus, (n) is the regular unipotent class of $\mathbf{PSL}_n(\Bbbk)$. Unipotent classes in $\mathbf{PSL}_n(q)$ with the same partition are isomorphic as racks.

TABLE 3. Kthulhu classes in $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$

$\boldsymbol{G}$	class	q
$\mathbf{PSL}_2(q)$	(2)	even, or 9, or odd not a square
$\mathbf{PSL}_3(2)$	(3)	2
$\mathbf{PSp}_{2n}(q)$, $n \geq 2$	$W(1)^{n-1} \oplus V(2)$	even
$\mathbf{PSp}_{2n}(q)$, $n \geq 2$	$(2, 1^{2n-2})$	9, or odd not a square
$\mathbf{PSp}_4(q)$	$W(2)$	even

(ii) Unipotent classes in $\mathbf{PSp}_{2n}(\Bbbk)$, for q odd, are also parametrized by suitable partitions.

(iii) Unipotent classes in $\mathbf{PSp}_{2n}(\Bbbk)$, for q even, are parametrized by their *label*, which is the decomposition of the standard representation as a module for the action of an element in the conjugacy class:

$$V = \bigoplus_{i=1}^{k} W(m_i)^{a_i} \oplus \bigoplus_{j=1}^{r} V(2k_j)^{b_j}, \qquad 0 < a_i,\ 0 < b_j \leq 2, \tag{3.2}$$

for $m_i, k_j \geq 1$. The block $W(m_i)$ corresponds to a unipotent class with partition (m_i, m_i), whereas the block $V(2k_j)$ corresponds to a unipotent class with partition $(2k_j)$.

(iv) The unipotent class in $\mathbf{PSp}_4(\Bbbk)$ with label $W(2)$, respectively, in $\mathbf{PSp}_{2n}(\Bbbk)$ with label $W(1)^{n-1} \oplus V(2)$ contains a unique unipotent class in $\mathbf{PSp}_4(q)$, respectively, $\mathbf{PSp}_{2n}(q)$.

Remark 3.4. Assume q is even. If $\mathcal{O}$ is a unipotent conjugacy class in $\mathbf{Sp}_{2n}(q)$ enjoying the $\alpha\beta$-property, for some α and β, then $\mathcal{O}$ is of type C, D, or F. Indeed, by Theorem 3.3 the kthulhu unipotent classes in $\mathbf{Sp}_{2n}(q)$ for q even consists of involutions. Remark 2.8 applies.

Remark 3.5. The conjugacy class $\mathcal{O}$ of involutions in $\mathbf{PSL}_2(7)$ is of type C, so line 1 of [ACGIII, Table 1] and the statement about involutions for $q = 7$ in [ACGIII, Corollary 3.5] are not correct. Indeed, this follows from [ACGIII, Lemma 2.12] because $\mathbf{PSL}_2(7) \simeq \mathbf{PSL}_3(2)$ and $\mathcal{O}$ is the class of involutions therein.

The proof of [ACGIII, Corollary 3.5] overlooks the possibility that $Y = \mathbb{S}_4 \cap \mathcal{O}$, which is the union of all involutions in $K = \mathbb{S}_4$ and it is decomposable.

3.3. **Further remarks.** If a product $X = X_1 \times X_2$ of racks has a factor X_1 that is not kthulhu, then neither is X. Indeed, pick $x \in X_2$; then $X_1 \times \{x\}$ is a subrack of X and Lemma 2.5 applies (here as usual X_2 can be realized as a subrack of a group, so that $x \triangleright x = x$). The following results will be needed in order to deal with products of possibly kthulhu racks.

Lemma 3.6. *Let $\mathcal{O}$ be a unipotent conjugacy class in Table 3.*

(a) *There exist* $x_1,\ x_2 \in \mathcal{O}$ *such that* $(x_1x_2)^2 \neq (x_2x_1)^2$.

(b) *If* $\boldsymbol{G} \neq \mathbf{PSL}_2(2), \mathbf{PSL}_2(3)$, *then there exist* $y_1, y_2 \in \mathcal{O}$ *such that* $y_1 \neq y_2$ *and* $y_1y_2 = y_2y_1$.

Proof. By the isogeny argument [ACGI, Lemma 1.2], we may reduce to classes in $\mathbf{SL}_n(q)$ or $\mathbf{Sp}_{2n}(q)$. Also, the classes in $\mathbf{PSp}_4(q)$ with label $W(2)$ and $W(1)\oplus V(2)$ are isomorphic as racks, [ACGII, Lemma 4.26], so we need not to deal with the last row in Table 3.

If $\mathcal{O}$ is the class in $\mathbf{SL}_3(2)$, then $x_1 = \mathrm{id} + e_{1,2} + e_{2,3}$ and $x_2 = \sigma \triangleright x_1$, where $\sigma = e_{1,2} + e_{2,1} + e_{3,3}$, do the job for (a). For (b), take $y_1 = x_1$ and $y_2 = x_1^3 = x_1^{-1}$, that belongs to $\mathcal{O}$ by [ACGI, Lemma 3.3].

If $\mathcal{O}$ is the class in $\mathbf{SL}_2(q)$, then $x_1 = \mathrm{id} + e_{1,2} \in \mathcal{O}$ and $x_2 = \sigma \triangleright x_1$, where $\sigma = e_{1,2} - e_{2,1}$ do the job for (a); while $y_1 = x_1$, and $y_2 = \mathrm{id} + a^2 e_{1,2}$, for $a \in \mathbb{F}_q$, $a^2 \neq 0, 1$, are as needed in (b) when $q > 3$.

Finally, let $\mathcal{O}$ be one of the classes in $\mathbf{Sp}_{2n}(q)$, cf. Table 3. Then $x_1 = \mathrm{id} + e_{1,2n} \in \mathcal{O}$ and $x_2 = \sigma \triangleright x_1$, where $\sigma = e_{1,2n} - e_{2n,1} + \sum_{j\neq 1,2n} e_{jj}$ do the job for (a). Let τ be the block-diagonal matrix $\tau = \mathrm{diag}(\mathtt{J}_2, \mathrm{id}_{2n-2}, \mathtt{J}_2)$, with $\mathtt{J}_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\tau \in \mathbf{Sp}_{2n}(q)$ and $y_1 := x_1$, $y_2 := \tau \triangleright y_1$ fulfil (b). □

Here are results on regular unipotent classes that will be needed later. Let $\mathbb{G}_{\mathrm{sc}}$ be a simply connected simple algebraic group and F a Steinberg endomorphism. Let $\boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$; here we do not assume that $\boldsymbol{G}$ is simple.

Proposition 3.7. [ACGII, 3.7, 3.8, 3.11] *Let* $\mathcal{O}$ *be a regular unipotent class in* $\boldsymbol{G}$. *If any of the conditions below is satisfied, then* $\mathcal{O}$ *is of type D, or F.*

(1) $\boldsymbol{G} \neq \mathbf{PSL}_2(q)$ *is Chevalley and* $q \neq 2, 4$;

(2) $\boldsymbol{G} = \mathbf{PSU}_3(q)$, *with* $q \neq 2, 8$;

(3) $\boldsymbol{G} = \mathbf{PSU}_4(q)$, *with* $q \neq 2, 4$;

(4) $\boldsymbol{G} = \mathbf{PSU}_n(q)$, *with* $n \geq 5$ *and* $q \neq 2$;

In addition, every regular unipotent class in $\mathbf{GU}_n(q)$, *where* $1 < n$ *is odd and* $q = 2^{2h+1}$, $h \in \mathbb{N}_0$, *is of type D.* □

Finally, we quote [ACGII, Lemma 4.8]:

Lemma 3.8. *Let* $\mathcal{O}$ *be a regular unipotent class in either* $\mathbf{SL}_n(q)$, $\mathbf{SU}_n(q)$ *or* $\mathbf{Sp}_{2n}(q)$, *q even. Then there are* $x_1, x_2 \in \mathcal{O}$ *such that* $(x_1x_2)^2 \neq (x_2x_1)^2$.

4. Unipotent classes in Chevalley groups

In this Section we deal with unipotent conjugacy classes in a finite simple Chevalley group $\boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$, different to $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$, treated in [ACGI, ACGII], see §3.2. For convenience, we shall work in $\mathbb{G}_{sc}^F$, cf. [ACGI, Lemma 1.2]. For $\beta \in \Phi$, set

$$\Psi(\beta) = \{\gamma \in \Phi \mid \beta \preceq \gamma\}. \tag{4.1}$$

Let $u \in \mathbb{U}$ and $\beta \in \Phi^+$. Then the support $\operatorname{supp} u$ depends on a fixed ordering of Φ^+, but the assertion $\operatorname{supp} u \subset \Psi(\beta)$ does not. Indeed, passing from one order to another boils down to successive applications of the Chevalley formula (2.1), that do not affect the claim.

We denote by $\mathcal{O}$ a non-trivial unipotent conjugacy class in $\boldsymbol{G}$.

4.1. **Unipotent classes in $\mathbf{P\Omega}_{2n}^+(q)$, $n \geq 4$; $E_6(q)$, $E_7(q)$ and $E_8(q)$.** We first deal with the case when Φ simply-laced, i.e. $\boldsymbol{G}$ is one of $\mathbf{P\Omega}_{2n}^+(q)$, $n \geq 4$; $E_6(q)$, $E_7(q)$ and $E_8(q)$.

Lemma 4.1. *Given $\beta \in \Phi^+ - \Delta$, there is $x \in \mathcal{O} \cap \mathbb{U}^F$ with $\operatorname{supp} x \not\subset \Psi(\beta)$.*

Proof. Let $u \in \mathcal{O} \cap \mathbb{U}^F$. If $\operatorname{supp} u \not\subset \Psi(\beta)$, then we are done. Assume that $\operatorname{supp} u \subset \Psi(\beta)$. We claim that there is $\tau \in N_{\mathbb{G}_{sc}^F}(\mathbb{T})$ such that

$$x := \tau \triangleright u \in \mathcal{O} \cap \mathbb{U}^F \qquad \text{and} \qquad \operatorname{supp} x \not\subset \Psi(\beta).$$

For every $\gamma \in \Psi(\beta)$ there is a unique k such that $\gamma = \beta + \alpha_{i_1} + \cdots + \alpha_{i_k}$ as in Lemma 2.9. Let m be the minimum k for $\gamma \in \operatorname{supp} u$. We call m the bound of u. We will prove the claim by induction on the bound m. If $m = 0$ then $\beta \in \operatorname{supp} u$ and since $\beta \notin \Delta$, there is a simple reflection s_i such that $s_i\beta = \beta - \alpha_i \in \Phi^+ - \Psi(\beta)$. Also, $s_i\gamma \in \Phi^+$ for every $\gamma \in \operatorname{supp} u$ because $s_i(\Phi^+ - \{\alpha_i\}) = \Phi^+ - \{\alpha_i\}$. In this case we take $\tau = \dot{s}_i$ to be any representative of s_i in $N_{\mathbb{G}_{sc}^F}(\mathbb{T})$.

Let now $m > 0$ and assume that the statement is proved for unipotent elements with bound $m-1$. Let $\gamma \in \operatorname{supp} u$ reach the minimum, i.e., be such that $\gamma = \beta + \alpha_{i_1} + \cdots + \alpha_{i_m}$ for some $\alpha_{i_j} \in \Delta$ chosen as in Lemma 2.9. Then $\gamma' = s_{i_m}\gamma = \beta + \alpha_{i_1} + \cdots + \alpha_{i_{m-1}} \in \Psi(\beta)$, and $s_{i_m}\alpha \in \Phi^+$ for every $\alpha \in \Psi(\beta)$ by construction. Let $\dot{s}_{i_m}$ be a representative of s_{i_m} in $N_{\mathbb{G}_{\text{sc}}^F}(\mathbb{T})$. Then $u' = \dot{s}_{i_m} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and either $\operatorname{supp} u' \not\subset \Psi(\beta)$, or $\operatorname{supp} u' \subset \Psi(\beta)$, with bound at most $m-1$. In the first case, we conclude by setting $x = u'$. In the second case, we use the inductive hypothesis. $\square$

Proposition 4.2. *$\mathcal{O}$ is not kthulhu.*

Proof. The basic idea of the proof is to apply Lemma 3.2 (c) to a series of standard F-stable parabolic subgroups $\mathbb{P}^i$ of $\mathbb{G}_{\text{sc}}$ for which (3.1) holds. We show that for every $\mathcal{O}$ and for every $\boldsymbol{G}$, we have $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$. This follows from Lemma 4.1 by observing that in each case $\cap_i V_i$ is a product of root subgroups corresponding to roots in $\Psi(\beta)$ for some $\beta \in \Phi^+ - \Delta$. We analyze the different cases according to Φ.

D_n, $n \geq 4$. We consider the parabolic subgroups $\mathbb{P}^1$ and $\mathbb{P}^2$ such that $\mathbb{L}_1$ and $\mathbb{L}_2$ have root systems A_{n-1}, generated respectively by $\Delta - \alpha_{n-1}$ and $\Delta - \alpha_n$. Since $n \geq 4$, (3.1) holds by Theorem 3.3. Let $u \in V_1 \cap V_2$. Then $\alpha \in \operatorname{supp} u$ if and only if α contains α_{n-1} and α_n in its expression, i.e. $\alpha \in \Psi(\beta)$ for $\beta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$.

E_6. We consider the parabolic subgroups $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^3$ such that $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{L}_3$ have root systems D_5, D_5 and A_5, generated respectively by $\Delta - \alpha_1$, $\Delta - \alpha_6$ and $\Delta - \alpha_2$. By Theorem 3.3 and the result for D_n, (3.1) holds. Let $u \in V_1 \cap V_2 \cap V_3$. Then $\alpha \in \operatorname{supp} u$ if and only if $\alpha \in \Psi(\beta)$ for $\beta = \sum_{i=1}^{6} \alpha_i$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$.

E_7. We consider the parabolic subgroups $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^3$ such that $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{L}_3$ have root systems D_6, E_6 and A_6, generated respectively by $\Delta - \alpha_1$, $\Delta - \alpha_7$ and $\Delta - \alpha_2$. By Theorem 3.3 and the results for D_n and E_6, (3.1) holds. Let $u \in V_1 \cap V_2 \cap V_3$. Then $\alpha \in \operatorname{supp} u$ if and only if $\alpha \in \Psi(\beta)$ for $\beta = \sum_{i=1}^{7} \alpha_i$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$.

E_8. We consider the parabolic subgroups $\mathbb{P}^1$, $\mathbb{P}^2$ and $\mathbb{P}^3$ such that $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{L}_3$ have root systems D_7, E_7 and A_7, generated respectively by $\Delta - \alpha_1$, $\Delta - \alpha_8$ and $\Delta - \alpha_2$. By Theorem 3.3 and the results for D_n and E_7, (3.1) holds. Let $u \in V_1 \cap V_2 \cap V_3$. Then $\alpha \in \operatorname{supp} u$ if and only if $\alpha \in \Psi(\beta)$ for $\beta = \sum_{i=1}^{8} \alpha_i$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$. □

4.2. **Unipotent classes in $\mathbf{P\Omega}_{2n+1}(q)$.** Here we deal with $\mathbf{P\Omega}_{2n+1}(q)$, i.e. Φ is of type B_n, $n \geq 3$. In this case, q is always odd.

Proposition 4.3. *$\mathcal{O}$ is not kthulhu.*

Proof. We consider the standard F-stable parabolic subgroups $\mathbb{P}^1$ and $\mathbb{P}^2$ such that $\mathbb{L}_1$ and $\mathbb{L}_2$ have root systems A_{n-1} and C_2, generated respectively by $\Pi_1 := \Delta - \alpha_n$ and $\Pi_2 = \{\alpha_{n-1}, \alpha_n\}$. By Lemma 3.2 (a) and Theorem 3.3, if $\mathcal{O} \cap \mathbb{U}^F \not\subset V_1$ then $\mathcal{O}$ is not kthulhu. Let us thus consider $u \in \mathcal{O} \cap V_1$. Then $\operatorname{supp} u \subset \Psi_{\Pi_1} = \{\varepsilon_i, \varepsilon_j + \varepsilon_l \mid i, j, l \in \mathbb{I}_n, j < l\}$, since it must contain α_n. We will apply the argument in 2.2.4.

Assume first that $\operatorname{supp} u \subset \{\varepsilon_j + \varepsilon_l \mid j, l \in \mathbb{I}_n, j < l\}$. Let ℓ be the maximum l such that $\varepsilon_j + \varepsilon_l \in \operatorname{supp} u$ for some $j \in \mathbb{I}_{n-1}$. Then $s_{\varepsilon_\ell}(\operatorname{supp} u) \subset \Phi^+$. Let $\dot{s}_{\varepsilon_\ell}$ be a representative of s_{ε_ℓ} in $N_{\mathbb{G}_{sc}^F}(\mathbb{T})$. Then $\dot{s}_{\varepsilon_\ell} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$

and $\varepsilon_j - \varepsilon_\ell \in \operatorname{supp}(\dot{s}_{\varepsilon_\ell} \triangleright u)$ for every j such that $\varepsilon_j + \varepsilon_\ell \in \operatorname{supp} u$. Hence $\dot{s}_{\varepsilon_\ell} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F - V_1$. By the previous argument, $\mathcal{O}$ is not kthulhu.

Assume next that there is some i such that $\varepsilon_i \in \operatorname{supp} u$. We can always assume $i = n$. Indeed, if $\varepsilon_n \notin \operatorname{supp} u$, we may replace u by $\dot{s}_{\varepsilon_i - \varepsilon_n} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$, where $\dot{s}_{\varepsilon_i-\varepsilon_n}$ is a representative of $s_{\varepsilon_i-\varepsilon_n}$ in $N_{\mathbb{G}_{sc}^F}(\mathbb{T})$. Then $\pi_2(u) \in M_2$ lies in a non-trivial unipotent conjugacy class in a group isomorphic to $\mathbf{Sp}_4(q)$ and the short simple root lies in the support. A direct computation shows that a representative of this class in $\mathbf{Sp}_4(q)$ is as follows:

$$\begin{pmatrix} 1 & a & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad\qquad a \neq 0.$$

Thus, its Jordan form has partition $(2, 2)$ and this class is not kthulhu by Theorem 3.3 (recall that q is odd). Then Lemma 3.2 applies. □

4.3. **Unipotent classes in $F_4(q)$.** Here we deal with unipotent classes in $F_4(q)$. In this case the approach in Section 4.1 is not effective. Indeed, in characteristic 2, (3.1) does not hold for any of the standard parabolic subgroups. For this reason we shall use explicit representatives of unipotent classes and apply results from Theorem 3.3 and Proposition 4.3 for B_3, where q is assumed to be odd.

We use the list of representatives of unipotent classes in $F_4(q)$ in [Sho, Tables 5,6] for q odd, see Table 4, respectively in [Shi, Theorem 2.1] for q even, see Table 5. We indicate the roots as in [Shi]: ε_i is indicated by i, $\varepsilon_i - \varepsilon_j$ is indicated by $i-j$, and $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ is indicated by $1\pm2\pm3\pm4$. Thus the simple roots are $\alpha_1 = 2-3$, $\alpha_2 = 3-4$, $\alpha_3 = 4$, $\alpha_4 = 1-2-3-4$. If q is odd, then the possible representatives are x_i, $i \in \mathbb{I}_{25}$, for $p \neq 3$, with two additional representatives x_{26}, x_{27} when $p = 3$.

Lemma 4.4. *If q is odd, then $\mathcal{O}$ is not kthulhu.*

Proof. A direct verification shows that all representatives for $i \geq 7$ enjoy the $\alpha\beta$-property with $(\alpha, \beta) \neq 0$; we list in Table 6 the roots α and β for each representative. By Proposition 2.7, $\mathcal{O}$ is of type D.

We next consider the representative x_1, that equals $x_\gamma(1)$ for a long root γ. By the discussion in §2.2.2, $\mathcal{O}_{x_1}$ contains an element in $\mathbb{U}_{\alpha_1}^F$, that lies in the subgroup of type A_2 generated by $\mathbb{U}_{\pm\alpha_1}, \mathbb{U}_{\pm\alpha_2}$. Theorem 3.3 applies.

Finally, we deal with the x_i's, $i = 2, 3, 4, 5, 6$. Let $\mathbb{L}_1$ be the standard Levi subgroup (of type B_3) generated by the root subgroups $\mathbb{U}_\gamma$, for $\gamma = \pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$. We claim that all x_i, $i = 2, 3, 4, 5, 6$, are conjugated to elements in M_1; then the result follows by Proposition 4.3. Indeed, x_2, x_3 lie in $\mathbb{U}_{1-2}^F \mathbb{U}_{1+2}^F$; thus conjugating by $s_{1-3}s_{2-4}$, we get a representative in

TABLE 4. Representatives of unipotent classes in $F_4(q)$ in odd characteristic; η, ξ and ζ are suitable elements in $\mathbb{F}_q^{\times}$

$x_1 = x_{1+2}(1)$
$x_2 = x_{1-2}(1)x_{1+2}(-1)$
$x_3 = x_{1-2}(1)x_{1+2}(-\eta)$
$x_4 = x_2(1)x_{3+4}(1)$
$x_5 = x_{2-3}(1)x_4(1)x_{2+3}(1)$
$x_6 = x_{2-3}(1)x_4(1)x_{2+3}(\eta)$
$x_7 = x_2(1)x_{1-2+3+4}(1)$
$x_8 = x_{2-3}(1)x_4(1)x_{1-2}(1)$
$x_9 = x_{2-3}(1)x_{3-4}(1)x_{3+4}(-1)$
$x_{10} = x_{2-3}(1)x_{3-4}(1)x_{3+4}(-\eta)$
$x_{11} = x_{2+3}(1)x_{1+2-3-4}(1)x_{1-2+3+4}(1)$
$x_{12} = x_{2-3}(1)x_4(1)x_{1-4}(1)$
$x_{13} = x_{2-3}(1)x_4(1)x_{1-4}(\eta)$
$x_{14} = x_{2-4}(1)x_{3+4}(1)x_{1-2}(-1)x_{1-3}(-1)$
$x_{15} = x_{2-4}(1)x_{3+4}(1)x_{1-2}(-\eta)x_{1-3}(-1)$
$x_{16} = x_{2-4}(1)x_{2+4}(-\eta)x_{1-2+3+4}(1)x_{1-3}(-1)$
$x_{17} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3+4}(1)x_{1-2}(-\eta)x_{1-3}(\xi)$
$x_{18} = x_2(1)x_{3+4}(1)x_{1-2+3-4}(1)x_{1-2}(-1)x_{1-3}(\zeta)$
$x_{19} = x_{2-3}(1)x_{3-4}(1)x_4(1)$
$x_{20} = x_2(1)x_{3+4}(1)x_{1-2-3-4}(1)$
$x_{21} = x_{2-4}(1)x_3(1)x_{2+4}(1)x_{1-2-3+4}(1)$
$x_{22} = x_{2-4}(1)x_3(1)x_{2+4}(\eta)x_{1-2-3+4}(1)$
$x_{23} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2}(1)$
$x_{24} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2}(\eta)$
$x_{25} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)$
$x_{26} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{1-2+3+4}(\zeta)$
$x_{27} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{1-2+3+4}(-\zeta)$

$\mathbb{U}^F_{3-4}\mathbb{U}^F_{3+4}$. Also x_5, x_6 lie in $\mathbb{U}^F_{2-3}\mathbb{U}^F_{2+3}\mathbb{U}^F_4$, and $x_4 = x_2(1)x_{3+4}(1)$, so they all lie in M_1. $\square$

Lemma 4.5. *If q is even, then $\mathcal{O}$ is not kthulhu.*

Proof. The representative x_1, respectively x_2, is equal to $x_\gamma(1)$ for a short, respectively long, root γ. By the discussion in §2.2.2, $\mathcal{O}_{x_1}$ intersects $\mathbb{U}^F_{\alpha_3}$ and $\mathcal{O}_{x_2}$ intersects $\mathbb{U}^F_{\alpha_1}$. Let $M = \langle \mathbb{U}^F_{\pm\alpha_3}, \mathbb{U}^F_{\pm\alpha_4}\rangle$ and $M' = \langle \mathbb{U}^F_{\pm\alpha_1}, \mathbb{U}^F_{\pm\alpha_2}\rangle$, both of type A_2. Then $\mathcal{O}_{x_1} \cap M$, respectively $\mathcal{O}_{x_2} \cap M'$, is a unipotent class corresponding to the partition $(2, 1)$ in M, respectively M'. By Theorem 3.3, these classes are not kthulhu.

TABLE 5. Representatives of unipotent classes in $F_4(q)$ in even characteristic; η and ζ are suitable elements in $\mathbb{F}_q^\times$

$x_1 = x_1(1)$
$x_2 = x_{1+2}(1)$
$x_3 = x_1(1)x_{1+2}(1)$
$x_4 = x_{2+3}(1)x_1(1)$
$x_5 = x_2(1)x_{2+3}(1)x_{1-3}(1)$
$x_6 = x_2(1)x_{2+3}(1)x_{1-3}(1)x_{1+3}(\eta)$
$x_7 = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)$
$x_8 = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1+4}(\eta)$
$x_9 = x_2(1)x_{1-2}(1)$
$x_{10} = x_2(1)x_{1-2}(1)x_{1+2}(\eta)$
$x_{11} = x_2(1)x_{3+4}(1)x_{1-4}(1)$
$x_{12} = x_2(1)x_{1-2+3+4}(1)x_{1-4}(1)$
$x_{13} = x_2(1)x_{2+3}(1)x_{1-2}(1)$
$x_{14} = x_2(1)x_{3+4}(1)x_{1-2}(1)$
$x_{15} = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1-3}(1)$
$x_{16} = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1-2}(1)$
$x_{17} = x_2(1)x_{2+3}(1)x_{1-2-3+4}(1)x_{1-2}(1)$
$x_{18} = x_2(1)x_{2+3}(1)x_{1-2-3+4}(1)x_{1-2}(1)x_{1-4}(\eta)$
$x_{19} = x_2(1)x_{3+4}(1)x_{1-2+3-4}(1)x_{1-2}(1)x_{1-3}(\zeta)$
$x_{20} = x_{1-2}(1)x_{2-3}(1)x_3(1)$
$x_{21} = x_{1-2}(1)x_{2-3}(1)x_3(1)x_{2+3}(\eta)$
$x_{22} = x_4(1)x_{2-4}(1)x_{1-2+3-4}(1)$
$x_{23} = x_4(1)x_{2-4}(1)x_{2+4}(\eta)x_{1-2+3-4}(1)$
$x_{24} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)$
$x_{25} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)x_{1-2}(\eta)$
$x_{26} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)x_{1-2}(\eta)x_{1-3}(\eta)$
$x_{27} = x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-2-3+4}(1)$
$x_{28} = x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-2-3+4}(1)x_{1-2}(\eta)$
$x_{29} = x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)$
$x_{30} = x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)x_{3+4}(\eta)$
$x_{31} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)$
$x_{32} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{3+4}(\eta)$
$x_{33} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{1-2}(\eta)$
$x_{34} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{3+4}(\eta)x_{1-2}(\eta)$

We consider now the classes labelled by $i \in \mathbb{I}_{20,34}$. Let $\mathbb{P}^1$ be the standard parabolic subgroup with standard Levi $\mathbb{L}_1$ as in the proof of Lemma 4.4. Set $y_i = \pi_1(x_i)$. Then the class $\mathcal{O}_{y_i}^{M_1}$ satisfies the $\alpha\beta$-property; we list in Table 7 the roots α and β for each representative. Since Φ_{Π_1} is of type B_3, the group

TABLE 6. $\mathcal{O}_{x_i}$ with the $\alpha\beta$-property

i	α	β
7	2	1-2+3+4
8	1-2	2-3
9,10	2-3	3-4
11	1+2-3-4	1-2+3+4
12,13	4	1-4
14,15	2-4	1-2
16	2-4	1-2+3+4
17,21,22	2-4	1-2-3+4
18	2	1-2
19,23,24,25,(26,27)	2-3	3-4
20	2	1-2-3-4

$[\mathbb{L}_1, \mathbb{L}_1]$ is isogenous to $\mathbf{Sp}_6(\Bbbk)$. By Remark 3.4, $\mathcal{O}_{y_i}^{M_1}$ is not kthulhu, hence neither is $\mathcal{O}$.

TABLE 7. $\mathcal{O}_{y_i}^{M_1}$ with the $\alpha\beta$-property.

i	α	β
$i \in \mathbb{I}_{20,21}$	α_1	$\alpha_2 + \alpha_3$
$i \in \mathbb{I}_{22,23}$	α_3	$\alpha_1 + \alpha_2$
$i \in \mathbb{I}_{24,28}$	$\alpha_1 + \alpha_2$	$\alpha_2 + 2\alpha_3$
$i \in \mathbb{I}_{29,34}$	α_1	α_2

We consider now the classes labelled by $i \in \mathbb{I}' = \{3, 4, 7, 8, 12\} \cup \mathbb{I}_{14,19}$. Let $\mathbb{P}^2$ be the standard parabolic subgroup with standard Levi $\mathbb{L}_2$ (of type C_3) associated with $\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4\}$; here $\Phi_{\Pi_2}^+$ consists of the roots $1-2$, 3, 4, 3 ± 4, $1-2 \pm 3 \pm 4$. Let $\beta_1 = \alpha_4$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_2$ be the simple roots of $\Phi_{\Pi_2}^+$. Set $z_i = \pi_2(x_i)$. Now $\mathcal{O}_{z_i}^{M_2}$ is a unipotent class in $\mathbf{Sp}_6(q)$. Let $\mathbb{I}'' = \mathbb{I}' - \{3, 4\}$. Table 8 lists the index $i \in \mathbb{I}''$, the support of z_i and the partition associated to $\mathcal{O}_{z_i}^{M_2}$, obtained from the Jordan form of z_i in $\mathbf{Sp}_6(\Bbbk)$. Since the partition is always different from $(2, 1^4)$, the label of the class in $\mathbf{Sp}_6(q)$ is never $W(1) \oplus V(2)$, whence $\mathcal{O}_{z_i}^{M_2}$ is not kthulhu by Theorem 3.3. The remaining classes in $\mathbb{I}'$ are represented by $x_3 = x_1(1)x_{1+2}(1)$ and $x_4 = x_{2+3}(1)x_1(1)$. Let $x = (\dot{s}_{1-3}\dot{s}_{2-4}) \triangleright x_3 \in \mathcal{O}_{x_3}^{\boldsymbol{G}}$ and $y = (\dot{s}_{2-3}\dot{s}_{1-2}\dot{s}_3) \triangleright$

$x_4 \in \mathcal{O}_{x_4}^{\boldsymbol{G}}$. Then $x \in \mathbb{U}_3\mathbb{U}_{3+4}$, so $x \in \mathbb{U}^F_{\beta_2+\beta_3}\mathbb{U}^F_{2\beta_2+\beta_3} \subset M_2$, $y \in \mathbb{U}_{1-2}\mathbb{U}_3$, so $y \in \mathbb{U}^F_{2\beta_1+2\beta_2+\beta_3}\mathbb{U}^F_{\beta_2+\beta_3} \subset M_2$. The partition associated to x, respectively y, as unipotent element in $\mathbf{Sp}_6(q)$ is $(2,2,1,1)$, respectively $(2,2,2)$. Hence, neither $\mathcal{O}_{x_3}^{\boldsymbol{G}}$ nor $\mathcal{O}_{x_4}^{\boldsymbol{G}}$ is kthulhu by Theorem 3.3.

TABLE 8. $\operatorname{supp} z_i$ and its partition

i	$\operatorname{supp} z_i$	partition
7,8,12,15	$\beta_1 + 2\beta_2 + \beta_3$	$(2,2,1,1)$
14	$2\beta_1 + 2\beta_2 + \beta_3,\ 2\beta_2 + \beta_3,$	$(2,2,1,1)$
16	$2\beta_1 + 2\beta_2 + \beta_3,\ \beta_1 + 2\beta_2 + \beta_3,$	$(2,2,1,1)$
17,18	$2\beta_1 + 2\beta_2 + \beta_3,\ \beta_1 + \beta_2,$	$(2,2,1,1)$
19	$2\beta_1 + 2\beta_2 + \beta_3,\ 2\beta_2 + \beta_3,\ \beta_1 + \beta_2 + \beta_3$	$(2,2,2)$

The x_i's for $i \in \mathbb{I}''' = \{5,6,9,10,11,13\}$ lie in the subgroup $\mathbb{K}$ of type B_4 generated by the subgroups $\mathbb{U}_{\pm\alpha}$, $\alpha \in \{1-2, 2-3, 3-4, 4\}$. If $i \in \mathbb{I}'''$, $\mathcal{O}_{x_i}^{\mathbb{K}^F}$ has the $\alpha\beta$-property, see Table 9. Since $\mathbf{SO}_9(\Bbbk)$ is isogenous to $\mathbf{Sp}_8(\Bbbk)$, Remark 3.4 applies.

TABLE 9. $\mathcal{O}_{x_i}^{\mathbb{K}^F}$ with the $\alpha\beta$-property, $i \in \mathbb{I}'' = \{5,6,9,10,11,13\}$.

x_i	α	β
$5,6$	2+3	1-3
$9,10,13$	2	1-2
11	1-4	3+4

□

4.4. **Unipotent classes in $G_2(q)$.** Here we deal with unipotent classes in $\boldsymbol{G} = G_2(q)$, $q > 2$. As for $F_4(q)$, we shall use explicit representatives of the classes, the parabolics being too small. The list of representatives can be found in [C] when $p > 3$ and in [E] otherwise; see (4.2), (4.3), (4.4). We show that all classes are not kthulhu. We split the proof in several cases, depending on q and the representatives on each class. We first prove the result for q odd, where we study the cases $q > 3$ and $q = 3$ separately. Then we proceed to deal with q even, where we treat first two conjugacy classes. For the remaining ones we split the proof for the cases $q > 4$ and $q = 4$.

4.4.1. *Unipotent classes in $G_2(q)$ for q odd.*

Lemma 4.6. *If $q > 3$ is odd, then $\mathcal{O}$ is not kthulhu.*

Proof. By [C, Theorems 3.1, 3.2, 3.9] every non-trivial class of p-elements in $\boldsymbol{G} = G_2(q)$ is either regular or can be represented by an element of the following form, for suitable $a, b, c \in \mathbb{F}_q^\times$:

$$\begin{aligned} &x_{\alpha_2}(1), && x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b), && x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(-1)x_{3\alpha_1+\alpha_2}(c), \\ &x_{\alpha_1+\alpha_2}(1), && x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a). && \end{aligned} \tag{4.2}$$

The regular classes are covered by Proposition 3.7 (1). We treat the remaining classes separately.

The elements $x_{\alpha_2}(1)$ and $x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b)$ lie in the subgroup of type A_2 generated by $\mathbb{U}^F_{\pm\alpha_2}$ and $\mathbb{U}^F_{\pm(3\alpha_2+\alpha_2)}$ and we apply Theorem 3.3. The classes represented by $x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(-1)x_{3\alpha_1+\alpha_2}(c)$ enjoy the $\alpha\beta$-property, so we invoke Proposition 2.7. We prove now that the class of $r = x_{\alpha_1+\alpha_2}(1)$ is of type D. First, we observe that there is an element $\sigma = \dot{s}_{\alpha_2} \in \boldsymbol{G} \cap N_{\mathbb{G}}(\mathbb{T})$ such that $s := \sigma \triangleright r = x_{\alpha_1}(\xi)$, $\xi \in \mathbb{F}_q^\times$. Then $sr \neq rs$ by the Chevalley commutator formula (2.1) and, as $rs, sr \in \mathbb{U}^F$ and p is odd, we have $(rs)^2 \neq (sr)^2$. In addition, $r, s \in \mathbb{P}_1^F$, for $\mathbb{P}_1$ the standard parabolic subgroup with Levi $\mathbb{L}_1$ associated with α_1. Since r lies in the unipotent radical $\mathbb{V}_1$ of $\mathbb{P}_1$ and s lies in $\mathbb{L}_1$, we have $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$.

Let $r = x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a)$; it lies in $\langle \mathbb{U}^F_{\pm\alpha_2}\rangle \times \langle \mathbb{U}^F_{\pm(2\alpha_1+\alpha_2)}\rangle$. We argue as in §3.3. As $q > 3$, Lemmata 3.6 and 2.1 apply whence $\mathcal{O}_r$ is of type D. □

Lemma 4.7. *If $q = 3$, then $\mathcal{O}$ is not kthulhu.*

Proof. By [E, 6.4] the non-trivial classes of p-elements in $\boldsymbol{G}$ are either regular or are represented by an element of the following form:

$$\begin{aligned} &x_{3\alpha_1+2\alpha_2}(1), && x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(a), \\ &x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1), && x_{2\alpha_1+\alpha_2}(1), \end{aligned} \tag{4.3}$$

for suitable $a \in \mathbb{F}_q^\times$. The regular classes are covered by Proposition 3.7 (1). The element $x_{3\alpha_1+2\alpha_2}(1)$ lies in the subgroup of type A_2 generated by $\mathbb{U}^F_{\pm\alpha_2}$ and $\mathbb{U}^F_{\pm(3\alpha_1+2\alpha_2)}$ and Theorem 3.3 applies.

Next we show that if $r = x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(a) \in \mathcal{O}$, then it is of type D. Indeed, let $s := \dot{s}_{\alpha_2} \triangleright r \in \mathbb{U}^F_{\alpha_1}\mathbb{U}^F_{3\alpha_1+2\alpha_2}$. Then $sr \neq rs$; since $sr, rs \in \mathbb{U}^F$, we have $(sr)^2 \neq (rs)^2$. Moreover, $r, s \in \mathbb{P}_1^F$ with $s \in \mathbb{L}_1$, $r \in \mathbb{V}_1$, with notation as for $p > 3$. Thus, $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$ and $\mathcal{O}$ is of type D.

Assume that $u = x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1) \in \mathcal{O}$. Conjugating by suitable elements in $N_{\boldsymbol{G}}(\mathbb{T})$ we find $r \in \mathcal{O} \cap \mathbb{U}_{\alpha_1}\mathbb{U}_{3\alpha_1+\alpha_2} \subset \mathbb{P}_1$, $r \notin \mathbb{V}_1$ and $s \in$

$\mathcal{O} \cap \mathbb{U}_{\alpha_1+\alpha_2}\mathbb{U}_{\alpha_2} \subset \mathbb{V}_1$. By repeated use of (2.1), we see that the coefficient of $x_{\alpha_1+\alpha_2}$ in srs^{-1} is $\neq 0$, hence $rs \neq sr$, $(rs)^2 \neq (sr)^2$ and $\mathcal{O}$ is of type D.

Assume finally that $u = x_{2\alpha_1+\alpha_2}(1) \in \mathcal{O}$. Let $r = \dot{s}_{\alpha_1} \triangleright u \in \mathcal{O}_u^{\boldsymbol{G}} \cap \mathbb{U}_{\alpha_1+\alpha_2}$ and $s = \dot{s}_{\alpha_1+\alpha_2} \triangleright u \in \mathcal{O}_u^{\boldsymbol{G}} \cap \mathbb{U}_{\alpha_1}$. Then $rs, sr \in \mathbb{U}$, $(rs)^2 \neq (sr)^2$, and $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$, as $s \in \mathbb{L}_1$ and $r \in \mathbb{V}_1$, so $\mathcal{O}$ is of type D. $\square$

4.4.2. *Unipotent classes in $G_2(q)$ for q even.* By [E, 2.6] all non-trivial classes of 2-elements in $G_2(q)$ can be represented by an element of the following form, for suitable $a, b, c \in \mathbb{F}_q$ with $c \neq 0$:

$$\begin{aligned} &x_{2\alpha_1+\alpha_2}(1), \quad x_{3\alpha_1+2\alpha_2}(1), \qquad x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a), \\ &x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b), \quad x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(c). \end{aligned} \tag{4.4}$$

We begin by studing the conjugacy classes corresponding to the first two representatives.

Lemma 4.8. *If $x = x_{2\alpha_1+\alpha_2}(1)$ or $x = x_{3\alpha_1+2\alpha_2}(1)$, then $\mathcal{O}_x^{\boldsymbol{G}}$ is not kthulhu.*

Proof. Let $\mathcal{O} = \mathcal{O}_x^{\boldsymbol{G}}$ and write $r = x_{2\alpha_1+\alpha_2}(1)$. It is enough to prove that $\mathcal{O}$ is of type C for $G_2(2)$, which is a non-simple subgroup of $G_2(q)$. We consider $\dot{s}_{\alpha_1+\alpha_2} \triangleright r = x_{\alpha_1}(1) \in \mathcal{O}$ and $s := x_{-\alpha_1}(1) \triangleright x_{\alpha_1}(1) = \dot{s}_{\alpha_1} \in \mathcal{O}_r^{G_2(2)}$. Let $H := \langle r, s, z = x_{\alpha_1+\alpha_2}(1)\rangle \leq \mathbb{P}_1$ (the parabolic subgroup associated with α_1), with $r \in \mathbb{V}_1$, $s \in \mathbb{L}_1$. Hence, $\mathcal{O}_r^H \neq \mathcal{O}_s^H$. By a direct computation,

$$\begin{aligned} s \triangleright r &= x_{\alpha_1+\alpha_2}(1) = z \neq r, & z \triangleright r &= r x_{3\alpha_1+2\alpha_2}(1), \\ r \triangleright s &= szr, & z \triangleright (szr) &= s x_{3\alpha_1+2\alpha_2}(1). \end{aligned}$$

So $H \leq \langle \mathcal{O}_r^H, \mathcal{O}_s^H \rangle \leq H$; $\{r, z, z \triangleright r\} \subset \mathcal{O}_r^H$ and $\{s, szr, sx_{3\alpha_1+2\alpha_2}(1)\} \subset \mathcal{O}_s^H$ hence $\mathcal{O}_r^{G_2(2)}$ is of type C by [ACGIII, Lemma 2.8].

Assume that $r = x_{3\alpha_1+2\alpha_2}(1) \in \mathcal{O}$. Now $r \in \mathbb{M} = \langle \mathbb{U}_{\pm\alpha_2}, \mathbb{U}_{\pm(3\alpha_1+\alpha_2)}\rangle$, which is of type A_2. Since $\mathcal{O}_r^{\mathbb{M}}$ has partition $(2,1)$, $\mathcal{O}$ is not kthulhu by Theorem 3.3 and [MT, Theorem 24.15]. $\square$

Now we deal with the remaining cases, dividing the proof for the cases $q > 4$ and $q = 4$. Thus, for the rest of the subsection we assume that x is either one of the following representatives, for suitable $a, b, c \in \mathbb{F}_q$ with $c \neq 0$:

$$\begin{aligned} &x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a), \qquad x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b), \\ &x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(c). \end{aligned}$$

Lemma 4.9. *If $q > 4$ is even, then $\mathcal{O}_x^{\boldsymbol{G}}$ is not kthulhu.*

Proof. The classes represented by the elements $x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a)$ for $a \in \mathbb{F}_q$ are regular, thus they are not kthulhu by Proposition 3.7. The classes of $x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b)$ and $x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(c)$ enjoy the $\alpha\beta$-property. By [ACGII, Proposition 3.6], these classes are of type F. $\square$

In order to deal with the remaining classes in $\boldsymbol{G} = G_2(4)$ we will need a precise version of (2.1) for all pairs of positive roots. We shall use the relations from [E, II.2], that we rewrite for convenience. They hold in general for q even, and we shall use them recalling that $a^3 = 1$ for every $a \in \mathbb{F}_4^\times$.

$$x_{\alpha_1}(a)x_{\alpha_2}(b) = x_{\alpha_2}(b)x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(ab)x_{2\alpha_1+\alpha_2}(a^2b)x_{3\alpha_1+\alpha_2}(a^3b) \tag{4.5}$$

$$x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(b) = x_{\alpha_1+\alpha_2}(b)x_{\alpha_1}(a)x_{3\alpha_1+\alpha_2}(a^2b)x_{3\alpha_1+2\alpha_2}(ab^2) \tag{4.6}$$

$$x_{\alpha_1}(a)x_{2\alpha_1+\alpha_2}(b) = x_{2\alpha_1+\alpha_2}(b)x_{\alpha_1}(a)x_{3\alpha_1+\alpha_2}(ab) \tag{4.7}$$

$$x_{\alpha_2}(a)x_{3\alpha_1+\alpha_2}(b) = x_{3\alpha_1+\alpha_2}(b)x_{\alpha_2}(a)x_{3\alpha_1+2\alpha_2}(ab) \tag{4.8}$$

$$x_{\alpha_1+\alpha_2}(a)x_{2\alpha_1+\alpha_2}(b) = x_{2\alpha_1+\alpha_2}(b)x_{\alpha_1+\alpha_2}(a)x_{3\alpha_1+2\alpha_2}(ab) \tag{4.9}$$

For all other pairs of positive roots the corresponding subgroups commute.

Fix ζ a generator of $\mathbb{F}_4^\times$, so $\zeta^2 + \zeta + 1 = 0$ and $\zeta^3 = 1$.

Lemma 4.10. *Let $x = x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a)$ with $a \in \mathbb{F}_4$. Then $\mathcal{O}_x^{\boldsymbol{G}}$ is not kthulhu.*

Proof. By [E] there are 2 regular unipotent classes, one represented by $x_{\alpha_1}(1)x_{\alpha_2}(1)$ and the other by $x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta)$. We shall apply Lemma 2.3 in order to show that these classes are of type C. For this, we need the following formula which can be retrieved applying (4.5) and (4.8).

$$\begin{aligned} x_{\alpha_1}(a)x_{\alpha_2}(b)x_{\alpha_1}(c)x_{\alpha_2}(d) &= x_{\alpha_1}(a+c)x_{\alpha_2}(b+d) \\ &\quad \times x_{3\alpha_1+\alpha_2}(b)x_{3\alpha_1+2\alpha_2}(bd)x_{2\alpha_1+\alpha_2}(c^2b)x_{\alpha_1+\alpha_2}(bc), \end{aligned} \tag{4.10}$$

$a, b, c, d \in \mathbb{F}_q$. Let $r = x_{\alpha_1}(1)x_{\alpha_2}(1)$, $t := \alpha_1^\vee(\zeta)$, $s := t \triangleright r = x_{\alpha_1}(\zeta^2)x_{\alpha_2}(1) \in \mathcal{O}_r^{\boldsymbol{G}}$. By direct computation using (4.10) we see that

$$\begin{aligned} r^2 &= x_{3\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1) \\ s^2 &= x_{3\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta)x_{\alpha_1+\alpha_2}(\zeta^2). \end{aligned}$$

Using (4.9) and that $\xi^2 \neq \xi$, we see $r^2s^2 \neq s^2r^2$, hence $r^2s \neq sr^2$ and $s^2r \neq rs^2$. In addition, $\langle r, s\rangle \subseteq \mathbb{U}^F$ and $\mathbb{U}^F \triangleright r \subset r\langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \Delta\rangle$ and $\mathbb{U}^F \triangleright s \subset s\langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \Delta\rangle$, so $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$, whence $\mathcal{O}_r^{\boldsymbol{G}}$ is of type C.

Similarly, we consider now $r = x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta)$, $t := \alpha_1^\vee(\zeta)$ and $s := t \triangleright r = x_{\alpha_1}(\zeta^2)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta^2) \in \mathcal{O}_r^{\boldsymbol{G}}$. In this case

$$\begin{aligned} r^2 &= x_{3\alpha_1+\alpha_2}(\zeta^2)x_{3\alpha_1+2\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1) \\ s^2 &= x_{3\alpha_1+\alpha_2}(\zeta^2)x_{3\alpha_1+2\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(\zeta)x_{\alpha_1+\alpha_2}(\zeta^2). \end{aligned}$$

As above we verify that $r^2 s \neq s^2 r$ and $s^2 r \neq r^2 s$ and that $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$ so $\mathcal{O}_r^{\boldsymbol{G}}$ is of type C. $\square$

Lemma 4.11. *If $x = x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b)$ with $b \in \mathbb{F}_4$, then $\mathcal{O}_x^{\boldsymbol{G}}$ is not kthulhu.*

Proof. Assume first that $x = x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b) \in \mathcal{O}$, with $b \neq 0$. By [E, Proposition 2.6, page 499], if $q = 4$ we can take $b = \zeta$. We prove that this class is of type C. Set $r_\alpha := x_\alpha(1)x_{-\alpha}(1)x_\alpha(1) = \dot{s}_\alpha$, $\alpha \in \Phi^+$, see [St, Lemma 19]. The elements

$$\begin{aligned} s = r_{\alpha_1} \triangleright x &= x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1)x_{\alpha_2}(\zeta) \\ &= x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_2}(\zeta)x_{3\alpha_1+2\alpha_2}(1), \\ r = r_{\alpha_2}r_{\alpha_1} \triangleright s &= x_{\alpha_1}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(\zeta) \end{aligned}$$

belong to $\mathcal{O}$. We claim that $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$. Indeed, r, $s \in \mathbb{P}_1$ with $r \notin \mathbb{V}_1$, $s \in \mathbb{V}_1$. A direct calculation shows that $r^2 = x_{3\alpha_1+\alpha_2}(1)$,

$$\begin{aligned} &r \triangleright s = x_{\alpha_1+\alpha_2}(1+\zeta)x_{2\alpha_1+\alpha_2}(1+\zeta)x_{\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta), \\ &r^2 \triangleright s = s x_{3\alpha_1+2\alpha_2}(\zeta), \\ &r^3 \triangleright s = r \triangleright (r^2 \triangleright s) = r \triangleright (s x_{3\alpha_1+2\alpha_2}(\zeta)) = (r \triangleright s)x_{3\alpha_1+2\alpha_2}(\zeta), \\ &s \triangleright (r \triangleright s) = x_{\alpha_1+\alpha_2}(1+\zeta)x_{2\alpha_1+\alpha_2}(1+\zeta)x_{\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+2\alpha_2}(1+\zeta). \end{aligned}$$

We see that all these are distinct, and different from s, by looking at the unique expression as a product of elements in root subgroups in the order:

$$\alpha_1 < \alpha_1 + \alpha_2 < 2\alpha_1 + \alpha_2 < \alpha_2 < 3\alpha_1 + \alpha_2 < 3\alpha_1 + 2\alpha_2.$$

Hence, $|\mathcal{O}_r^{\langle r,s\rangle}| \geq 5$ and $\mathcal{O}$ is of type C, by [ACGIII, Lemma 2.8], with $H = \langle r, s\rangle$.

Assume now that $x = x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)$. Let $t_2 := \alpha_1^\vee(\zeta)\alpha_2^\vee(\zeta)$, $t_3 := \alpha_2^\vee(\zeta)$, $t_4 := \alpha_1^\vee(\zeta)\alpha_2^\vee(\zeta^2)$ and set

$$\begin{aligned} x_1 &= r_{\alpha_2} \triangleright x = x_{\alpha_1}(1)x_{2\alpha_1+\alpha_2}(1) \in \mathcal{O}_x; \\ x_2 &= t_2 \triangleright x_1 = x_{\alpha_1}(\zeta)x_{2\alpha_1+\alpha_2}(\zeta), \\ x_3 &= t_3 \triangleright x_1 = x_{\alpha_1}(\zeta^2)x_{2\alpha_1+\alpha_2}(1), \\ x_4 &= t_4 \triangleright x_1 = x_{\alpha_1}(1)x_{2\alpha_1+\alpha_2}(\zeta). \end{aligned}$$

Let $Y_i = \mathbb{U}^F \triangleright x_i$, $i \in \mathbb{I}_4$. A direct computation shows that

$$Y_1 = \bigcup_{f,\ell \in \mathbb{F}_4} x_{\alpha_1}(1) x_{\alpha_1+\alpha_2}(\ell) x_{2\alpha_1+\alpha_2}(\ell+1) x_{3\alpha_1+2\alpha_2}(f^2+f) \mathbb{U}^F_{3\alpha_1+\alpha_2},$$
$$Y_2 = \bigcup_{f,\ell \in \mathbb{F}_4} x_{\alpha_1}(\zeta) x_{\alpha_1+\alpha_2}(\ell\zeta) x_{2\alpha_1+\alpha_2}(\ell\zeta^2+\zeta) x_{3\alpha_1+2\alpha_2}(f^2\zeta+f\zeta) \mathbb{U}^F_{3\alpha_1+\alpha_2},$$
$$Y_3 = \bigcup_{f,\ell \in \mathbb{F}_4} x_{\alpha_1}(\zeta^2) x_{\alpha_1+\alpha_2}(\ell^2\zeta) x_{2\alpha_1+\alpha_2}(\ell\zeta+1) x_{3\alpha_1+2\alpha_2}(f^2\zeta^2+f) \mathbb{U}^F_{3\alpha_1+\alpha_2},$$
$$Y_4 = \bigcup_{f,\ell \in \mathbb{F}_4} x_{\alpha_1}(1) x_{\alpha_1+\alpha_2}(\ell) x_{2\alpha_1+\alpha_2}(\ell+\zeta) x_{3\alpha_1+2\alpha_2}(f^2+f\zeta) \mathbb{U}^F_{3\alpha_1+\alpha_2}.$$

The union $Y = \bigcup_{i \in \mathbb{I}_4} Y_i$ is disjoint and a subrack of $\mathcal{O}_x$. We take

$$\begin{aligned}
&r_1 = x_1, \\
&r_2 = x_{\alpha_1}(\zeta) x_{\alpha_1+\alpha_2}(\zeta) x_{2\alpha_1+\alpha_2}(1) \in \mathbb{U}^F \triangleright x_2, && (\ell = 1, f = 0), \\
&r_3 := x_{\alpha_1}(\zeta^2) x_{2\alpha_1+\alpha_2}(1) \in \mathbb{U}^F \triangleright x_3, && (\ell = f = 0), \\
&r_4 := x_{\alpha_1}(1) x_{\alpha_1+\alpha_2}(1) x_{2\alpha_1+\alpha_2}(\zeta^2) \in \mathbb{U}^F \triangleright x_4, && (\ell = 1, f = 0).
\end{aligned}$$

We claim that $x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(b)x_{2\alpha_1+\alpha_2}(c)$ and $x_{\alpha_1}(\widetilde{a})x_{\alpha_1+\alpha_2}(\widetilde{b})x_{2\alpha_1+\alpha_2}(\widetilde{c})$ do not commute, for $a, b, c, \widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathbb{F}_q$ such that $c\widetilde{a} + \widetilde{a}^2 b \neq \widetilde{c}a + a^2\widetilde{b}$. This follows from the formula:

$$x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(b)x_{2\alpha_1+\alpha_2}(c)x_{\alpha_1}(\widetilde{a})x_{\alpha_1+\alpha_2}(\widetilde{b})x_{2\alpha_1+\alpha_2}(\widetilde{c}) =$$
$$x_{\alpha_1}(a+\widetilde{a})x_{\alpha_1+\alpha_2}(b+\widetilde{b})x_{2\alpha_1+\alpha_2}(c+\widetilde{c})x_{3\alpha_1+\alpha_2}(c\widetilde{a}+\widetilde{a}^2b)x_{3\alpha_1+2\alpha_2}(b^2\widetilde{a}+c\widetilde{b}).$$

Hence, $r_i r_j \neq r_j r_i$ for $i \neq j$, $i, j \in \mathbb{I}_4$ and the class $\mathcal{O}_x$ is of type F. □

Lemma 4.12. *If $x = x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(c)$ with $c \neq 0$, then $\mathcal{O}^{\boldsymbol{G}}_x$ is not kthulhu.*

Proof. We show that this class is of type F. By [E], this class can be represented by any of

$$r_1 = x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(\zeta),$$
$$r_2 = r_{\alpha_1} \triangleright r_1 = x_{\alpha_2}(\zeta)x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(\zeta).$$

Let $t := \alpha_1^\vee(\zeta)\alpha_2^\vee(\zeta^2)$ and

$$x := t \triangleright r_1 = x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta^2),$$
$$y := t \triangleright r_2 = x_{\alpha_2}(\zeta^2)x_{\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+2\alpha_2}(1).$$

It is easier now to work with a different ordering of the positive roots:

$$\alpha_1 < \alpha_2 < 2\alpha_1 + \alpha_2 < \alpha_1 + \alpha_2 < 3\alpha_1 + \alpha_2 < 3\alpha_1 + 2\alpha_2.$$

Let $Y_i = \mathbb{U}^F \triangleright r_i$, $i = 1, 2$, $Y_3 = \mathbb{U}^F \triangleright x$, $Y_4 = \mathbb{U}^F \triangleright y$. A direct computation shows that

$$Y_1 = \bigcup_{\ell \in \mathbb{F}_4} x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1+\ell^2)x_{\alpha_1+\alpha_2}(\ell)x_{3\alpha_1+\alpha_2}(\zeta+\ell^3+\ell)\mathbb{U}^F_{3\alpha_1+2\alpha_2},$$

$$Y_2 = \bigcup_{\ell \in \mathbb{F}_4} x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(\ell^2\zeta)x_{\alpha_1+\alpha_2}(1+\ell\zeta)x_{3\alpha_1+\alpha_2}(\ell^2+\ell^3\zeta+1)\mathbb{U}^F_{3\alpha_1+2\alpha_2},$$

$$Y_3 = \bigcup_{\ell \in \mathbb{F}_4} x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(\zeta+\ell^2\zeta)x_{\alpha_1+\alpha_2}(\ell\zeta)x_{3\alpha_1+\alpha_2}(\ell^3\zeta+\ell\zeta+\zeta^2)\mathbb{U}^F_{3\alpha_1+2\alpha_2},$$

$$Y_4 = \bigcup_{\ell \in \mathbb{F}_4} x_{\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(\ell^2\zeta^2)x_{\alpha_1+\alpha_2}(\ell\zeta^2+\zeta)x_{3\alpha_1+\alpha_2}(\ell^3\zeta^2+\ell^2\zeta+\zeta)\mathbb{U}^F_{3\alpha_1+2\alpha_2}.$$

The union $Y = \bigcup_{i \in \mathbb{I}_4} Y_i$ is disjoint and a subrack of $\mathcal{O}$. We take

$$r_3 := x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1) \in Y_3, \qquad (\ell = \zeta^2),$$

$$r_4 := x_{\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(\zeta^2)x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(\zeta^2) \in Y_4, \qquad (\ell = 1).$$

By looking at the coefficient of $x_{3\alpha_1+2\alpha_2}$ in the expression of each product, we verify that $r_i \triangleright r_j \neq r_j \triangleright r_i$ if $i \neq j$, hence $\mathcal{O}$ is of type F. □

5. Unipotent classes in Steinberg groups

In this Section we deal with unipotent classes in Steinberg groups, i.e. $\mathbf{PSU}_n(q)$, $n \geq 3$; $\mathbf{P\Omega}^-_{2n}(q)$, $n \geq 4$; ${}^3D_4(q)$ and ${}^2E_6(q)$. In order to apply inductive arguments as in Section 4, we first need information about the unitary groups $\mathbf{PSU}_n(q)$, including the non-simple group $\mathbf{PSU}_3(2)$.

5.1. **Unipotent classes in unitary groups.** Here $\boldsymbol{G} = \mathbf{PSU}_n(q)$, $G = \mathbf{SU}_n(q)$, $n \geq 3$ and $\mathbb{G} = \mathbf{SL}_n(\Bbbk)$, for $n \geq 2$. For a clearer visibility of the behaviour of the conjugacy classes in small rank, we use the language of matrices and partitions. Here we choose $\mathbb{B}$, $\mathbb{U}$, as the subgroups of upper triangular, respectively unipotent upper triangular, matrices. We start by some notation and basic facts.

- ◇ $\mathtt{J}_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} = \mathtt{J}_n^{-1} \in \mathbf{GL}_n(\Bbbk)$.
- ◇ Fr_q is the Frobenius endomorphism of $\mathbf{GL}_n(\Bbbk)$ raising all entries of the matrix to the q-th power.
- ◇ $F : \mathbf{GL}_n(\Bbbk) \to \mathbf{GL}_n(\Bbbk)$, $F(X) = \mathtt{J}_n \, {}^t(\mathrm{Fr}_q(X))^{-1} \mathtt{J}_n$, $X \in \mathbf{GL}_n(\Bbbk)$.
- ◇ $\mathbf{GU}_n(q) = \mathbf{GL}_n(\Bbbk)^F$, $\mathbf{SU}_n(q) = \mathbf{SL}_n(\Bbbk)^F \leq \mathbf{SL}_n(q^2)$, [MT, 21.14(2), 23.10(2)].
- ◇ To every unipotent class in $\mathbf{SU}_n(q)$ we assign the partition of n corresponding to the class in $\mathbf{GL}_n(q)$ it is embedded into.
- ◇ Every unipotent class in $\mathbf{GL}_n(\Bbbk)$ meets $\mathbf{GU}_n(q)$ in exactly one class, since $C_{\mathbf{GL}_n(\Bbbk)}(x)$ is connected for every x [HuCC, 8.5], [SpSt, I.3.5]. In other words, every partition comes from a class in $\mathbf{SU}_n(q)$.

- ⋄ By construction, $\mathbf{SU}_n(q) \leq \mathbf{SU}_n(q^{1+2h})$ for every $h \geq 0$, so for any partition λ of n, there is a unipotent class in $\mathbf{SU}_n(q^{1+2h})$ corresponding to λ and represented by an element in $\mathbf{SU}_n(q)$.
- ⋄ Since $\mathbf{SU}_n(q)$ is normal in $\mathbf{GU}_n(q)$, [ACGI, Remark 2.1] says that all unipotent classes in $\mathbf{SU}_n(q)$ with the same partition are isomorphic as racks.
- ⋄ In the spirit of Subsection 3 from which we adopt notation, we consider the standard parabolic subgroups $\mathbb{P}^1$ associated with the simple roots $\Delta - \{\alpha_d, \alpha_{d+1}\}$ if $n = 2d+1$ and $\Delta - \{\alpha_d\}$ if $n = 2d$, so $M_1 \simeq \mathbf{SL}_d(q^2)$; and $\mathbb{P}^2$ associated with the simple roots $\Delta - \{\alpha_1, \alpha_{n-1}\}$, so $M_2 \simeq \mathbf{SU}_{n-2}(q)$. More precisely, M_1 and M_2, respectively, consists of matrices of diagonal block form $\operatorname{diag}(A, \mathrm{id}_{n-2d}, \mathtt{J}\,{}^tA\mathtt{J})$, for $A \in \mathbf{SL}_d(q^2)$ and $\operatorname{diag}(1, B, 1)$, for $B \in \mathbf{SU}_{n-2}(q)$. Observe that for $d > 2$ no unipotent class in M_1 is kthulhu.
- ⋄ We denote by $\mathbb{M} \leq \mathbb{G}$ the subgroup of matrices $\begin{pmatrix} A & & B \\ & \mathrm{id}_{n-2c} & \\ C & & D \end{pmatrix}$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{SL}_{2c}(\Bbbk)$, where $c = d-1$ if $n = 2d$ and $c = d$ if $n = 2d+1$. It is the semisimple part of an F-stable non-standard Levi subgroup. Let $M = \mathbb{M}^F$. We have: $M \simeq \mathbf{SU}_{n-2}(q)$ for n even and $M \simeq \mathbf{SU}_{n-1}(q)$ for n odd. The group $\mathbb{M}$ contains $\mathbb{T}$ and its root system has basis
$$\Delta \cup \{-(\alpha_1 + \cdots + \alpha_{n-1})\} - \{\alpha_{d-1}, \alpha_d, \alpha_{d+1}\} \text{ if } n = 2d,$$
$$\Delta \cup \{-(\alpha_1 + \cdots + \alpha_{n-1})\} - \{\alpha_d, \alpha_{d+1}\} \text{ if } n = 2d+1.$$
- ⋄ If q is odd, then $G^{\mathrm{Fr}_q} = \mathbf{SO}_n(q)$. If q and n are even, then $G^{\mathrm{Fr}_q} = \mathbf{Sp}_n(q)$.

Here is the main result of this Subsection:

Proposition 5.1. *Let $\mathcal{O} \neq \{e\}$ be a unipotent class in $\boldsymbol{G} = \mathbf{PSU}_n(q)$ with partition λ. Then, $\mathcal{O}$ is kthulhu if and only if $\lambda = (2, 1, \dots)$ and q is even.*

Proof. First, we reduce our analysis to $G = \mathbf{SU}_n(q)$ by the isogeny argument [ACGI, Lemma 1.2]. Thus, from now on $\mathcal{O}$ is a unipotent class in G. We first deal with the classes associated with the partition $(2, 1, \dots)$ when q is even.

Lemma 5.2. *If q is even and $\lambda = (2, 1^a)$ for $a \geq 1$, then $\mathcal{O}$ is austere, hence kthulhu.*

Proof. We show that any subrack generated by two elements is either abelian or indecomposable. Let $r, s \in \mathcal{O}$, $rs \overset{\star}{\neq} sr$. We may assume $r = \mathrm{id}_n + ae_{1,n} = x_\beta(a)$ where $\beta = \alpha_1 + \cdots + \alpha_{n-1}$, the highest positive root in Φ and $a \in \mathbb{F}_q^\times$. Let $g \in G$ be such that $s = grg^{-1}$. By [MT, 24.1] there are $u, v \in \mathbb{U}^F$, and $\sigma \in G \cap N(\mathbb{T})$ such that $g = u\sigma v$. As $F(\sigma) = \sigma$, the coset $\overline{\sigma} = \sigma\mathbb{T} \in W$ lies in $W^F \simeq \mathbb{S}_n^F$ which is the centralizer of the permutation
$$(1, n)(2, n-1) \cdots ([\frac{n}{2}], n+1-[\frac{n}{2}]);$$

hence, either $\overline{\sigma}(\{1,n\}) = \{1,n\}$ or $\overline{\sigma}(\{1,n\}) \cap \{1,n\} = \emptyset$. Since r is central in $\mathbb{U}^F$, $s = u\sigma r\sigma^{-1}u^{-1} = ux_{\overline{\sigma}(\beta)}(a')u^{-1}$ for some $a' \in \mathbb{F}_q$. Since $ru = ur$ and $rv = vr$, $\star$ holds if and only if $\overline{\sigma}\beta + \beta \in \Phi \cup \{0\}$. Thus, $\overline{\sigma}(1) = n$ and $\overline{\sigma}(n) = 1$, so σ is of the form

$$\sigma = \begin{pmatrix} 0 & 0 & \xi \\ 0 & A & 0 \\ \xi^{-q} & 0 & 0 \end{pmatrix}, \qquad \text{where } A \in \mathbf{GU}_{n-2}(q), \qquad \xi \in \mathbb{F}_q^\times,\ \xi^{q-1} = \det A.$$

Then $\sigma r \sigma^{-1} = \mathrm{id}_n + a\xi^{-1-q}e_{n,1}$, so

$$H := \langle r, s\rangle \simeq \langle \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} 1 & 0 \\ \xi^{-q-1}a & 1 \end{pmatrix}\rangle \subset \mathbf{SL}_2(q).$$

Since the non-trivial unipotent class in $\mathbf{SL}_2(q)$ is sober [ACGI, 3.5], $\mathcal{O}_r^H = \mathcal{O}_s^H$. □

We now prove Proposition 5.1 by induction in a series of Lemmata, dealing with the cases of $n = 3, 4, 5$ separately. The reader should be alert that sometimes we use formulas or matrices that are independent of the parity of q, with the understanding that -1 should be treated as 1 when q is even.

Lemma 5.3. *The statement of Proposition 5.1 holds for $n = 3$.*

Proof. There are, up to isomorphism, two nontrivial unipotent conjugacy classes, corresponding to the partitions (3) and (2, 1). If $\lambda = (3)$, then $\mathcal{O}$ is regular. For $q \neq 2, 8$ this situation is covered by Proposition 3.7. We show that regular unipotent classes in $\mathbf{SU}_3(2^{2h+1})$, $h \in \mathbb{N}_0$, are of type D. It suffices to prove the claim for $G = \mathbf{SU}_3(2)$. Let ω be a generator of $\mathbb{F}_4^\times$. Consider the class $\mathcal{O}$ represented by $r = \begin{pmatrix} 1 & 1 & \omega \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Let $t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \omega^2 \\ 1 & \omega & \omega \end{pmatrix} \in \mathbf{SU}_3(2)$ and $s := t \triangleright r = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \omega^2 & 1 & 1 \end{pmatrix} \in \mathcal{O}$. By direct verification, $(rs)^2 \neq (rs)^2$. A computation with GAP shows that $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$.

Let now $\lambda = (2, 1)$ for $q > 3$ odd. Let $r = \begin{pmatrix} 1 & 0 & a \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in \mathcal{O}$ with $a \in \mathbb{F}_{q^2}^\times$, $a^q = -a$. As $\mathbb{F}_q^\times = \{\xi^{q+1} | \xi \in \mathbb{F}_{q^2}^\times\}$ and $q > 3$, we may pick $\xi \in \mathbb{F}_{q^2}^\times$ such that $-a^2\xi^{q+1} \in \mathbb{F}_q^\times - (\{2\} \cup (\mathbb{F}_q^\times)^2)$. Let $t \in G$ be the diagonal matrix $(\xi, \xi^{q-1}, \xi^{-q})$, $\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in G$ and

$$s := (\sigma t) \triangleright r = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ a\xi^{1+q} & 0 & 1 \end{pmatrix} \in \mathcal{O}.$$

Since $2 \neq -a^2\xi^{q+1}$, $(rs)^2 \neq (sr)^2$. Let $\eta \in \Bbbk$ be such that $\eta^2 = a^{-1}$. Conjugating by the diagonal matrix (η, η^{-1}) we have

$$H := \langle r, s\rangle \simeq \left\langle \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} 1 & 0 \\ a\xi^{q+1} & 1 \end{pmatrix} \right\rangle \simeq \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} 1 & 0 \\ a^2\xi^{q+1} & 1 \end{pmatrix} \right\rangle.$$

By [Su, Theorem 6.21, page 409], $H \simeq \mathbf{SL}_2(q)$. Since $-a^2\xi^{q+1}$ is not a square, $\mathcal{O}_r^H \neq \mathcal{O}_s^H$. Thus $\mathcal{O}$ is of type D.

Finally, let $\lambda = (2,1)$ and $q = 3$. We will show that $\mathcal{O}$ is of type C. Let $\mathbb{F}_9^\times = \langle\zeta\rangle$. Without loss of generality we may assume that

$$r = \begin{pmatrix} 1 & & \zeta^2 \\ & 1 & \\ & & 1 \end{pmatrix} = \mathrm{id}_3 + \zeta^2 e_{1,3} \in \mathcal{O}.$$

We consider the following elements of $\mathbf{SU}_3(3)$:

$$\sigma := \begin{pmatrix} 0 & 0 & \zeta \\ 0 & -\zeta^2 & 0 \\ \zeta^{-3} & 0 & 0 \end{pmatrix},$$

$$s := \sigma \triangleright r = \begin{pmatrix} 1 & & \\ & 1 & \\ \zeta^{-2} & & 1 \end{pmatrix} = \mathrm{id}_3 + \zeta^{-2} e_{3,1} \in \mathcal{O}.$$

Then $rs \neq sr$. Let

$$H := \langle r, s\rangle \simeq \langle \begin{pmatrix} 1 & \zeta^2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \zeta^{-2} & 1 \end{pmatrix}\rangle.$$

Conjugation by $\mathrm{diag}(\zeta^{-1}, \zeta)$ and [Su, Theorem 6.21, page 409] give $H \simeq \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\rangle \simeq \mathbf{SL}_2(3)$, so $\mathcal{O}_r^H \neq \mathcal{O}_s^H$. We conclude by [ACGIII, Lemma 2.7]. □

Lemma 5.4. *The statement of Proposition 5.1 holds for $n = 4$.*

Proof. Here we need to consider the classes associated with the partitions (4), (3, 1), (2, 2) for all q and (2, 1, 1) for q odd. If $\lambda = (4)$, then $\mathcal{O}$ is regular so for $q \neq 2, 4$ it is covered by Proposition 3.7. For the remaining cases we observe that by the Jordan form theory, $\mathcal{O}$ is represented by an element of a regular class in $\mathbf{Sp}_4(q) = \mathbf{SU}_4(q)^{\mathrm{Fr}_q}$, so $\mathcal{O}$ is not kthulhu by Theorem 3.3.

Let $\lambda = (3,1)$ and $q > 3$ odd. Then, by the Jordan form theory, $\mathcal{O}$ has a representative which is regular in $\mathbf{SO}_4(q) = \mathbf{SU}_4(q)^{\mathrm{Fr}_q}$. Now $\mathbf{SO}_4(\Bbbk)$ is isogenous to $H = \mathbf{SL}_2(\Bbbk) \times \mathbf{SL}_2(\Bbbk)$ and the class $\mathcal{O}_u^H$ is isomorphic as a rack to the product $X \times X$ for X the non-trivial unipotent class in $\mathbf{SL}_2(q)$. By Lemma 2.1, $\mathcal{O}_u^H$ is of type D.

Let $\lambda = (3,1)$ and $q = 3$. We show that $\mathcal{O}$ is of type D. Let ζ be a generator of $\mathbb{F}_9^\times$. We may assume that $r := \begin{pmatrix} 1 & \zeta & \zeta & 1 \\ 0 & 1 & 0 & \zeta^7 \\ 0 & 0 & 1 & \zeta^7 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$. Let $t := \begin{pmatrix} 2 & \zeta^6 & \zeta^2 & \zeta \\ 2 & \zeta^5 & 0 & 0 \\ 0 & \zeta^2 & 2 & \zeta^5 \\ 0 & \zeta^6 & 1 & \zeta^7 \end{pmatrix} \in \mathbf{SU}_4(3)$ and $s = t \triangleright r = \left(\begin{smallmatrix} 0 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{smallmatrix}\right) \in \mathcal{O}$. A direct computation shows that $(rs)^2 \neq (sr)^2$. Clearly $H = \langle r, s\rangle \subset \{\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \mid A, D \in \mathbf{SL}_2(9)\}$. If $s \in \mathcal{O}_r^H$, then $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 2 \end{smallmatrix}\right)$ and $\begin{pmatrix} 1 & \zeta^7 \\ 0 & 1 \end{pmatrix}$ would be conjugate in $\mathbf{SL}_2(9)$. But $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \triangleright \left(\begin{smallmatrix} 0 & 1 \\ 2 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ which is not conjugate to $\begin{pmatrix} 1 & \zeta^7 \\ 0 & 1 \end{pmatrix}$ because ζ^7 is not a square. Hence, $\mathcal{O}_r^H \neq \mathcal{O}_s^H$ and $\mathcal{O}$ is of type D.

Let $\lambda = (3,1)$ and q even. Here, either $\mathbf{SU}_n(2) \leq \mathbf{SU}_n(q)$ or $\mathbf{SU}_n(4) \leq \mathbf{SU}_n(q)$, so it is enough to prove the statement for $q = 2, 4$.

Let $q = 2$ and let ω be a generator of $\mathbb{F}_4^\times$. We may assume that $r = \begin{pmatrix} 1 & 1 & \omega & \omega \\ 0 & 1 & 0 & \omega^2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$. Let $t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \in \mathbf{SU}_4(2)$ and $s := t \triangleright r = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \omega^2 & \omega & 1 & 1 \\ 0 & \omega & 0 & 1 \end{pmatrix}$. Then $(rs)^2 \neq (rs)^2$. By GAP we see that $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$ so $\mathcal{O}$ is of type D.

Let now $q = 4$ and let η be a generator of $\mathbb{F}_{16}^\times$. We may assume that $r = \begin{pmatrix} 1 & 1 & \eta & \eta \\ 0 & 1 & 0 & \eta^4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$. Let $t = \begin{pmatrix} \eta^{11} & \eta^2 & \eta^5 & \eta^{14} \\ \eta^{11} & \eta^2 & \eta^8 & \eta^2 \\ 0 & \eta^{14} & \eta^9 & \eta^{11} \\ 0 & \eta^{14} & \eta^9 & \eta^6 \end{pmatrix} \in \mathbf{SU}_4(4)$ and $s := t \triangleright r = \begin{pmatrix} 0 & 1 & 0 & \eta^{12} \\ 1 & 0 & \eta^3 & \eta^{10} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{O}$. We check at once that $(rs)^2 \neq (rs)^2$, and with GAP that $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$, so $\mathcal{O}$ is of type D.

Assume $q > 3$ is odd and $\lambda = (2,2)$ or $(2,1,1)$. We take $r = \begin{pmatrix} 1 & 0 & 0 & a \\ & 1 & b & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in \mathcal{O}$ for a, $b \in \mathbb{F}_{q^2}$ satisfying $a^q = -a$ and $b^q = -b$, $a \neq 0$ always and $b = 0$ if and only if $\lambda = (2,1,1)$. Let $t \in G$ be the diagonal matrix $(\xi, \xi^{-1}, \xi^q, \xi^{-q})$ for $\xi \in \mathbb{F}_{q^2}^\times$, such that $-a^2\xi^{q+1} \in \mathbb{F}_q^\times$ is not a square in $\mathbb{F}_q^\times$, and let $\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^{-q} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in G$ with $\zeta \in \mathbb{F}_{q^2}^\times$ such that $\zeta^q = -\zeta$. We consider

$$s := (\sigma t) \triangleright r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & b\zeta^{q+1}\xi^{-1-q} & 0 \\ 0 & 0 & 1 & 0 \\ a\xi^{1+q} & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}.$$

We observe that $H = \langle r, s\rangle$ is contained in the subgroup of matrices of the form $\begin{pmatrix} a_{11} & 0 & a_{12} \\ 0 & B & 0 \\ a_{21} & 0 & a_{22} \end{pmatrix}$, where $B \in \mathbf{SU}_2(q)$ and $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a\xi^{1+q} & 1 \end{pmatrix} \rangle$. We proceed then as we did for $n = 3$ and $\lambda = (2,1)$.

Assume now $q = 3$ and $\lambda = (2,2)$. We show that $\mathcal{O}$ is of type D. Let ζ be a generator of $\mathbb{F}_9^\times$. Let

$$r = \begin{pmatrix} 1 & \zeta & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -\zeta^3 \\ & & & 1 \end{pmatrix} \in \mathcal{O},\ \sigma = \begin{pmatrix} \zeta^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix} \in G,\ s = \sigma \triangleright r = \begin{pmatrix} 1 & 0 & \zeta^3 & 0 \\ & 1 & 0 & -\zeta \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

It is easy to see that $(rs)^2 \neq (sr)^2$. In addition, $\langle r, s\rangle \subset \mathbb{U}^F$ and $\mathcal{O}_r^{\mathbb{U}^F} \neq \mathcal{O}_s^{\mathbb{U}^F}$, whence the statement.

Let $q = 3$ and $\lambda = (2,1,1)$. We show that $\mathcal{O}$ is of type C. Let $\mathbb{F}_9^\times = \langle\zeta\rangle$. We take $r = \begin{pmatrix} 1 & 0 & \zeta^2 \\ & \mathrm{id}_2 & 0 \\ & & 1 \end{pmatrix} = \mathrm{id}_4 + \zeta^2 e_{1,4} \in \mathcal{O}$. We consider the following elements of $\mathbf{SU}_4(3)$:

$$\tau := \begin{pmatrix} 0 & 0 & 0 & \zeta \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^{-3} & 0 \\ \zeta^{-3} & 0 & 0 & 0 \end{pmatrix},$$

$$s := \tau \triangleright r = \begin{pmatrix} 1 & & \\ 0 & \mathrm{id}_2 & \\ \zeta^{-2} & 0 & 1 \end{pmatrix} = \mathrm{id}_4 + \zeta^{-2} e_{4,1} \in \mathcal{O}$$

and we proceed as we did for $q = 3$, $n = 3$, $\lambda = (2, 1)$.

Let $\lambda = (2, 2)$ for q even. Such a class is represented by an element u of a unipotent class with label $V(2) \oplus V(2)$ in $\mathbf{Sp}_4(q) = \mathbf{SU}_4(q)^{\mathrm{Fr}_q} \leq \mathbf{SU}_4(q)$. By Theorem 3.3, $\mathcal{O}_u^{\mathbf{Sp}_4(q)}$ is not kthulhu. □

The above Lemmata are the basis of our induction for n odd or even. However, we need to deal with $\mathbf{PSU}_5(q)$ separately because of the presence of kthulhu unipotent classes in M_1 for $n = 5$.

Lemma 5.5. *The statement of Proposition 5.1 holds for $n = 5$.*

Proof. In this case $M_2 \simeq \mathbf{SU}_3(q)$, $M \simeq \mathbf{SU}_4(q)$ and the partitions to be considered are (5), $(4, 1)$, $(3, 2)$, $(3, 1, 1)$, $(2, 2, 1)$ and $(2, 1, 1, 1)$. The regular class, corresponding to (5), is represented by $u \in \mathbb{U}^F \subset P_2$ with $u = \prod_{\alpha \in \Phi^+} x_\alpha(\xi_\alpha)$ in some order, with $\xi_\alpha \neq 0$ for $\alpha \in \Delta$. Thus, $\pi_2(u)$ is regular in M_2 and the statement follows from Lemma 5.3. If $\lambda = (4, 1)$, then $\mathcal{O}$ is represented by an element of the form $u = \begin{pmatrix} 1 & a & 0 & g & f \\ & 1 & 0 & e & h \\ & & 1 & 0 & 0 \\ & & & 1 & b \\ & & & & 1 \end{pmatrix}$ for a, b, e, f, g, $h \in \mathbb{F}_{q^2}$, $a^q = -b$, $b^q = -a$, $e^q = -e$, $g^q = eb - g$, $h^q = ae - h$ and $f^q = -f - abe + bh + ga$ and $abe \neq 0$. Then $u \in M$ and it corresponds to the partition (4) therein. Hence, $\mathcal{O}$ is not kthulhu.

If $\lambda = (3, 2)$ or $(3, 1, 1)$, then $\mathcal{O}$ is represented by an element of the form $u = \begin{pmatrix} 1 & 0 & 0 & 0 & f \\ & 1 & a & e & 0 \\ & & 1 & b & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix} \in \mathbb{U}^F \subset P_2$ for a, b, e, $f \in \mathbb{F}_{q^2}$, $a^q = -b$, $b^q = -a$, $(ba - e)^q = e$, $f^q = -f$, $ba \neq 0$ and $f \neq 0$ for $(3, 2)$ and $f = 0$ for $(3, 1, 1)$. Then $\pi_2(u)$ corresponds to the partition (3) for $\mathbf{SU}_3(q)$ and we have the statement.

If $\lambda = (2, 2, 1)$ or $(2, 1, 1, 1)$, then $\mathcal{O}$ is represented by an element of the form $u = \begin{pmatrix} 1 & 0 & 0 & 0 & f \\ & 1 & 0 & e & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$ for f, $e \in \mathbb{F}_{q^2}$, $e^q = -e$, $f^q = -f$, $e \neq 0$ always and $f \neq 0$ for $(2, 2, 1)$ whereas $f = 0$ for $(2, 1, 1, 1)$. Then $u \in M$ and it corresponds to the partition $(2, 2)$ when $f \neq 0$ and $(2, 1, 1)$ when $f = 0$. Hence, it is not kthulhu unless $\lambda = (2, 1, 1, 1)$ and q is even. This case is covered by Lemma 5.2. □

Proof of Proposition 5.1. Assume first that q is odd. We show by induction on $n \geq 3$ that no nontrivial class is kthulhu, the basis of the induction being Lemmata 5.3 and 5.4. Observe that $n = 5$ is dealt with in Lemma 5.5, so here $n \geq 6$ and by induction no nontrivial class in $M_1 \simeq \mathbf{SL}_d(q^2)$ and $M_2 \simeq \mathbf{SU}_{n-2}(q)$ is kthulhu. By Lemma 3.2, it is enough to show that $\mathcal{O} \cap \mathbb{U}^F \not\subset V_1 \cap V_2$. Assume $u \in V_1 \cap V_2 \cap \mathcal{O}$. With notation as in (4.1),

$$\operatorname{supp} u \subset (\Psi(\alpha_1) \cup \Psi(\alpha_{n-1})) \cap \Psi(\alpha_d) \qquad \text{if } n = 2d,$$
$$\operatorname{supp} u \subset (\Psi(\alpha_1) \cup \Psi(\alpha_{n-1})) \cap ((\Psi(\alpha_d) \cup \Psi(\alpha_{d+1})) \qquad \text{if } n = 2d+1.$$

For any representative σ of $s_1 s_{n-1} \in N_{\mathbb{G}^F}(\mathbb{T})$, we have $v = \sigma \triangleright u \in \mathbb{V}_1 \cap \mathcal{O} \subset \mathbb{U}^F \cap \mathcal{O}$. Also, the only possible roots in $\operatorname{supp} u$ that can be mapped by $s_1 s_{n-1}$ to $\Psi(\alpha_1) \cup \Psi(\alpha_{n-1})$ are $\alpha_1 + \cdots + \alpha_{n-2}$ and $\alpha_2 + \cdots + \alpha_{n-1}$. Therefore, either $v \notin V_1 \cap V_2$, so we are done, or else or $\operatorname{supp} u \subset \{\alpha_1 + \cdots + \alpha_{n-2}, \alpha_2 + \cdots + \alpha_{n-1}\}$. In this case invariance of u with respect to F forces $\operatorname{supp} u = \{\alpha_1 + \cdots + \alpha_{n-2}, \alpha_2 + \cdots + \alpha_{n-1}\}$. Thus, $u \in M$ and it is nontrivial therein. If n is even, then $M \simeq \mathbf{SU}_{n-2}(q)$ and the proof follows by induction for all even $n \geq 4$. If n is odd, then $M \simeq \mathbf{SU}_{n-1}(q)$ and the proof follows by induction and the case of n even.

Assume now that q is even. The proof is once more by induction on $n \geq 3$, where the cases $n = 3, 4, 5$ have been settled, so we assume $n \geq 6$. Thus, no nontrivial class in $M_1 \simeq \mathbf{SL}_d(q^2)$ is kthulhu and the only kthulhu classes in M_2 and M are those corresponding to the partition $(2, 1, 1, ...)$. Let $u \in \mathbb{U}^F \cap \mathcal{O}$ and assume $\mathcal{O}$ is kthulhu. Then, $u \in V_1 \subset \mathbb{U}^F \subset P^2$ by Lemma 3.2. The projection π_2 corresponds to removing the first and last rows and columns and $\mathcal{O}^{M_2}_{\pi_2(u)}$ is kthulhu because $\mathcal{O}$ is so. Therefore u is conjugate by an element of M_2 to some $u' = \mathrm{id}_n + \xi e_{2,n-1} + \sum_{j=d+1}^{n-1} \xi_j e_{1,j} + \sum_{l=2}^{n-d} \zeta_l e_{l,n} + \zeta e_{1,n}$, for suitable ξ, $\xi_j, \zeta_l, \zeta \in \mathbb{F}_{q^2}$. For any representative σ of $s_1 s_{n-1} \in N_{\mathbb{G}^F}(\mathbb{T})$, we have

$$\begin{aligned} v &= \sigma \triangleright u' \\ &= \mathrm{id}_n + \xi' e_{1,n} + \sum_{j=d+1}^{n-2} \xi'_j e_{2,j} + \xi'_{n-1} e_{2,n} + \zeta'_2 e_{1,n-1} + \sum_{l=3}^{n-d} \zeta'_l e_{l,n-1} + \zeta' e_{2,n-1} \end{aligned}$$

for $\xi', \zeta', \xi'_j, \zeta'_l \in \mathbb{F}_{q^2}$. Now, $v \in \mathcal{O} \cap \mathbb{U}^F \subset P^2$ and $\pi_2(v)$ represents a kthulhu class in M_2 only if $\xi'_j = 0$ for $d+1 \leq j \leq n-2$ and $\zeta'_l = 0$ for $3 \leq l \leq n-d$, so $v = \mathrm{id}_n + \xi' e_{1,n} + \xi'_{n-1} e_{2,n} + \zeta'_2 e_{1,n-1} + \zeta' e_{2,n-1} \in M$. By induction, $\mathcal{O}^M_v$ is kthulhu only if $\xi'\zeta' = 0$, $\zeta'_2 = 0$, $\xi'_{n-1} = 0$, i.e., only if the partition associated with $\mathcal{O}$ is $(2, 1, \ldots)$. □

5.2. **Unipotent classes in $\mathbf{P\Omega}^-_{2n}(q)$, $n \geq 4$.** In this subsection $\boldsymbol{G} = \mathbf{P\Omega}^-_{2n}(q)$, $n \geq 4$. We shall use the knowledge of unipotent conjugacy classes in $\mathbf{PSL}_n(q)$ and $\mathbf{PSU}_n(q)$ and apply inductive arguments.

Here $\mathbb{G}$ is assumed simply connected. The root system of $\mathbb{G}$ is of type D_n, and the Dynkin diagram automorphism ϑ interchanges α_{n-1} and α_n;

it fixes the basis vectors ε_j for $j \in \mathbb{I}_{n-1}$, and maps ε_n to $-\varepsilon_n$. Here is the main result of this Subsection:

Proposition 5.6. *Let $\mathcal{O}$ be a non-trivial unipotent class in $\mathbf{P\Omega}^-_{2n}(q)$ with $n \geq 4$. Then $\mathcal{O}$ is not kthulhu.*

We split the proof for q odd in §5.2.1 and for q even in §5.2.2.

5.2.1. *Proof of Proposition 5.6 when q is odd.*

Proof. Let $\mathbb{P}^1$ and $\mathbb{P}^2$ be the standard F-stable parabolic subgroups with F-stable Levi factors $\mathbb{L}_1$ and $\mathbb{L}_2$ associated respectively with $\Pi_1 := \Delta - \{\alpha_{n-1}, \alpha_n\}$ (of type A_{n-2}), and $\Pi_2 := \{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$ (of type A_3). Then

$$\Phi^+_{\Pi_1} = \{\varepsilon_i - \varepsilon_j \mid i < j \in \mathbb{I}_{n-1}\},\ \Phi^+_{\Pi_2} = \{\varepsilon_i \pm \varepsilon_j \mid i < j \in \mathbb{I}_{n-2,n}\},$$
$$\Psi_{\Pi_1} = \Phi^+ \smallsetminus \Phi^+_{\Pi_1} = \{\varepsilon_i + \varepsilon_j, \varepsilon_k - \varepsilon_n \mid i < j \in \mathbb{I}_n,\ k \in \mathbb{I}_{n-1}\},$$
$$\Psi_{\Pi_2} = \{\varepsilon_i \pm \varepsilon_j \mid i < j,\ i \in \mathbb{I}_{n-3}, j \in \mathbb{I}_n\}.$$

By Lemma 3.2, Theorem 3.3 and Proposition 5.1, it is enough to show that $\mathcal{O} \cap \mathbb{U}^F \not\subset V_1 \cap V_2$. Assume that there is $u \in \mathcal{O} \cap V_1 \cap V_2$; then

$$\operatorname{supp} u \subset \Psi_{\Pi_1} \cap \Psi_{\Pi_2} = \{\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_n \mid i < j,\ i \in \mathbb{I}_{n-3}, j \in \mathbb{I}_n\}.$$

We consider various different cases.

(i) $\varepsilon_i - \varepsilon_n \in \operatorname{supp} u$ for some $i \in \mathbb{I}_{n-3}$.

Then $s_{\varepsilon_i - \varepsilon_{n-2}}(\operatorname{supp} u) \subseteq \Phi^+$. Since $s_{\varepsilon_i - \varepsilon_{n-2}} \in W^F$, it has a representative $\dot{s}_{\varepsilon_i - \varepsilon_{n-2}} \in N_{\mathbb{G}^F}(\mathbb{T})$; hence $\dot{s}_{\varepsilon_i - \varepsilon_{n-2}} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and $\varepsilon_{n-2} - \varepsilon_n \in \operatorname{supp}(\dot{s}_{\varepsilon_i - \varepsilon_{n-2}} \triangleright u)$. Thus $\dot{s}_{\varepsilon_i - \varepsilon_{n-2}} \triangleright u \in \mathbb{U}^{\mathbb{F}} \cap \mathcal{O} - V_2$.

(ii) $\varepsilon_i - \varepsilon_n \notin \operatorname{supp} u$ for all $i \in \mathbb{I}_{n-3}$.

Then there exist $k \in \mathbb{I}_{n-3}$ and j such that $\varepsilon_k + \varepsilon_j \in \operatorname{supp} u$. Let

$$\ell = \max\{j \mid\ \varepsilon_k + \varepsilon_j \in \operatorname{supp} u \text{ for some } k\}.$$

If $\ell = n$, then pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T}) \cap \mathbb{L}_2$ of $s_{\varepsilon_{n-1} - \varepsilon_n} s_{\varepsilon_{n-1} + \varepsilon_n} \in W^F$. Thus $\sigma \triangleright u \in \mathcal{O} \cap V_2$ and $\varepsilon_k - \varepsilon_n \in \operatorname{supp}(\sigma \triangleright u)$ for all k such that $\varepsilon_k + \varepsilon_n \in \operatorname{supp} u$. Therefore, either $\operatorname{supp}(\sigma \triangleright u) \not\subset V_1$, and we are done, or $\operatorname{supp}(\sigma \triangleright u) \subset V_1$ and $\varepsilon_k - \varepsilon_n \in \operatorname{supp}(\sigma \triangleright u)$, and we fall in (i).

If $\ell = n-1$, then pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T}) \cap \mathbb{L}_2$ of $s_{\varepsilon_{n-2} + \varepsilon_{n-1}} \in W^F$. As above, $\sigma \triangleright u \in \mathcal{O} \cap V_2$, and $\varepsilon_i - \varepsilon_{n-2} \in \operatorname{supp}(\sigma \triangleright u) \cap \Phi_{\Pi_1}$ for some $i < n-2$. That is, $\operatorname{supp}(\sigma \triangleright u) \not\subset V_1$.

Finally, if $\ell < n-1$, then we pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T})$ of $s_{\varepsilon_\ell - \varepsilon_{n-1}}$. Then $\operatorname{supp} \sigma \triangleright u \subset V_1 \cap V_2$, and we fall in the case $\ell = n-1$. $\square$

5.2.2. *Proof of Proposition 5.6, q even.* Here Lemma 3.2 does not apply in its full strength because of the existence of kthulhu classes in $\mathbf{PSU}_4(q)$, q even, and in $\mathbf{PSL}_3(2)$. We proceed by induction on n. The case $n = 4$, Lemma 5.7 below, requires a special treatment.

Lemma 5.7. *If $\boldsymbol{G} = \mathbf{P\Omega}_8^-(q)$ with q even, then $\mathcal{O}$ is not kthulhu.*

Proof. Let us consider the F-stable standard parabolic subgroups $\mathbb{P}^1$, $\mathbb{P}^2$ with standard Levi subgroups $\mathbb{L}_1$ and $\mathbb{L}_2$ associated with the sets $\Pi_1 = \{\alpha_1, \alpha_2\}$ and $\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4\}$, respectively. Let $u \in \mathcal{O} \cap \mathbb{U}^F$. We analyse different situations, according to $\Delta \cap \operatorname{supp} u$. Recall that, u being F-invariant, the simple root $\alpha_3 \in \operatorname{supp} u$ if and only if $\alpha_4 \in \operatorname{supp} u$.

(i) $\alpha_2, \alpha_3, \alpha_4 \in \operatorname{supp} u$.

The projection $\pi_2(u) \in L_2$ is regular, thus $\mathcal{O}^{M_2}_{\pi_2(u)}$ is isomorphic as a rack to a unipotent class in $\mathbf{SU}_4(q)$ of partition (4) and Proposition 5.1 applies.

(ii) $\Delta \cap \operatorname{supp} u = \{\alpha_1, \alpha_3, \alpha_4\}$ or $\Delta \cap \operatorname{supp} u = \{\alpha_3, \alpha_4\}$.

Then $\mathcal{O}^{M_2}_{\pi_2(u)}$ has partition $(2, 2)$ or $(3, 1)$ and Proposition 5.1 applies.

(iii) $\Delta \cap \operatorname{supp} u = \{\alpha_1\}$ or $\Delta \cap \operatorname{supp} u = \{\alpha_2\}$.

Here $\pi_1(u) \in L_1$ is not regular, hence $\mathcal{O}^{M_1}_{\pi_1(u)}$ is isomorphic as a rack to a unipotent class in $\mathbf{SL}_3(q)$ with partition $\neq (3)$; Theorem 3.3 applies.

(iv) $\Delta \cap \operatorname{supp} u = \{\alpha_1, \alpha_2\}$: either $\alpha_2 + \alpha_3 + \alpha_4 \in \operatorname{supp} u$ or not.

We may assume that $\alpha_2 + \alpha_3 \notin \operatorname{supp} u$, by conjugating with a suitable element in $(\mathbb{U}_{\alpha_3}\mathbb{U}_{\alpha_4})^F$ and using (2.1). If $\alpha_2 + \alpha_3 + \alpha_4 \in \operatorname{supp} u$, then $\mathcal{O}^{M_2}_{\pi_2(u)} \simeq \mathcal{O}^{\mathbf{SU}_4(q)}_v$, where $\operatorname{rk}(v - \mathrm{id}) = 2$ and $(v - \mathrm{id})^2 = 0$, which is not kthulhu since its partition is $(2, 2)$. If $\alpha_2 + \alpha_3 + \alpha_4 \notin \operatorname{supp} u$, then pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T})$ of $s_3 s_4 \in W$. Then $\sigma \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and $\Delta \cap \operatorname{supp}(\sigma \triangleright u) = \{\alpha_1\}$ so we reduce to (iii).

(v) $\Delta \cap \operatorname{supp} u = \emptyset$ and $\alpha_1 + \alpha_2 \in \operatorname{supp} u$ or $\alpha_2 + \alpha_3 \in \operatorname{supp} u$.

In the first case, $\mathcal{O}^{M_1}_{\pi_1(u)}$ has type $(2, 1)$, and Theorem 3.3 applies. In the second, also $\alpha_2 + \alpha_4 \in \operatorname{supp} u$ and $\mathcal{O}^{M_2}_{\pi_2(u)}$ has type $(2, 2)$. Indeed, $\mathcal{O}^{M_2}_{\pi_2(u)} \simeq \mathcal{O}^{\mathbf{SU}_4(q)}_v$, where $\operatorname{rk}(v - \mathrm{id}) = 2$ and $(v - \mathrm{id})^2 = 0$. We invoke Proposition 5.1.

(vi) $(\Delta \cup \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\}) \cap \operatorname{supp} u = \emptyset$.

Let $\dot{s}_i \in N_{\mathbb{G}^F}(\mathbb{T})$ be a representative of s_i, $i = 1, 2$. If $\alpha_1 + \alpha_2 + \alpha_3 \in \operatorname{supp} u$, then also $\alpha_1 + \alpha_2 + \alpha_4 \in \operatorname{supp} u$. Now $\dot{s}_1 \triangleright u \in \mathbb{U}^F \cap \mathcal{O}$, $\Delta \cap \operatorname{supp}(\dot{s}_1 \triangleright u) = \emptyset$ and $\alpha_2 + \alpha_3 \in \operatorname{supp}(\dot{s}_1 \triangleright u)$, so we fall in (v). Let σ be as in (iv). If $\alpha_2 + \alpha_3 + \alpha_4 \in \operatorname{supp} u$, then $\sigma \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and $\alpha_2 \in \operatorname{supp}(\sigma \triangleright u)$ and we are in case (iii).

(vii) $\operatorname{supp} u \subset \{\alpha_1+\alpha_2+\alpha_3+\alpha_4, \alpha_1+2\alpha_2+\alpha_3+\alpha_4\}$.

If $\alpha_1+\alpha_2+\alpha_3+\alpha_4 \in \operatorname{supp} u$, then $\dot{s}_1 \triangleright u$ is as in case (vi); while if $\operatorname{supp} u = \{\alpha_1+2\alpha_2+\alpha_3+\alpha_4\}$, then $\operatorname{supp}(\dot{s}_2 \triangleright u) = \{\alpha_1+\alpha_2+\alpha_3+\alpha_4\}$. □

We now proceed with the recursive step and assume that all non-trivial unipotent classes in a twisted group with root system D_{n-1} are not kthulhu.

Let $\mathbb{P}^1$ and $\mathbb{P}^2$ be the standard parabolic subgroups with F-stable standard Levi subgroups $\mathbb{L}_1$ and $\mathbb{L}_2$ associated with the sets $\Pi_1 = \{\alpha_i \mid i \in \mathbb{I}_{n-2}\}$ and $\Pi_2 = \{\alpha_i \mid i \in \mathbb{I}_{2,n}\}$, of type A_{n-2} and D_{n-1} respectively. By Lemma 3.2 in order to prove the inductive step, it is enough to show that no non-trivial unipotent class $\mathcal{O}$ in $\mathbb{G}^F$ satisfies $\mathcal{O} \cap \mathbb{U}^F \subset V_1 \cap V_2$. As usual let

$$\Phi_{\Pi_1} = \{\varepsilon_i - \varepsilon_j \mid i < j \in \mathbb{I}_{n-1}\}, \qquad \Phi_{\Pi_2} = \{\varepsilon_i \pm \varepsilon_j \mid i < j \in \mathbb{I}_{2,n}\},$$
$$\Psi_{\Pi_1} = \{\varepsilon_i - \varepsilon_n,\, \varepsilon_j + \varepsilon_k \mid i \in \mathbb{I}_{n-1}, j < k \in \mathbb{I}_n\}, \; \Psi_{\Pi_2} = \{\varepsilon_1 \pm \varepsilon_j \mid j \in \mathbb{I}_{2,n}\}.$$

Let $u \in \mathcal{O} \cap V_1 \cap V_2$. Then $\operatorname{supp} u \subset \Psi_{\Pi_1} \cap \Psi_{\Pi_2} = \{\varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_j \mid j \in \mathbb{I}_{2,n}\}$. Let $\dot{s}_i \in N_{\mathbb{G}^F}(\mathbb{T})$ be a representative of $s_i \in W^F$, $i = 1, 2$. If $\operatorname{supp} u \neq \{\varepsilon_1 + \varepsilon_2\}$ then $\dot{s}_1 \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$, but $\dot{s}_1 \triangleright u \notin V_2$ (look at its support). If, instead, $\operatorname{supp} u = \{\varepsilon_1 + \varepsilon_2\}$ then $\dot{s}_1\dot{s}_2 \triangleright u \in \mathcal{O} \cap \mathbb{U}^F \cap \mathbb{U}_{\varepsilon_2+\varepsilon_3}$, so $\dot{s}_1\dot{s}_2 \triangleright u \notin V_2$.

This finishes the proof for q even and Proposition 5.6 is now proved. □

5.3. **Unipotent classes in ${}^2E_6(q)$.** We deal now with the group ${}^2E_6(q)$. Here the Dynkin diagram automorphism ϑ interchanges α_1 with α_6 and α_3 with α_5. Here is the main result of this Subsection:

Proposition 5.8. *Let $\mathcal{O} \neq \{e\}$ be a unipotent class in ${}^2E_6(q)$. Then $\mathcal{O}$ is not kthulhu.*

We give the proof for q odd in §5.3.1 and for q even in §5.3.2. Let $\mathbb{P}^1$ and $\mathbb{P}^2$ be the F-stable standard parabolic subgroups with standard Levi subgroups $\mathbb{L}_1$ and $\mathbb{L}_2$ associated with $\Pi_1 = \Delta - \{\alpha_2\}$ (of type A_5) and $\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ (of type D_4). Then Ψ_{Π_1}, respectively Ψ_{Π_2}, consists of all positive roots containing α_2, respectively at least one of α_1 and α_6.

5.3.1. *Proof of Proposition 5.8, q odd.* Here Lemma 3.2 (c) applies directly to the F-stable parabolic subgroups.

Proof. By Lemma 3.2, Propositions 5.1 and 5.6, it is enough to show that $\mathcal{O} \cap \mathbb{U}^F \not\subset V_1 \cap V_2$. Let $\beta = \sum_{i=1}^4 \alpha_i$, $\gamma = \sum_{i=1}^6 \alpha_i$; thus $\vartheta(\beta) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$. Let $u \in \mathcal{O} \cap \mathbb{U}^F$ lying in $V_1 \cap V_2$. Then

$$\operatorname{supp} u \subset \Psi_{\Pi_1} \cap \Psi_{\Pi_2} = \Psi(\beta) \cup \Psi(\vartheta(\beta)) = \Sigma \cup \vartheta(\Sigma) \cup \Psi(\gamma);$$

here $\Sigma = \{\beta_j \mid j \in \mathbb{I}_{0,3}\}$ and $\Psi(\gamma) = \{\gamma_j \mid j \in \mathbb{I}_{0,6}\}$, where

$$\begin{aligned} &\beta_0 = \beta, && \beta_1 = s_5\beta_0; && \beta_2 = s_4\beta_1; && \beta_3 = s_3\beta_2;\\ &\gamma_0 = \gamma, && \gamma_1 = s_4\gamma_0; && \gamma_2 = s_3\gamma_1; && \gamma_3 = s_5\gamma_1;\\ &\gamma_4 = s_5\gamma_2 = s_3\gamma_3; && \gamma_5 = s_4\gamma_4; && \gamma_6 = s_2\gamma_5. \end{aligned}$$

Let $\dot{w} \in N_{\mathbb{G}^F}(\mathbb{T})$ be a representative of $w \in W^F$. If either β_j or $\vartheta(\beta_j) \in \operatorname{supp} u$ for $j \in \mathbb{I}_{0,3}$, then $\dot{w_j} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F - V_1$, where $w_0 = w_1 = s_2$, $w_2 = s_2s_4$, $w_3 = s_2s_4s_5s_3$. Thus we may assume that $\operatorname{supp} u \subset \Psi(\gamma)$.

If $\gamma_0 \in \operatorname{supp} u$, then $\dot{s}_2 \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$, $\gamma - \alpha_2 \in \operatorname{supp}(\dot{s}_2 \triangleright u) - \Psi_{\Pi_1}$. Now we argue inductively. Suppose that $\gamma_i \in \operatorname{supp} u$ for some $i \in \mathbb{I}_{0,j-1}$ implies that $\mathcal{O}$ is not kthulhu. Assume that $\gamma_i \notin \operatorname{supp} u$ for $i \in \mathbb{I}_{0,j-1}$ and $\gamma_j \in \operatorname{supp} u$. We claim that there is $\omega_j \in W^F$ with $\dot{\omega}_j \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and either $\operatorname{supp}(\dot{\omega}_j \triangleright u) \not\subset \Psi(\gamma)$ (a case settled above), or $\gamma_l \in \operatorname{supp}(\dot{\omega}_j \triangleright u)$ for some $l \in \mathbb{I}_{0,j-1}$, where the recursive hypothesis applies. The claim holds, taking $\omega_1 = \omega_5 = s_4$, $\omega_2 = \omega_3 = s_1s_6$, $\omega_4 = s_3s_5$, $w_6 = s_2$. □

5.3.2. *Proof of Proposition 5.8, q even.* Here, the use of Lemma 3.2 is hampered by the presence of kthulhu classes in $\mathbf{PSU}_6(q)$.

Proof. As we have shown in the odd case, §5.3.1, there is $u \in \mathcal{O} \cap \mathbb{U}^F$ such that $u \notin V_1 \cap V_2$. If $u \notin V_2$, then the result follows from Proposition 5.6. Let us assume that $u \in V_2 - V_1$. In particular, α_3, α_4, α_5, $\alpha_3 + \alpha_4$, $\alpha_4 + \alpha_5$, $\alpha_3 + \alpha_4 + \alpha_5 \notin \operatorname{supp} u$. Then $\mathcal{O}^{M_1}_{\pi_1(u)}$ is non-trivial.

If $\operatorname{supp} u \cap \Phi_{\Pi_1} \neq \{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$, then $\mathcal{O}^{M_1}_{\pi_1(u)} \simeq \mathcal{O}^{\mathbf{SU}_6(q)}_v$ where $\operatorname{rk}(v - \mathrm{id}) = 2$, hence its associated partition is not $(2, 1, 1, 1, 1)$. By Lemma 5.1, $\mathcal{O}^{M_1}_v$ and $\mathcal{O}$ are not kthulhu.

If $\operatorname{supp} u \cap \Phi_{\Pi_1} = \{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\}$, then $\dot{w} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ where $w = s_1s_6 \in W^F$. But $\alpha_3 + \alpha_4 + \alpha_5 \in \operatorname{supp}(\dot{w} \triangleright u)$, hence $\dot{w} \triangleright u \notin V_2$ and we are done. □

5.4. **Unipotent classes in $^3D_4(q)$.** We deal now with triality; F arises from the graph automorphism ϑ of order 3 determined by $\vartheta(\alpha_1) = \alpha_3$. We assume that $\mathbb{G} = \mathbb{G}_{\mathrm{sc}}$. We fix and ordering of the ϑ-orbits in Φ^+. Let

$$y_\alpha(\xi) := x_\alpha(\xi)x_{\vartheta\alpha}(\xi^q)x_{\vartheta^2\alpha}(\xi^{q^2}), \qquad \alpha \in \Phi, \vartheta(\alpha) \neq \alpha, \qquad \xi \in \mathbb{F}_{q^3}.$$

Every element in $\mathbb{U}^F$ can be uniquely written as a product of elements $y_\alpha(\xi)$, $\vartheta\alpha \neq \alpha$, $\xi \in \mathbb{F}_{q^3}$, and $x_\beta(\zeta)$, $\vartheta\beta = \beta$, $\zeta \in \mathbb{F}_q$. Let

$$\Upsilon = \langle x_{\pm\gamma}(\xi), y_{\pm\delta}(\xi) \mid \vartheta(\gamma) = \gamma, \vartheta(\delta) \neq \delta, \xi \in \mathbb{F}_q^\times\rangle \leq \mathbb{G}^F. \tag{5.1}$$

The generators in (5.1) are the non-trivial elements in the root subgroups with respect to $\mathbb{T}^F \cap \mathbb{T}^{\mathrm{Fr}_q}$. It is known that $\Upsilon \simeq G_2(q) \simeq G^{\mathrm{Fr}_q}$.

Proposition 5.9. *Every unipotent class $\mathcal{O} \neq \{e\}$ in ${}^3D_4(q)$ is not kthulhu.*

Proof. By the isogeny argument we work in $G = \mathbb{G}_{sc}^F$ [ACGI, Lemma 1.2]. We analyse different cases separately, according to q being odd, even and > 2, or 2.

(i) *q is odd.*

A list of representatives of the unipotent classes in ${}^3D_4(q)$ appears in [Ge, Table 3.1]; they all have one of the following forms:

$$\begin{aligned} &x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1), && x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(-1), && u = x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(\zeta), \\ &y_{\alpha_1+\alpha_2+\alpha_3}(1), && y_{\alpha_1}(1)x_{\alpha_2}(1), && r = y_{\alpha_1}(1)y_{\alpha_1+\alpha_2}(a), \end{aligned}$$

where $\zeta \in \mathbb{F}_{q^3}$ is not a square and $a \in \mathbb{F}_{q^3} - \mathbb{F}_q$. So all classes but those of u and r have a representative in $\Upsilon \simeq G_2(q)$, hence they are not kthulhu by Lemmata 4.6 and 4.7. Now $u \in H = \langle \mathbb{U}_{\pm\alpha_2}^F, y_{\pm(\alpha_1+\alpha_2+\alpha_3)}(b) \mid b \in \mathbb{F}_{q^3}^\times \rangle$, which is isogeneous to $\mathbf{SL}_2(q) \times \mathbf{SL}_2(q^3)$. Since $\mathcal{O}_u^H$ is the product of two non-trivial racks and $q^3 > 3$, $\mathcal{O}_u^H$ is of type D by Lemmata 2.1 and 3.6.

Assume that $r \in \mathcal{O}$. Let ξ be a generator of $\mathbb{F}_{q^3}^\times$,

$$\eta = \xi^{q-1}, \quad t = \alpha_1^\vee(\eta)\alpha_3^\vee(\eta^q)\alpha_4^\vee(\eta^{q^2}), \quad s = t \triangleright r = y_{\alpha_1}(\eta^2)y_{\alpha_1+\alpha_2}(a\eta^2).$$

By [Ge, Table 3.2], for every $b, c \in \mathbb{F}_{q^3}^\times$ we have

$$\begin{aligned} y_{\alpha_1+\alpha_2}(b)y_{\alpha_1}(c) = y_{\alpha_1}(c)&y_{\alpha_1+\alpha_2}(b)y_{\alpha_1+\alpha_2+\alpha_3}(bc^q + cb^q) \\ &\times x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(-(bc^{q^2+q} + b^q c^{q^2+1} + b^{q^2}c^{q+1})) \\ &\times x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(-(cb^{q^2+q} + c^q b^{q^2+1} + c^{q^2}b^{q+1})). \end{aligned} \tag{5.2}$$

Using (5.2) we verify that the coefficient of $y_{\alpha_1+\alpha_2+\alpha_3}$ in the expression of rs, respectively sr, equals $a\eta^{2q} + a^q\eta^2$, respectively $a^q\eta^{2q} + a\eta^2$. These coefficients are equal if and only if $(a^q - a)(\eta^{2q} - \eta^2) = 0$. As $\eta^{2(q-1)} \neq 1$ and $a^q \neq a$, we have $rs \neq sr$, with $rs, sr \in \mathbb{U}^F$. Thus, $(sr)^2 \neq (rs)^2$, as q is odd. Comparing the coefficients of x_{α_1} in the expressions of r and s as products of elements in root subgroups, we see that

$$\mathbb{U}^F \triangleright r \subset x_{\alpha_1}(1)\langle \mathbb{U}_\beta | \beta \in \Phi^+ - \{\alpha_1\}\rangle, \, \mathbb{U}^F \triangleright s \subset x_{\alpha_1}(\eta^2)\langle \mathbb{U}_\beta | \beta \in \Phi^+ - \{\alpha_1\}\rangle.$$

So $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$, whence $\mathcal{O}_r$ is of type D.

(ii) *$q > 2$ is even.*

The list of representatives of the unipotent classes in G appears in [DM], see [Hi, Table A2]. For suitable $\zeta, \zeta' \in \mathbb{F}_q$, the representatives are of the form

$$\begin{aligned} u_1 &= x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1), & u_2 &= x_{\alpha_2}(1)x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(1),\\ u_3 &= y_{\alpha_1+\alpha_2+\alpha_3}(1), & u_4 &= y_{\alpha_1+\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(1)x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(\zeta),\\ u_5 &= y_{\alpha_1}(1)x_{\alpha_2}(1), & u_6 &= y_{\alpha_1}(1)x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(\zeta'),\\ u_7 &= y_{\alpha_1}(1)y_{\alpha_1+\alpha_2}(a), & & a\in \mathbb{F}_{q^3}-\mathbb{F}_q. \end{aligned}$$

All classes except $\mathcal{O}_{u_7}$ are represented by $v \in \Upsilon \simeq G_2(q)$; thus, these are not kthulhu by Lemmata 4.8, 4.9, 4.10, 4.11, 4.12. We deal now with $\mathcal{O}_{u_7}$, we shall show that it is of type F. Let $\gamma_j = \sum_{i=1}^{j}\alpha_i$ for shortness. We use (5.2) and the following relations from [Ge], cf. [Hi]:

$$\begin{aligned} y_{\alpha_1}(b)y_{\gamma_3}(c) &= y_{\gamma_3}(c)y_{\alpha_1}(b)x_{\gamma_4}(c^qb + c^{q^2}b^q + cb^{q^2})\\ y_{\gamma_2}(b)y_{\gamma_3}(c) &= y_{\gamma_3}(c)y_{\gamma_2}(b)x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(c^qb + c^{q^2}b^q + cb^{q^2}),\\ x_{\alpha_2}(d)x_{\gamma_4}(e) &= x_{\gamma_4}(e)x_{\alpha_2}(d)x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(de),\\ y_{\alpha_1}(b)x_{\alpha_2}(d) &= x_{\alpha_2}(d)y_{\alpha_1}(b)y_{\gamma_2}(bd)y_{\gamma_3}(db^{q+1})x_{\gamma_4}(db^{q^2+q+1}); \end{aligned}$$

here $b, c \in \mathbb{F}_{q^3}^{\times}$ and $d, e \in \mathbb{F}_q^{\times}$. Let $\mathbf{C} \leq \mathbb{F}_{q^3}^{\times}$ be the cyclic subgroup of order q^2+q+1 and $\mathbf{D} := \mathbf{C}\cap\mathbb{F}_q^{\times}$, a cyclic group of order $(q-1,3)$. Thus $|\mathbf{C}/\mathbf{D}| \geq 4$. Let ξ_i, $i \in \mathbb{I}_4$, be representatives of 4 distinct cosets in $\mathbf{C}/\mathbf{D}$ and let

$$t_i := \alpha_1(\xi_i)\alpha_3(\xi_i^q)\alpha_4(\xi_i^{q^2}),\quad r_i := t_i \triangleright u_7 = y_{\alpha_1}(\xi_i^2)y_{\alpha_1+\alpha_2}(a\xi_i^2) \in \mathcal{O}_r \cap \mathbb{U}^F.$$

Since $\mathbb{U}^F \triangleright r_i \subset y_{\alpha_1}(\xi_i^2)\langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \Delta\rangle$, we have $\mathcal{O}_{r_i}^{\langle r_1,r_2,r_3,r_4\rangle} \neq \mathcal{O}_{r_j}^{\langle r_1,r_2,r_3,r_4\rangle}$ for $i \neq j$. In addition by (5.2) we see that

$$r_ir_j \in y_{\alpha_1}(\xi_i^2 + \xi_j^2)y_{\gamma_2}(a(\xi_i^2 + \xi_j^2))y_{\gamma_3}(a\xi_i^2\xi_j^{2q} + a^q\xi_i^{2q}\xi_j^2)\, \mathbb{U}_{\gamma_4}^F \mathbb{U}_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}^F$$

The coefficients of y_{γ_3} in the expressions of r_ir_j and r_jr_i are equal iff $(a + a^q)(\xi_i^2\xi_j^{2q} + \xi_j\xi_i^{2q}) = 0$, iff $(\xi_i\xi_j^{-1})^{2(q-1)} = 1$ (since $a \notin \mathbb{F}_q$), iff $i = j$ by our choice of the ξ_i's. Hence, $r_i \triangleright r_j \neq r_j$ for $i \neq j$ and $\mathcal{O}_{u_7}$ is of type F.

(iii) $q = 2$.

The description of the representatives is the same as in (ii) with $\zeta = 0$ and $\zeta' = 1$, see [Hi, §3], so that

$$u_4 = y_{\alpha_1+\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(1), \qquad u_6 = y_{\alpha_1}(1)x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(1).$$

We do not have information on the unipotent classes of $G_2(2)$ yet, so we have to argue differently. We proceed case by case. The argument for u_7 is exactly as for $q > 2$, Thus, $\mathcal{O}_{u_7}$ is of type F. Now $u_1 \in \langle \mathbb{U}_{\pm\alpha_2}^F, \mathbb{U}_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}^F\rangle$, a subgroup of type A_2, but it is not regular there. Hence $\mathcal{O}_{u_1}$ is not kthulhu by Theorem 3.3 and [MT, Theorem 24.15].

By [Hi, Tables A.8], we have $r := y_{\alpha_1}(1)y_{\gamma_3}(1)x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1) \in \mathcal{O}_{u_2}$. Let $\xi \in \mathbb{F}_8^\times$ such that $\xi^3 = \xi + 1$. Then the roots in $\mathbb{F}_8^\times$ of the polynomial $X^4 + X^2 + X$ are ξ, ξ^2 and ξ^4. Their inverses, together with 1, are the roots of the polynomial $X^4 + X^2 + X + 1$. Let $\mathbb{P}_1$ be the parabolic subgroup with standard Levi subgroup associated with $\{\alpha_1, \alpha_3, \alpha_4\}$, and, for $i \in \mathbb{I}_4$, let

$$\begin{aligned}
&t_i := \alpha_1^\vee(\xi^i)\alpha_3^\vee(\xi^{2i})\alpha_4^\vee(\xi^{4i}),\\
&r_i := t_i \triangleright r = y_{\alpha_1}(\xi^{2i})y_{\gamma_3}(\xi^{6i})x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1) \in \mathcal{O}_{u_2}, \text{ so}\\
&\mathbb{U}^F \triangleright r_a \subset y_{\alpha_1}(\xi^{2i})V_1.
\end{aligned}$$

Then, $\mathcal{O}_{r_i}^{\langle r_1,r_2,r_3,r_4\rangle} \neq \mathcal{O}_{r_j}^{\langle r_1,r_2,r_3,r_4\rangle}$ for $i \neq j$. In addition,

$$r_ir_j = y_{\alpha_1}(\xi^{2i} + \xi^{2j})y_{\gamma_3}(\xi^{6i} + \xi^{6j})x_{\gamma_4}(\xi^{4(j-i)} + \xi^{2(j-i)} + \xi^{j-i}).$$

Let $i \neq j$. The coefficient of x_{γ_4} in the expression of r_ir_j is 0 if and only if $\xi^{j-i} \in \{\xi, \xi^2, \xi^4\}$ if and only if the coefficient of x_{γ_4} in the expression of r_jr_i is 1. Thus, $\mathcal{O}_{u_2}$ is of type F.

Let now $r_1 = u_3$. Let σ and τ in Υ be representatives of $s_1s_3s_4, s_2 \in W^F$, respectively. Let $\mathbb{P}_2$ be the F-stable parabolic subgroup with standard Levi subgroup associated with α_2. We consider the following elements in $\mathcal{O} \cap V_2$:

$$\begin{aligned}
&r_2 = \sigma \triangleright r_1 = y_{\alpha_1+\alpha_2}(1), \qquad r_3 = \tau \triangleright r_2 = y_{\alpha_1}(1)\\
&r_4 = x_{\alpha_2}(1) \triangleright r_3 = y_{\alpha_1}(1)y_{\alpha_1+\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(1)x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(1).
\end{aligned}$$

Let $Z = \langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}\rangle$. Then

$$\begin{aligned}
&V_2 \triangleright r_1 \subset y_{\alpha_1+\alpha_2+\alpha_3}(1)Z, && V_2 \triangleright r_2 \subset y_{\alpha_1+\alpha_2}(1)Z,\\
&V_2 \triangleright r_3 \subset y_{\alpha_1}(1)Z, && V_2 \triangleright r_4 \subset y_{\alpha_1}(1)y_{\alpha_1+\alpha_2}(1)Z.
\end{aligned}$$

Hence, the classes $\mathcal{O}_{r_i}^{\langle r_1,r_2,r_3,r_4\rangle}$ for $i \in \mathbb{I}_4$ are disjoint. A direct computation shows that $r_ir_j \neq r_jr_i$ for $i \neq j$, so $\mathcal{O}_{u_3}$ is of type F.

We deal now with u_4. Let ξ, $\mathbb{P}_1$ and $\mathbb{P}_2$ be as above and let

$$\begin{aligned}
&t_1 := \alpha_1^\vee(\xi^3)\alpha_3^\vee(\xi^6)\alpha_4^\vee(\xi^5), && t_2 := \alpha_1^\vee(\xi)\alpha_3^\vee(\xi^2)\alpha_4^\vee(\xi^4),\\
&r_1 := t_1 \triangleright u_4 = y_{\gamma_2}(\xi^6)y_{\gamma_3}(\xi^4), && r_2 := x_{\alpha_2}(1)y_{\gamma_3}(1)y_{\gamma_4}(1),\\
&r_3 := y_{\alpha_1}(1)y_{\gamma_3}(1), && r_4 := t_2 \triangleright r_3 = y_{\alpha_1}(\xi^2)y_{\gamma_3}(\xi^{-1}).
\end{aligned}$$

Then, $r_i \in \mathcal{O}_{u_4} \cap \mathbb{U}^F$, [Hi, Tables A.2, A.4, A.8, A.12]. In addition,

$$\begin{aligned}
&\mathbb{U}^F \triangleright r_1 \subset V_1 \cap V_2, && \mathbb{U}^F \triangleright r_2 \subset x_{\alpha_2}(1)V_1 \cap V_2,\\
&\mathbb{U}^F \triangleright r_3 \subset y_{\alpha_1}(1)V_1 \cap V_2, && \mathbb{U}^F \triangleright r_4 \subset y_{\alpha_1}(\xi^2)V_1 \cap V_2.
\end{aligned}$$

Hence, for $H = \langle r_i \mid i \in \mathbb{I}_4\rangle$ we have $\mathcal{O}_{r_i}^H \neq \mathcal{O}_{r_j}^H$ for $i, j \in \mathbb{I}_4$, with $i \neq j$. A direct computation shows that $r_ir_j \neq r_jr_i$, for $i \neq j$, so $\mathcal{O}_{u_4}$ is of type F.

Finally, we treat simultaneously the classes of u_5 and u_6, that are of the form $x = y_{\alpha_1}(1)x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(\epsilon)$ with $\epsilon \in \{0,1\}$ respectively. Let $\mathbf{C}$ be as in the odd case and let $(\xi_i)_{i\in\mathbb{I}_4}$ be a family of distinct elements in $\mathbf{C}$. Set

$$t_i := \alpha_1^\vee(\xi_i)\alpha_3^\vee(\xi_i^q)\alpha_4^\vee(\xi_i^{q^2}),$$
$$r_i := t_i \triangleright x = y_{\alpha_1}(\xi_i^2)x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(\epsilon\xi_i^{1+q-q^2}) \in \mathcal{O}_x \cap \mathbb{U}^F.$$

Let $Q = \langle r_1, r_2, r_3, r_4\rangle$. Since $\mathbb{U}^F \triangleright r_i \subset y_{\alpha_1}(\xi_i^2)x_{\alpha_2}(1)\langle \mathbb{U}_\gamma |\ \gamma \in \Phi^+ - \Delta\rangle$, we have $\mathcal{O}_{r_i}^Q \neq \mathcal{O}_{r_j}^Q$ for $i \neq j$. The coefficient of $y_{\alpha_1+\alpha_2}$ in the expression of r_ir_j equals ξ_i^2, hence $r_ir_j \neq r_jr_i$ for $i \neq j$. Hence $\mathcal{O}_{u_5}$ and $\mathcal{O}_{u_6}$ are of type F. $\square$

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