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THE t-STRUCTURE INDUCED BY AN n-TILTING MODULE

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Abstract. We study the t-structure induced by an n-tilting module T in the derived category $\mathcal{D}(R)$ of a ring R. Our main objective is to determine when the heart of the t-structure is a Grothendieck category. We obtain characterizations in terms of properties of the module category over the endomorphism ring of T and, as a main result, we prove that the heart is a Grothendieck category if and only if T is a pure projective R-module.

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Introduction

The notion of t-structure in a triangulated category was introduced by Beĭlinson, Bernstein, and Deligne [BBD82] in a geometric context. Its impact and relevance in the algebraic setting has become more and more apparent, and many constructions of t-structures are now available. A first important example is provided by the t-structure associated with torsion pairs in abelian categories [HRS96].

One of the key results about t-structures proved in [BBD82] is that their heart is an abelian category, and a lot of work has been done to determine when the heart of some classes of t-structures is a particularly nice category, like a Grothendieck or even a module category. For instance it is well known that the heart of the t-structure induced by a finitely generated tilting module is equivalent to the module category over the endomorphism ring of the module. Colpi, Gregorio, and Mantese [CGM07] proved that a faithful torsion pair in a module category with torsion free class closed under direct limits induces a t-structure whose heart is a Grothendieck category. In particular, this applies to 1-cotilting torsion pairs.

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Šťovíček [Šťo14] generalized this result to an arbitrary n-cotilting module by using powerful tools from model structures.

A detailed study of properties of the heart of t-structures induced by torsion pairs in a Grothendieck category has been carried out by Parra and Saorín [CS14]. In particular, they prove that if the torsion class is cogenerating, then the heart is a Grothendieck category if and only if the torsion free class is closed under direct limits. This applies to the case of a tilting torsion class, and a natural question posed by Saorín was to decide if the closure under direct limits of the tilting torsion free class implies necessarily that the tilting module is finitely generated. This has been answered in [BHP$^+$16], where it is shown that a tilting torsion free class is closed under direct limits if and only if the tilting module is pure projective and examples of non–finitely generated pure projective tilting modules are exhibited.

In this paper we consider the t-structure induced by an n-tilting module T over a ring R, and our main interest is to determine when the heart of the t-structure is a Grothendieck category. We obtain characterizations in terms of properties of the module category over the endomorphism ring of T and, as a main result, we prove that the heart is a Grothendieck category if and only if T is a pure projective R-module.

The paper is organized as follows. After the necessary preliminaries, in section 2 we recall the basic definitions and results about model structures and the relations developed by Hovey between cotorsion pairs in Mod-R and model structures on the category $\mathcal{C}h(R)$ of unbounded complexes of R-modules. In section 3 we use the model structure on $\mathcal{C}h(R)$ corresponding to an n-tilting cotorsion pair to describe the t-structure in the derived category $\mathcal{D}(R)$ of R, induced by an n-tilting module T. In section 4 we study the heart of the t-structure, its objects and their cohomologies. In particular, we show that T is a projective generator of the heart and, imitating the arguments on the theory of derivators used by Šťovíček [Šťo14], we show that the inclusion of the heart $\mathcal{H}$ in $\mathcal{D}(R)$ extends to an equivalence between $\mathcal{D}(\mathcal{H})$ and $\mathcal{D}(R)$ (Theorem 4.5).

In section 5 we apply the celebrated Gabriel-Popescu theorem and a result proved in [BMT11] about the derived equivalence induced by a good n-tilting R-module T between $\mathcal{D}(R)$ and a localization of $\mathcal{D}(S)$, where S is the endomorphism ring of T. We characterize the case in which the heart is a Grothendieck category in terms of the properties of S-modules. In particular, we show that the heart is a Grothendieck category if and only if for every right S-module M the derived tensor product $M \otimes^{\mathbf{L}}_S T$ is an object of the heart (Theorem 5.10). Moreover, the heart is a Grothendieck category if and only if S admits a two-sided idempotent ideal A, projective as right S-module, such that the canonical morphism $S \to S/A$ is a homological ring epimorphism such that S/A acts as a "generalized universal localization" (Theorem 5.12).

These characterizations allow us to describe the direct limits in the heart, and in section 6 we show how it is possible to compute direct limits in the heart by means of direct limits of complexes of modules.

Finally in section 7 we study the consequences of the Grothendieck condition on the heart in terms of closure properties of classes of R-modules. This allows us to prove our main result (Theorem 7.5); that is, we show that the heart $\mathcal{H}$ is a Grothendieck category if and only if the tilting module T is pure projective, generalizing to the case of $n > 1$, the result proved in [BHP$^+$16].

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We end by studying properties of the trace functor corresponding to an n-tilting module and properties of a pure projective n-tilting module.

1. Preliminaries

R will be an associative ring with a unit. Mod-R (R-Mod) will denote the category of right (left) R-modules and mod-R (R-mod) the subcategory of finitely presented right (left) R-modules.

For more details about the terminology and the results stated in this section we refer the reader to [GT12].

Given a class $\mathcal{M}$ of objects of an abelian category $\mathcal{C}$ and an index $i \geq 0$, we denote

$$\mathcal{M}^{\perp_i} = \{X \in \mathcal{C} \mid \mathrm{Ext}^i_{\mathcal{C}}(M, X) = 0 \text{ for all } M \in \mathcal{M}\}.$$

$$\mathcal{M}^{\perp} = \{X \in \mathcal{C} \mid \mathrm{Ext}^i_{\mathcal{C}}(M, X) = 0 \text{ for all } M \in \mathcal{M}, \text{ for all } i \geq 1\}.$$

The classes ${}^{\perp_i}\mathcal{M}$ and ${}^{\perp}\mathcal{M}$ are defined symmetrically.

A pair $(\mathcal{A}, \mathcal{B})$ of classes of objects of $\mathcal{C}$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$.

Recall that a full subcategory $\mathcal{C}'$ of an abelian category $\mathcal{C}$ is *resolving* if it is closed under summands, extensions, and kernels of epimorphisms (in $\mathcal{C}'$) and is generating; that is, for every $X \in \mathcal{C}$ there is an epimorphism $C \to X$ with $C \in \mathcal{C}'$.

If moreover the epimorphism $C \to X$ can be chosen functorially in X, then $\mathcal{C}'$ is called *functorially resolving*.

The $\mathcal{C}'$-resolution dimension of an object $X \in \mathcal{C}$ with respect to a resolving subcategory $\mathcal{C}'$ is the minimum integer $n \geq 0$ for which there is an exact sequence $0 \to C_n \to \cdots \to C_1 \to C_0 \to X \to 0$, with $C_i \in \mathcal{C}'$ or ∞, if such an n does not exist.

The notions of coresolving subcategories, functorially coresolving, and coresolution dimension are defined dually.

Note that for any subcategory $\mathcal{C}$ of Mod-R, ${}^{\perp}\mathcal{C}$ is resolving and, in particular, syzygy closed. Dually, $\mathcal{C}^{\perp}$ is coresolving and, in particular, cosyzygy closed.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called a *hereditary cotorsion pair* if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$.

A (hereditary) cotorsion pair $(\mathcal{A}, \mathcal{B})$ in an abelian category $\mathcal{C}$ is *complete* provided that every object $X \in \mathcal{C}$ admits a *special* $\mathcal{B}$*–pre-envelope*; that is, there exists an exact sequence of the form $0 \to X \to B \to A \to 0$, with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. Equivalently, every object X admits a *special* $\mathcal{A}$*-precover*; that is, there exists an exact sequence of the form $0 \to B \to A \to X \to 0$, with $B \in \mathcal{B}$ and $A \in \mathcal{A}$.

Pre-envelopes and precovers are also called left and right approximations. For a class $\mathcal{M}$ of objects of an abelian category $\mathcal{C}$, the pair $({}^{\perp}(\mathcal{M}^{\perp}), \mathcal{M}^{\perp})$ is a (hereditary) cotorsion pair; it is called the cotorsion pair *generated* by $\mathcal{M}$. Symmetrically, the pair $({}^{\perp}\mathcal{M}, ({}^{\perp}\mathcal{M})^{\perp})$ is a (hereditary) cotorsion pair called the cotorsion pair *cogenerated* by $\mathcal{M}$. Every cotorsion pair generated by a *set* of objects is complete [Qui73], [ET01]. Moreover, every cotorsion pair cogenerated by a class of pure injective objects is generated by a set of objects and hence is complete [ET00].

For every R-module M, Add$M$ will denote the class of modules isomorphic to summands of direct sums of copies of M, and Gen M will denote the class of all epimorphic images of direct sums of copies of M. p.d.M will denote the projective dimension of M.

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Definition 1.1. A right R-module T is n tilting if it satisfies the following conditions:

(T1) p.d.$T \leq n$;
(T2) $\mathrm{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every cardinal λ and every $i \geq 1$;
(T3) there exists an $r \geq 0$ and an exact sequence

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0,$$

where $T_i \in \mathrm{Add}T$, for every $0 \leq i \leq r$.

A finitely generated n-tilting module is called *classic*.

If T is an n-tilting module, $T^{\perp}$ is called n-tilting class and the cotorsion pair $(\mathcal{A}, T^{\perp})$ generated by T is called an n-tilting cotorsion pair. The kernel $\mathcal{A} \cap T^{\perp}$ of the cotorsion pair coincides with $\mathrm{Add}T$. Two n-tilting modules T and U are said to be *equivalent* if $T^{\perp} = U^{\perp}$ or, equivalently, if $\mathrm{Add}T = \mathrm{Add}U$.

By [BH08, BŠ07] an n-tilting cotorsion pair $(\mathcal{A}, T^{\perp})$ is of *finite type*—that is, there is a set $\mathcal{S}$ of modules in $\mathcal{A}$ with a projective resolution consisting of finitely generated projective modules such that $\mathcal{S}^{\perp} = T^{\perp}$. In Crawley-Boevey terminology (see [CB98]) this means that, for every $S \in \mathcal{S}$ the functors $\mathrm{Ext}^i_R(S, -)$ are coherent, and hence that n-tilting classes are *definable*; that is, they are closed under direct products, direct limits, and pure submodules.

Note that all the syzygies of a classical n-tilting module are finitely generated (see [BH09, Corollary 3.9]).

Recall that a module is pure projective if and only if it has the projective property with respect to pure exact sequences.

By Warfield [War69] a module M is pure projective if and only if every pure exact sequence $0 \to A \to B \to M \to 0$ splits or, equivalently, if and only if M is a direct summand of a direct sum of finitely presented modules.

2. Model structures

We describe some *model structures* on the category $\mathcal{C}h(R)$ of unbounded complexes of R-modules whose homotopy category is the derived category $\mathcal{D}(R)$ of R.

For the definition of a model structure we refer to the book by Hoevy [Hov99] or to the survey [Šťo13].

We recall only that a model structure on a category $\mathcal{C}$ consists of three classes of morphisms Cof, W, Fib called *cofibrations*, *weak equivalences*, and *fibrations*, respectively, satisfying certain axioms.

A *model category* $\mathcal{C}$ is a cocomplete category with a model structure. An object X in a pointed model category $\mathcal{C}$ is called *cofibrant (trivial)* if the unique morphism from the initial object to X is a cofibration (a weak equivalence), and it is called *fibrant* if the unique morphism from X to the terminal object is a fibration.

The homotopy category Ho$\mathcal{C}$ is obtained by formally inverting all morphisms in W.

An *abelian model structure* on an abelian category $\mathcal{C}$ is a model structure such that cofibrations (fibrations) are the monomorphisms (epimorphisms) with cofibrant (fibrant) cokernels (kernels).

We recall a method discovered by Hovey [Hov02, Hov07] and developed by Gillespie [Gil06, Gil04] and other authors which allows one to define a model structure on the category $\mathcal{C}h(\mathcal{C})$ of unbounded complexes over $\mathcal{C}$ starting from a complete

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cotorsion pair on $\mathcal{C}$. We state Hovey's result only in the situation needed in the sequel.

If $\mathcal{C}$ is a subclass of Mod-R closed under extensions, following the notations used by Gillespie (see [Gil06, Gil04] or [Šťo13]), we denote by $\tilde{\mathcal{C}}$ the class of all acyclic complexes of $\mathcal{C}h(R)$ with terms in $\mathcal{C}$ and cocycles in $\mathcal{C}$.

Proposition 2.1 ([Hov07, Gil06, Gil04]). *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in* Mod-R *generated by a set of modules, there is an abelian model structure on $\mathcal{C}h(R)$ given as follows:*

(1) *Weak equivalences are quasi isomorphisms.*
(2) *Cofibrations (trivial cofibrations) are the monomorphism f such that $\mathrm{Ext}^1_{\mathcal{C}h(R)}(\mathrm{Coker}\, f, X) = 0$ for every $X \in \tilde{\mathcal{B}}$ ($\mathrm{Coker}\, f \in \tilde{\mathcal{A}}$), and C is a cofibrant object if and only if $\mathrm{Ext}^1_{\mathcal{C}h(R)}(C, X) = 0$ for every $X \in \tilde{\mathcal{B}}$.*
(3) *Fibrations (trivial fibrations) are the epimorphisms g such that $\mathrm{Ext}^1_{\mathcal{C}h(R)}(X, \mathrm{Ker}\, g) = 0$ for every $X \in \tilde{\mathcal{A}}$ ($\mathrm{Ker}\, g \in \tilde{\mathcal{B}}$), and F is a fibrant object if and only if $\mathrm{Ext}^1_{\mathcal{C}h(R)}(X, F) = 0$ for every $X \in \tilde{\mathcal{A}}$.*

The homotopy category of this model structure is the derived category $\mathcal{D}(R)$ of R.

Moreover, if $\mathcal{C}$, $\mathcal{W}$, $\mathcal{F}$ are the classes of cofibrant, trivial (acyclic), and fibrant objects, respectively, then $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{C}h(R)$.

This allows us to describe the morphisms in $\mathcal{D}(R)$. In fact, if X is a cofibrant object in $\mathcal{C}h(R)$ and Y is a fibrant object in $\mathcal{C}h(R)$, then

$$\mathrm{Hom}_{\mathcal{K}(R)}(X, Y) = \mathrm{Hom}_{\mathcal{D}(R)}(X, Y),$$

where $\mathcal{K}(R)$ is the homotopy category of R.

To describe the cofibrant and fibrant objects in the model structure induced by a complete cotorsion pair, we will make use of the following well-known formula:

$$(*) \qquad \mathrm{Ext}^1_{dw}(X[1], Y) \cong \mathrm{Hom}_{\mathcal{K}(R)}(X, Y),$$

where Ext^1_{dw} denotes the subgroup of $\mathrm{Ext}^1_{\mathcal{C}h(R)}$ consisting of the degreewise splitting short exact sequences.

Example 2.2. If $\mathcal{P}$ is the class of projective R-modules, the model structure induced by the cotorsion pair $(\mathcal{P}, \text{Mod-}R)$ is called the canonical projective model structure: The trivial objects are the acyclic complexes, the cofibrant objects (also called K-projective) are the complexes X with projective terms such that $\mathrm{Hom}_{\mathcal{K}(R)}(X, N) = 0$ for every acyclic complex N, and every complex is a fibrant object. Moreover, if $\mathcal{KP}$ is the class of K-projective complexes and $\mathcal{N}$ the class of acyclic complexes, the pair $(\mathcal{KP}, \mathcal{N})$ is a complete cotorsion pair in $\mathcal{C}h(R)$.

Example 2.3. Symmetrically, if $\mathcal{I}$ is the class of injective R-modules, the model structure induced by the cotorsion pair $(\text{Mod-}R, \mathcal{I})$ is called the canonical injective model structure: The trivial objects are the acyclic complexes, the fibrant objects (also called K-injective) are the complexes Y with injective terms such that $\mathrm{Hom}_{\mathcal{K}(R)}(N, Y) = 0$ for every acyclic complex N, and every complex is a cofibrant object. Moreover, if $\mathcal{KI}$ is the class of K-injective complexes and $\mathcal{N}$ the class of acyclic complexes, the pair $(\mathcal{N}, \mathcal{KI})$ is a complete cotorsion pair in $\mathcal{C}h(R)$.

Remark 2.4. If $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair in Mod-R and X is a bounded above complex with terms in $\mathcal{A}$, then X is cofibrant in the model structure induced

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by $(\mathcal{A}, \mathcal{B})$ (the proof is similar to the proof that a bounded above complex with projective terms is K-projective (see [Hov99, Lemma 2.3.6]). Dually, if X is a bounded below complex with terms in $\mathcal{B}$, then X is fibrant.

We are now in a position to describe some particular model structures: the model structure induced by a module of finite homological dimension. The following results are obtained by imitating the arguments used by Šťovíček [Šťo14, Theorem 3.17] to describe the model structure induced by a cotilting module.

Theorem 2.5. *Let M be an R-module with p.d.$M \leq n$. Let $(\mathcal{A}, \mathcal{B})$ be the hereditary cotorsion pair generated by M. There is an abelian model structure on $\mathcal{C}h(R)$ described as follows:*

(1) *Cofibrations (trivial cofibrations) are the monomorphism f such that $\operatorname{Ext}^1_{\mathcal{C}h(R)}(\operatorname{Coker} f, X) = 0$ for every $X \in \tilde{\mathcal{B}}$ ($\operatorname{Coker} f \in \tilde{\mathcal{A}}$).*
(2) *Fibrations (trivial fibrations) are the epimorphisms g such that $\operatorname{Ker} g$ has terms in $\mathcal{B}$ ($\operatorname{Ker} g \in \tilde{\mathcal{B}}$).*

Proof. In view of Proposition 2.1 the only thing which remains to be proved is the description of the fibrant objects. The statement will follow from the next lemma. □

Lemma 2.6. *Let M be as in Theorem 2.5, and let $Y \in \mathcal{C}h(R)$. Then Y is a fibrant object in the model structure induced by the hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ generated by M if and only if Y has all the terms in $\mathcal{B}$.*

Proof. The proof is inspired by [Šťo14, Theorem 3.17], but we give the details for the sake of completeness.

Let Y be a fibrant object. Then $\operatorname{Ext}^1_{\mathcal{C}h(R)}(X, Y) = 0$, for all $X \in \tilde{\mathcal{A}}$. For every $A \in \mathcal{A}$ let $D^n(A)$ be the complex defined by $0 \to A \stackrel{1_A}{\to} A \to 0$, with A in degrees n and $n+1$. Then $D^n(A) \in \tilde{\mathcal{A}}$ and by [Gil04, Lemma 3.1.5]) $\operatorname{Ext}^1_{\mathcal{C}h(R)}(D^n(A), Y) \cong \operatorname{Ext}^1_R(A, Y^n)$; hence, $Y^n \in \mathcal{B}$.

Conversely, assume that all of the terms Y^i of a complex Y are in $\mathcal{B}$. We claim that Y is fibrant—that is, $\operatorname{Ext}^1_{\mathcal{C}h(R)}(X, Y) = 0$ for every $X \in \tilde{\mathcal{A}}$. Clearly $\operatorname{Ext}^1_{\mathcal{C}h(R)}(X, Y) \cong \operatorname{Ext}^1_{dw}(X, Y)$ and by formula $(*)$, $\operatorname{Ext}^1_{dw}(X[1], Y) \cong \operatorname{Hom}_{\mathcal{K}(R)}(X, Y)$.

Consider a special pre-envelope

$$0 \to Y \to I_0 \to N \to 0$$

of Y with respect to the complete cotorsion pair $(\mathcal{N}, \mathcal{KI})$ in $\mathcal{C}h(R)$ (described in Example 2.3).

Then N is an exact complex and, since $\mathcal{B}$ contains the injective modules, all of the terms I_0^i of I_0 are in $\mathcal{B}$; hence, the terms N^i are also in $\mathcal{B}$ since $\mathcal{B}$ is coresolving. For every $i \in \mathbb{Z}$ consider the short exact sequences $0 \to \operatorname{Ker} d_N^i \to N^i \to \operatorname{Ker} d_N^{i+1} \to 0$. By dimension shifting and by the condition p.d.$M \leq n$ we conclude that $\operatorname{Ext}^j_R(M, \operatorname{Ker} d_N^i) = 0$ for every $i, j \in \mathbb{Z}$. So $N \in \tilde{\mathcal{B}}$. Thus, for every $X \in \tilde{\mathcal{A}}$,

$$\text{(a)} \quad \operatorname{Ext}^1_{\mathcal{C}h(R)}(X, N) \cong \operatorname{Ext}^1_{dw}(X, N) \cong \operatorname{Hom}_{\mathcal{K}(R)}(X[-1], N)) = 0.$$

Consider the triangle

$$N[-1] \to Y \to I_0 \to N,$$

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and let $X \in \tilde{\mathcal{A}}$. We have an exact sequence

$$\mathrm{Hom}_{\mathcal{K}(R)}(X, N[-1]) \to \mathrm{Hom}_{\mathcal{K}(R)}(X, Y) \to \mathrm{Hom}_{\mathcal{K}(R)}(X, I_0),$$

where the last term vanishes since I_0 is K injective (and X is acyclic) and the first term vanishes by (a). So Y is fibrant. □

Corollary 2.7. *In the notations of Theorem 2.5 let Y be a complex with terms in $\mathcal{B}$, and let Z be cofibrant in the model structure induced by M. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(R)}(Z, Y) \cong \mathrm{Hom}_{\mathcal{D}(R)}(Z, Y).$$

In particular, this applies to the complexes Z bounded above and with terms in $\mathcal{A}$.

If C is a pure injective module, then by Auslander's result every cosyzygy of C is pure injective and thus, by [ET00], the hereditary cotorsion pair $({}^{\perp}C, ({}^{\perp}C,)^{\perp})$ cogenerated by C is complete.

This observation allows us to state also a dual of Theorem 2.5.

Theorem 2.8. *Let C be a pure injective R-module with injective dimension $C \leq n$. Let $(\mathcal{A}, \mathcal{B})$ be the hereditary cotorsion pair cogenerated by C. There is an abelian model structure on $\mathcal{C}h(R)$ described as follows:*

(1) *Cofibrations (trivial cofibrations) are the monomorphism f such that* Coker f *has terms in $\mathcal{A}$* (Coker $f \in \tilde{\mathcal{A}}$).
(2) *Fibrations (trivial vibrations) are the epimorphisms g such that* $\mathrm{Ext}^1_{\mathcal{C}h(R)}(X, \mathrm{Ker}\, g) = 0$ *for every $X \in \tilde{\mathcal{A}}$* ($\mathrm{Ker}\, g \in \tilde{\mathcal{B}}$).

Proof. It is enough to dualize the proof of Theorem 2.5. To prove that every complex Z with terms in $\mathcal{A}$ is cofibrant, one uses special precovers with respect to the complete cotorsion pair $(\mathcal{KP}, \mathcal{N})$ in $\mathcal{C}h(R)$ (described in Example 2.2). □

Corollary 2.9. *In the notations of Theorem 2.8 let Z be a complex with terms in $\mathcal{A}$, and let Y be fibrant in the model structure induced by C. Then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{K}(R)}(Z, Y) \cong \mathrm{Hom}_{\mathcal{D}(R)}(Z, Y).$$

In particular, this applies to the complexes Y bounded below and with terms in $\mathcal{B}$.

3. t-STRUCTURES

Definition 3.1 ([BBD82])**.** A t-structure in a triangulated category $(\mathcal{D}, [-])$ is a pair $(\mathcal{U}, \mathcal{V})$ of full subcategories such that

(1) $\mathcal{U}[1] \subseteq \mathcal{U}$;
(2) $\mathcal{V} = \mathcal{U}^{\perp}[1]$, where $\mathcal{U}^{\perp} = \{Y \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(\mathcal{U}, Y) = 0\}$;
(3) for every object $D \in \mathcal{D}$ there is a triangle $U \to D \to Y \to U[1]$, with $U \in \mathcal{U}$ and $Y \in \mathcal{U}^{\perp}$.

Theorem 3.2 ([BBD82])**.** *The heart $\mathcal{H} = \mathcal{U} \cap \mathcal{V}$ of a t-structure $(\mathcal{U}, \mathcal{V})$ is an abelian category.*

Notation 3.3. Let T be an n-tilting module. We denote by $\mathcal{U}$ and $\mathcal{V}$ the following full subcategories of $\mathcal{D}(R)$:

(1) $\mathcal{U} = \{X \in \mathcal{D}(R) \mid \mathrm{Hom}_{\mathcal{D}(R)}(T[i], X) = 0 \text{ for all } i < 0\}$.
(2) $\mathcal{V} = \{Y \in \mathcal{D}(R) \mid \mathrm{Hom}_{\mathcal{D}(R)}(T[i], Y) = 0 \text{ for all } i > 0\}$.

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Our aim is to prove that the pair $(\mathcal{U}, \mathcal{V})$ in Notation 3.3 is a t-structure in $\mathcal{D}(R)$. Following the pattern of the proof of [Šťo14, Lemma 4.4], we can give a description of the complexes in $\mathcal{U}$.

Lemma 3.4. *Let T be an n-tilting module, $T^\perp$ the corresponding tilting class, and $\mathcal{U}$ as in Notation 3.3. For a complex $X \in \mathcal{D}(R)$ the following are equivalent:*

(1) *$X \in \mathcal{U}$.*
(2) *X is isomorphic in $\mathcal{D}(R)$ to a complex of the form*

$$\cdots \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^0 \to 0 \to 0 \ldots,$$

with $X^{-i} \in T^\perp$ for every $i \geq 0$.
(3) *X is isomorphic in $\mathcal{D}(R)$ to a complex as in (2), with $X^{-i} \in \mathrm{Add}T$ for every $i \geq 0$.*

Proof. (3) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (1) follows from Corollary 2.7, since it is obvious that $\mathrm{Hom}_{\mathcal{K}(R)}(T[i], X) = 0$ for every $i < 0$.

It remains for us to show that (1) $\Rightarrow$ (3). Let $X \in \mathcal{D}(R)$. By Theorem 2.5 we can assume that X has terms in $T^\perp$. By induction we construct a complex

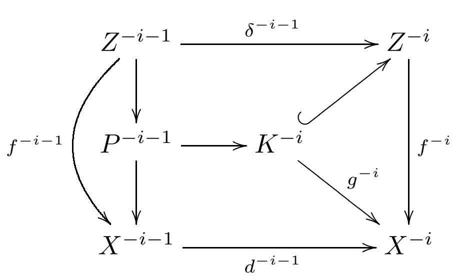
$$Z = \cdots \to Z^{-n} \to Z^{-n+1} \to \cdots \to Z^{-1} \to Z^0 \to 0 \to 0 \ldots,$$

with terms $Z^i \in \mathrm{Add}T$ and a cochain map $f\colon Z \to X$, which becomes an isomorphism in $\mathcal{D}(R)$.

Let $d^{-i}\colon X^{-i} \to X^{-i+1}$ be the ith-differential of X. Consider an $\mathrm{Add}T$-precover of $\mathrm{Ker}\, d^0$, like for instance the canonical morphism

$$Z^0 = T^{(\mathrm{Hom}_R(T, \mathrm{Ker}\, d^0))} \xrightarrow{\phi} \mathrm{Ker}\, d^0,$$

and let f^0 be the composition of ϕ with the inclusion $\mathrm{Ker}\, d^0 \to X^0$. By induction construct $f^{-i-1}\colon Z^{-i-1} \to X^{-i-1}$ in the following way. Having defined f^{-i} and $\delta^{-i}\colon Z^{-i} \to Z^{-i+1}$, let K^{-i} be the kernel of δ^{-i}, and let g^{-i} be the composition $K^{-i} \to Z^{-i} \xrightarrow{f^{-i}} X^{-i}$. Consider the pullback P^{-i-1} of the maps g^{-i} and d^{-i-1}, and let $Z^{-i-1} \to P^{-i-1}$ be an $\mathrm{Add}T$-precover of P^{-i-1}. Then let $f^{-i-1}\colon Z^{-i-1} \to X^{-i-1}$ be the obvious composition. We then have



visually. We claim that the cochain map $f = (f^{-i})_i$ is an isomorphism in $\mathcal{D}(R)$. By Corollary 2.7 f induces a morphism

$$\mathrm{Hom}_{\mathcal{K}(R)}(T[i], Z) \xrightarrow{\mathrm{Hom}_{\mathcal{K}(R)}(T[i], f)} \mathrm{Hom}_{\mathcal{K}(R)}(T[i], X),$$

with $\mathrm{Hom}_{\mathcal{K}(R)}(T[i], f) = 0$ for every $i < 0$ by the assumption on X and by the fact that $Z^j = 0$ for every $j > 0$. If $i \geq 0$, then $\mathrm{Hom}_{\mathcal{K}(R)}(T[i], f) = 0$ by construction, since Z^{-j} indicates $\mathrm{Add}T$-precovers for every $j \geq 0$.

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From the mapping cone $Z \xrightarrow{f} X \to \operatorname{cone} f \to Z[1]$ we obtain

$$\operatorname{Hom}_{\mathcal{K}(R)}(T[i], \operatorname{cone} f) \cong \operatorname{Hom}_{\mathcal{D}(R)}(T[i], \operatorname{cone} f) = 0$$

for all $i \in \mathbb{Z}$ since $\operatorname{cone} f$ is fibrant. Thus, we conclude that $\operatorname{cone} f = 0$, since, by (T3) of Definition 1.1, T is a generator of $\mathcal{D}(R)$ which yields f as a quasi isomorphism. □

Theorem 3.5. *The pair* $(\mathcal{U}, \mathcal{V})$ *defined in Notation* 3.3 *is a* t*-structure called the* t*-structure induced by* T*, and its heart* $\mathcal{H}$ *is given by*

$$\mathcal{H} = \{Y \in \mathcal{D}(R) \mid \operatorname{Hom}_{\mathcal{D}(R)}(T[i], Y) = 0 \text{ for all } i \neq 0\}.$$

Proof. From the description of the objects in the subcategory $\mathcal{U}$ it follows that $\mathcal{U}$ is a *pre-aisle*; that is, if $X \in \mathcal{U}$, then also $X[1] \in \mathcal{U}$ and, for every triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}(R)$, if $X, Z \in \mathcal{U}$, then $Y \in U$.

Then $\mathcal{U}$ is the smallest cocomplete subcategory of $\mathcal{D}(R)$ containing T. By [ATJLSS03, Lemma 3.1, Proposition 3.2] $(\mathcal{U}, \mathcal{U}^{\perp}[1])$ is a t-structure and $\mathcal{U}^{\perp}[1] = \mathcal{V}$. The description of the heart is now obvious. □

Remark 3.6. If T is a 1-tilting module and $\mathcal{F} = \{N_R \mid \operatorname{Hom}_R(T, N) = 0\}$, then $(T^{\perp}, \mathcal{F})$ is a torsion pair in Mod-R and the t-structure induced by T defined in Theorem 3.5 coincides with the t-structure induced by the torsion pair $(T^{\perp}, \mathcal{F})$ as defined in [HRS96]; that is,

$$\mathcal{U} = \{X \in \mathcal{D}(R) \mid H^0(X) \in T^{\perp} \text{ and } H^i(X) = 0 \text{ for all } i > 0\},$$

$$\mathcal{V} = \{Y \in \mathcal{D}(R) \mid H^{-1}(X) \in \mathcal{F} \text{ and } H^i(Y) = 0 \text{ for all } i < -1\},$$

so the objects of $\mathcal{H}$ are isomorphic to complexes of the form $0 \to X^{-1} \xrightarrow{d^{-1}} X^0 \to 0$, with $\operatorname{Ker} d^{-1} \in \mathcal{F}$ and $\operatorname{Coker} d^{-1} \in T^{\perp}$.

4. The heart of the t-structure induced by an n-tilting module

In this section $\mathcal{H}$ will always denote the heart of the t-structure induced by an n-tilting module T.

It is well known that $\mathcal{H}$ satisfies the following properties:

(1) If $X, Z \in \mathcal{H}$ and $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{D}(R)$, then $Y \in \mathcal{H}$.
(2) A sequence $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{H}$ if and only if $X \to Y \to Z \to X[1]$ is a triangle in $\mathcal{D}(R)$.
(3) For every $X, Y \in \mathcal{H}$, $\operatorname{Ext}^1_{\mathcal{H}}(X, Y) \cong \operatorname{Hom}_{\mathcal{D}(R)}(X, Y[1])$.

Remark 4.1 ([PS15, Lemma 3.1]). The inclusion functor $\iota \colon \mathcal{H} \to \mathcal{U}$ admits a left adjoint which can be defined using the cohomological functor $\tilde{H} \colon \mathcal{D}(R) \to \mathcal{H}$ constructed in [BBD82]. Given $X \in \mathcal{U}$, let

$$U \to X[-1] \to Z \to U[1]$$

be a triangle in $\mathcal{D}(R)$ with $U \in \mathcal{U}$ and $Z \in \mathcal{U}^{\perp}$. Then, a left adjoint $b \colon \mathcal{U} \to \mathcal{H}$ is defined by letting $b(X) = Z[1]$.

When applicable, we will make use of Corollary 2.7 without explicitly mentioning it.

First of all, we note the following.

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Proposition 4.2. *Let T be an n-tilting module. Then T is a projective object of the heart $\mathcal{H}$ and, for every complex X in $\mathcal{H}$, $H^i(X) = 0$ for every $i > 0$ and $i < -n$.*

Proof. By property (3) above, $\mathrm{Ext}^1_{\mathcal{H}}(T, X) \cong \mathrm{Hom}_{\mathcal{D}(R)}(T, X[1])$, so $\mathrm{Ext}^1_{\mathcal{H}}(T, X) = 0$, by the description of the objects in $\mathcal{H}$. Hence, T is a projective object of $\mathcal{H}$. By [Baz04, Proposition 3.5] we can chose a sequence E as in (T3) of Definition 1.1 with $r = n$. Apply the functor $\mathrm{Hom}_{\mathcal{D}(R)}(T[i], -)$ to the triangles in $\mathcal{D}(R)$ corresponding to the short exact sequences in which the exact sequence E splits, and consider the long exact sequences in cohomology associated with these short exact sequences. For every $X \in \mathcal{H}$ we get $\mathrm{Hom}_{\mathcal{D}(R)}(R[i], X) \cong \mathrm{Hom}_{\mathcal{K}(R)}(R[i], X) = 0$ for every $i \neq 0, 1, 2, \ldots, n$. Hence, X has cohomology only in degrees $0, -1, -2, \ldots, -n$. □

Let

$$\mathcal{H}_i = \{X \in \mathcal{H} \mid H^{-j}(X) = 0 \text{ for every } j > i\}.$$

Thus, $\mathcal{H}_0 = T^{\perp}[0]$, $\mathcal{H}_i \subseteq \mathcal{H}_{i+1}$ and, by Proposition 4.2, $\mathcal{H} = \mathcal{H}_n$.

The next result is obtained by dualizing the proofs of [Što14, Lemma 5.18, Proposition 5.20].

Proposition 4.3. *For every $X \in \mathcal{H}_i$ there is an exact sequence*

$$0 \to Y \to T_0[0] \to X \to 0$$

in $\mathcal{H}$, with $T_0 \in \mathrm{Add}T$ and $Y \in \mathcal{H}_{i-1}$. In particular, T is a projective generator of $\mathcal{H}$, $\mathrm{Add}T$ is equivalent to the full subcategory of projective objects of $\mathcal{H}$, and the $T^{\perp}$-resolution dimension of an object in $\mathcal{H}$ is at most n.

Proof. Let $X \in \mathcal{H}_i$. By Lemma 3.4 we may assume that

$$X = \cdots \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^0 \to 0,$$

with $X^i \in \mathrm{Add}T$. Consider the complex $X^0[0]$ and the obvious chain map $f\colon X^0[0] \to X$. We have a triangle in $\mathcal{D}(R)$

$$X^0[0] \to X \to Z \to X^0[1],$$

where Z is fibrant, and we may assume that

$$Z : \cdots \to X^{-n} \overset{d^{-n}}{\to} X^{-n+1} \to \cdots \to X^{-1} \to 0 \to 0,$$

with X^{-i} in degrees $-i$. Applying the cohomological functor $\mathrm{Hom}_{\mathcal{D}(R)}(T, -)$ to the triangle and using condition (T2) of the tilting modules, we obtain that $\mathrm{Hom}_{\mathcal{D}(R)}(T[i], Z) = 0$ for every $i \neq 0, 1$. Moreover, $\mathrm{Hom}_{\mathcal{D}(R)}(T[i], Z) \cong \mathrm{Hom}_{\mathcal{K}(R)}(T[i], Z)$ (by Corollary 2.7) and by the choice of Z we have $\mathrm{Hom}_{\mathcal{K}(R)}(T[0], Z) = 0$. Thus, $Z[-1] \in \mathcal{H}$ and, computing the homologies of the terms in the triangle, we see that $Z[-1] \in \mathcal{H}_{i-1}$. Thus, the triangle $Z[-1] \to X^0[0] \to X \to Z$ gives the wanted exact sequence $0 \to Z[-1] \to X^0[0] \to X \to 0$ in $\mathcal{H}$. The last statements are now obvious. □

Proposition 4.4. $\mathrm{Add}T$ *and* $T^{\perp}$ *are functorially resolving subcategories of* $\mathcal{H}$.

Proof. $\mathrm{Add}T$ and $T^{\perp}$ are closed under summands and extensions and by Proposition 4.3 they are generating subcategories in $\mathcal{H}$. We need to prove their closure under kernels of epimorphisms. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in

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$\mathcal{H}$, with $Y, Z \in T^{\perp}$, and consider the triangle $X \to Y \xrightarrow{f} Z \to X[1]$ in $\mathcal{D}(R)$. Then X is quasi isomorphic to

$$\ldots 0 \to 0 \to Y \to Z \to 0 \to 0 \ldots,$$

with Y in degree 0 and Z in degree 1; hence, by Lemma 2.6 X is a fibrant object. Let $0 \to B_Z \to A_Z \xrightarrow{\pi} Z \to 0$ be a special $\mathcal{A}$-precover of Z in the cotorsion pair $(\mathcal{A}, T^{\perp})$ in Mod-R. Then $A_Z \in \mathcal{A} \cap T^{\perp} = \text{Add}T$. By Theorem 3.5 and Corollary 2.7 $\text{Hom}_{\mathcal{D}(R)}(A_Z[-1], X) = \text{Hom}_{\mathcal{K}(R)}(A_Z[-1], X) = 0$. Hence, there is a $g\colon A_Z \to Y$ such that $f \circ g = \pi$, showing that f is an epimorphism in Mod-R. Thus, X is isomorphic to an object in $\mathcal{H} \cap \text{Mod-}R = T^{\perp}$.

In a case where Y, Z are in Add$T$, the previous argument shows that π splits, and so does f.

To prove the functoriality, note that for every $X \in \mathcal{H}$, we have a functorial epimorphism $T^{(\text{Hom}_{\mathcal{H}}(T,X))} \to X \to 0$. □

Using the theory of derivators as explained in [Šťo14, section 5], the previous results yield the following.

Theorem 4.5. *Let $\mathcal{H}$ be the heart of the t-structure induced by an n-tilting R-module T. Then the inclusion $\mathcal{H} \subseteq \mathcal{D}(R)$ extends to a triangle equivalence*

$$F\colon \mathcal{D}(\mathcal{H}) \to \mathcal{D}(R).$$

Proof. It is well known that $T^{\perp}$ is a functorially coresolving subcategory of Mod-R and that the $T^{\perp}$-coresolution dimension of every module is bounded by the projective dimension of T (see [GT12, Chap. 13]).

By Propositions 4.3 and 4.4 $T^{\perp}$ is a functorially resolving subcategory of $\mathcal{H}$ and the $T^{\perp}$-resolution dimension of an object in $\mathcal{H}$ is bounded by n.

By [Šťo14, Proposition 5.14] and [Šťo14, Remark 5.15] we can argue as in the proof of [Šťo14, Theorem 5.21] to get the conclusion. □

As noted in Remark 3.6, in the case of a 1-tilting module, the objects of the heart $\mathcal{H}$ can be described in terms of properties of their cohomology modules. This is no longer true if $n > 1$, but we show a characterization of the complexes in $\mathcal{H}$ in terms of their cycles and boundaries. The description will be very useful in section 7.

Lemma 4.6. *Let $\mathcal{H}$ be the heart of the t-structure induced by an n-tilting module T. A complex $X \in \mathcal{D}(R)$ belongs to $\mathcal{H}$ if and only if it satisfies the following two conditions:*

(1) *X is quasi isomorphic to a complex*

$$\cdots \to 0 \to X^{-n} \xrightarrow{d^{-n}} X^{-n+1} \to \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \to 0,$$

with $X^{-i} \in T^{\perp}$ for all $0 \leq i \leq n$.

(2) *For every $1 \leq i \leq n$, the following hold true:*
   (a) $\text{Hom}_R(T, \text{Ker}\, d^{-i}) = \text{Hom}_R(T, \text{Im}\, d^{-i-1})$*; that is, the trace of T in $\text{Ker}\, d^{-i}$ coincides with $\text{Im}\, d^{-i-1}$.*
   (b) $\text{Ext}^1_R(T, \text{Ker}\, d^{-i-1}) = 0$.

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In particular, if $X \in \mathcal{H}$ and $H^{-i}(X) = 0$ for every $i > j$, then

(i) *X is quasi isomorphic to*

$$\cdots \to 0 \to X^{-j} \stackrel{d^{-j}}{\to} X^{-j+1} \to \cdots \to X^{-1} \stackrel{d^{-1}}{\to} X^0 \to 0,$$

with X^{-n} in $T^\perp$ for all $0 \le n \le j$;

(ii) $H^{-j}(X) \in T^{\perp_0}$.

Proof. Let $X \in \mathcal{H}$. By Lemma 3.4 we can assume that X is of the form

$$\cdots \to X^{-n-1} \to X^{-n} \to X^{-n+1} \to \cdots \to X^{-1} \to X^0 \to 0 \to 0 \ldots,$$

with X^i in $T^\perp$. By Proposition 4.2 the $-n-1$ truncation of X is an exact complex, so X is isomorphic to

$$0 \to X^{-n}/\operatorname{Im} d^{-n-1} \to X^{-n+1} \to \cdots \to X^{-1} \to X^0 \to 0 \to 0 \ldots,$$

where $X^{-n}/\operatorname{Im} d^{-n-1}$ is in $T^\perp$ since $\operatorname{Ker} d^{-n-1} \in T^\perp$. This establishes condition (1).

Condition (2) is the translation of the fact that $\operatorname{Hom}_{\mathcal{K}(R)}(T[i], X) = 0$ for every $i \neq 0$.

Conversely, it is easy to check that a complex satisfying conditions (1) and (2) belongs to $\mathcal{H}$.

The last two statements follow easily from (1) and (2). □

On the basis of the above description we exhibit some objects of $\mathcal{H}$ using special $T^\perp$–pre-envelopes of R-modules.

Notation 4.7. If T is an n-tilting module and $N \in$ Mod-R is an arbitrary R-module, we consider the following:

(1) See [GT12, Chap. 13]. A $T^\perp$-coresolution of N, that is, an exact sequence

$$0 \to N \to B^0 \to B^1 \to B^2 \to \cdots \to B^n \to 0,$$

where B^0 is a special $T^\perp$–pre-envelope of N and for every $i \ge 0$, B^{i+1} is a special $T^\perp$–pre-envelope of $A_i = \operatorname{Coker} B^{i-1} \to B^i$ (let $B^{-1} = N$).

(2) A short exact sequence $0 \to K \to T^{(\operatorname{Hom}(T,N))} \to \operatorname{tr}_T(N) \to 0$, where $\operatorname{tr}_T(N)$ denotes the trace of T in N and $K \in T^{\perp_1}$.

Moreover, a $T^\perp$-coresolution of N as in (1) can be chosen functorially in N, since the cotorsion pair generated by T is functorially complete thanks to Quillen's small object argument and the sequence in (2) is functorial in N by construction.

Proposition 4.8. *Let N be an R-module. Consider a functorial $T^\perp$-coresolution of N as in Notation 4.7 (1) and a list of exact sequences as in Notation 4.7 (2) starting with $0 \to K_2 \to T^{(\alpha_2)} \to tr_T(N) \to 0$ and continuing with $0 \to K_{i+1} \to T^{(\alpha_{i+1})} \to \operatorname{tr}_T(K_i) \to 0$ for every $i \ge 2$. Glue them together to construct the complex*

$$X_N = \cdots \to T^{(\alpha_i)} \to \cdots \to T^{(\alpha_2)} \to B^0 \to B^1 \to 0,$$

in degrees ≤ 0 with differentials given by the obvious compositions of the morphisms involved in the short exact sequences. Then $X_N \in \mathcal{H}$.

Moreover,

(1) *If $N \in T^{\perp_0}$, the complex $X_N = 0 \to B^0 \to B^1 \to 0$ (in degrees $-1, 0$) belongs to $\mathcal{H}$.*

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(2) *If $N \in T^{\perp_1}$, the complex*

$$X_N = \cdots \to T^{(\alpha_n)} \to \cdots \to T^{(\alpha_3)} \to B^0 \to B^1 \to B^2 \to 0,$$

in degrees ≤ 0 obtained by glueing the short exact sequence $0 \to K_3 \to T^{(\alpha_3)} \to \operatorname{tr}_T(N) \to 0$ and the sequences $0 \to K_{i+1} \to T^{(\alpha_{i+1})} \to \operatorname{tr}_T(K_i) \to 0$ for every $i \geq 3$, belongs to $\mathcal{H}$.

(3) *If $N \in T^{\perp_1} \cap T^{\perp_2} \cap \cdots \cap T^{\perp_{n-1}}$, the complex*

$$X_N = \cdots \to T^{(\alpha_i)} \to \cdots \to T^{(\alpha_{n+1})} \to B^0 \to B^1 \to B^2 \to \cdots \to B^n \to 0,$$

in degrees ≤ 0 obtained by glueing the short exact sequences $0 \to K_{n+1} \to T^{(\alpha_{n+1})} \to \operatorname{tr}_T(N) \to 0$ and $0 \to K_{i+1} \to T^{(\alpha_{i+1})} \to \operatorname{tr}_T(K_i) \to 0$ for every $i \geq n+1$, belongs to $\mathcal{H}$.

Proof. The proof follows easily by the characterization of the complexes in $\mathcal{H}$ stated in Lemma 4.6. □

We can apply the previous proposition to obtain information about the torsion radical of the torsion pair induced by an n-tilting module T.

Corollary 4.9. *Let T be an n-tilting module T and consider the torsion pair $({}^{\perp_0}(T^{\perp_0}), T^{\perp_0})$ associated with T.*

If $N \in T^{\perp_1} \cap T^{\perp_2} \cap \cdots \cap T^{\perp_{n-1}}$, then the torsion submodule of N in the torsion pair is given by $\operatorname{tr}_T(N)$. Moreover, $\operatorname{tr}_T(N) \in T^{\perp_n} \cap T^{\perp_{n-1}}$ and $N/\operatorname{tr}_T(N) \in T^{\perp_0} \cap T^{\perp_{n-1}}$.

In particular, if $n = 2$ and $N \in T^{\perp_1}$, $\operatorname{tr}_T(N) \in T^{\perp}$ and $N/\operatorname{tr}_T(N) \in T^{\perp_0} \cap T^{\perp_1}$.

Proof. Given N, consider the complex X_N constructed in Proposition 4.8 (3). Since $X_N \in \mathcal{H}$, $H^{-n}(X) \in T^{\perp_0}$, by Lemma 4.6 (ii) and by construction $H^{-n}(X) \cong N/\operatorname{tr}_T(N)$. □

Another application of Proposition 4.8 is given by the following.

Proposition 4.10. *The heart $\mathcal{H}$ is closed under coproducts in $\mathcal{D}(R)$ if and only if $T^{\perp_1}$ is closed under direct sums in* Mod-R.

Proof. Let $(X_\alpha : \alpha \in \Lambda)$ be a family of objects of $\mathcal{H}$. Up to isomorphisms the X_α's are represented by complexes in $\mathcal{C}h(R)$ as described in Lemma 4.6. Let X be the coproduct of the X_α's in $\mathcal{C}h(R)$. The cycles and the boundaries of the complex X are the coproducts of the cycles and the boundaries of the complexes X_α. Thus, if $T^{\perp_1}$ is closed under direct sums, X satisfies conditions (1) and (2) of Lemma 4.6; hence, $X \in \mathcal{H}$.

Conversely, assume that $\mathcal{H}$ is closed under coproducts in $\mathcal{D}(R)$ and let $(N_\alpha : \alpha \in \Lambda)$ be a family of modules in $T^{\perp_1}$. For each α consider the complex $X_{N_\alpha} \in \mathcal{H}$ constructed in Proposition 4.8 (2). By assumption the coproduct in $\mathcal{D}(R)$ of the X_{N_α}'s belongs to $\mathcal{H}$ and—again by Lemma 4.6, conditions (1) and (2)—we conclude that $\operatorname{Ext}^1_R(T, \oplus N_\alpha) = 0$. □

Remark 4.11. The condition that $T^{\perp_1}$ be closed under direct sums is automatically true for a 1-tilting module T, but in general it is not true for an n-tilting module with $n > 1$.

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5. The heart $\mathcal{H}$ and the module category over $\mathrm{End}(T)$

We will make use of the results for the derived equivalence induced by *good* n-tilting modules proved in [BMT11] and [BP13].

Definition 5.1. An n-tilting module T_R is *good* if the terms in the exact sequence (T3) in Definition 1.1 can be chosen to be direct summands of finite direct sums of copies of T. By [BMT11, Proposition 1.3] every n-tilting module T_R is equivalent to a good n-tilting module.

We recall the following facts about good n-tilting modules.

Fact 5.2. Let T_R be a good n-tilting module with $S = \mathrm{End}(T_R)$. The following hold:

(1) See [BMT11] and [Miy86].
   (a) $\mathrm{p.d.}_S T \leq n$, and ${}_S T$ has a finite projective resolution consisting of finitely generated projective left S-modules.
   (b) $\mathrm{Ext}^i_S(T, T^{(\alpha)}) = 0$ for every $i \geq 1$ and every cardinal α.

(2) See [BMT11, Theorem 2.2], [BP13, Proposition 5.2].
   (i) The pair $(\mathbb{L}G, \mathbb{R}H)$

$$\mathcal{D}(R) \underset{\mathbb{R}H=\mathbf{R}\mathrm{Hom}_R(T,-)}{\overset{\mathbb{L}G=-\otimes^{\mathbf{L}}_S T}{\rightleftarrows}} \mathcal{D}(S)$$

   is an adjoint pair.
   (ii) The functor $\mathbb{R}H : \mathcal{D}(R) \to \mathcal{D}(S)$ is fully faithful.
   (iii) The essential image of $\mathbf{R}\mathrm{Hom}_R(T, -)$ is $\mathrm{Ker}(\mathbb{L}G)^{\perp}$ where

$$\mathrm{Ker}(\mathbb{L}G)^{\perp} = \{Z \in \mathcal{D}(S) \mid \mathrm{Hom}_{\mathcal{D}(S)}(Y, Z) = 0 \text{ for all } Y \in \mathrm{Ker}(\mathbb{L}G)\}.$$

We illustrate a property of the functor $\mathbb{L}G$ which will be useful in section 6.

Lemma 5.3. *Let $(M_\alpha; g_{\beta\alpha})_{\alpha\in\Lambda}$ be a direct system of right S-modules. There are projective resolutions P_α of M_α such that the direct system $(M_\alpha; g_{\beta\alpha})$ can be lifted to a direct system $(P_\alpha; \tilde{g}_{\beta\alpha})_{\alpha\in\Lambda}$ in $\mathcal{C}h(S)$ giving rise to the following isomorphisms in $\mathcal{D}(R)$:*

$$M_\alpha \otimes^{\mathbf{L}}_S T \cong P_\alpha \otimes_S T; \ \varinjlim_{\mathcal{C}h(R)} (P_\alpha \otimes_S T) \cong (\varinjlim_{\mathcal{C}h(S)} P_\alpha) \otimes_S T \cong (\varinjlim_{\text{Mod-}S} M_\alpha) \otimes^{\mathbf{L}}_S T.$$

Proof. The observation that for every module $A \in \text{Mod-}S$ the canonical epimorphism $S^{(\mathrm{Hom}_S(S,A))} \to A$ is functorial in A implies that, for each $M_\alpha \in \text{Mod-}S$, we can choose functorially a projective resolution P_α so that the direct system $(M_\alpha; g_{\beta\alpha})_{\alpha\in\Lambda}$ in Mod-S can be lifted to a direct system $(P_\alpha; \tilde{g}_{\beta\alpha})_{\alpha\in\Lambda}$ in $\mathcal{C}h(S)$. We have $M_\alpha \otimes^{\mathbf{L}}_S T \cong P_\alpha \otimes_S T$ and, since $\varinjlim_{\mathcal{C}h(S)} P_\alpha$ is a flat resolution of $M = \varinjlim_{\text{Mod-}S} M_\alpha$, we also get $M \otimes^{\mathbf{L}}_S T \cong (\varinjlim_{\mathcal{C}h(S)} P_\alpha) \otimes_S T$. Thus, the following isomorphisms hold in $\mathcal{D}(R)$:

$$\varinjlim_{\mathcal{C}h(R)} (P_\alpha \otimes_S T) \cong (\varinjlim_{\mathcal{C}h(S)} P_\alpha) \otimes_S T \cong (\varinjlim_{\text{Mod-}S} M_\alpha) \otimes^{\mathbf{L}}_S T.$$

□

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From now on in this section, $\mathcal{H}$ will always denote the heart of the t-structure induced by a good n-tilting module T_R with endomorphism ring S.

A characterization of the objects in $\mathcal{H}$ is given by the following.

Lemma 5.4. *A complex in $\mathcal{D}(R)$ belongs to $\mathcal{H}$ if and only if it is isomorphic to a complex X with terms in the tilting class $T_R^{\perp}$ such that $\mathbf{R}\mathrm{Hom}_R(T, X)$ has cohomology concentrated in degree 0 and isomorphic to $\mathrm{Hom}_{\mathcal{D}(R)}(T, X)$ (and thus also to $\mathrm{Hom}_{\mathcal{K}(R)}(T, X)$ by Corollary 2.7).*

Proof. By Lemma 2.6 every complex in $\mathcal{D}(R)$ is isomorphic to a complex X with terms in $T^{\perp}$. Thus, X is a $\mathrm{Hom}_R(T, -)$-acyclic object and since p.d.$T \leq n$, $\mathbf{R}\mathrm{Hom}_R(T, X) \cong \mathrm{Hom}_R(T, X)$ (see, e.g., [Har66, Theorem I.5.1]). Moreover, $H^{-i}(\mathrm{Hom}_R(T, X) \cong \mathrm{Hom}_{\mathcal{K}(R)}(T[i], X)$ and, by Corollary 2.7, $\mathrm{Hom}_{\mathcal{K}(R)}(T[i], X) \cong \mathrm{Hom}_{\mathcal{D}(R)}(T[i], X)$.

Hence, if $X \in \mathcal{H}$, then $\mathbf{R}\mathrm{Hom}_R(T, X)$ has cohomology concentrated in degree 0 and it is isomorphic to $\mathrm{Hom}_{\mathcal{K}(R)}(T, X) \cong \mathrm{Hom}_{\mathcal{D}(R)}(T, X)$.

Conversely, if $\mathbf{R}\mathrm{Hom}_R(T, X) \cong \mathrm{Hom}_{\mathcal{D}(R)}(T, X)$, then $\mathrm{Hom}_{\mathcal{K}(R)}(T[i], X) = 0$ for every $i \neq 0$; hence, $X \in \mathcal{H}$. □

Proposition 5.5. *The following condition hold true:*

1. *The restriction to $\mathcal{H}$ of the functor $\mathbf{R}\mathrm{Hom}_R(T, -)$ has an essential image in Mod-S, and it gives a functor $H_T = \mathrm{Hom}_{\mathcal{H}}(T, -)\colon \mathcal{H} \to$ Mod-S which is exact and fully faithful.*
2. *The essential image of $\mathrm{Hom}_{\mathcal{H}}(T, -)$ is given by $\mathrm{Ker}(\mathbb{L}G)^{\perp} \cap$ Mod-S.*
3. *H_T has a left adjoint F given by $b \circ G$, where G is the restriction to Mod-S of the functor $\mathbb{L}G$ and b is left adjoint of the inclusion functor $\iota\colon \mathcal{H} \to \mathcal{U}$.*
4. *There is an equivalence $\mathcal{H} \cong$ Mod-$S[\Sigma^{-1}]$, where $\Sigma = \{g \in$ Mod-$S \mid F(g)$ is an isomorphism$\}$.*

Proof. (1) The functor $\mathrm{Hom}_{\mathcal{H}}(T, -)$ has image in Mod-S by Lemma 5.4 and it is exact since T is a projective object of $\mathcal{H}$, by Proposition 4.2. Let $X \in \mathcal{H}$; we can assume that X has terms in $T^{\perp}$. By Lemma 5.4 $\mathbf{R}\mathrm{Hom}_R(T, X)$ is isomorphic to $\mathrm{Hom}_{\mathcal{D}(R)}(T, X)$, and $\mathrm{Hom}_{\mathcal{D}(R)}(T, X) \cong \mathrm{Hom}_{\mathcal{H}}(T, X)$ since $\mathcal{H}$ is a full subcategory of $\mathcal{D}(R)$. Thus, the functor $H_T = \mathrm{Hom}_{\mathcal{H}}(T, -)$ is isomorphic to the restriction to $\mathcal{H}$ of the functor $\mathbf{R}\mathrm{Hom}_R(T, -)$; hence, H_T is fully faithful, by Fact 5.2 (ii).

(2) It follows easily from Fact 5.2 (iii) and Lemma 5.4.

(3) The functor $\mathbb{L}G$ is left adjoint to $\mathbb{R}H$ and, for every right S-module M_S, $\mathbb{L}G(M) \cong P_M \otimes_S T$, where P_M is a projective resolution of M_S. Thus, $\mathbb{L}G(M)$ is isomorphic to a complex of R-modules with terms in Add$T$ and in degrees $i \leq 0$. By Lemma 3.4 $\mathbb{L}G(M) \in \mathcal{U}$. By Remark 4.1 the inclusion $\iota\colon \mathcal{H} \to \mathcal{U}$ admits a left adjoint b. We show now that the functor $F = b \circ G$, where G is the restriction of $\mathbb{L}G$ to Mod-S, is left adjoint to H_T. Let $X \in \mathcal{H}$ and $M \in$ Mod-S; then

$$\mathrm{Hom}_S(M, H_T(X)) \cong \mathrm{Hom}_{\mathcal{D}(S)}(M, \mathbf{R}\mathrm{Hom}_R(T, X)) \cong \mathrm{Hom}_{\mathcal{D}(R)}(G(M), X) \cong$$
$$\cong \mathrm{Hom}_{\mathcal{U}}((G(M), X) \cong \mathrm{Hom}_{\mathcal{H}}(b(G(M), X).$$

(4) It follows from [GZ67, Proposition 1.3]. □

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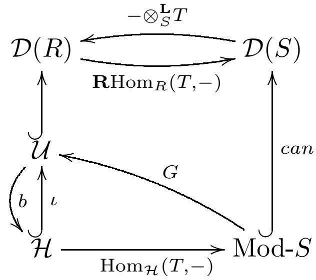
The diagram



can depict the situation described by Proposition 5.5.

By Proposition 5.5 (1) and (2), the functor $\mathrm{Hom}_{\mathcal{H}}(T,-)$ induces an equivalence between $\mathcal{H}$ and

$$\mathrm{Ker}(\mathbb{L}G)^{\perp} \cap \text{Mod-}S.$$

For the rest of this section we deal with our main concern, which is to characterize the case in which the heart $\mathcal{H}$ of the t-structure induced by a good n-tilting module is a Grothendieck category. In Theorems 5.10 and 5.12 we will give characterizations in terms of properties of subcategories of Mod-S.

A first observation is obtained by an application of the Gabriel–Popescu theorem [PG64].

Proposition 5.6. *Let* $F\colon \text{Mod-}S \to \mathcal{H}$ *be the left adjoint of the functor* $\mathrm{Hom}_{\mathcal{H}}(T,-)$ *given by Proposition* 5.5 *(3). The following are equivalent:*

(1) *$\mathcal{H}$ is a Grothendieck category.*
(2) *F is an exact functor.*
(3) $\mathrm{Ker}\, F$ *is a hereditary torsion class in* Mod-S*.*

Proof.
(1) ⇒ (2) follows from Gabriel–Popescu's theorem [PG64], since T is a generator of $\mathcal{H}$ by Proposition 4.3.

(2) ⇒ (3) From (2) it follows that $\mathrm{Ker}\, F$ is a Serre subcategory of Mod-S; that is, for every short exact sequence $0 \to N \to M \to L \to 0$ in Mod-S, $M \in \mathrm{Ker}\, F$ if and only if N and L are in $\mathrm{Ker}\, F$. Since F is a left adjoint, it sends coproducts in Mod-S to coproducts in $\mathcal{H}$. So $\mathrm{Ker}\, F$ is a hereditary torsion class.

(3) ⇒ (1) As in Proposition 5.5, let

$$\Sigma = \{g \in \text{Mod-}S \mid F(g) \text{ is an isomorphism}\}.$$

When $\mathrm{Ker}\, F$ is a hereditary torsion class, then by [Gab62, Chap. III], $g \in \Sigma$ if and only if $\mathrm{Ker}\, g$ and $\mathrm{Coker}\, g$ belong to $\mathrm{Ker}\, F$. Thus,

$$\text{Mod-}S[\Sigma^{-1}] \cong \text{Mod-}S/\,\mathrm{Ker}\, F$$

and the latter category is well known to be a Grothendieck category. The conclusion follows from Proposition 5.5 (4). □

If $\mathcal{C}$ is a subcategory of an abelian category $\mathcal{A}$, its perpendicular category, denoted by $\mathcal{C}_{\perp}$, is defined by

$$\mathcal{C}_{\perp} = \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(C,X) = \mathrm{Ext}^1_{\mathcal{A}}(C,X) = 0 \text{ for all } C \in \mathcal{C}\}.$$

We define also

$$\mathcal{C}_{\perp_\infty} = \{X \in \mathcal{A} \mid \mathrm{Ext}^i_{\mathcal{A}}(C,X) = 0 \text{ for all } C \in \mathcal{C} \text{ and for all } i \geq 0\}.$$

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Definition 5.7. If T_R is a good n-tilting module with endomorphism ring S, we let

$$\mathcal{E} = \{M_S \in \text{Mod-}S \mid \text{Tor}_i^S(M, T) = 0 \text{ for all } i \geq 0\}.$$

Lemma 5.8. *The essential image of the functor* $\text{Hom}_{\mathcal{H}}(T, -)$ *given in Proposition* 5.5 *is contained in* $\mathcal{E}_{\perp_\infty}$*, and they coincide in the case in which* $\mathcal{E}$ *is a hereditary torsion class in* Mod-S.

Proof. Let $M \cong \text{Hom}_{\mathcal{H}}(T, X)$ for some $X \in \mathcal{H}$. For every $E \in \mathcal{E}$ and every $j \in \mathbb{Z}$, $E[j]$ belongs to $\text{Ker}(\mathbb{L}G)$; hence, by Proposition 5.5 (2), $\text{Hom}_{\mathcal{D}(S)}(E[j], M) = 0$. But $\text{Hom}_{\mathcal{D}(S)}(E[j], M) \cong \text{Ext}_S^{-j}(E, M)$ for every $j \in \mathbb{Z}$; hence, $M \in \mathcal{E}_{\perp_\infty}$.

To prove the other statement, we have to show that, if $\mathcal{E}$ is hereditary and $M \in \mathcal{E}_{\perp_\infty}$, then $\text{Hom}_{\mathcal{D}(S)}(Y, M) = 0$ for every $Y \in \text{Ker}(\mathbb{L}G)$. Note that a complex belongs to $\text{Ker}(\mathbb{L}G)$ if and only if it is quasi isomorphic to a complex with terms in $\mathcal{E}$: This has been proved in [CX12, Proposition 4.6] for the case of a good 1-tilting module, but using [BMT11, Lemma 1.5], everything goes through for the $n > 1$ case. The cycles and the boundaries of every complex with terms in $\mathcal{E}$ are again in $\mathcal{E}$, since we are assuming that $\mathcal{E}$ is a hereditary torsion class.

(a) We first prove that if Z is a bounded complex with terms in $\mathcal{E}$ and $M \in \mathcal{E}_{\perp_\infty}$, then $\text{Hom}_{\mathcal{D}(S)}(Z, M) = 0$. We make induction on the number k of nonzero terms in Z. If $k = 1$, Z is of the form $E[j]$ for some $j \in \mathbb{Z}$ and some $E \in \mathcal{E}$. Thus, $\text{Hom}_{\mathcal{D}(S)}(E[j], M) \cong \text{Ext}_S^{-j}(E, M) = 0$. Let now $Z = 0 \to E^i \to E^{i+1} \to \cdots \to E^{i+k} \to 0$, and let K be the kernel of the ith-differential of Z. Since $\mathcal{E}$ is hereditary, we have a triangle $K \to Z \to Z' \to K[1]$ where $K \in \mathcal{E}$ and Z' is a bounded complex with at most $k-1$ terms in $\mathcal{E}$. Thus, by induction $\text{Hom}_{\mathcal{D}(S)}(Z', M) = 0$, hence also $\text{Hom}_{\mathcal{D}(S)}(Z, M) = 0$.

(b) Let now $Y \in \text{Ker}(\mathbb{L}G)$ be a bounded below complex. Then Y is a homotopy colimit of its truncation subcomplexes Z_n which is bounded and with terms in $\mathcal{E}$, again by the hereditary condition on $\mathcal{E}$. Hence, from the triangle $\coprod_i Z_i \to \coprod_i Z_i \to Y \to \coprod_i Z_i[1]$ and by (a), we conclude that $\text{Hom}_{\mathcal{D}(S)}(Y, M) = 0$. It remains to consider the case of a bounded above complex $Y \in \text{Ker}(\mathbb{L}G)$. (Note that Y is a homotopy limit of its quotient complexes obtained from truncations, which are bounded and with terms in $\mathcal{E}$, but the triangle of the homotopy limit does not help us to conclude.)

(c) To prove that $\text{Hom}_{\mathcal{D}(S)}(Y, M) = 0$ for a bounded above complex, we consider a suitable model structure on $\mathcal{D}(S)$ described as follows. Let W be an injective cogenerator of Mod-R and let $(-)^d = \text{Hom}_R(-, W)$ denote the dual of any right R-module. Then $T^d = C$ is a pure injective right S-module and, by well-known homological formulas, we have $\text{Ext}_S^i(N, C) \cong [\text{Tor}_i^S(N, T)]^d$ for every right S-module N and every $i \geq 0$. Hence, $\mathcal{E} \subseteq {}^{\perp}C$. We consider the model structure on $\mathcal{D}(S)$ induced by the complete cotorsion pair $({}^{\perp}C, ({}^{\perp}C)^{\perp})$, as described in Theorem 2.8. Let Y be a bounded above complex with terms in $\mathcal{E}$ (hence, $Y \in \text{Ker}(\mathbb{L}G)$). By Theorem 2.8 Y is a cofibrant object in the model structure induced by C. For every $N \in \text{Mod-}S$ a fibrant replacement of N is a $({}^{\perp}C)^{\perp}$-coresolution of N constructed by taking special $({}^{\perp}C)^{\perp}$–pre-envelopes. Now let $M \in \mathcal{E}_{\perp_\infty}$ and let I be its fibrant replacement. By Corollary 2.9 $\text{Hom}_{\mathcal{D}(S)}(Y, M) \cong \text{Hom}_{\mathcal{K}(S)}(Y, I)$ and by (a) and the hereditary condition on $\mathcal{E}$ we can assume that Y has nonzero terms only in

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degrees $i \leq 0$. If $f\colon Y \to I$ is a cochain map, we have

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & Y^{-2} & \longrightarrow & Y^{-1} & \longrightarrow & Y^{0} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & & & & \downarrow & & \downarrow{\scriptstyle f^0} & & \downarrow & & \\
 & & & & \cdots & \longrightarrow & 0 & \longrightarrow & I^{0} & \xrightarrow{d^0} & I^{1} & \longrightarrow & I^2 \longrightarrow \cdots,
\end{array}$$

where $\operatorname{Im} f^0 \subseteq \operatorname{Ker} d^0 = M$. Since $\operatorname{Hom}_S(E, M) = 0$ for every $E \in \mathcal{E}$, we conclude that $f^0 = 0$ and thus also that $\operatorname{Hom}_{\mathcal{D}(S)}(Y, M) = 0$. □

Proposition 5.9. *Assume that $\mathcal{H}$ is a Grothendieck category. Let $\mathcal{E}$ be as in Definition 5.7, and let F be the left adjoint of the functor $\operatorname{Hom}_{\mathcal{H}}(T, -)$ given by Proposition 5.5 (3). Then the following hold true:*

1. (1) $\operatorname{Ker} F$ *is a hereditary torsion class and* $\operatorname{Ker} F = \mathcal{E}$.
2. (2) $(\operatorname{Ker} F)_\perp$ *is the essential image of the functor* $\operatorname{Hom}_{\mathcal{H}}(T,)\colon \mathcal{H} \to \text{Mod-}S$.
3. (3) $\mathcal{E}_{\perp_\infty} = \mathcal{E}_\perp = (\operatorname{Ker} F)_\perp$.

Proof.

(1) $\operatorname{Ker} F$ is a hereditary torsion class by Proposition 5.6. The inclusion $\mathcal{E} \subseteq \operatorname{Ker} F$ is immediate since if $M \in \mathcal{E}$, then $\mathbb{L}G(M) = 0$; hence, $F(M) = 0$.

For the converse let $M \in \operatorname{Ker} F$, and let

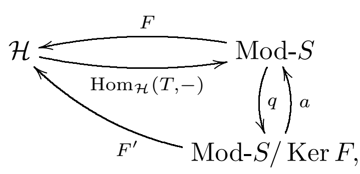
$$P_M = \cdots P^{-i} \overset{d^{-i}}{\to} P^{-i+1} \to \cdots \to P^{-1} \overset{d^{-1}}{\to} P^0 \to M \to 0$$

be a projective resolution of M. By Proposition 5.6 F is an exact functor; hence, the sequence

$$\ldots F(P^{-i}) \overset{F(d^{-i})}{\longrightarrow} F(P^{-i+1}) \to \cdots \to F(P^{-1}) \overset{F(d^{-1})}{\longrightarrow} F(P^0) \to F(M) = 0 \to 0$$

is exact in $\mathcal{H}$ and $F(P^{-i}) \cong P^{-i} \otimes_S T$ belongs to $\operatorname{Add}T$, so $F(d^{-i})$ is naturally isomorphic to $d^{-i} \otimes_S 1_T$. We infer that $P_M \otimes_S T = 0$; hence, $M \in \mathcal{E}$.

(2) By (1) and by [Gab62, Chap. III], the canonical quotient functor $q\colon \text{Mod-}S \to \text{Mod-}S/\operatorname{Ker} F$ is exact and admits a fully faithful right adjoint a whose essential image is the perpendicular category $(\operatorname{Ker} F)_\perp$ consisting of the closed objects. By Proposition 5.5 we have the following diagram:



where F' is the unique functor such that $F' \circ q = F$ and F' is an equivalence of categories. Let D be an inverse of F'; then (F', D) is an adjoint pair, so (F, aD) is an adjoint pair. Hence, the functor $H_T = \operatorname{Hom}_{\mathcal{H}}(T, -)$ is naturally isomorphic to the functor aD and thus also $a \cong H_T \circ F'$. We conclude that the essential images of a and H_T coincide, so they coincide with $(\operatorname{Ker} F)_\perp$.

(3) Clearly $\mathcal{E}_{\perp_\infty} \subseteq \mathcal{E}_\perp$. In view of conditions (1) and (2), to show the reverse inclusion, it is enough to prove that the essential image of the functor $\operatorname{Hom}_{\mathcal{H}}(T,)$ is contained in $\mathcal{E}_{\perp_\infty}$. Let M_S be of the form $\operatorname{Hom}_{\mathcal{H}}(T, X)$ for some $X \in \mathcal{H}$. By Proposition 5.5 (2) we have $\operatorname{Hom}_{\mathcal{D}(S)}(Y, M) = 0$ for every $Y \in \operatorname{Ker}(\mathbb{L}G)$. In particular, for every $E \in \mathcal{E}$, $\operatorname{Hom}_{\mathcal{D}(S)}(E[j], M) = 0$ for every $j \in \mathbb{Z}$; hence, $\operatorname{Ext}^j_S(E, M) = 0$ for every $j \geq 0$. That is, $M \in \mathcal{E}_{\perp_\infty}$. □

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Theorem 5.10. *Let $\mathcal{E}$ be as in Definition 5.7. The following are equivalent:*

(1) *$\mathcal{H}$ is a Grothendieck category.*
(2) *$\mathcal{E}$ is a hereditary torsion class in* Mod-S *and $\mathcal{E}_{\perp} = \mathcal{E}_{\perp_\infty}$.*
(3) *For every $M \in$ Mod-S, $\mathbb{L}G(M) \in \mathcal{H}$.*
(4) *If G is the restriction of $\mathbb{L}G$ to* Mod-S*, then G is naturally isomorphic to the left adjoint F of the fully faithful functor $\mathrm{Hom}_{\mathcal{H}}(T, -)\colon \mathcal{H} \to$ Mod-S constructed in Proposition* 5.6 *and G is an exact functor.*

Proof.

(1) $\Rightarrow$ (2) This follows from Proposition 5.9.

(2) $\Rightarrow$ (3) Let $M \in$ Mod-S and let $M_{\mathcal{E}}$ be the torsion submodule of M with respect to the torsion pair $(\mathcal{E}, \mathcal{E}^{\perp_0})$. Then, clearly, $\mathbb{L}G(M/M_{\mathcal{E}}) \cong \mathbb{L}G(M)$, so, w.l.o.g. we may assume that M is $\mathcal{E}$-torsion free.

By [GL91, Proposition 2.2] M has an $\mathcal{E}_{\perp}$-reflection. That is, there is a short exact sequence $0 \to M \to Y \to E \to 0$, with $Y \in \mathcal{E}_{\perp}$ and $E \in \mathcal{E}$. Thus, $\mathbb{L}G(Y) \cong \mathbb{L}G(M)$. Now, by assumption and Lemma 5.8, there is an $X \in \mathcal{H}$ such that $\mathrm{Hom}_{\mathcal{H}}(T, X) \cong Y$. By Lemma 5.4 $\mathrm{Hom}_{\mathcal{H}}(T, X) \cong \mathrm{Hom}_{\mathcal{D}(R)}(T, X) \cong \mathbf{R}\mathrm{Hom}_R(T, X)$; hence, $\mathbb{L}G(Y) \cong \mathbb{L}G(\mathbf{R}\mathrm{Hom}_R(T, X)) \cong X$ since $\mathrm{Hom}_{\mathcal{H}}(T, -)$ is fully faithful. We conclude that $\mathbb{L}G(M) \in \mathcal{H}$.

(3) $\Rightarrow$ (4) Let F be the left adjoint of the functor $\mathrm{Hom}_{\mathcal{H}}(T, -)\colon \mathcal{H} \to$ Mod-S given by Proposition 5.5 (3). Condition (3) implies that for every $M \in$ Mod-S, $F(M) \cong G(M)$. To show that G is an exact functor we prove that $\mathrm{Ker}\, G = \mathrm{Ker}\, F$ is a hereditary torsion class in Mod-S. Since G is right exact it is enough to show that $\mathrm{Ker}\, G$ is closed under submodules. Let $0 \to M \to L \to N \to 0$ be an exact sequence in Mod-S, with $L \in \mathrm{Ker}\, G$. From the triangle $N[-1] \to M \to L \to N$ we have the triangle

$$N \otimes^{\mathbf{L}}_S T[-1] \to M \otimes^{\mathbf{L}}_S T \to L \otimes^{\mathbf{L}}_S T \to N \otimes^{\mathbf{L}}_S T$$

in $\mathcal{D}(R)$. By assumption $L \otimes^{\mathbf{L}}_S T = G(L) = 0$ and $N \otimes^{\mathbf{L}}_S T = G(N) = 0$. Then, also, $N \otimes^{\mathbf{L}}_S T[-1]$ is zero, and by the above triangle we conclude that $M \otimes^{\mathbf{L}}_S T$ is zero. Hence $G(M) = 0$.

(4) $\Rightarrow$ (1) This follows from Proposition 5.6. $\square$

We can now interpret the previous characterization in terms of homological epimorphisms and a generalized universal localization, which is a generalization of the well-known concept of universal localization in Schofield's sense [Sch85].

Definition 5.11.

(1) See [GL91]. A ring homomorphism $f\colon S \to U$ is a homological ring epimorphism if the associated restriction functor $f_*\colon \mathcal{D}(U) \to \mathcal{D}(S)$ is fully faithful. Equivalently, $f\colon S \to U$ is a ring epimorphism and $\mathrm{Tor}_i^S(U, U) = 0$ for every $i \geq 1$.
(2) See [Kra05, section 15]. Let S be a ring and Σ a set of perfect complexes $P \in \mathcal{K}(S^{\mathrm{op}})$. A ring U is a generalized universal localization of S at the set Σ if there is a ring homomorphism $\lambda\colon S \to U$ such that $U \underset{S}{\otimes} P$ is acyclic and λ satisfies the universal property with respect to this property. That is, for every ring homomorphism $\mu\colon S \to R$ such that $R \underset{S}{\otimes} P$ is acyclic, there exists a unique ring homomorphism $\nu\colon U \to R$ such that $\nu \circ \lambda = \mu$.

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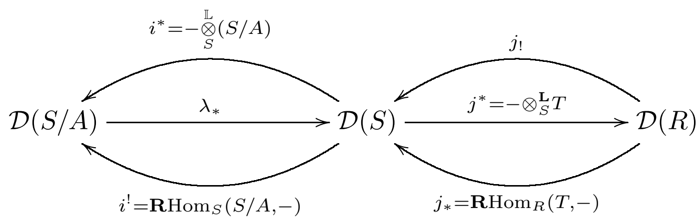
Theorem 5.12. *Let $\mathcal{H}$ be the heart of the t-structure induced by a good n-tilting module T_R with endomorphism ring S, and let $\mathcal{E}$ be as in Definition 5.7. The following are equivalent:*

(1) *$\mathcal{H}$ is a Grothendieck category.*
(2) *There is an idempotent two-sided ideal A of S, projective as a right S-module, such that $\mathcal{E} = \text{Mod-}S/A$.*

If the above conditions are satisfied, then the canonical morphism $S \to S/A$ is a homological epimorphism and S/A is a generalized universal localization at a projective resolution of ${}_ST$.

In particular,



is a recollement.

Proof. (1) $\Rightarrow$ (2) By Theorem 5.10 $\mathcal{E}$ is a hereditary torsion class. Since ${}_ST$ has a projective resolution consisting of finitely generated projective S-modules, $\mathcal{E}$ is closed under direct products. Thus, $\mathcal{E}$ is a torsion-torsion free class. By [Ste75, Proposition 6.11] there is an idempotent two-sided ideal A of S such that $\mathcal{E} = \text{Mod-}S/A$. By [BP13, Theorem 6.1] the canonical morphism $S \to S/A$ is a homological epimorphism; hence, by [GL91, Theorem 4.4], $\mathrm{Ext}^i_S(S/A, E) = 0$ for every $E \in \mathcal{E}$ and every $i \geq 1$. We show now that A_S is moreover a projective module.

By Theorem 5.10, we know that $\mathcal{E}_\perp = \mathcal{E}_{\perp_\infty}$. Hence, for every module $Y \in \mathcal{E}_\perp$, $\mathrm{Ext}^i_S(S/A, Y) = 0$ for every $i \geq 1$. Let $M \in \text{Mod-}S$ and let $M_\mathcal{E}$ be the $\mathcal{E}$-torsion submodule of M. As in the proof of (3) $\Rightarrow$ (4) in Theorem 5.10, consider an $\mathcal{E}$-reflection of $M/M_\mathcal{E}$ that is a module $Y \in \mathcal{E}_\perp$ such that there is a short exact sequence $0 \to M/M_\mathcal{E} \to Y \to E \to 0$, with $E \in \mathcal{E}$. By the above remarks we have $\mathrm{Ext}^i_S(S/A, M/M_\mathcal{E}) = 0$ for every $i \geq 2$ and, from the exact sequence $0 \to M_\mathcal{E} \to M \to M/M_\mathcal{E} \to 0$, we conclude that $\mathrm{Ext}^i_S(S/A, M) = 0$ for every $i \geq 2$; hence, p.d.$S/A \leq 1$, and A is projective as a right S-module.

(2) $\Rightarrow$ (1) $\mathcal{E} = \text{Mod-}S/A$ implies that $\mathcal{E}$ is a hereditary torsion class, and the projectivity of A_S implies that $\mathcal{E}_\perp = \mathcal{E}_{\perp_\infty}$. Hence, the conclusion follows from Theorem 5.10.

The last statement follows from [BP13, Theorem 6.1, Proposition 7.3]. □

For the case of a good 1-tilting module, condition (2) in Theorem 5.10 can be weakened, since the assumption on $\mathcal{E}$ to be hereditary is enough. To see this, we use the following result [Baz10, section 4].

Remark 5.13 ([Baz10, section 4]). If ${}_ST_R$ is a good 1-tilting module and W is an injective cogenerator of Mod-R, we let $C = \mathrm{Hom}_R(T, W)$ be the dual of T. Then $\mathcal{T}_C = \{M_S \mid \mathrm{Hom}_S(M, C) = 0\} = \{M_S \mid M \otimes_S T = 0\}$ is a torsion class in Mod-S with corresponding torsion free class $\mathrm{Cogen}\, C \subseteq^\perp C$. Let $(\mathcal{T}, \mathcal{F})$ be the torsion pair

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in Mod-R induced by T_R. It follows that

$$\mathcal{T} \underset{\mathrm{Hom}_R(T,-)}{\overset{-\otimes_S T}{\leftrightarrows}} \operatorname{Cogen} C \cap \mathcal{E}_\perp$$

$$\mathcal{F} \underset{\mathrm{Ext}^1_R(T,-)}{\overset{\mathrm{Tor}^S_1(-,T)}{\leftrightarrows}} \mathcal{T}_C \cap \mathcal{E}_\perp$$

are equivalences.

Proposition 5.14. *Let $\mathcal{H}$ be the heart of the t-structure induced by a good 1-tilting module T_R with endomorphism ring S, and let $\mathcal{E}$ be as in Definition 5.7. The following are equivalent:*

(1) *$\mathcal{H}$ is a Grothendieck category.*
(2) *$\mathcal{E}$ is a hereditary torsion class in Mod-S.*
(3) *For every $M \in$ Mod-S, $\mathbb{L}G(M) \in \mathcal{H}$.*

Proof. In view of Theorem 5.10 it is enough to prove the implication (2) $\Rightarrow$ (3).

Let M be a right S-module. Since $\mathrm{Tor}^S_i(-,T) = 0$ for every $i \geq 2$, $\mathbb{L}G(M) \in \mathcal{H}$ if and only if $\mathrm{Tor}^S_1(M,T) \in T^{\perp_0}$. Let $M_\mathcal{E}$ be the torsion submodule of M with respect to the torsion pair $(\mathcal{E}, \mathcal{E}^{\perp_0})$. Then $\mathrm{Tor}^S_1(M,T) \cong \mathrm{Tor}^S_1(M/M_\mathcal{E},T)$; hence, w.l.o.g. we may assume that M is $\mathcal{E}$-torsion free. Let now M_C be the torsion submodule of M with respect to the torsion class $\mathcal{T}_C$ (see Remark 5.13). Then, $M/M_C \in \operatorname{Cogen} C \subseteq^\perp C$; hence, $\mathrm{Tor}^S_1(M/M_C,T) = 0$, so we can assume even that $M \in \mathcal{T}_C \cap \mathcal{E}^{\perp_0}$. As in the proof of (2) $\Rightarrow$ (3) in Theorem 5.10, if there is an exact sequence $0 \to M \to Y \to Y/M \to 0$, with $Y \in \mathcal{E}_\perp$ and $Y/M \in \mathcal{E}$, then $\mathrm{Tor}^S_1(M,T) \cong \mathrm{Tor}^S_1(Y,T)$. Since $E \otimes_S T = 0$ for every $E \in \mathcal{E}$ we have $\mathcal{E} \subseteq \mathcal{T}_C$; hence, $Y/M \in \mathcal{T}_C$. Thus, $Y \in \mathcal{E}_\perp \cap \mathcal{T}_C$ and, by Remark 5.13, $\mathrm{Tor}^S_1(Y,T) \in \mathcal{F}$. Thus, also $\mathrm{Tor}^S_1(M,T) \in \mathcal{F}$. □

6. Computing direct limits in the heart

In this section $\mathcal{H}$ will always denote the heart of the t-structure induced by an n-tilting module T_R.

We apply the characterization proved by Theorem 5.10 to show some properties of the category $\mathcal{H}$.

Proposition 6.1. *Assume that $\mathcal{H}$ is a Grothendieck category. Then the following hold true:*

(1) *$\mathcal{H}$ is closed under coproducts in $\mathcal{D}(R)$.*
(2) *The classes $T^{\perp_i}$ are closed under direct sums in Mod-R for every $i \geq 0$.*

Proof. (1) Let $(X_\alpha, \alpha \in \Lambda)$ be a family of objects in $\mathcal{H}$. By Lemma 5.4, $\mathrm{Hom}_\mathcal{H}(T, X_\alpha) = M_\alpha \in$ Mod-S. By Theorem 5.10 we have $\mathbb{L}G(M_\alpha) \cong X_\alpha$ and $\mathbb{L}G(\bigoplus_{\alpha\in\Lambda} M_\alpha) \cong \coprod_{\alpha\in\Lambda} \mathbb{L}G(M_\alpha)$ belongs to $\mathcal{H}$; hence, the coproduct $\coprod_{\alpha\in\Lambda} X_\alpha$ in $\mathcal{D}(R)$ belongs to $\mathcal{H}$.

(2) It is clear that $T^{\perp_0}$ is closed under direct sums. By Proposition 4.10 condition (1) implies that $T^{\perp_1}$ is closed under direct sums. We prove the statement by

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induction. Let $(N_\alpha, \alpha \in \Lambda)$ be a family of R-modules in $T^{\perp_i}$ with $i > 1$ and for every α consider a special $T^\perp$–pre-envelope $0 \to N_\alpha \to B_\alpha \to A_\alpha \to 0$ of N_α; then $A_\alpha \in T^{\perp_{i-1}}$. Consider the exact sequence $0 \to \oplus N_\alpha \to \oplus B_\alpha \to \oplus A_\alpha \to 0$. $\oplus B_\alpha$ belongs to the tilting class; hence, $\mathrm{Ext}^i_R(T, \oplus N_\alpha) \cong \mathrm{Ext}^{i-1}_R(T, \oplus A_\alpha)$ and the latter is zero by induction. □

Proposition 6.2. *Assume that $\mathcal{H}$ is a Grothendieck category. Consider a direct system $(X_\alpha; f_{\beta\alpha})_{\alpha\in\Lambda}$ of objects of $\mathcal{H}$, and let $(M_\alpha; g_{\beta\alpha})_{\alpha\in\Lambda}$ be the corresponding direct system of right S-modules obtained by applying the functor $\mathrm{Hom}_{\mathcal{H}}(T, -)$ (see Proposition 5.5). Let $M = \varinjlim_{\text{Mod-}S} M_\alpha$; then $\varinjlim_{\mathcal{H}} X_\alpha \cong \mathbb{L}G(M)$.*

In particular, for every $i \in \mathbb{Z}$ there are canonical isomorphisms

$$H^{-i}(\varinjlim_{\mathcal{H}} X_\alpha) \cong \varinjlim_{\text{Mod-}R} H^{-i}(X_\alpha).$$

Proof. By Theorem 5.10 the restriction of the functor $\mathbb{L}G$ to Mod-S is exact and left adjoint of the fully faithful functor $\mathrm{Hom}_{\mathcal{H}}(T, -)$. Thus, $\mathbb{L}G(\varinjlim_{\text{Mod-}S} M_\alpha) \cong \varinjlim_{\mathcal{H}} \mathbb{L}G(M_\alpha)$ and for every α we have $\mathbb{L}G(M_\alpha) \cong X_\alpha$. Hence, the conclusion.

In particular,

$$H^{-i}(\varinjlim_{\mathcal{H}} X_\alpha) \cong \mathrm{Tor}^S_i(\varinjlim_{\text{Mod-}S} M_\alpha, T) \cong \varinjlim_{\text{Mod-}R} \mathrm{Tor}^S_i(M_\alpha, T) \cong \varinjlim_{\text{Mod-}R} H^{-i}(X_\alpha).$$

□

We show now that the last statement in Proposition 6.2 gives indeed a characterization of the Grothendieck condition of $\mathcal{H}$.

Theorem 6.3. *The heart $\mathcal{H}$ is a Grothendieck category if and only if for every direct system $(X_\alpha; f_{\beta\alpha})$ of objects of $\mathcal{H}$*

$$(*) \quad H^{-i}(\varinjlim_{\mathcal{H}} X_\alpha) \cong \varinjlim_{\text{Mod-}R} H^{-i}(X_\alpha)$$

for every $i \in \mathbb{Z}$.

Proof. Only the sufficiency needs to be proved. We follow the arguments as in the proof of [PS15, Proposition 3.4]. Let $0 \to \{X_\alpha\} \to \{Y_\alpha\} \to \{Z_\alpha\} \to 0$ be an exact sequence of direct systems of objects in $\mathcal{H}$. Since the direct limit functor is right exact being a left adjoint, there is an exact sequence

$$\varinjlim_{\mathcal{H}} X_\alpha \xrightarrow{f} \varinjlim_{\mathcal{H}} Y_\alpha \xrightarrow{g} \varinjlim_{\mathcal{H}} Z_\alpha \to 0$$

giving rise to short exact sequences: $0 \to \mathrm{Im}\, f \to \varinjlim_{\mathcal{H}} Y_\alpha \to \varinjlim_{\mathcal{H}} Z_\alpha \to 0$ and $0 \to \mathrm{Ker}\, f \to \varinjlim_{\mathcal{H}} X_\alpha \xrightarrow{p} \mathrm{Im}\, f \to 0$. Applying the cohomological functor H to the triangles in $\mathcal{D}(R)$ corresponding to the above exact sequences and using the fact that direct limits are exact in Mod-R, we obtain a commutative diagram of

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R-modules:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \varinjlim_{\text{Mod-}R} H^{-i-1}(Z_\alpha) & \longrightarrow & \varinjlim_{\text{Mod-}R} H^{-i}(X_\alpha) & \longrightarrow & \varinjlim_{\text{Mod-}R} H^{-i}(Y_\alpha) & \longrightarrow \cdots \\ & \cong \downarrow & & h \downarrow & & \downarrow \cong & \\ \cdots \longrightarrow & H^{-i-1}(\varinjlim_{\mathcal{H}}(Z_\alpha) & \longrightarrow & H^{-i}(\operatorname{Im} f) & \longrightarrow & H^{-i}(\varinjlim_{\mathcal{H}} Y_\alpha) & \longrightarrow \cdots ; \end{array}$$

thus, h is an isomorphism. Note that h factors as

$$\varinjlim_{\text{Mod-}R} H^{-i}(X_\alpha) \to H^{-i}(\varinjlim_{\mathcal{H}}(X_\alpha) \overset{H^{-i}(p)}{\to} H^{-i}(\operatorname{Im} f),$$

showing that $H^{-i}(p)$ is an isomorphism for every $i \in \mathbb{Z}$; hence, p is an isomorphism and thus $\operatorname{Ker} f = 0$. □

If $\mathcal{H}$ is a Grothendieck category, we show that some direct limits in $\mathcal{H}$ can be computed in $\mathcal{C}h(R)$.

Proposition 6.4. *Let $(X_\alpha; f_{\beta\alpha})_{\alpha\in\Lambda}$ be a direct system in $\mathcal{C}h(R)$ such that $X_\alpha \in \mathcal{H}$ for every $\alpha \in \Lambda$. If $\mathcal{H}$ is a Grothendieck category, then*

$$\varinjlim_{\mathcal{C}h(R)} X_\alpha \cong \varinjlim_{\mathcal{H}} X_\alpha$$

in $\mathcal{D}(R)$.

Proof. Consider the direct system $(X_\alpha; q(f_{\beta\alpha}))_{\alpha\in\Lambda}$ in $\mathcal{H}$, where q is the canonical quotient functor $q\colon \mathcal{C}h(R) \to \mathcal{D}(R)$. Let $(M_\alpha; g_{\beta\alpha})_{\alpha\in\Lambda}$ be the direct system of right S-modules obtained by applying the functor $\operatorname{Hom}_{\mathcal{H}}(T, -)$ to $(X_\alpha; q(f_{\beta\alpha}))_{\alpha\in\Lambda}$. By Proposition 6.2 $\varinjlim_{\mathcal{H}} X_\alpha \cong \mathbb{L}G(M)$, where $M = \varinjlim_{\text{Mod-}S} M_\alpha$. By Lemma 5.3 there are projective resolutions P_α of M_α and a direct system $(P_\alpha; \tilde{g}_{\beta\alpha})_{\alpha\in\Lambda}$ in $\mathcal{C}h(S)$ such that $P_\alpha \otimes_S T \cong M_\alpha \otimes^{\mathbf{L}}_S T$ and

$$\varinjlim_{\mathcal{H}} X_\alpha \cong \mathbb{L}G(M) \cong (\varinjlim_{\mathcal{C}h(S)} P_\alpha) \otimes_S T \cong \varinjlim_{\mathcal{C}h(R)} (P_\alpha \otimes_S T).$$

By Proposition 5.5 and its proof, the functor $\operatorname{Hom}_{\mathcal{H}}(T, -)$ is isomorphic to $\mathbf{R}\operatorname{Hom}_R(T, -)$, and it is fully faithful. Thus, the counit morphism

$$- \otimes^{\mathbf{L}}_S T \circ \operatorname{Hom}_{\mathcal{H}}(T, -)$$

is invertible (see Fact 5.2), showing that $X_\alpha \cong M_\alpha \otimes^{\mathbf{L}}_S T \cong P_\alpha \otimes_S T$ in $\mathcal{H}$ for every $\alpha \in \Lambda$. Let $\phi_\alpha\colon X_\alpha \to P_\alpha \otimes_S T$ be an isomorphism and let ψ_α be a chain map in $\mathcal{C}h(R)$ such that $q(\psi_\alpha) = \phi_\alpha$. The map

$$\psi = \varinjlim_{\alpha} \psi_\alpha =\colon \varinjlim_{\mathcal{C}h(R)} X_\alpha \to \varinjlim_{\mathcal{C}h(R)} (P_\alpha \otimes_S T)$$

is a chain map in $\mathcal{C}h(R)$. Consider the morphisms

$$\varinjlim_{\text{Mod-}R} H^{-i}(X_\alpha) \overset{g}{\to} H^{-i}(\varinjlim_{\mathcal{C}h(R)} X_\alpha) \overset{H^{-i}(\psi)}{\to} H^{-i}(\varinjlim_{\mathcal{H}} X_\alpha),$$

where g is a canonical isomorphism by the exactness of the direct limit in Mod-R and the composition $H^{-i}(\psi) \circ g$ is an isomorphism by Proposition 6.2. Hence, $H^{-i}(\psi)$ is an isomorphism for every $i \in \mathbb{Z}$, implying that ψ is an isomorphism in $\mathcal{D}(R)$. □

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7. The pure projectivity

In this section we translate the Grothendieck condition on the category $\mathcal{H}$ in terms of properties of subcategories of Mod-R in order to be able to pin down conditions on the tilting module T_R itself.

First, we prove a result which is a consequence of Proposition 6.4, where tr_T denotes the trace in the module T.

Proposition 7.1. *Assume that $\mathcal{H}$ is a Grothendieck category. The following hold true:*

(1) *If $(N_\alpha; f_{\beta\alpha})_{\alpha\in\Lambda}$ is a direct system of R-modules, then*

$$(*) \qquad \mathrm{tr}_T(\varinjlim_{i\in I} N_\alpha) \cong \varinjlim_{i\in I} \mathrm{tr}_T(N_\alpha).$$

In particular, the torsion free class $T^{\perp_0}$ is closed under direct limits; hence, it is a definable class.

(2) *For every $i \geq 1$ the classes $T^{\perp_i}$ are closed under direct limits.*

Proof.

(1) For each module N_α choose functorially a complex $X_\alpha \in \mathcal{H}$, as constructed in Proposition 4.8 (1). Then $\mathrm{Ker}\, d^{-1}_{X_\alpha} = N_\alpha$ and $\mathrm{Im}\, d^{-2}_{X_\alpha} = \mathrm{tr}_T(N_\alpha)$. By functoriality we obtain a direct system $(X_\alpha; \tilde{f}_{\beta\alpha})_{\alpha\in\Lambda}$ in $\mathcal{C}h(R)$, and also a direct system $(X_\alpha; q(\tilde{f}_{\beta\alpha}))_{\alpha\in\Lambda}$ in $\mathcal{H}$, where q is the canonical quotient functor $q\colon \mathcal{C}h(R) \to \mathcal{D}(R)$. Let $X = \varinjlim_{\mathcal{C}h(R)} X_\alpha$. By Proposition 6.4 $\varinjlim_{\mathcal{H}} X_\alpha \cong X$.

X is a complex with terms in $T^\perp$, since an n-tilting class is closed under direct limits (see [BŠ07]). We have $\mathrm{Ker}\, d^{-1}_X = \varinjlim_{\text{Mod-}R} N_\alpha$; $\mathrm{Im}\, d^{-2}_X = \varinjlim_{\text{Mod-}R} \mathrm{tr}_T(N_\alpha)$ and, by Lemma 4.6 (2), $\mathrm{Im}\, d^{-2}_X = \mathrm{tr}_T(\mathrm{Ker}\, d^{-1}_X) = \mathrm{tr}_T(\varinjlim_{\text{Mod-}R} N_\alpha)$ since $X \in \mathcal{H}$. Hence, the conclusion.

The last statement follows immediately from $(*)$.

(2) We first prove that the class $T^{\perp_1}$ is closed under direct limits.

Let $(N_\alpha; f_{\beta\alpha})_{\alpha\in\Lambda}$ be a direct system of R-modules in $T_1^\perp$. For each module N_α choose functorially a complex $X_\alpha \in \mathcal{H}$ as constructed in Proposition 4.8 (2) so that $N_\alpha = \mathrm{Ker}\, d^{-2}_{X_\alpha}$. Arguing as in part (1), we get a direct system $\{X_\alpha\}_{\alpha\in\Lambda}$ both in $\mathcal{C}h(R)$ and in $\mathcal{H}$, whose direct limit in $\mathcal{H}$ is isomorphic to $X = \varinjlim_{\mathcal{C}h(R)} X_\alpha$, from Proposition 6.4. Now $\mathrm{Ker}\, d^{-2}_X \cong \varinjlim_{\text{Mod-}R} \mathrm{Ker}\, d^{-2}_{X_\alpha}$ and, by Lemma 4.6 (2), the latter belongs to $T_1^\perp$.

By induction we get that $T^{\perp_i}$ is closed under direct limits for every $i \geq 1$. In fact, let $(N_\alpha; f_{\beta\alpha})_{\alpha\in\Lambda}$ be a direct system of R-modules in $T^{\perp_i}$, with $i > 1$, and choose functorially special $T^\perp$–pre-envelopes of N_α of the form $0 \to N_\alpha \to B_\alpha \to A_\alpha \to 0$, with $B_\alpha \in T^\perp$ and $A_\alpha \in {}^\perp(T^\perp)$. Then $A_\alpha \in T^{\perp_{i-1}}$. We obtain a short exact sequence

$$0 \to \varinjlim_\alpha N_\alpha \to \varinjlim_\alpha B_\alpha \to \varinjlim_\alpha A_\alpha \to 0.$$

Since $T^\perp$ is closed under direct limits, $\varinjlim_\alpha N_\alpha \in T^{\perp_i}$ if and only if $\varinjlim_\alpha A_\alpha \in T^{\perp_{i-1}}$. Thus, the conclusion follows from induction. $\square$

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We show now that if the heart $\mathcal{H}$ is a Grothendieck category, then the n-tilting module T must be pure projective.

Proposition 7.2. *Let $\mathcal{H}$ be the heart of the t-structure induced by an n-tilting module T. If $\mathcal{H}$ is a Grothendieck category, then T is a pure projective module.*

Proof. Write T as a direct limit of a direct system $\{A_i \colon f_{ji}\}_{i\leq j\in I}$ of finitely presented modules. By Proposition 7.1 $T \cong \varinjlim_i \mathrm{tr}_T(A_i)$ and for every $i \in I$, we have a functorial presentation of $\mathrm{tr}_T(A_i)$ given by

$$0 \to K_i \to T^{(\mathrm{Hom}(T,A_i))} \to \mathrm{tr}_T(A_i) \to 0,$$

where $K_i \in T^{\perp_1}$. By the functoriality of the presentation we get direct systems $\{K_i\}_{i\in I}$, $\{T^{(\mathrm{Hom}(T,A_i))}\}_{i\in I}$ and $\{\mathrm{tr}_T(A_i)\}_{i\in I}$, giving rise to the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D_1 & \longrightarrow & D & \longrightarrow & D_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{i\in I} K_i & \longrightarrow & \bigoplus_{i\in I} T^{(\mathrm{Hom}(T,A_i))} & \xrightarrow{\ \sigma\ } & \bigoplus_{i\in I} \mathrm{tr}_T(A_i) & \longrightarrow & 0 \\
 & & \downarrow & & {\scriptstyle\phi}\downarrow & \overset{\beta}{\nwarrow} & \downarrow{\scriptstyle\pi} & & \\
0 & \longrightarrow & \varinjlim_{i\in I} K_i & \longrightarrow & \varinjlim_{i\in I} T^{(\mathrm{Hom}(T,A_i))} & \underset{\rho}{\overset{\alpha}{\leftrightarrows}} & \varinjlim_{i\in I} \mathrm{tr}_T(A_i) \cong T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

By Proposition 7.1 (2), $\varinjlim_{i\in I} K_i$ is in $T^{\perp_1}$; hence, the last row splits. That is, there is a morphism $\alpha\colon T \to \varinjlim_{i\in I} T^{(\mathrm{Hom}(T,A_i))}$ such that $\rho \circ \alpha = 1_T$. The second column is a pure exact sequence; hence, $D \in T^{\perp}$ since the tilting class $T^{\perp}$ is definable (by [BŠ07]). This implies that the morphism α can be lifted to a morphism β such that $\phi \circ \beta = \alpha$. Now we infer that $\pi \circ \sigma \circ \beta = \rho \circ \phi \circ \beta = \rho \circ \alpha = 1_T$, showing that the morphism $\sigma \circ \beta$ gives a splitting map for the third column. We then conclude that T is isomorphic to a direct summand of $\bigoplus_{i\in I} \mathrm{tr}_T(A_i)$. We also have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_1 & \longrightarrow & \bigoplus_{i\in I} \mathrm{tr}_T(A_i) & \longrightarrow & \varinjlim_{i\in I} \mathrm{tr}_T(A_i) \cong T & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & D' & \longrightarrow & \bigoplus_{i\in I} A_i & \longrightarrow & \varinjlim_{i\in I} A_i \cong T & \longrightarrow & 0
\end{array}$$

where the first row splits, showing that also the second row splits. This proves that T is pure projective. □

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Remark 7.3. Note that the proof of Proposition 7.2 shows that if T is an n-tilting module, the following two conditions, (1) the functor tr_T commutes with the direct limit and (2) $T^{\perp_1}$ is closed under direct limits, are sufficient to conclude that T is pure projective.

We use an argument communicated by Herzog to prove the converse of the preceding proposition.

Proposition 7.4. *Let T be a pure projective n-tilting module. Then the heart $\mathcal{H}$ of the t-structure induced by T is a Grothendieck category.*

Proof. Consider the functor category $\mathcal{A} = ((\text{mod-}R)^{op}, \mathcal{A}b)$ consisting of the contravariant additive functors from the category of finitely presented right R-modules to the category of abelian groups. It is well known that the Yoneda functor $Y\colon \text{Mod-}R \to \mathcal{A}$; $\quad M \mapsto \mathrm{Hom}_R(-, M)$ yields a left exact full embedding and that $\mathrm{Hom}_R(-, M)$ is a projective object of $\mathcal{A}$ provided that M is a pure projective R-module. Thus, by assumption the functor $H^T = \mathrm{Hom}_R(-, T)$ is a projective object of $\mathcal{A}$ and the class

$$\mathcal{C} = \{G \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(H^T, G) = 0\}$$

is a torsion-torsion free class, so we can form the quotient category $\mathcal{A}/\mathcal{C}$. By [Gab62, Chap. III] $\mathcal{A}/\mathcal{C}$ is a Grothendieck category and the quotient functor $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$ is exact. The group of morphisms $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(q(G), q(F)$ between two objects in $\mathcal{A}/\mathcal{C}$ is defined as $\varinjlim \mathrm{Hom}_{\mathcal{A}}(G', F/F')$, where G' and F' vary among the subobjects of G and F such that $G/G', F' \in \mathcal{C}$. Thus, by the definition of $\mathcal{C}$ and by the projectivity of H^T we infer that $\mathrm{Hom}_{\mathcal{A}/\mathcal{C}}(q(H^T), q(F)) \cong \mathrm{Hom}_{\mathcal{A}}(H^T, F)$, which indicates that $q(H^T)$ is a projective object of $\mathcal{A}/\mathcal{C}$. Moreover, from the definition of $\mathcal{C}$ it is clear that $q(H^T)$ is a generator for $\mathcal{A}/\mathcal{C}$. Thus, $\mathrm{Add}\, q(H^T)$ is the class of projective objects of the Grothendieck category $\mathcal{A}/\mathcal{C}$, and the composition of functors

$$\text{Mod-}R \xrightarrow{Y} \mathcal{A} \xrightarrow{q} \mathcal{A}/\mathcal{C}$$

induces an equivalence between $\mathrm{Add}T$ and $\mathrm{Add}\, q(H^T)$.

By Proposition 4.3 the full subcategory of projective objects of $\mathcal{H}$ is equivalent to $\mathrm{Add}T$, so we have an equivalence between the full subcategories of projective objects of $\mathcal{H}$ and of $\mathcal{A}/\mathcal{C}$. It is well known that the equivalence extends to the entire categories (see, e.g., [ARS95, section IV]); thus, we conclude that $\mathcal{H}$ is a Grothendieck category. □

Combining Propositions 7.2 and 7.4, we obtain the main result of this section.

Theorem 7.5. *Let $\mathcal{H}$ be the heart of the t-structure induced by an n-tilting module T. $\mathcal{H}$ is a Grothendieck category if and only the tilting module T is pure projective.*

We illustrate now some properties of the trace functor corresponding to an n-tilting module. If $n = 1$, tr_T is the torsion radical of the tilting torsion class $T^{\perp} = \mathrm{Gen}\, T$.

For $n > 1$ we have the following lemma.

Lemma 7.6. *Let T be an n-tilting R-module and consider a special $T^{\perp}$–pre-envelope of R of the form*

$$(*) \quad 0 \to R \xrightarrow{\varepsilon} T \to T/R \to 0,$$

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and let $\varepsilon(1) = w$*. Then, for every module* N *there are two exact sequences:*

$$(1) \quad 0 \to \mathrm{Hom}_R(T/R, N) \to \mathrm{Hom}_R(T, N) \to tr_T(N) \to 0,$$

$$(2) \quad 0 \to N/tr_T(N) \to \mathrm{Ext}^1_R(T/R, N) \to \mathrm{Ext}^1_R(T, N) \to 0.$$

In particular, the following hold:

(i) *For every module* N, $x \in \mathrm{tr}_T(N)$ *if and only if there is a morphism* $g\colon T \to N$ *such that* $g(w) = x$*. In other words,* tr_T *is isomorphic to the matrix functor* $\mathrm{Hom}_R(T, -)(w)$.

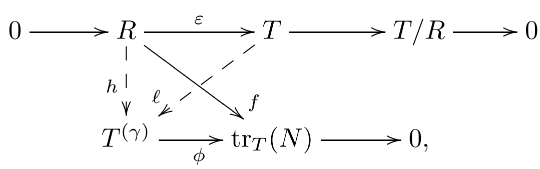
(ii) *The trace* tr_T *commutes with direct limits if and only if there is a finitely presented module* A *and an element* $a \in \mathrm{tr}_T(A)$ *such that* tr_T *is isomorphic to the finite matrix functor* $\mathrm{Hom}_R(A, -)(a)$.

Proof. Possibly passing to an equivalent tilting module, it is easy to see that there exists a special $T^{\perp}$–pre-envelope of R as in the statement.

From $(*)$ we obtain the exact sequence:

$$(a) \quad 0 \to \mathrm{Hom}_R(T/R, N) \to \mathrm{Hom}_R(T, N) \to \mathrm{Hom}_R(R, N) \to$$
$$\to \mathrm{Ext}^1_R(T/R, N) \to \mathrm{Ext}^1_R(T, N) \to 0.$$

Identifying $\mathrm{Hom}_R(R, N)$ with N, it is obvious that the image of the map $\mathrm{Hom}_R(T, N) \to N$ is contained in the trace of T in N. To prove the other inclusion, pick $x \in \mathrm{tr}_T(N)$, let $f\colon R \to \mathrm{tr}_T(N)$ be a morphism satisfying $f(1) = x$, and let $T^{(\gamma)} \xrightarrow{\phi} \mathrm{tr}_T(N)$ be an epimorphism. Consider the diagram



where the dotted arrow $h\colon R \to T^{(\gamma)}$ is a lift of f and the dotted arrow $\ell\colon T \to T^{(\gamma)}$ satisfying $\ell \circ \varepsilon = h$ exists from the pre-envelope property. Then the morphism $g = \phi \circ \ell\colon T \to \mathrm{tr}_T(N)$ satisfies $g \circ \varepsilon = f$; hence, $g(w) = x$ and sequence (1) is established.

Sequence (2) follows from (a) and (1).

(i) This follows immediately from (1).

(ii) Assume that tr_T commutes with direct limits, and write T as a direct limit of a direct system $\{A_i\}_{i \in I}$ of finitely presented modules. By assumption $T \cong \varinjlim_{i \in I} \mathrm{tr}_T(A_i)$. Let $\mu_i\colon A_i \to T$ be the canonical morphisms. There is an index $j \in I$ and an element $a_j \in \mathrm{tr}_T(A_j)$ such that $\mu_j(a_j) = w$. Let N be an R-module and let $x \in \mathrm{tr}_T(N)$. By (i) and the above remarks, there is an $f\colon A_j \to N$ such that $f(a_j) = x$. Hence, $\mathrm{Hom}_R(T, N)(w) \leq \mathrm{Hom}_R(A_j, N)(a_j)$.

On the other hand, $\mathrm{Hom}_R(A_j, N)(a_j) \leq \mathrm{Hom}_R(\mathrm{tr}_T(A_j), N)(a_j)$ and the latter is contained in $\mathrm{tr}_T(N)$ since $\mathrm{tr}_T(A_j)$ is generated by T. Hence tr_T is isomorphic to $\mathrm{Hom}_R(A_j, -)(a_j)$.

The converse follows immediately by recalling that, for every finitely presented module A, the functor $\mathrm{Hom}_R(A, -)$ commutes with direct limits. □

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Remark 7.7. The condition on tr_T to commute with direct limits does not seem to imply the pure projectivity of T. In fact, in Proposition 7.2 to prove that T is pure projective, we used also that $T^{\perp_1}$ is closed under direct limits.

We illustrate now some features of a pure projective n-tilting module.

Proposition 7.8. *Let T be a pure projective n-tilting module. Then every jth-syzygy of T, $j \geq 1$ is pure projective.*

Proof. First of all, we show that T may be assumed to be countably presented. By assumption T is a direct summand of a direct sum $\bigoplus_{i\in I} E_i$ of finitely presented modules E_i, and hence, in particular, countably generated. By Kaplansky's theorem [Kap58, Theorem 1] T is a direct sum of countably generated submodules. Thus, $T = \bigoplus_{\alpha\in\Lambda} X_\alpha$ where, for every α, X_α is a countably generated, and hence also countably presented, direct summand of $\bigoplus_{i\in I} E_i$. Let A be a countably presented module in the left component $\mathcal{A}$ of the cotorsion pair $(\mathcal{A}, T^\perp)$ generated by T, and consider a special $T^\perp$–pre-envelope of A,

$$0 \to A \xrightarrow{\epsilon} B \to A_1 \to 0.$$

W.l.o.g. we may assume that $B = T^{(\gamma)}$ for some cardinal γ so that $\varepsilon(A)$ is contained in a summand $U_0 = (\bigoplus_{\beta\in F} X_\beta)^{(\nu)}$ of $T^{(\gamma)}$ where F is a countable subset of Λ and ν is a countable subset of γ. Thus,

$$0 \to A \to U_0 \to U_0/A \to 0$$

is also a special pre-envelope of A, and U_0/A is a countably presented module in $\mathcal{A}$. Starting with $R \in \mathcal{A}$, the above arguments show that we can construct an iteration of $T^\perp$ pre-envelopes of R of the form

$$0 \to R \to U_0 \to U_1 \to \cdots \to U_n \to 0,$$

with U_i countably presented modules in $\mathrm{Add}T$; hence, by [GT12, Ch 13] we obtain that $U_0 \oplus U_1 \oplus \cdots \oplus U_n$ is an n-tilting module equivalent to T.

Second, we observe that all the syzygies of T are countably presented. Indeed this follows from recalling that every module in $\mathcal{A}$ is an R Mittag-Leffler one (see for instance [AHH08, Theorem 9.5]), and by applying [BH09, Proposition 3.8] to the syzygies of T. Since countably generated modules are pure projective if and only if they are Mittag-Leffler ones [RG71], we are led to show that every syzygy $\Omega_j(T)$ $(j \geq 1)$ of T is a Mittag-Leffler module provided that T is pure projective. This is equivalent to showing that the canonical morphism $\rho_j\colon \mathrm{Tor}_j^R(T, \prod_{i\in I} Q_i) \to \prod_{i\in I} \mathrm{Tor}_j^R(T, Q_i)$ is a monomorphism for every $j \geq 1$ and every set $\{Q_i\}_{i\in I}$ of left R-modules (see, e.g., [AHH08, Proposition 1.10]). By dimension shifting it is enough to prove that ρ_1 is a monomorphism. By assumption T is a summand of a direct sum $\bigoplus_{n\in\mathbb{N}} E_n$ of finitely presented modules E_n. So

$$\mathrm{Tor}_1^R(T, \prod_{i\in I} Q_i) \underset{\oplus}{\leq} \mathrm{Tor}_1^R(\bigoplus_{n\in\mathbb{N}} E_n, \prod_{i\in I} Q_i) \cong \bigoplus_{n\in\mathbb{N}} \left(\prod_{i\in I} \mathrm{Tor}_1^R(E_n, Q_i) \right),$$

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and the latter can be embedded in $\prod_{i\in I}(\bigoplus_{n\in\mathbb{N}} \mathrm{Tor}_1^R(E_n, Q_i)$. Thus, by the naturality of ρ_1 we conclude that ρ_1 is a monomorphism. □

We can characterize the pure projectivity of an n-tilting module in terms of properties of its Ext-orthogonal classes. First, we prove the following lemma.

Lemma 7.9. *Let M be a pure projective module. The following hold true:*

(1) *$M^{\perp_0}$ is a definable class and $M^{\perp_1}$ is closed under direct sums.*
(2) *If all of the syzygies $\Omega_j(M)$ of M are pure projective, then the classes $M^{\perp_i}$ are definable for every $i \geq 0$.*

Proof.

(1) The pure projectivity of M yields easily that $M^{\perp_0}$ is closed under direct limits, and thus $M^{\perp_0}$ is a definable class.

Let X_α be a family of modules in $M^{\perp_1}$. From the pure exact sequence

$$\text{(a)} \qquad 0 \to \oplus_\alpha X_\alpha \to \prod_\alpha X_\alpha \xrightarrow{\pi} \frac{\prod_\alpha X_\alpha}{\oplus_\alpha X_\alpha} \to 0,$$

and from the pure projectivity of M we obtain that $\mathrm{Hom}_R(M,\pi)$ is surjective and thus that $\oplus_\alpha X_\alpha \in M^{\perp_1}$.

(2) We prove the statement by induction. The case $i = 0$ holds by (1). Let $i = 1$. We first show that $M^{\perp_1}$ is closed under pure submodules. Let $0 \to Y \to X \xrightarrow{\pi} X/Y \to 0$ be a pure exact sequence with $X \in M^{\perp_1}$. Then $\mathrm{Hom}_R(M,\pi)$ is surjective; hence, the exact sequence $0 \to \mathrm{Ext}_R^1(M,Y) \to \mathrm{Ext}_R^1(M,X) = 0$ shows that $Y \in M^{\perp_1}$.

Let X_α be a direct system of modules in $M^{\perp_1}$ and consider a pure exact sequence

$$0 \to K \to \oplus_\alpha X_\alpha \to \varinjlim_\alpha X_\alpha \to 0.$$

By (1) $\oplus_\alpha X_\alpha \in M^{\perp_1}$ and we have an exact sequence

$$0 \to \mathrm{Ext}_R^1(M, \varinjlim_\alpha X_\alpha) \to \mathrm{Ext}_R^2(M,K) \xrightarrow{f} \mathrm{Ext}_R^2(M, \oplus_\alpha X_\alpha).$$

By dimension shifting we have a canonical isomorphism $\mathrm{Ext}_R^2(M,-) \cong \mathrm{Ext}_R^1(\Omega_1(M),-)$. Using the pure projectivity of $\Omega_1(M)$, we have a monomorphism $\mathrm{Ext}_R^1(\Omega_1(M),K) \to \mathrm{Ext}_R^1(\Omega_1(M),\oplus_\alpha X_\alpha)$, and by the naturality of the isomorphism we conclude that f is a monomorphism too.

Hence, $\mathrm{Ext}_R^1(M, \varinjlim_\alpha X_\alpha) = 0$.

To conclude the proof, it is enough to note that $\mathrm{Ext}_R^{i+1}(M,-) \cong \mathrm{Ext}_R^1(\Omega_i(M),-)$ for every $i \geq 1$ and to apply the previous arguments. □

Proposition 7.10. *Let T be an n-tilting module. The following are equivalent:*

(1) *T is pure projective.*
(2) *tr_T commutes with direct limits, and the classes $T^{\perp_i}$ are definable for every case in which $0 \leq i$.*
(3) *tr_T commutes with direct limits, and the class $T^{\perp_1}$ is closed under direct limits.*

Proof.

(1) ⇒ (2) The first statement follows from the definition of pure projectivity and by the canonical presentation of a direct limit by means of a pure exact sequence. In particular, $T^{\perp_0}$ is closed under direct limits and thus is definable.

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For the closure under direct limits of the classes $T^{\perp_i}$, $i \geq 1$ we could invoke Theorem 7.5 and Proposition 7.1 and then apply [AHST15, Theorem 6.1]. Alternatively, we can use Proposition 7.8 and Lemma 7.9.

(2) $\Rightarrow$ (3) This is obvious.

(3) $\Rightarrow$ (1) This follows from the proof of Proposition 7.2. See also Remark 7.3. □

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