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Rank-one theorem and subgraphs of BV functions in Carnot groups ☆

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We prove a rank-one theorem *à la* G. Alberti for the derivatives of vector-valued maps with bounded variation in a class of Carnot groups that includes Heisenberg groups $\mathbb{H}^n$ for $n \geq 2$. The main tools are properties relating the horizontal derivatives of a real-valued function with bounded variation and its subgraph.

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1. Introduction

One of the main results in the theory of functions with bounded variation (BV) is the rank-one theorem. Recall that a function $u \in L^1(\Omega, \mathbb{R}^d)$ has bounded variation in an open set $\Omega \subset \mathbb{R}^n$ ($u \in BV(\Omega, \mathbb{R}^d)$) if the derivatives Du of u in the sense of distributions are represented by a (matrix-valued) measure with finite total variation. The measure Du can then be decomposed as the sum $Du = D^a u + D^s u$ of a measure $D^a u$, that is absolutely continuous with respect to $\mathscr{L}^n$, and a measure $D^s u$ that is singular with respect to $\mathscr{L}^n$. The Radon–Nikodym derivative $\frac{D^s u}{|D^s u|}$ of $D^s u$ with respect to its total variation $|D^s u|$ is a $|D^s u|$-measurable map from Ω to $\mathbb{R}^{d\times n}$. The rank-one theorem states that $|D^s u|$-a.e. this map takes values in the space of rank-one matrices. We refer to [3] for more details on BV functions.

The rank-one theorem was first conjectured by L. Ambrosio and E. De Giorgi in [7] and it has important applications to vectorial variational problems and systems of PDEs. It was proved by G. Alberti in [1] (see also [2,8]): due to its complexity, Alberti's proof is generally regarded as a *tour de force* in measure theory. Two different proofs of the rank-one theorem were recently found. One is due to G. De Philippis and F. Rindler and follows from a profound PDE result [9], where a rank-one property for maps with bounded deformation (BD) was also proved for the first time. At the same time another proof, of a geometric flavor and considerably simpler than those in [1,9], was provided by the second- and third-named authors in [27].

Motivated by these results, in this paper we consider the following natural generalization. Let $X_1, \dots, X_m$ be linearly independent vector fields in $\mathbb{R}^n$, $m \le n$, and let $u : \Omega \to \mathbb{R}^d$ be a function with *bounded H-variation* in an open set $\Omega \subset \mathbb{R}^n$, i.e., a vector valued function such that the distributional *horizontal* derivatives $D_H u := (X_1 u, \dots, X_m u)$ are represented by a $d\times m$-matrix valued measure with finite total variation in Ω; consider the singular part $D^s_H u$ of $D_H u$ with respect to $\mathscr{L}^n$. Is it true that the Radon–Nikodym derivative $\frac{D^s_H u}{|D^s_H u|}$ is a rank-one matrix $|D^s_H u|$-a.e.?

We investigate this question in the setting of *Carnot groups* $\mathbb{G} \equiv \mathbb{R}^n$ (see Section 2) endowed with a left-invariant basis $X_1, \dots, X_m$ of the first layer $\mathfrak{g}_1$ in the stratification of their Lie algebra. In particular, we find two assumptions on $\mathbb{G}$, that we call properties $\mathscr{C}_2$ and $\mathscr{R}$ (see Definitions 2.2 and 5.1, respectively), that ensure the rank-one property for BV_H functions in $\mathbb{G}$. We will discuss later the role played by these properties in our argument. Our first main result is the following

Theorem 1.1. *Let $\mathbb{G}$ be a Carnot group satisfying properties $\mathscr{C}_2$ and $\mathscr{R}$; let $\Omega \subset \mathbb{G}$ be an open set and $u \in BV_{H,loc}(\Omega, \mathbb{R}^d)$ be a function with locally bounded H-variation. Then the singular part $D^s_H u$ of $D_H u$ is a rank-one measure, i.e., the matrix-valued function $\frac{D^s_H u}{|D^s_H u|}(x)$ has rank one for $|D^s_H u|$-a.e. $x \in \Omega$.*

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It is worth pointing out that Theorem 1.1 applies to the n-th *Heisenberg group* $\mathbb{H}^n$ provided $n \geq 2$. Recall that Heisenberg groups, defined in Example 2.1 below, are the most notable examples of Carnot groups.

Corollary 1.2. *Let u be as in Theorem 1.1 and assume that $\mathbb{G}$ is the Heisenberg group $\mathbb{H}^n$, $n \geq 2$; then $D_H^s u$ is a rank-one measure. More generally, the same holds if $\mathbb{G}$ is a Carnot group of step 2 satisfying property $\mathscr{C}_2$.*

Corollary 1.2 is an immediate consequence of Theorem 1.1, see Remarks 2.4 and 5.3.

Theorem 1.1 does not directly follow from the outcomes of [9], see Remark 5.5. Its proof follows the geometric strategy devised in [27] and it is based on the relations between a (real-valued) BV_H function u in $\mathbb{G}$ and the *H-perimeter* of its subgraph $E_u := \{(x,t) : t < u(x)\} \subset \mathbb{G} \times \mathbb{R}$. Recall that a set $E \subset \mathbb{G} \times \mathbb{R}$ has finite H-perimeter if its characteristic function χ_E has bounded H-variation with respect the vector fields of a basis of the first layer in the Lie algebra stratification of the Carnot group $\mathbb{G} \times \mathbb{R}$. Our second main result is the following

Theorem 1.3. *Suppose that $\Omega \subset \mathbb{G}$ is open and bounded and let $u \in L^1(\Omega)$. Then u belongs to $BV_H(\Omega)$ if and only if its subgraph E_u has finite H-perimeter in $\Omega \times \mathbb{R}$.*

Actually, the proof of Theorem 1.1 requires much finer properties than the one stated in Theorem 1.3. Such properties are stated in Theorems 4.2 and 4.3 in a much more general context than Carnot groups, i.e., for maps with bounded H-variation with respect to a generic fixed family of linearly independent vector fields $X_1, \dots, X_m$ on $\mathbb{R}^n$. Theorem 4.2, from which Theorem 1.3 immediately follows, focuses on the relations between the horizontal (in $\mathbb{R}^n$) derivatives of u and the horizontal (in $\mathbb{R}^n \times \mathbb{R}$) derivatives of χ_{E_u}. Theorem 4.3 instead deals with the relations between the *horizontal normal* to E_u and the *polar vector* σ_u in the decomposition $D_H u = \sigma_u |D_H u|$, and it also deals with the relations between $D_H^a u, D_H^s u$ and the horizontal derivatives of χ_{E_u}. When $m = n$ and $X_i = \partial_{x_i}$ one recovers some results that belong to the folklore of Geometric Measure Theory and are scattered in the literature (see e.g. [28], [11, 4.5.9] and [17, Section 4.1.5]); we tried here to collect them in a more systematic way. We were not able to find references for some of the results we stated.

Property $\mathscr{R}$ ("rectifiability") intervenes in ensuring that the horizontal derivatives of χ_{E_u} are a "rectifiable" measure, see Definition 5.1. This is a non-trivial technical obstruction one has to face when following the strategy of [27]: the rectifiability of sets with finite H-perimeter in Carnot groups is indeed a major open problem, which has been solved only in step 2 Carnot groups (see [14,15]) and in the class of Carnot groups *of type* $\star$ ([26]). See also [4] for a partial result in general Carnot groups.

Once the rectifiability of E_u is ensured, the proof of Theorem 1.1 follows rather easily from the technical Lemma 3.2 below, which is the natural counterpart of the Lemma in [27]. The latter, however, was proved by utilizing the area formula for maps between rectifiable subsets of $\mathbb{R}^n$, see e.g. [3]. A similar tool is not available in the context of Carnot groups, a fact which forces us to follow a different path. The proof of Lemma 3.2 is indeed achieved by a covering argument that is based on the following result: we state it and postpone to Section 2 the definitions of property $\mathscr{C}_k$, the Hausdorff measure $\mathcal{H}^d$, the homogeneous dimension Q of $\mathbb{G}$ and of hypersurfaces of class C^1_H with their horizontal normal.

Theorem 1.4. *Let $k \geq 1$ be an integer, $\mathbb{G}$ a Carnot group satisfying property $\mathscr{C}_k$ and let $\Sigma_1, \dots, \Sigma_k$ be hypersurfaces of class C^1_H with horizontal normals $\nu_1, \dots, \nu_k$. Let also $x \in \Sigma := \Sigma_1 \cap \dots \cap \Sigma_k$ be such that $\nu_1(x), \dots, \nu_k(x)$ are linearly independent. Then, there exists an open neighborhood U of x such that*

$$0 < \mathcal{H}^{Q-k}(\Sigma \cap U) < \infty.$$

In particular, the measure $\mathcal{H}^{Q-k}$ is σ-finite on the set

$$\Sigma^{\pitchfork} := \{x \in \Sigma : \nu_1(x), \dots, \nu_k(x) \text{ are linearly independent}\}.$$

Theorem 1.4, that we prove in Appendix A, is an easy consequence of Theorems A.3 and A.5 proved, respectively, in [12] and [22]. Theorem A.5, in particular, states the much deeper property that $\Sigma^{\pitchfork}$ is locally an *intrinsic Lipschitz graph*. To this aim, one needs the intersection $T_x\Sigma_1 \cap \dots \cap T_x\Sigma_k$ of the *tangent subgroups* to Σ_i at x to admit a (necessarily commutative) complementary homogeneous subgroup that is horizontal, i.e., contained in $\exp(\mathfrak{g}_1)$. This algebraic property is guaranteed by property $\mathscr{C}_k$ ("complementability"), see Remark 2.3. We will provide in Appendix A a proof of Theorem A.5 which does not rely on the homotopy invariance of the topological degree and is then simpler and shorter than the one in [22].

For the validity of Theorem 1.4, property $\mathscr{C}_k$ might seem a restrictive one. We however point out that Theorem 1.4 is no longer valid already when $k = 2$ and $\mathbb{G}$ is the first Heisenberg group $\mathbb{H}^1$, which does not satisfy $\mathscr{C}_2$: indeed, in this setting the measure $\mathcal{H}^{Q-2}(\Sigma^{\pitchfork})$ might be either 0 or $+\infty$ (even locally) as shown by A. Kozhevnikov [19]. See also [20,25].

The fact that Theorem 1.4 does not apply to $\mathbb{H}^1$ (actually, to $\mathbb{H}^1 \times \mathbb{R} \times \mathbb{R}$, see the proof of Lemma 3.2) prevents us from proving the rank-one Theorem 1.1 for $\mathbb{G} = \mathbb{H}^1$. This does not follow from [9] either (see Remark 5.6) and, thus, it remains a very interesting open problem.

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2. Preliminaries on Carnot groups

2.1. Algebraic facts

A *Carnot* (or *stratified*) *group* is a connected, simply connected and nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is *stratified*, i.e., it has a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$ such that

$$\forall j = 1, \ldots, s-1 \quad \mathfrak{g}_{j+1} = [\mathfrak{g}_j, \mathfrak{g}_1], \qquad \mathfrak{g}_s \neq \{0\} \qquad \text{and} \qquad [\mathfrak{g}_s, \mathfrak{g}] = \{0\}.$$

We refer to the integer s as the *step* of $\mathbb{G}$ and to $m := \dim \mathfrak{g}_1$ as its *rank*; apart from the case in which $\mathbb{G}$ is a Heisenberg group (see Example 2.1), n denotes the topological dimension of $\mathbb{G}$. The group identity is denoted by 0.

The exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a diffeomorphism and, given a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, we often identify $\mathbb{G}$ with $\mathbb{R}^n$ by means of exponential coordinates:

$$\mathbb{R}^n \ni x = (x_1, \ldots, x_n) \longleftrightarrow \exp(x_1 X_1 + \cdots + x_n X_n) \in \mathbb{G}.$$

A one-parameter family $\{\delta_\lambda\}_{\lambda>0}$ of *dilations* $\delta_\lambda : \mathfrak{g} \to \mathfrak{g}$ is defined by $\delta_\lambda(X) := \lambda^j X$ for any $X \in \mathfrak{g}_j$; notice that $\delta_{\lambda\mu} = \delta_\lambda \circ \delta_\mu$. By composition with exp one can then define a one-parameter family, for which we use the same symbol δ, of group isomorphisms $\delta_\lambda : \mathbb{G} \to \mathbb{G}$.

Example 2.1. Apart from Euclidean spaces, which are the only commutative Carnot groups, the most basic examples of Carnot groups are Heisenberg groups. Given an integer $n \geq 1$, the n-th Heisenberg group $\mathbb{H}^n$ is the $2n+1$ dimensional Carnot group of step 2 whose Lie algebra is generated by $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ and the only non-vanishing commutation relations among these generators are given by

$$[X_j, Y_j] = T \qquad \text{for any } j = 1, \ldots, n.$$

The stratification of the Lie algebra is given by $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, where $\mathfrak{g}_1 := \mathrm{span}\{X_j, Y_j : j = 1, \ldots n\}$ and $\mathfrak{g}_2 := \mathrm{span}\{T\}$. In exponential coordinates

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \ni (x, y, t) \longleftrightarrow \exp(x_1 X_1 + \cdots + y_n Y_n + tT)$$

one has

$$X_j = \partial_{x_j} - \frac{y_j}{2} \partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2} \partial_t, \quad T = \partial_t.$$

In this paper, given a Carnot group $\mathbb{G}$ we will frequently deal with products like $\mathbb{G} \times \mathbb{R}^N$. Needless to say, this is the Carnot group with algebra $\mathfrak{g} \times \mathbb{R}^N$ with product defined by $[(X, t), (Y, s)] = ([X, Y], 0)$ for any $X, Y \in \mathfrak{g}, t, s \in \mathbb{R}^N$ and whose stratification is given by $(\mathfrak{g}_1 \times \mathbb{R}^N) \oplus (\mathfrak{g}_2 \times \{0\}) \oplus \cdots \oplus (\mathfrak{g}_s \times \{0\})$.

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Definition 2.2. Let $\mathbb{G}$ be a Carnot group with rank m and let $1 \leq k \leq m$ be an integer. We say that $\mathbb{G}$ satisfies the *property* $\mathscr{C}_k$ if the first layer $\mathfrak{g}_1$ of its Lie algebra has the following property: for any linear subspace $\mathfrak{w}$ of $\mathfrak{g}_1$ of codimension k there exists a commutative complementary subspace in $\mathfrak{g}_1$, i.e., a k-dimensional subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ such that $[\mathfrak{h}, \mathfrak{h}] = 0$ and $\mathfrak{g}_1 = \mathfrak{w} \oplus \mathfrak{h}$.

Remark 2.3. As customary in the literature, we say that $\mathbb{W} \subset \mathbb{G}$ is a *vertical plane* of codimension k (for some $1 \leq k \leq m$) if $\mathbb{W} = \exp(\mathfrak{w} \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s)$ for some linear subspace $\mathfrak{w}$ of $\mathfrak{g}_1$ of codimension k (possibly $\mathfrak{w} = \{0\}$); such a $\mathbb{W}$ is a homogeneous normal subgroup of $\mathbb{G}$ of topological dimension $n - k$, see also Section 3 below. Then, a Carnot group has the property $\mathscr{C}_k$ if and only if, for any vertical plane $\mathbb{W}$ in $\mathbb{G}$, there exists a complementary homogeneous subgroup $\mathbb{H}$ that is horizontal, i.e., such that $\mathbb{H} \subset \exp(\mathfrak{g}_1)$. Notice also that, in this case, $\mathbb{H}$ is necessarily commutative.

Remark 2.4. The Heisenberg group $\mathbb{H}^n$ has the property $\mathscr{C}_k$ if and only if $1 \leq k \leq n$.

All Carnot groups have the property $\mathscr{C}_1$. *Free* Carnot groups (see e.g. [18]) have the property $\mathscr{C}_k$ if and only if $k = 1$.

A Carnot group of rank m has the property $\mathscr{C}_m$ if and only if $\mathbb{G}$ is Abelian (i.e., $\mathbb{G} \equiv \mathbb{R}^m$).

Remark 2.5. It is an easy exercise to show that, if $k \geq 2$ and $\mathbb{G}$ has the property $\mathscr{C}_k$, then $\mathbb{G}$ has also the property $\mathscr{C}_h$ for any $1 \leq h \leq k$.

Lemma 2.6. *Let $N \geq 1$ be an integer and $\mathbb{G}$ be a Carnot group. Then $\mathbb{G}$ has the property $\mathscr{C}_k$ if and only if $\mathbb{G} \times \mathbb{R}^N$ has the property $\mathscr{C}_k$.*

Proof. It is clearly enough to prove the statement for $N = 1$.

Assume first that $\mathbb{G}$ has the property $\mathscr{C}_k$ and let $\mathfrak{w}$ be a k-codimensional subspace of the first layer $\mathfrak{g}_1 \times \mathbb{R}$ of the Lie algebra of $\mathbb{G} \times \mathbb{R}$. We have two cases according to the dimension of $\mathfrak{w}' := \mathfrak{w} \cap (\mathfrak{g}_1 \times \{0\})$:

- if $\dim \mathfrak{w}' = m - k$, using the $\mathscr{C}_k$ property of $\mathbb{G}$ one can find a k-dimensional commutative subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 \times \{0\} = \mathfrak{w}' \oplus (\mathfrak{h} \times \{0\})$. In particular, $\mathfrak{g}_1 \times \mathbb{R} = \mathfrak{w} \oplus (\mathfrak{h} \times \{0\})$;
- if $\dim \mathfrak{w}' = m + 1 - k$, then $\mathfrak{w} = \mathfrak{w}' \subset \mathfrak{g}_1 \times \{0\}$ and, by Remark 2.5, one can find a $(k-1)$-dimensional commutative subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 \times \{0\} = \mathfrak{w} \oplus (\mathfrak{h} \times \{0\})$. In particular, $\mathfrak{g}_1 \times \mathbb{R} = \mathfrak{w} \oplus (\mathfrak{h} \times \mathbb{R})$.

In both cases we have found a commutative complementary subspace of $\mathfrak{w}$.

Assume now that $\mathbb{G} \times \mathbb{R}$ has the property $\mathscr{C}_k$ and let $\mathfrak{w}$ be a k-codimensional linear subspace of $\mathfrak{g}_1$. Then $\mathfrak{w} \times \mathbb{R}$ is a k-codimensional linear subspace of $\mathfrak{g}_1 \times \mathbb{R}$, hence it admits a k-dimensional commutative complementary subspace $\mathfrak{h}$ in $\mathfrak{g}_1 \times \mathbb{R}$. Denoting by

$\pi : \mathfrak{g}_1 \times \mathbb{R} \to \mathfrak{g}_1$ the canonical projection, it is readily noticed that $\pi(\mathfrak{h})$ is a k-dimensional commutative subspace of $\mathfrak{g}_1$ such that $\mathfrak{g}_1 = \mathfrak{w} \oplus \pi(\mathfrak{h})$. This concludes the proof. □

2.2. Metric facts

Let $\mathbb{G}$ be a Carnot group with stratified algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$. We endow $\mathfrak{g}$ with a positive definite scalar product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{g}_i \perp \mathfrak{g}_j$ whenever $i \neq j$. We also let $|\cdot| := \langle \cdot, \cdot \rangle^{1/2}$. We fix an orthonormal basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ adapted to the stratification, i.e., such that $\mathfrak{g}_j = \operatorname{span}\{X_{m_{j-1}+1}, \ldots, X_{m_j}\}$ for any $j = 1, \ldots, s$, where $m_j := \dim(\mathfrak{g}_1) + \cdots + \dim(\mathfrak{g}_j)$ and $m_0 := 0$ (in particular, $m_1 = m$).

We will frequently use the *homogeneous* (pseudo-)*norm* $\|\cdot\|$ on $\mathbb{G}$ defined in this way: if $x = \exp(Y_1 + \cdots + Y_s)$ for $Y_j \in \mathfrak{g}_j$, then

$$\|x\| := \sum_{j=1}^{s} |Y_j|^{1/j}.$$

Clearly one has $\|\delta_\lambda(x)\| = \lambda \|x\|$ for any $x \in \mathbb{G}, \lambda > 0$. Homogeneous pseudo-norms arising from different choices of the scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{G}$ are equivalent.

The group $\mathbb{G}$ is endowed with the *Carnot–Carathéodory (CC) distance* d induced by the family $X_1, \ldots, X_m$, as we now introduce. Given an interval $I \subset \mathbb{R}$, a Lipschitz curve $\gamma : I \to \mathbb{G}$ is said to be *horizontal* if there exist functions $h_1, \ldots, h_m \in L^\infty(I)$ such that for a.e. $t \in I$ we have

$$\dot{\gamma}(t) = \sum_{i=1}^{m} h_i(t) X_i(\gamma(t)). \tag{2.1}$$

Letting $|h| := (h_1^2 + \ldots + h_m^2)^{1/2}$, the length of γ is defined as

$$L(\gamma) := \int_I |h(t)|\, dt.$$

It is well-known that for any pair of points $x, y \in \mathbb{G}$ there exists a horizontal curve joining x to y. We can therefore define a distance function d letting

$$d(x,y) := \inf \big\{ L(\gamma) : \gamma : [0,T] \to M \text{ horizontal with } \gamma(0) = x \text{ and } \gamma(T) = y \big\}.$$

It is also well-known that, for any pair $x, y \in \mathbb{G}$, there exists a geodesic joining x and y, i.e., a horizontal curve γ realizing the infimum in the previous formula. Notice that

$$d(zx, zy) = d(x,y) \quad \text{and} \quad d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x,y) \qquad \forall\, x, y, z \in G, \lambda > 0$$

and that $d(x,y)$ is equivalent to $\|x^{-1}y\|$.

We denote by $B(x,r)$ open balls of center $x \in \mathbb{G}$ and radius $r > 0$ with respect to the CC distance; we also write B_r instead of $B(0,r)$, so that $B(x,r) = xB_r$. The diameter diam E of $E \subset \mathbb{G}$ and the distance $d(E_1, E_2)$ between $E_1, E_2 \subset \mathbb{G}$ is understood with respect to the CC distance.

As customary, for $E \subset \mathbb{G}, d > 0$ and $\delta > 0$ we set

$$\mathcal{H}^d_\delta(E) := \inf\left\{\sum_{i=1}^\infty (\text{diam } E_i)^d : E \subset \bigcup_{i=1}^\infty E_i,\ \text{diam } E_i < \delta\right\}$$

$$\mathcal{S}^d_\delta(E) := \inf\left\{\sum_{i=1}^\infty (\text{diam } B_i)^d : B_i \text{ are open balls}, E \subset \bigcup_{i=1}^\infty B_i,\ \text{diam } B_i < \delta\right\}$$

and we define the *d-dimensional Hausdorff measure* and *d-dimensional spherical Hausdorff measure* of E respectively as

$$\mathcal{H}^d(E) := \lim_{\delta \downarrow 0} \mathcal{H}^d_\delta(E) = \sup_{\delta > 0} \mathcal{H}^d_\delta(E)$$

$$\mathcal{S}^d(E) := \lim_{\delta \downarrow 0} \mathcal{S}^d_\delta(E) = \sup_{\delta > 0} \mathcal{S}^d_\delta(E).$$

The Hausdorff dimension of E is $\inf\{d : \mathcal{H}^d(E) = 0\} = \sup\{d : \mathcal{H}^d(E) = \infty\}$. It is well-known that the metric space $(\mathbb{G}, d)$ has Hausdorff dimension $Q := \sum_{j=1}^s j \dim \mathfrak{g}_j$ and that, in exponential coordinates and up to multiplicative constants, the measures $\mathcal{H}^Q$, $\mathcal{S}^Q$ and $\mathscr{L}^n$ coincide, all of them being Haar measures on $\mathbb{G}$.

3. Intrinsic regular hypersurfaces in Carnot groups

We say that a continuous real function f on an open set $\Omega \subset \mathbb{G}$ is *of class* C^1_H if its *horizontal derivatives* $X_1 f, \ldots, X_m f$ are continuous in Ω. In this case we write $f \in C^1_H(\Omega)$ and we set $\nabla_H f := (X_1 f, \ldots, X_m f)$.

A set $S \subset \mathbb{G}$ is a C^1_H *hypersurface* if for any $x \in S$ there exist an open neighborhood U of x and $f \in C^1_H(U)$ such that

$$S \cap U = \{y \in U : f(y) = 0\} \quad \text{and} \quad \nabla_H f \neq 0 \text{ on } U.$$

In this case, we define the *horizontal normal* to x as $\nu_S(x) := \frac{\nabla_H f(x)}{|\nabla_H f(x)|} \in \mathbb{R}^m$. The normal $\nu_S(x) = ((\nu_S(x))_1, \ldots, (\nu_S(x))_m)$ is defined up to sign and it can be canonically identified with a horizontal vector at x by

$$\nu_S(x) = (\nu_S(x))_1 X_1(x) + \cdots + (\nu_S(x))_m X_m(x).$$

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A C^1_H hypersurface has locally finite $\mathcal{H}^{Q-1}$-measure, see e.g. [30] and the references therein.[1]

The hyperplane $\nu_S(x)^\perp$ in $\mathfrak{g}$ is a Lie subalgebra. The associated subgroup $T_xS := \exp(\nu_S(x)^\perp)$ is called *tangent subgroup* to S at x: we point out the well-known property that

$$\forall\, \varepsilon > 0\ \exists\, \bar r = \bar r(x,\varepsilon) > 0 \text{ such that } \forall\, r \in (0,\bar r) \quad (x^{-1}S) \cap B_r \subset (T_xS)_{\varepsilon r} \cap B_r, \tag{3.2}$$

where for $E \subset \mathbb{G}$ and $\delta > 0$ we denote by E_δ the δ-neighborhood of E. A proof of (3.2), using the fact that in exponential coordinates $T_xS = \{(\xi,\eta) \in \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} : \xi \perp \nu_S(x)\}$, is implicitly contained in the proof of Lemma A.4. Notice also that

$$T_xS = \exp(\{X \in \mathfrak{g}_1 : Xf(x) = 0\} \oplus \mathfrak{g}_2 \cdots \oplus \mathfrak{g}_s);$$

in particular, while $\nu_S(x)$ depends on the scalar product $\langle \cdot,\cdot \rangle$ on $\mathfrak{g}$, the subgroup T_xS is intrinsic.

The tangent group T_xS is a vertical plane of codimension 1 (or *vertical hyperplane*), where we say that $\mathbb{W} \subset \mathbb{G}$ is a *vertical plane* of codimension $k, 1 \le k \le m$, if $\mathbb{W} = \exp(\mathfrak{w} \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s)$ for some linear subspace $\mathfrak{w}$ of $\mathfrak{g}_1$ of codimension k (possibly $\mathfrak{w} = \{0\}$). Such a $\mathbb{W}$ is a homogeneous normal subgroup of $\mathbb{G}$ of topological dimension $n-k$ and Hausdorff dimension $Q-k$. The intersection of vertical planes is always a vertical plane (of possibly higher codimension).

The following simple lemma will be used in the proof of Lemma 3.2.

Lemma 3.1. *Let $\mathbb{W} \subset \mathbb{G}$ be a vertical plane of codimension k and let $x \in \mathbb{W}$, $r > 0$ and $\varepsilon \in (0,1)$ be fixed. Then, the set $\mathbb{W} \cap B(x,r)$ can be covered by a family of balls $\{B(y_\ell,\varepsilon r)\}_{\ell\in L}$ of radius εr with cardinality $\#L \le (4/\varepsilon)^{Q-k}$.*

Proof. By dilation and translation invariance, it is not restrictive to assume that $x = 0$ and $r = 1$. Let $\{y_\ell\}_{\ell\in L}$ be a maximal family of points of $\mathbb{W} \cap B(0,1)$ such that the balls $B(y_\ell,\varepsilon/2)$ are pairwise disjoint; working by contradiction, it can be easily seen that the family $\{B(y_\ell,\varepsilon)\}_{\ell\in L}$ covers $\mathbb{W} \cap B(0,1)$. The measure $\mathcal{H}^{Q-k}$ is locally finite on $\mathbb{W}$ (see e.g. [21,23,24]), is left-invariant and it is $(Q-k)$-homogeneous with respect to dilations. In particular, setting $M := \mathcal{H}^{Q-k}(\mathbb{W} \cap B(0,1))$, we have

$$\left(\frac{\varepsilon}{2}\right)^{Q-k} M\, \#L = \sum_{\ell\in L} \mathcal{H}^{Q-k}(\mathbb{W} \cap B(y_\ell,\varepsilon/2)) \le \mathcal{H}^{Q-k}(\mathbb{W} \cap B(0,2)) = 2^{Q-k} M,$$

which proves the claim. □

[1] Actually, this also follows from Theorem 1.4 with $k = 1$.

A key tool in the proof of the rank-one Theorem 1.1 is the following Lemma 3.2 which, in turn, uses Theorem 1.4, whose proof is instead postponed to Appendix A. We denote by $\pi : \mathbb{G} \times \mathbb{R} \to \mathbb{G}$ the canonical projection $\pi(x,t) = x$.

Lemma 3.2. *Let $\mathbb{G}$ be a Carnot group satisfying property $\mathscr{C}_2$. Let Σ_1, Σ_2 be C^1_H hypersurfaces in $\mathbb{G} \times \mathbb{R}$ with unit normals $\nu_{\Sigma_1}, \nu_{\Sigma_2}$. Then, the set*

$$R := \left\{ p \in \Sigma_1 : \exists\, q \in \Sigma_2 \text{ such that } \begin{array}{l} \pi(q) = \pi(p), \\ (\nu_{\Sigma_1}(p))_{m+1} = (\nu_{\Sigma_2}(q))_{m+1} = 0, \\ \nu_{\Sigma_1}(p) \neq \pm \nu_{\Sigma_2}(q) \end{array} \right\}$$

is $\mathcal{H}^Q$-negligible.

Proof. Let us consider the distances $d_{\mathbb{G}\times\mathbb{R}}$ and $d_{\mathbb{G}\times\mathbb{R}\times\mathbb{R}}$ on (respectively) $\mathbb{G} \times \mathbb{R}$ and $\mathbb{G} \times \mathbb{R} \times \mathbb{R}$ defined by

$$\begin{aligned} &d_{\mathbb{G}\times\mathbb{R}}((x,t),(x',t')) := d(x,x') + |t-t'| && \forall\, x,x' \in \mathbb{G}, t,t' \in \mathbb{R} \\ &d_{\mathbb{G}\times\mathbb{R}\times\mathbb{R}}((x,t,s),(x',t',s')) := d(x,x') + |t-t'| + |s-s'| && \forall\, x,x' \in \mathbb{G}, t,t',s,s' \in \mathbb{R}, \end{aligned}$$

where d is the Carnot–Carathéodory distance on $\mathbb{G}$. Such distances are left-invariant and homogeneous, hence they are equivalent to the Carnot–Carathéodory distances on $\mathbb{G}\times\mathbb{R}$ and $\mathbb{G} \times \mathbb{R} \times \mathbb{R}$; in particular, it is enough to prove the statement when the Hausdorff measure $\mathcal{H}^Q$ is the one induced by $d_{\mathbb{G}\times\mathbb{R}}$ on $\mathbb{G} \times \mathbb{R}$. We use the same notation $B(a,r)$ for balls of radius $r > 0$ in either $\mathbb{G}, \mathbb{G} \times \mathbb{R}$ or $\mathbb{G} \times \mathbb{R} \times \mathbb{R}$, according to which group the center a belongs to.

The sets

$$\begin{aligned} \widetilde{\Sigma}_1 &:= \{(x,t,s) \in \mathbb{G} \times \mathbb{R} \times \mathbb{R} : (x,t) \in \Sigma_1, s \in \mathbb{R}\} \\ \widetilde{\Sigma}_2 &:= \{(x,t,s) \in \mathbb{G} \times \mathbb{R} \times \mathbb{R} : (x,s) \in \Sigma_2, t \in \mathbb{R}\} \end{aligned}$$

are clearly C^1_H hypersurfaces in $\mathbb{G} \times \mathbb{R} \times \mathbb{R}$ and, moreover,

$$\begin{aligned} \nu_{\widetilde{\Sigma}_1}(x,t,s) &= \big((\nu_{\Sigma_1}(x,t))_1, \dots, (\nu_{\Sigma_1}(x,t))_m, (\nu_{\Sigma_1}(x,t))_{m+1},\ 0\ \big) \\ \nu_{\widetilde{\Sigma}_2}(x,t,s) &= \big((\nu_{\Sigma_2}(x,s))_1, \dots, (\nu_{\Sigma_2}(x,s))_m,\ 0\ , (\nu_{\Sigma_2}(x,s))_{m+1}\big). \end{aligned}$$

Let us define

$$\begin{aligned} \widetilde{R} &:= \{P \in \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2 : (\nu_{\widetilde{\Sigma}_1}(P))_{m+1} = (\nu_{\widetilde{\Sigma}_2}(P))_{m+2} = 0 \text{ and } \nu_{\widetilde{\Sigma}_1}(P) \neq \pm \nu_{\widetilde{\Sigma}_2}(P)\} \\ &= \{(x,t,s) \in \widetilde{\Sigma}_1 \cap \widetilde{\Sigma}_2 : (\nu_{\Sigma_1}(x,t))_{m+1} = (\nu_{\Sigma_2}(x,s))_{m+1} = 0 \\ &\qquad \text{and } \nu_{\Sigma_1}(x,t) \neq \pm \nu_{\Sigma_2}(x,s)\}. \end{aligned}$$

By construction we have $\widetilde{\pi}(\widetilde{R}) = R$, where $\widetilde{\pi} : \mathbb{G} \times \mathbb{R} \times \mathbb{R} \to \mathbb{G} \times \mathbb{R}$ is the group homomorphism defined by $\widetilde{\pi}(x, t, s) := (x, t)$; moreover the measure $\mathcal{H}^Q \llcorner \widetilde{R}$ is σ-finite by Theorem 1.4 (notice that we are also using Lemma 2.6). We are going to show that $\mathcal{H}^Q(\widetilde{\pi}(T)) = 0$ for any fixed $T \subset \widetilde{R}$ such that $\mathcal{S}^Q(T) < \infty$; this is clearly enough to conclude.

For any $P \in T$ and $i = 1, 2$, the tangent space $T_P\widetilde{\Sigma}_i$ equals $\mathbb{W}_i \times \mathbb{R} \times \mathbb{R}$ for a suitable vertical hyperplane $\mathbb{W}_i$ of $\mathbb{G}$. In particular, setting $\mathbb{W} = \mathbb{W}(P) := \mathbb{W}_1 \cap \mathbb{W}_2$, we have by (3.2) that for any $P \in T$ and any $\varepsilon \in (0, 1)$ there exists $\bar{r} = \bar{r}(\varepsilon, P) > 0$ such that

$$\begin{aligned} (P^{-1}T) \cap B(0, r) &\subset (\mathbb{W} \times \mathbb{R} \times \mathbb{R})_{\varepsilon r} \cap B(0, r) \\ &= (\mathbb{W}_{\varepsilon r} \times \mathbb{R} \times \mathbb{R}) \cap B(0, r) \qquad \text{for any } r \in (0, \bar{r}). \end{aligned} \tag{3.3}$$

Notice also that $\mathbb{W}$ is a vertical plane of codimension 2 in $\mathbb{G}$. Let $\varepsilon > 0$ be fixed and set

$$T_j := \{P \in T : \bar{r}(\varepsilon, P) \geq \tfrac{1}{j}\}, \qquad j = 1, 2, \ldots$$

Since $T_j \uparrow T$, the proof will be accomplished by showing that for any fixed j

$$\mathcal{H}^Q(\widetilde{\pi}(T_j)) < C\varepsilon, \tag{3.4}$$

where $C > 0$ is a constant that will be determined in the sequel.

Let us prove (3.4). Fix $\delta \in (0, \frac{1}{j})$; since $\mathcal{H}^Q(T_j) \leq \mathcal{H}^Q(T) < +\infty$, one can find a (countable or finite) family $\{B(\widetilde{P}_i, r_i/2)\}_i$ of balls in $\mathbb{G} \times \mathbb{R} \times \mathbb{R}$ such that $0 < r_i < \delta$,

$$T_j \subset \bigcup_i B(\widetilde{P}_i, r_i/2) \qquad \text{and} \qquad \sum_i (r_i/2)^Q \leq \sum_i (\operatorname{diam} B(\widetilde{P}_i, r_i/2))^Q \leq C_1$$

where $C_1 := \mathcal{H}^Q(T) + 1$. We can also assume that $T_j \cap B(\widetilde{P}_i, r_i/2)$ is non-empty for any i. Choosing $P_i \in T_j \cap B(\widetilde{P}_i, r_i/2)$, for any i the balls $B(P_i, r_i)$ have then the following properties:

$$P_i \in T_j, \quad 0 < r_i < \delta, \quad T_j \subset \bigcup_i B(P_i, r_i) \quad \text{and} \quad \sum_i r_i^Q \leq 2^Q C_1. \tag{3.5}$$

Setting $\mathbb{W}_i := \mathbb{W}(P_i)$, by (3.3) we have

$$\begin{aligned} (P_i^{-1}T_j) \cap B(0, r_i) &\subset ((\mathbb{W}_i)_{\varepsilon r_i} \times \mathbb{R} \times \mathbb{R}) \cap B(0, r_i) \\ &= ((\mathbb{W}_i)_{\varepsilon r_i} \cap B(0, r_i)) \times (-r_i, r_i) \times (-r_i, r_i). \end{aligned} \tag{3.6}$$

By Lemma 3.1, for any i we can find a family of balls $\{B(y_{i,\ell}, \varepsilon r_i)\}_{\ell \in L_i}$ such that

$$\forall\, \ell \in L_i \ y_{i,\ell} \in \mathbb{W}_i, \quad \# L_i \leq (8/\varepsilon)^{Q-2} \quad \text{and} \quad \mathbb{W}_i \cap B(0, 2r_i) \subset \bigcup_{\ell \in L_i} B(y_{i,\ell}, \varepsilon r_i).$$

In particular

$$(\mathbb{W}_i)_{\varepsilon r_i} \cap B(0,r_i) \subset (\mathbb{W}_i \cap B(0,r_i+\varepsilon r_i))_{\varepsilon r_i} \subset \bigcup_{\ell \in L_i} B(y_{i,\ell}, 2\varepsilon r_i). \tag{3.7}$$

Let us also fix points $\{\tau_k\}_{k\in K_i} \subset (-r_i, r_i)$ such that $\# K_i \le 2\varepsilon^{-1}$ and

$$(-r_i, r_i) \subset \bigcup_{k\in K_i} (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \tag{3.8}$$

By (3.6), (3.7) and (3.8) we get

$$(P_i^{-1} T_j) \cap B(0,r_i) \subset \bigcup_{\substack{\ell\in L_i\\ k,h\in K_i}} B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_h - 2\varepsilon r_i, \tau_h + 2\varepsilon r_i).$$

For any $\ell \in L_i$ and $k, h, h' \in K_i$ one has

$$\begin{aligned} &\widetilde{\pi}\big(B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_h - 2\varepsilon r_i, \tau_h + 2\varepsilon r_i)\big) \\ &= \widetilde{\pi}\big(B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_{h'} - 2\varepsilon r_i, \tau_{h'} + 2\varepsilon r_i)\big) \\ &= B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \\ &\subset B((y_{i,\ell}, \tau_k), 4\varepsilon r_i) \end{aligned}$$

which, using (3.5), implies that

$$\begin{aligned} \widetilde{\pi}(T_j) &\subset \bigcup_i \widetilde{\pi}\big(T_j \cap B(P_i, r_i)\big) \\ &\subset \bigcup_i \bigcup_{\substack{\ell\in L_i\\ k,h\in K_i}} \widetilde{\pi}\big(P_i(B(y_{i,\ell}, 2\varepsilon r_i) \times (\tau_k - 2\varepsilon r_i, \tau_k + 2\varepsilon r_i) \times (\tau_h - 2\varepsilon r_i, \tau_h + 2\varepsilon r_i))\big) \\ &\subset \bigcup_i \bigcup_{\substack{\ell\in L_i\\ k\in K_i}} \widetilde{\pi}(P_i) B((y_{i,\ell}, \tau_k), 4\varepsilon r_i) \\ &= \bigcup_i \bigcup_{\substack{\ell\in L_i\\ k\in K_i}} B(p_{i\ell k}, 4\varepsilon r_i) \end{aligned}$$

where $p_{i\ell k} := \widetilde{\pi}(P_i)(y_{i,\ell}, \tau_k) \in \mathbb{G} \times \mathbb{R}$. Using again (3.5) we obtain that

$$\mathcal{H}^Q_{4\varepsilon\delta}(T_j) \le \sum_i \# L_i \; \# K_i \; (8\varepsilon r_i)^Q \le \varepsilon \sum_i 2^{6Q-5} r_i^Q \le 2^{7Q-5} C_1 \varepsilon$$

which, by the arbitrariness of $\delta \in (0, \frac{1}{j})$, gives the claim (3.4). □

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4. Functions with bounded H-variation and subgraphs

Let $X = (X_1, \dots, X_m)$ be an m-tuple of linearly independent vector fields in $\mathbb{R}^n$; for $i = 1, \dots, m$ and $j = 1, \dots, n$ we consider smooth functions a_{ij} such that

$$X_i(x) = \sum_{j=1}^{n} a_{ij}(x)\partial_{x_j}.$$

The model case is of course that of a Carnot group $\mathbb{G} \equiv \mathbb{R}^n$ endowed with a left-invariant basis $X_1, \dots, X_m$ of the first layer $\mathfrak{g}_1$ in the Lie algebra stratification; in the present section, however, we work in higher generality.

One of the main purposes of this paper is the study of *functions with bounded H-variation* ([6,13]), that we are going to introduce only very briefly. In this section, Ω is an open subset of $\mathbb{R}^n$ and, given $\varphi \in C^1(\Omega, \mathbb{R}^m)$, we let $\operatorname{div}_X \varphi := \sum_{i=1}^m X_i^* \varphi_i$ where X_i^* denotes the formal adjoint operator of the vector field X_i. Given a $\mathbb{R}^m$-valued function f on Ω and a $\mathbb{R}^m$-valued measure μ on Ω we use the compact notation $\int_\Omega f \cdot d\mu$ for the sum $\int_\Omega f_1\, d\mu_1 + \dots + \int_\Omega f_m\, d\mu_m$.

Definition 4.1. We say that $u \in L^1_{loc}(\Omega)$ is a function of *locally bounded H-variation* in Ω, and we write $u \in BV_{H,loc}(\Omega)$, if there exists a vector valued Radon measure $D_H u = (D_{X_1}u, \dots, D_{X_m}u)$ with locally finite total variation such that for every $\varphi \in C^1_c(\Omega; \mathbb{R}^m)$ we have

$$\int_\Omega \varphi \cdot d\, D_H u = -\int_\Omega u \operatorname{div}_X \varphi \, d\mathscr{L}^n. \tag{4.9}$$

Moreover, if $u \in L^1(\Omega)$, we say that u has *bounded H-variation* in Ω ($u \in BV_H(\Omega)$) if $D_H u$ has finite total variation $|D_H u|$ on Ω.

We say that $E \subset \Omega$ has *finite H-perimeter* in Ω if its characteristic function χ_E belongs to $BV_H(\Omega)$.

We recall that the total variation $|\mu|$ of a $\mathbb{R}^d$-valued measure $\mu = (\mu_1, \dots, \mu_d)$ is defined for Borel sets B as

$$\begin{aligned} |\mu|(B) &:= \sup\left\{ \sum_{\ell=1}^{\infty} |\mu(B_\ell)| : (B_\ell)_\ell \text{ disjoint Borel subsets of } B \right\} \\ &= \sup\left\{ \int_B \varphi \cdot d\mu : \ \varphi : B \to \mathbb{R}^d \text{ Borel function, } |\varphi| \le 1 \right\}. \end{aligned}$$

If $A \Subset \Omega$ is open and $u \in BV_{H,loc}(\Omega)$, one can easily prove that

$$|D_H u|(A) = \sup\left\{ \int_A u \, \mathrm{div}_X \varphi \, d\mathscr{L}^n : \varphi \in C^1_c(A;\mathbb{R}^m), |\varphi| \le 1 \right\};$$

actually, $u \in BV_H(A)$ if and only if the supremum on the right-hand side is finite. The total variation is lower-semicontinuous with respect to the L^1_{loc} convergence; moreover (see [16,13]), for any $u \in BV_H(\Omega)$ there exists a sequence $(u_h)_h$ in $C^\infty(\Omega) \cap BV_H(\Omega)$ such that

$$\begin{aligned}
&u_h \to u \text{ in } L^1(\Omega)\\
&|D_H u_h|(\Omega) \to |D_H u|(\Omega)\\
&|D_{X_i} u_h|(\Omega) \to |D_{X_i} u|(\Omega) \quad \forall\, i = 1,\dots,m\\
&|(D_H u_h, \mathscr{L}^n)|(\Omega) \to |(D_H u, \mathscr{L}^n)|(\Omega).
\end{aligned} \tag{4.10}$$

The aim of this section is the study of the relations occurring between a function $u \in BV_H(\Omega)$ and its *subgraph*

$$E_u := \{(x,t) \in \Omega \times \mathbb{R} : t < u(x)\} \subset \Omega \times \mathbb{R}.$$

We introduce the family $\widetilde{X} = (\widetilde{X}_1,\dots,\widetilde{X}_{m+1})$ of linearly independent vector fields in $\mathbb{R}^{n+1}$ defined for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ by

$$\begin{aligned}
&\widetilde{X}_i(x,t) := (X_i(x), 0) \in \mathbb{R}^{n+1} \equiv \mathbb{R}^n \times \mathbb{R} \qquad \text{if } i = 1,\dots,m\\
&\widetilde{X}_{m+1}(x,t) := \partial_t.
\end{aligned}$$

If $U \subset \mathbb{R}^{n+1}$ is open and $u \in BV_{H,loc}(U)$ with respect to the family $\widetilde{X}$ we write $D_{\widetilde{H}} u := (D_{\widetilde{X}_1} u,\dots,D_{\widetilde{X}_{m+1}} u)$.

The following result is the natural generalization of some classical facts about Euclidean functions of bounded variation, see e.g. [17, Section 4.1.5]. We denote by $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the canonical projection $\pi(x,t) = x$; $\pi_\#$ denotes the associated push-forward of measures.

Theorem 4.2. *Suppose Ω is bounded in $\mathbb{R}^n$ and let $u \in L^1(\Omega)$. Then u belongs to $BV_H(\Omega)$ if and only if its subgraph E_u has finite H-perimeter (with respect to the family $\widetilde{X}$) in $\Omega \times \mathbb{R}$.*

Moreover, writing $D'_{\widetilde{H}}\chi_{E_u} := (D_{\widetilde{X}_1}\chi_{E_u},\dots,D_{\widetilde{X}_m}\chi_{E_u})$, then the following statements hold:

(i) $\pi_\# D_{\widetilde{X}_i}\chi_{E_u} = D_{X_i} u$ *for any* $i = 1,\dots,m$;
(ii) $\pi_\# \partial_t \chi_{E_u} = -\mathscr{L}^n$;
(iii) $\pi_\# |D_{\widetilde{X}_i}\chi_{E_u}| = |D_{X_i} u|$ *for any* $i = 1,\dots,m$;
(iv) $\pi_\# |\partial_t \chi_{E_u}| = \mathscr{L}^n$;

(v) $\pi_\#|D'_{\widetilde{H}}\chi_{E_u}| = |D_H u|$;
(vi) $\pi_\#|D_{\widetilde{H}}\chi_{E_u}| = |(D_H u, -\mathscr{L}^n)|$.

Proof. Suppose first that $\chi_{E_u} \in BV_H(\Omega \times \mathbb{R})$ with respect to the family $\widetilde{X}$. We need to fix a sequence $(g_h)_h$ in $C_c^\infty(\mathbb{R})$ such that g_h is even, $g_h \equiv 1$ on $[0, h]$, $g_h \equiv 0$ on $[h+1, +\infty)$ and $\int_{\mathbb{R}} g_h(t)dt = 2h + 1$. Let $\varphi \in C_c^1(\Omega, \mathbb{R}^m)$ with $|\varphi| \le 1$ be fixed. By the Dominated Convergence Theorem we have

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} \varphi(x) \cdot d(D'_{\widetilde{H}}\chi_{E_u})(x,t) &= \lim_{h\to+\infty} \int_{\Omega\times\mathbb{R}} g_h(t)\varphi(x) \cdot d(D'_{\widetilde{H}}\chi_{E_u})(x,t) \\ &= -\lim_{h\to+\infty} \int_{\Omega\times\mathbb{R}} \chi_{E_u}(x,t) g_h(t) \operatorname{div}_X \varphi(x) d\mathscr{L}^{n+1}(x,t) \\ &= -\lim_{h\to+\infty} \int_\Omega \left(\int_{-\infty}^{u(x)} g_h(t)dt \right) \operatorname{div}_X \varphi(x) d\mathscr{L}^n(x).\end{aligned}$$

For every $z \in \mathbb{R}$ and every $h \in \mathbb{N}$ we have

$$\int_{-\infty}^{z} g_h(t)dt \le |z| + h + \frac{1}{2} \quad \text{and} \quad \lim_{h\to+\infty} \left(\int_{-\infty}^{z} g_h(t)dt - h - \frac{1}{2} \right) = z;$$

using the fact that $\int_\Omega \operatorname{div}_X \varphi(x) d\mathscr{L}^n(x) = 0$, by the Dominated Convergence Theorem we obtain

$$\begin{aligned}\int_{\Omega\times\mathbb{R}} \varphi(x) \cdot d(D'_{\widetilde{H}}\chi_{E_u})(x,t) &= -\lim_{h\to+\infty} \int_\Omega \left(\int_{-\infty}^{u(x)} g_h(t)dt - h - \frac{1}{2} \right) \operatorname{div}_X \varphi(x) d\mathscr{L}^n(x) \\ &= -\int_\Omega u(x) \operatorname{div}_X \varphi(x) d\mathscr{L}^n(x) \qquad (4.11) \\ &= \int_\Omega \varphi(x) \cdot d(D_H u)(x).\end{aligned}$$

In particular, $u \in BV_H(\Omega)$ and, for any open set $A \subset \Omega$,

$$\begin{aligned} &|D_H u|(A) \le |D'_{\widetilde{H}}\chi_{E_u}|(A \times \mathbb{R}) \\ &|D_{X_i} u|(A) \le |D_{\widetilde{X}_i}\chi_{E_u}|(A \times \mathbb{R}) \quad \text{for any } i = 1, \dots, m. \end{aligned} \tag{4.12}$$

Before passing to the reverse implication we observe two facts. First, for any $\varphi \in C_c^1(\Omega)$ one has

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$$\begin{aligned}\int\limits_{\Omega\times\mathbb{R}} \varphi(x) d\left(\partial_t \chi_{E_u}\right)(x,t) &= \lim_{h\to+\infty} \int\limits_{\Omega\times\mathbb{R}} \varphi(x) g_h(t) d\left(\partial_t \chi_{E_u}\right)(x,t) \\ &= -\lim_{h\to+\infty} \int\limits_{\Omega\times\mathbb{R}} \varphi(x) g_h'(t) \chi_{E_u}(x,t) d\mathscr{L}^{n+1}(x,t) \\ &= -\lim_{h\to+\infty} \int\limits_{\Omega} \varphi(x) \left(\int\limits_{-\infty}^{u(x)} g_h'(t)dt \right) d\mathscr{L}^n(x) \qquad (4.13) \\ &= -\lim_{h\to+\infty} \int\limits_{\Omega} \varphi(x) g_h(u(x)) d\mathscr{L}^n(x) \\ &= -\int\limits_{\Omega} \varphi d\mathscr{L}^n\end{aligned}$$

whence, for any open set $A \subset \Omega$,

$$\mathscr{L}^n(A) \leq |\partial_t \chi_{E_u}|(A\times\mathbb{R}). \qquad (4.14)$$

Second, if $\varphi \in C_c^1(\Omega, \mathbb{R}^{m+1})$ one has by (4.11) and (4.13)

$$\int\limits_{\Omega\times\mathbb{R}} \varphi(x) \cdot d(D_{\widetilde{H}} \chi_{E_u})(x,t) = \int\limits_{\Omega} \varphi(x) \cdot d(D_H u, -\mathscr{L}^n)(x)$$

which gives for any open set $A \subset \Omega$

$$|(D_H u, -\mathscr{L}^n)|(A) \leq |D_{\widetilde{H}} \chi_{E_u}|(A\times\mathbb{R}). \qquad (4.15)$$

Suppose now that $u \in BV_H(\Omega)$. Let $A \subset \Omega$ be open and let $\varphi \in C_c^1(A\times\mathbb{R})$ and $i = 1, \dots, m$ be fixed. Let $(u_h)_h$ be a sequence in $C^\infty(A) \cap BV_H(A)$ satisfying (4.10) (with A in place of Ω); then

$$\begin{aligned}&\int\limits_{A\times\mathbb{R}} \varphi\, d(D_{\widetilde{X}_i} \chi_{E_{u_h}}) \\ &= -\int\limits_{A\times\mathbb{R}} \chi_{E_{u_h}}(x,t) \widetilde{X}_i^* \varphi(x,t) d\mathscr{L}^{n+1}(x,t) \\ &= -\int\limits_{A} \left(\int\limits_{-\infty}^{u_h(x)} \sum_{j=1}^{n} \partial_{x_j}\left(a_{ij}(x)\varphi(x,t)\right) dt \right) d\mathscr{L}^n(x) \qquad (4.16)\end{aligned}$$

$$= -\int_A \left(\sum_{j=1}^n \partial_{x_j} \int_{-\infty}^{u_h(x)} a_{ij}(x)\varphi(x,t)dt - \sum_{j=1}^n a_{ij}(x)\varphi(x,u_h(x))\partial_{x_j} u_h(x) \right) d\mathscr{L}^n(x)$$
$$= \int_A \varphi(x,u_h(x)) X_i u_h(x) d\mathscr{L}^n(x),$$

where we used the fact that $x \mapsto a_{ij}(x) \int_{-\infty}^{u_h(x)} \varphi(x,t)dt$ is in $C_c^1(A)$. In a similar way

$$\begin{aligned} \int_{A\times\mathbb{R}} \varphi \, d\big(\partial_t \chi_{E_{u_h}}\big) &= -\int_A \left(\int_{-\infty}^{u_h(x)} \partial_t \varphi(x,t)dt \right) d\mathscr{L}^n(x) \\ &= -\int_A \varphi(x,u_h(x)) d\mathscr{L}^n(x). \end{aligned} \tag{4.17}$$

Formulas (4.16) and (4.17) imply that for any $\varphi \in C_c^1(A\times\mathbb{R}, \mathbb{R}^{m+1})$

$$\int_{A\times\mathbb{R}} \varphi \cdot d(D_{\widetilde{H}} \chi_{E_{u_h}}) = \int_A \varphi(x,u_h(x)) \cdot d(D_H u_h, -\mathscr{L}^n)(x).$$

Since $\chi_{E_{u_h}} \to \chi_{E_u}$ in $L^1(A\times\mathbb{R})$ we obtain

$$\begin{aligned} |D_{\widetilde{H}} \chi_{E_u}|(A\times\mathbb{R}) &\le \liminf_{h\to+\infty} |D_{\widetilde{H}} \chi_{E_{u_h}}|(A\times\mathbb{R}) \le \lim_{h\to+\infty} |(D_H u_h, -\mathscr{L}^n)|(A) \\ &= |(D_H u, -\mathscr{L}^n)|(A) < +\infty, \end{aligned} \tag{4.18}$$

which proves that $\chi_{E_u} \in BV_{\widetilde{H}}(\Omega\times\mathbb{R})$, as desired. Notice that, using the lower semicontinuity in a similar way, one also gets

$$\begin{aligned} &|D'_{\widetilde{H}} \chi_{E_u}|(A\times\mathbb{R}) \le |D_H u|(A) \\ &|D_{\widetilde{X}_i} \chi_{E_u}|(A\times\mathbb{R}) \le |D_{X_i} u|(A) \quad \text{for any } i = 1,\dots,m \\ &|\partial_t \chi_{E_u}|(A\times\mathbb{R}) \le \mathscr{L}^n(A) < +\infty. \end{aligned} \tag{4.19}$$

Eventually, statements (i) and (ii) follow from (4.11) and (4.13), while statements (iii)–(vi) are consequences of formulas (4.12), (4.14), (4.15), (4.18) and (4.19). □

Let us introduce some further notation. For $u \in BV_{H,loc}(\Omega)$ we decompose its distributional horizontal derivatives as $D_H u = D_H^a u + D_H^s u$, where $D_H^a u$ is absolutely continuous with respect to $\mathscr{L}^n$ and $D_H^s u$ is singular with respect to $\mathscr{L}^n$. We also write $D_H^a u = Xu\,\mathscr{L}^n$ for some function $Xu \in L^1_{loc}(\Omega,\mathbb{R}^m)$.

We also consider the polar decomposition $D_H u = \sigma_u |D_H u|$, where $\sigma_u : \Omega \to \mathbb{S}^{m-1}$ is a $|D_H u|$-measurable function. In case $u = \chi_E$ is the characteristic function of a set

$E \subset \Omega \times \mathbb{R}$ of locally finite $\widetilde{H}$-perimeter in $\Omega \times \mathbb{R}$ we write $D_{\widetilde{H}}\chi_E = \nu_E |D_{\widetilde{H}}\chi_E|$ for some Borel function $\nu_E = ((\nu_E)_1, \dots, (\nu_E)_{m+1})$ called *horizontal inner normal* to E.

The following result is basically a consequence of Theorem 4.2.

Theorem 4.3. *Let $u \in BV_H(\Omega)$ and define*

$$S := \{(x,t) \in \Omega \times \mathbb{R} : (\nu_{E_u})_{m+1}(x,t) = 0\}$$
$$T := \{(x,t) \in \Omega \times \mathbb{R} : (\nu_{E_u})_{m+1}(x,t) \neq 0\}\,.$$

Then, the following identities hold

$$\nu_{E_u}(x,t) = (\sigma_u(x), 0) \quad \textit{for } |D_{\widetilde{H}}\chi_{E_u}|\textit{-a.e. } (x,t) \in S; \tag{4.20}$$

$$\nu_{E_u}(x,t) = \frac{(Xu(x), -1)}{\sqrt{1+|Xu(x)|^2}} \quad \textit{for } |D_{\widetilde{H}}\chi_{E_u}|\textit{-a.e. } (x,t) \in T; \tag{4.21}$$

$$\pi_\#(D_{\widetilde{H}}\chi_{E_u} \llcorner S) = (D_H^s u, 0); \tag{4.22}$$

$$\pi_\#(D_{\widetilde{H}}\chi_{E_u} \llcorner T) = (D_H^a u, -\mathscr{L}^n). \tag{4.23}$$

Proof. Thanks to Theorem 4.2 (vi) we can disintegrate the measure $|D_{\widetilde{H}}\chi_{E_u}|$ with respect to $|(D_H u, -\mathscr{L}^n)|$ (see e.g. [3, Theorem 2.28]): for every $x \in \Omega$ there exists a probability measure μ_x on $\mathbb{R}$ such that for every Borel function $g \in L^1(\Omega \times \mathbb{R}, |D_{\widetilde{H}}\chi_{E_u}|)$

$$\int\limits_{\Omega\times\mathbb{R}} g(x,t)d|D_{\widetilde{H}}\chi_{E_u}|(x,t) = \int\limits_{\Omega} \left(\int\limits_{\mathbb{R}} g(x,t)d\mu_x(t) \right) d|(D_H u, -\mathscr{L}^n)|(x).$$

It follows that for any Borel function $\varphi : \Omega \to \mathbb{R}$

$$\begin{aligned} \int\limits_{\Omega} \varphi(x)d(D_H u, -\mathscr{L}^n)(x) &= \int\limits_{\Omega} \varphi(x)d\pi_\#(\nu_{E_u}|D_{\widetilde{H}}\chi_{E_u}|)(x) \\ &= \int\limits_{\Omega\times\mathbb{R}} \varphi(x)\nu_{E_u}(x,t)d|D_{\widetilde{H}}\chi_{E_u}|(x,t) \\ &= \int\limits_{\Omega} \varphi(x) \left(\int\limits_{\mathbb{R}} \nu_{E_u}(x,t)d\mu_x(u) \right) d|(D_H u, -\mathscr{L}^n)|(x). \end{aligned} \tag{4.24}$$

Since $D_H^a u$ and $D_H^s u$ are mutually singular we have

$$|(D_H u, -\mathscr{L}^n)| = |(D_H^a u, -\mathscr{L}^n)| + |(D_H^s u, 0)| = \sqrt{1+|Xu|^2}\mathscr{L}^n + |D_H^s u|$$

and (4.24) gives

$$\int_\Omega \varphi \, d\Big((Xu, -1)\mathscr{L}^n + (\sigma_u, 0)|D_H^s u|\Big) \tag{4.25}$$

$$= \int_\Omega \varphi(x) \left(\int_{\mathbb{R}} \nu_{E_u}(x,t) d\mu_x(t) \right) d\left(\sqrt{1+|Xu|^2}\mathscr{L}^n + |D_H^s u|\right)(x). \tag{4.26}$$

Denote by I a subset of Ω such that $\mathscr{L}^n(I) = 0$ and $|D_H^s u|(\Omega \setminus I) = 0$. Considering Borel test functions φ such that $\varphi = 0$ in $\Omega \setminus I$, we deduce that for $|D_H^s u|$-a.e. $x \in I$ one has

$$(\sigma_u(x), 0) = \int_{\mathbb{R}} \nu_{E_u}(x,t) d\mu_x(t).$$

Taking on both sides the scalar product with $(\sigma_u(x), 0)$ we get

$$\left\langle (\sigma_u(x), 0), \int_{\mathbb{R}} \nu_{E_u}(x,t) d\mu_x(t) \right\rangle = 1,$$

and, since $\mu_x(\mathbb{R}) = 1$ and (for $|(D_H u, -\mathscr{L}^n)|$-a.e. $x \in \Omega$) $|\nu_{E_u}(x,t)| = 1$ for μ_x-a.e. t, we deduce that

$$\nu_{E_u}(x,t) = (\sigma_u(x), 0) \qquad \text{for } |D_H^s u|\text{-a.e. } x \in I \text{ and } \mu_x\text{-a.e. } t \in \mathbb{R},$$

i.e.,

$$\nu_{E_u}(x,t) = (\sigma_u(x), 0) \qquad \text{for } |D_{\widetilde{H}} \chi_{E_u}|\text{-a.e. } (x,t) \in I \times \mathbb{R}. \tag{4.27}$$

Taking into account again (4.25) and letting φ be such that $\varphi = 0$ on I we instead obtain

$$\int_\Omega \varphi \, \frac{(Xu, -1)}{\sqrt{1+|Xu|^2}} \sqrt{1+|Xu|^2} d\mathscr{L}^n$$

$$= \int_\Omega \varphi(x) \left(\int_{\mathbb{R}} \nu_{E_u}(x,t) d\mu_x(t) \right) \sqrt{1+|Xu(x)|^2} d\mathscr{L}^n(x)$$

Consequently, for $\mathscr{L}^n$-a.e. $x \in \Omega \setminus I$ we have

$$\int_{\mathbb{R}} \nu_{E_u}(x,t) d\mu_x(t) = \frac{(Xu(x), -1)}{\sqrt{1+|Xu(x)|^2}}.$$

Reasoning as before we deduce that

$$\nu_{E_u}(x,t) = \frac{(Xu(x),-1)}{\sqrt{1+|Xu(x)|^2}} \quad \text{for } \mathscr{L}^n\text{-a.e. } x \in \Omega \setminus I \text{ and } \mu_x\text{-a.e. } t \in \mathbb{R},$$

or equivalently

$$\nu_{E_u}(x,t) = \frac{(Xu(x),-1)}{\sqrt{1+|Xu(x)|^2}} \quad \text{for } |D_{\widetilde{H}}\chi_{E_u}|\text{-a.e. } (x,t) \in (\Omega \setminus I) \times \mathbb{R}. \tag{4.28}$$

Formula (4.27) implies that $|D_{\widetilde{H}}\chi_{E_u}|$-a.e. $(x,t) \in I \times \mathbb{R}$ belongs to S and that $|D_{\widetilde{H}}\chi_{E_u}|$-a.e. $(x,t) \in T$ belongs to $(\Omega \setminus I) \times \mathbb{R}$. Similarly, (4.28) says that $|D_{\widetilde{H}}\chi_{E_u}|$-a.e. $(x,t) \in (\Omega \setminus I) \times \mathbb{R}$ belongs to T and that $|D_{\widetilde{H}}\chi_{E_u}|$-a.e. $(x,t) \in S$ belongs to $I \times \mathbb{R}$. Since S and T are disjoint, this is enough to conclude (4.20) and (4.21). Statement (4.22) now easily follows because

$$\pi_\#(D_{\widetilde{H}}\chi_{E_u} \llcorner S) = \pi_\#(\nu_{E_u}|D_{\widetilde{H}}\chi_{E_u}| \llcorner (I \times \mathbb{R})) = (\sigma_u, 0)|(D_H u, -\mathscr{L}^n)| \llcorner I = (D_H^s u, 0).$$

Similarly, one has

$$\begin{aligned} \pi_\#(D_{\widetilde{H}}\chi_{E_u} \llcorner T) =& \pi_\#(\nu_{E_u}|D_{\widetilde{H}}\chi_{E_u}| \llcorner ((\Omega \setminus I) \times \mathbb{R})) \\ =& \frac{(Xu,-1)}{\sqrt{1+|Xu|^2}}|(D_H u, -\mathscr{L}^n)| \llcorner (\Omega \setminus I) = (Xu,-1)\mathscr{L}^n, \end{aligned}$$

which gives (4.23). □

5. The rank-one theorem for BV_H functions in Carnot groups

We now use the results of the previous section in the setting of a Carnot group $\mathbb{G}$. We utilize the notation of Section 2; in particular, we identify $\mathbb{G} \equiv \mathbb{R}^n$ by exponential coordinates and a left-invariant basis $X_1, \dots, X_m$ of $\mathfrak{g}_1$ is fixed. The vector fields $\widetilde{X}_1, \dots, \widetilde{X}_{m+1}$ on $\mathbb{G} \times \mathbb{R}$ are defined as in the previous section; notice that they form a basis of the first layer of the Lie algebra of $\mathbb{G} \times \mathbb{R}$. The homogeneous dimension of $\mathbb{G} \times \mathbb{R}$ is $Q + 1$.

A set $R \subset \mathbb{G}$ is *H-rectifiable* if $\mathcal{H}^{Q-1}(R) < \infty$ and there exists a (finite or countable) family $(\Sigma_i)_i$ of C^1_H hypersurfaces in $\mathbb{G}$ such that

$$\mathcal{H}^{Q-1}\Big(R \setminus \bigcup_i \Sigma_i\Big) = 0.$$

We define the *horizontal normal* ν_R to R as

$$\nu_R(x) := \nu_{\Sigma_i}(x) \qquad \text{if } x \in R \cap \Sigma_i \setminus \cup_{j<i}\Sigma_j.$$

The normal ν_R is well-defined (up to sign) $\mathcal{H}^{Q-1}$-a.e. on R.[2]

[2] The key property to prove this assertion is that the set of points where two C^1_H hypersurfaces intersect transversally is $\mathcal{H}^{Q-1}$-negligible: this fact holds true in any *equiregular Carnot–Carathéodory space*, see

Definition 5.1. We say that a Carnot group $\mathbb{G}$ satisfies property $\mathscr{R}$ if the following holds. For any bounded open set $\Omega \subset \mathbb{G}$ and any $u \in BV_H(\Omega)$, the distributional $\widetilde{X}$-derivatives $D_{\widetilde{H}}\chi_{E_u}$ of the characteristic function of the subgraph E_u of u can be represented as

$$D_{\widetilde{H}}\chi_{E_u} = \nu_{\partial^*_H E_u}\theta\mathcal{S}^Q \llcorner \partial^*_H E_u \tag{5.29}$$

for some H-rectifiable set $\partial^*_H E_u$ in $\Omega \times \mathbb{R}$ and some positive density $\theta \in L^1(\partial^*_H E_u, \mathcal{S}^Q)$. We call $\partial^*_H E_u$ the *H-reduced boundary of E_u*.

Notice that, in Definition 5.1, the measure $D_{\widetilde{H}}\chi_{E_u}$ has finite total variation by Theorem 4.2.

Remark 5.2. In view of Theorem 1.3, for the validity of property $\mathscr{R}$ in $\mathbb{G}$ it is enough that a rectifiability theorem holds for sets with finite H-perimeter in $\mathbb{G}\times\mathbb{R}$; namely, it suffices that any set E with finite H-perimeter in $\mathbb{G} \times \mathbb{R}$ satisfies $D_{\widetilde{H}}\chi_E = \nu_{\partial^*_H E}\theta\mathcal{S}^Q \llcorner \partial^*_H E$ for some H-rectifiable set $\partial^*_H E$ and some positive density $\theta \in L^1(\partial^*_H E, \mathcal{S}^Q)$. We conjecture that this, in turn, is equivalent to the validity of a rectifiability theorem for sets with finite H-perimeter in $\mathbb{G}$; in particular, we conjecture that property $\mathscr{R}$ is equivalent to the rectifiability theorem in $\mathbb{G}$.

Remark 5.3. If $\mathbb{G}$ is a Carnot group of step 2, then $\mathbb{G}$ satisfies property $\mathscr{R}$: this follows from the fact that $\mathbb{G}\times\mathbb{R}$ is also a step 2 Carnot group and that the rectifiability theorem holds in any step 2 Carnot group, see [15].

Remark 5.4. If (5.29) holds, then

$$|D_{\widetilde{H}}\chi_{E_u}| = \theta\mathcal{S}^Q \llcorner \partial^*_H E_u \quad \text{and} \quad \nu_{E_u} = \nu_{\partial^*_H E_u} \ \mathcal{S}^Q\text{-a.e. on } \partial^*_H E_u.$$

Proof of Theorem 1.1. Without loss of generality one can assume that $u = (u_1, \ldots, u_d) \in BV_H(\Omega, \mathbb{R}^d)$. It is not restrictive to assume that Ω is bounded. For any $i = 1, \ldots, d$ we write $D^s_H u_i = \sigma_i |D^s_H u_i|$ for a $|D^s_H u_i|$-measurable map $\sigma_i : \Omega \to \mathbb{S}^{m-1}$; notice that, using the notation of Section 4, the equality $\sigma_i = \sigma_{u_i}$ holds $|D^s u_i|$-almost everywhere. We also let $E_i := \{(x,t) \in \Omega \times \mathbb{R} : t < u_i(x)\}$ be the subgraph of u_i, that has finite H-perimeter in $\Omega\times\mathbb{R}$ by Theorem 4.2. Denoting by $\partial^*_H E_i$ the H-reduced boundary of E_i and writing $\nu_i = \nu_{E_i}$ for the measure theoretic inner normal to E_i, we have by Theorem 4.3 and Remark 5.4 that

$$|D^s_H u_i| = \pi_\#(\theta_i \mathcal{S}^Q \llcorner S_i) \quad \text{for some positive } \theta_i \in L^1(\partial^*_H E_i, \mathcal{S}^Q),$$

e.g. [10]. Actually, in view of Theorem 1.1 we could restrict to the setting of Carnot groups satisfying property $\mathscr{C}_2$, where the claim follows from Theorem 1.4.

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where $S_i := \{p \in \partial^*_H E_i : (\nu_i(p))_{m+1} = 0\}$ and $\pi_\#$ denotes push-forward of measures through the projection π defined by $\mathbb{G} \times \mathbb{R} \ni (x,t) \mapsto x \in \mathbb{G}$. By rectifiability, we can assume that $\partial^*_H E_i$ is contained in the union $\cup_{\ell\in\mathbb{N}}\Sigma^i_\ell$ of C^1_H hypersurfaces Σ^i_ℓ in $\mathbb{G} \times \mathbb{R}$.

Using Theorem 4.3, Remark 5.4 and Lemma 3.2 the following properties hold for $\mathcal{S}^Q$-a.e. $p \in S_1 \cup \dots \cup S_d$:

$$\text{if } p \in S_i, \text{ then } \nu_i(p) = (\sigma_i(\pi(p)), 0) \tag{5.30}$$

$$\text{if } p \in \Sigma^i_\ell, \text{ then } \nu_i(p) = \pm\nu_{\Sigma^i_\ell}(p) \tag{5.31}$$

$$\text{if } p \in \Sigma^i_\ell \text{ and } \exists\, q \in S_j \cap \Sigma^j_k \cap \pi^{-1}(\pi(p)), \text{ then } \nu_{\Sigma^i_\ell}(p) = \pm\nu_{\Sigma^j_k}(q). \tag{5.32}$$

Up to modifying each S_i on a $\mathcal{S}^Q$-negligible set and each σ_i on a $|D^s_H u_i|$-negligible set, we can assume that (5.30), (5.31) and (5.32) hold for any $p \in S_1 \cup \dots \cup S_d$ and that, for any $i = 1, \dots, d$, $\sigma_i = 0$ on $\Omega \setminus \pi(S_i)$.

Since $D^s_H u = (\sigma_1 |D^s_H u_1|, \dots, \sigma_d |D^s_H u_d|)$ and $|D^s_H u|$ is concentrated on $\pi(S_1) \cup \dots \cup \pi(S_d)$, it is enough to prove that the matrix-valued function $(\sigma_1, \dots, \sigma_d)$ has rank 1 on $\pi(S_1) \cup \dots \cup \pi(S_d)$. This follows if we prove that the implication

$$i, j \in \{1, \dots, d\},\ i \neq j,\ x \in \pi(S_i) \implies \sigma_j(x) \in \{0, \sigma_i(x), -\sigma_i(x)\}$$

holds. If i, j, x are as above and $x \notin \pi(S_j)$, then $\sigma_j(x) = 0$. Otherwise, $x \in \pi(S_i) \cap \pi(S_j)$, i.e., there exist $p \in S_i$ and $\ell \in \mathbb{N}$ such that $\pi(p) = x$ and $\sigma_i(x) = \pm\nu_{\Sigma^i_\ell}(p)$ and there exist $q \in S_j$ and $k \in \mathbb{N}$ such that $\pi(q) = x$ and $\sigma_j(x) = \pm\nu_{\Sigma^j_k}(p)$. By (5.32) we obtain $\sigma_j(x) = \pm\sigma_i(x)$, as wished. □

Remark 5.5. As an easy consequence of Remark 2.4 and Remark 5.3, Theorem 1.1 holds for the Heisenberg group $\mathbb{H}^n$ provided $n \geq 2$. This result does not directly follow from [9], as we now briefly explain using the notation of Example 2.1 and restricting for simplicity to $n = 2$, the general case $n \geq 2$ being a straightforward generalization.

Let $u \in BV_H(\Omega, \mathbb{R}^m)$ for some open set $\Omega \subset \mathbb{H}^2$. It can be easily seen that the matrix-valued measure $(\mu_1, \mu_2, \mu_3, \mu_4) := D_H u = (X_1 u, X_2 u, Y_1 u, Y_2 u)$ satisfies the equations

$$\mathscr{A}\mu := \begin{pmatrix} X_1\mu_2 - X_2\mu_1 \\ Y_1\mu_4 - Y_2\mu_3 \\ X_1\mu_4 - Y_2\mu_1 \\ Y_1\mu_2 - X_2\mu_3 \\ X_1\mu_3 - Y_1\mu_1 + Y_2\mu_2 - X_2\mu_4 \end{pmatrix} = 0$$

in the sense of distributions. Write the first-order differential operator $\mathscr{A}$ (the *horizontal curl* in $\mathbb{H}^2$, see [5, Example 3.12]) in the form

$$\mathscr{A} = A_1\partial_{x_1} + A_2\partial_{x_2} + A_3\partial_{y_1} + A_4\partial_{y_2} + A_5\partial_t$$

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for suitable $A_j = A_j(x,y,t)$ and consider the *wave cone* $\Lambda_{\mathscr{A}}(x,y,t)$ (see [9]) associated with $\mathscr{A}$

$$\Lambda_{\mathscr{A}}(x,y,t) := \bigcup_{\xi\in\mathbb{R}^5\setminus\{0\}} \ker \mathbb{A}_{x,y,t}(\xi), \qquad \text{where } \mathbb{A}_{x,y,t}(\xi) := 2\pi i \sum_{j=1}^{5} A_j(x,y,t)\xi_j.$$

One can readily check that

$$\mathbb{A}_{x,y,t}(\xi) = 0 \quad \text{for } \xi := (\tfrac{y}{2}, -\tfrac{x}{2}, 1) \in \mathbb{R}^5 \setminus \{0\},$$

i.e., the wave cone $\Lambda_{\mathscr{A}}(x,y,t)$ is the full space for any $(x,y,t) \in \mathbb{H}^2$. In particular, [9, Theorem 1.1] gives no information on the polar decomposition of $D_H^s u$.

Remark 5.6. The rank-one property for BV functions in the first Heisenberg group remains a very interesting open question, since it does not follow either from Theorem 1.1 (because property $\mathscr{C}_2$ fails for $\mathbb{H}^1$) or from [9, Theorem 1.1], as we now explain.

Let $u \in BV_H(\Omega, \mathbb{R}^m)$ for some open set $\Omega \subset \mathbb{H}^1$; we use again the notation of Example 2.1 and we set $p = (x,y,t) \in \mathbb{H}^1 \equiv \mathbb{R}^3$. One can check that $(\mu_1, \mu_2) := D_H u = (Xu, Yu)$ satisfies

$$\mathscr{A}\mu := \begin{pmatrix} YX\mu_1 - 2XY\mu_1 + XX\mu_2 \\ YY\mu_1 - 2YX\mu_2 + XY\mu_2, \end{pmatrix} = 0$$

in the sense of distributions. Now $\mathscr{A}$ (the horizontal curl in $\mathbb{H}^1$, see [5, Example 3.11]) is a second-order differential operator that one can write as

$$\mathscr{A} = \sum_{|\alpha|=2} A_\alpha(p)\partial^\alpha,$$

where $\alpha \in \mathbb{N}^3$ is a multi-index and $\partial^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3}$. As before, one can define the wave cone

$$\Lambda_{\mathscr{A}}(p) = \bigcup_{\xi\in\mathbb{R}^3\setminus\{0\}} \ker \mathbb{A}_p(\xi), \qquad \text{where } \mathbb{A}_p(\xi) = (2\pi i)^2 \sum_{|\alpha|=2} A_\alpha(p)\xi^\alpha.$$

Again, one has

$$\mathbb{A}_p(\xi) = 0 \qquad \text{for } \xi := (\tfrac{y}{2}, -\tfrac{x}{2}, 1) \in \mathbb{R}^3 \setminus \{0\}$$

and the wave cone $\Lambda_{\mathscr{A}}(x,y,t)$ is the full space.

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Appendix A. Intersection of regular hypersurfaces vs. intrinsic Lipschitz graphs

A.1. Intrinsic Lipschitz graphs

We follow [12]. Let $\mathbb{W}, \mathbb{H}$ be homogeneous (i.e., invariant under dilations) complementary subgroups of $\mathbb{G}$, i.e., such that $\mathbb{W} \cap \mathbb{H} = \{0\}$ and $\mathbb{G} = \mathbb{W}\mathbb{H}$. In particular, for any $x \in \mathbb{G}$ there exist unique $x_{\mathbb{W}} \in \mathbb{W}$ and $x_{\mathbb{H}} \in \mathbb{H}$ such that $x = x_{\mathbb{W}} x_{\mathbb{H}}$. Recall (see e.g. [12, Remark 2.3]) that any homogeneous subgroup $\mathbb{W}$ is stratified, that is, its Lie algebra $\mathfrak{w}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{w} = \mathfrak{w}_1 \oplus \cdots \oplus \mathfrak{w}_s$ where $\mathfrak{w}_i = \mathfrak{w} \cap \mathfrak{g}_i$. Moreover, the metric (Hausdorff) dimension of $\mathbb{W}$ is $Q_{\mathbb{W}} := \sum_{i=1}^{s} i \dim \mathfrak{w}_i$.

The *intrinsic graph* of a function $\phi : \mathbb{W} \to \mathbb{H}$ is defined by

$$\mathrm{gr}\ \phi := \{w\phi(w) : w \in \mathbb{W}\}.$$

We introduce the homogeneous cones $C_{\mathbb{W},\mathbb{H}}(x, \alpha)$ of center $x \in \mathbb{G}$ and aperture $\alpha > 0$ as

$$C_{\mathbb{W},\mathbb{H}}(x, \alpha) := xC_{\mathbb{W},\mathbb{H}}(0, \alpha) \quad \text{where} \quad C_{\mathbb{W},\mathbb{H}}(0, \alpha) := \{y \in \mathbb{G} : \|x_{\mathbb{W}}\| \le \alpha \|x_{\mathbb{H}}\|\}.$$

Definition A.1. A function $\phi : \mathbb{W} \to \mathbb{H}$ is *intrinsic Lipschitz* if there exists $\alpha > 0$ such that

$$\forall\, x \in \mathrm{gr}\ \phi \qquad \mathrm{gr}\ \phi \cap C_{\mathbb{W},\mathbb{H}}(x, \alpha) = \{x\}.$$

We say that $S \subset \mathbb{G}$ is an *intrinsic Lipschitz graph* if there exists an intrinsic Lipschitz map $\phi : \mathbb{W} \to \mathbb{H}$ such that $S = \mathrm{gr}\ \phi$.

Remark A.2. We will later use the following equivalent definition of intrinsic Lipschitz continuity: $\phi : \mathbb{W} \to \mathbb{H}$ is intrinsic Lipschitz if and only if there exists $\beta > 0$ such that

$$\forall\, x \in \mathrm{gr}\ \phi \qquad \mathrm{gr}\ \phi \cap D(x, \mathbb{H}, \beta) = \{x\}$$

where the homogeneous cone $D(x, \mathbb{H}, \beta)$ is defined by

$$D(x, \mathbb{H}, \beta) := xD(\mathbb{H}, \beta) \quad \text{and} \quad D(\mathbb{H}, \beta) := \bigcup_{h \in \mathbb{H}} \overline{B(h, \beta d(h, 0))}.$$

Indeed, it is enough to observe that, for any $\alpha > 0$ and $\beta > 0$, there exist $\beta_\alpha > 0$ and $\alpha_\beta > 0$ such that

$$C_{\mathbb{W},\mathbb{H}}(x, \alpha) \supset D(\mathbb{H}, \beta_\alpha) \quad \text{and} \quad D(\mathbb{H}, \beta) \supset C_{\mathbb{W},\mathbb{H}}(x, \alpha_\beta).$$

This, in turn, is a consequence of a homogeneity argument based on the following fact: if $S := \{x \in \mathbb{G} : \|x\| = 1\}$ and

$$A_\alpha := S \cap \mathrm{int}(C_{\mathbb{W},\mathbb{H}}(x,\alpha)), \qquad B_\beta := S \cap \mathrm{int}(D(\mathbb{H},\beta)),$$

then $\{A_\alpha\}_{\alpha>0}$ and $\{B_\beta\}_{\beta>0}$ are monotone families of (relatively) open subsets of S such that the intersection

$$\bigcap_{\alpha>0} A_\alpha = \bigcap_{\beta>0} B_\beta = \mathbb{H} \cap S$$

is a compact set.

The following result will be used in the proof of Theorem 1.4.

Theorem A.3 *([12, Theorem 3.9]). Let $\mathbb{W}, \mathbb{H}$ be homogeneous complementary subgroups of $\mathbb{G}$, let $\phi : \mathbb{W} \to \mathbb{H}$ be intrinsic Lipschitz and let $\alpha > 0$ be as in Definition A.1. Then there exists a positive $C = C(\mathbb{W}, \mathbb{H}, \alpha)$ such that*

$$\frac{1}{C} r^{Q_{\mathbb{W}}} \le \mathcal{H}^{Q_{\mathbb{W}}}(\mathrm{gr}\ \phi \cap B(x,r)) \le C r^{Q_{\mathbb{W}}} \qquad \forall\, x \in \mathrm{gr}\ \phi, r > 0.$$

A.2. Transversal intersections of C^1_H hypersurfaces are intrinsic Lipschitz graphs

The aim of this section is proving Theorem A.5, due to V. Magnani [22], for which we need the preparatory Lemma A.4. Actually, its use could be avoided by utilizing a local version of Theorem A.3 which, even though not explicitly stated there, would easily follow adapting the techniques of [12]. We note however that Lemma A.4, and (A.33) in particular, provides also a proof of (3.2).

Lemma A.4. *Let $\Omega \subset \mathbb{G}$ be open, $f \in C^1_H(\Omega)$, $\bar{x} \in \Omega$ and let $A := \nabla_H f(\bar{x})$. Then, for any $\varepsilon > 0$ there exist an open set $U \subset \Omega$ with $\bar{x} \in U$ and a function $g \in C^1_H(\mathbb{G})$ such that*

(i) *$g = f$ on U;*
(ii) *$|\nabla_H g - A| < \varepsilon$ on $\mathbb{G}$.*

Proof. Without loss of generality we can assume that $\bar{x} = 0$. We preliminarily fix a smooth function $\chi : \mathbb{G} \to [0,1]$ such that $\chi \equiv 1$ on B_1 and $\chi \equiv 0$ on $\mathbb{G} \setminus B_2$. For any $r > 0$, the functions $\chi_r := \chi \circ \delta_{1/r}$ satisfy

$$0 \le \chi_r \le 1, \quad \chi \equiv 1 \text{ on } B_r, \quad \chi \equiv 0 \text{ on } \mathbb{G} \setminus B_{2r}, \quad |\nabla_H \chi_r| \le \frac{C}{r}$$

for some positive C independent of r.

Let $\varepsilon > 0$ be fixed. We fix $r > 0$ such that $|\nabla_H f - A| < \varepsilon$ on B_{2r}. With this choice, setting $\lambda(x) := A_1 x_1 + \cdots + A_m x_m$ (where x is represented in exponential coordinates) we prove that

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$$|f(x) - \lambda(x)| < 2\varepsilon r \quad \text{for any } x \in B_{2r}. \tag{A.33}$$

Indeed, for any $x \in B_{2r}$ there exists a horizontal curve $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma(0) = 0$, $\gamma(1) = x$ and $L(\gamma) < 2r$. By definition, there exists $h \in L^\infty([0,1], \mathbb{R}^m)$ such that

$$\dot\gamma(t) = \sum_{i=1}^m h_i(t) X_i(\gamma(t)) \quad \text{for a.e. } t \in [0,1].$$

Moreover, for any $i = 1, \dots, m$ we have $\int_0^1 h_i = x_i$, because in exponential coordinates one has $X_i(x) = \partial_{x_i} + \sum_{\ell > m+1} a_{i\ell} \partial_{x_\ell}$ (see e.g. [29]). It follows that

$$\begin{aligned}
|f(x) - \lambda(x)| &= \left| \int_0^1 \sum_{i=1}^m h_i(t) X_i f(\gamma(t)) dt - \int_0^1 \sum_{i=1}^m A_i h_i(t) dt \right| \\
&\le \int_0^1 |h(t)| \, \|\nabla_H f(\gamma(t)) - A\| dt \\
&< 2\varepsilon r.
\end{aligned}$$

We now define $g := \chi_r f + (1 - \chi_r)\lambda$; statement (i) is readily checked, while for (ii)

$$\begin{aligned}
|\nabla_H g - A| &= |\chi_r \nabla_H f + (1 - \chi_r) A + (f - \lambda) \nabla_H \chi_r - A| \\
&\le \chi_r |\nabla_H f - A| + |f - \lambda| |\nabla_H \chi_r| \\
&\le \varepsilon + 2C\varepsilon.
\end{aligned}$$

The proof is then accomplished. □

We can now prove the main result of this section. Since property $\mathscr{C}_1$ holds in any Carnot group, when $k = 1$ Theorem A.5 states in particular that hypersurfaces of class C^1_H in a Carnot group $\mathbb{G}$ are locally intrinsic Lipschitz graphs of codimension 1.

Theorem A.5 *([22, Theorem 1.4]). Let $\mathbb{G}$ be a Carnot group of rank m and let $\Sigma_1, \dots, \Sigma_k$, $k \le m$, be hypersurfaces of class C^1_H with horizontal normals $\nu_1, \dots, \nu_k$; let $x \in \Sigma := \Sigma_1 \cap \dots \cap \Sigma_k$ be such that $\nu_1(x), \dots, \nu_k(x)$ are linearly independent. Consider the vertical plane $\mathbb{W} := T_x\Sigma_1 \cap \dots \cap T_x\Sigma_k$ of codimension k and assume that there exists a complementary homogeneous horizontal subgroup $\mathbb{H}$ such that $\mathbb{G} = \mathbb{W}\mathbb{H}$. Then, there exists an open neighborhood U of x and an intrinsic Lipschitz $\phi : \mathbb{W} \to \mathbb{H}$ such that*

$$\Sigma \cap U = \text{gr } \phi \cap U.$$

Proof. We work in exponential coordinates associated with an adapted basis $X_1, \dots, X_n$ of $\mathfrak{g}$ such that

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$$\mathbb{H} = \exp(\mathrm{span}\,\{X_1, \dots, X_k\}), \qquad \mathbb{W} = \exp((\mathrm{span}\,\{X_{k+1}, \dots, X_s\}) \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s).$$

By definition we can find an open neighborhood U of x and $f = (f_1, \dots, f_k) \in C^1_H(U, \mathbb{R}^k)$ such that $\Sigma \cap U = \{x \in U : f(x) = 0\} \cap U$ and the $m \times k$ matrix-valued function $\nabla_H f$ has rank k in U. Actually, by our choice of the basis the $k \times k$ minor $M := (X_1 f(x), \dots, X_k f(x))$ has rank k.

Let ε be a positive number, to be fixed later and only depending on M. By Lemma A.4, possibly restricting U we can assume that f is defined on the whole $\mathbb{G}$, that $f \in C^1_H(\mathbb{G}, \mathbb{R}^k)$ and $|\nabla_H f - \nabla_H f(x)| < \varepsilon$; in particular,

$$|(X_1 f, \dots, X_k f) - M| < \varepsilon \quad \text{on } \mathbb{G}.$$

It is enough to prove that the level set $R := \{x \in \mathbb{G} : f(x) = 0\}$ is an intrinsic Lipschitz graph. We divide the proof of this claim into two steps.

Step 1: R is the intrinsic graph of some $\phi : \mathbb{W} \to \mathbb{H}$. It is enough to show that, for any $w \in \mathbb{W}$, there exists a unique $h \in \mathbb{H}$ such that $f(wh) = 0$; in particular, this allows to define the map ϕ by $\phi(w) := h$.

The map $(h_1, \dots, h_k) \longleftrightarrow \exp(h_1 X_1 + \cdots + h_k X_k)$ is a group isomorphism between $\mathbb{H}$ and $\mathbb{R}^k$. Upon identifying $\mathbb{H}$ and $\mathbb{R}^k$ in this way, for any $w \in \mathbb{W}$ we can consider $f_w : \mathbb{R}^k \to \mathbb{R}^k$ defined by $f_w(h) := f(wh)$. This map is of class C^1 and

$$\nabla f_w(h) = (X_1 f(wh), \dots, X_k f(wh)).$$

We have $|\nabla f_w - M| < \varepsilon$ which, if ε is small enough, implies that f_w is a C^1 diffeomorphism of $\mathbb{R}^k$: see e.g. the argument in [11, 3.1.1].[3] This concludes the proof of Step 1; we notice also that, possibly reducing ε, there exists $c > 0$ such that (see again in [11, 3.1.1])

$$|f(wh_1) - f(wh_2)| = |f_w(h_1) - f_w(h_2)| \geq c|h_1 - h_2| \quad \forall\, h_1, h_2 \in \mathbb{R}^k. \tag{A.34}$$

Step 2: ϕ is intrinsic Lipschitz. By Remark A.2 it is enough to prove that

$$\mathrm{gr}\ \phi \cap D(x, \mathbb{H}, \beta) = \{x\} \qquad \text{for any } x \in \mathbb{G}$$

for a suitable $\beta > 0$ that we will choose in a moment.

Let then $x \in \mathrm{gr}\ \phi$ be fixed; consider $x' \in D(x, \mathbb{H}, \beta)$, so that $x' = xy$ for some $y \in D(\mathbb{H}, \beta)$. By definition, there exists $h \in \mathbb{H}$ such that

$$d(0, h^{-1} y) = d(h, y) \leq \beta d(h, 0).$$

Denoting by L the Lipschitz constant of f we deduce using (A.34) that

[3] The careful reader will notice that the argument in [11, 3.1.1] works also when the parameter δ introduced therein is $+\infty$.

$$\begin{aligned}|f(x')| &= |f(xhh^{-1}y) - f(x)| \\ &\geq |f(xh) - f(x)| - |f(xhh^{-1}y) - f(xh)| \geq c\|h\| - Ld(h,y) \geq (\tilde{c} - \beta L)d(0,h)\end{aligned}$$

for some $\tilde{c} > 0$. In particular, if β is small enough, one can have $f(x') = 0$ only if $h = 0$, which immediately gives $x' = x$. This concludes the proof. $\square$

We can eventually prove Theorem 1.4.

Proof of Theorem 1.4. By property $\mathscr{C}_k$ and Remark 2.3, the vertical plane $\mathbb{W} := T_x\Sigma_1 \cap \cdots \cap T_x\Sigma_k$ admits a complementary horizontal homogeneous subgroup $\mathbb{H}$. One can then easily conclude using Theorems A.3 and A.5. $\square$

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