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Real analytic families of harmonic functions in a planar domain with a small hole

Q1 M. Dalla Riva [a,*], P. Musolino [b]

[a] Centro de Investigação e Desenvolvimento em Matemática e Aplicações (CIDMA), Universidade de Aveiro, Portugal
[b] Dipartimento di Matematica, Università degli Studi di Padova, Italy

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ABSTRACT

We consider a Dirichlet problem in a planar domain with a hole of diameter proportional to a real parameter ϵ and we denote by u_ϵ the corresponding solution. The behavior of u_ϵ for ϵ small and positive can be described in terms of real analytic functions of two variables evaluated at $(\epsilon, 1/\log\epsilon)$. We show that under suitable assumptions on the geometry and on the boundary data one can get rid of the logarithmic behavior displayed by u_ϵ for ϵ small and describe u_ϵ by real analytic functions of ϵ. Then it is natural to ask what happens when ϵ is negative. The case of boundary data depending on ϵ is also considered. The aim is to study real analytic families of harmonic functions which are not necessarily solutions of a particular boundary value problem.

1. Introduction

This paper continues the work begun by the authors in [1]. Indeed, in [1], the case of harmonic function in a perforated domain of $\mathbb{R}^n$, with $n \geq 3$, has been investigated. Here instead we focus on the two-dimensional case. We begin by introducing some notation. We fix once for all

$$\alpha \in]0,1[.$$

Then we fix two sets Ω^o and Ω^i in the two-dimensional Euclidean space $\mathbb{R}^2$. The letter 'o' stands for 'outer domain' and the letter 'i' stands for 'inner domain'. We assume that Ω^o and Ω^i satisfy the following condition.

* Corresponding author at: Departamento de Matemática, Universidade de Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
E-mail addresses: matteo.dallariva@gmail.com (M. Dalla Riva), musolinopaolo@gmail.com (P. Musolino).
URLs: http://https://sites.google.com/site/matteodallariva/ (M. Dalla Riva), http://https://sites.google.com/site/musolinopaolo/ (P. Musolino).

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$$\begin{aligned}&\Omega^o \text{ and } \Omega^i \text{ are open bounded connected subsets of } \mathbb{R}^2 \text{ of}\\&\text{class } C^{1,\alpha} \text{ such that } \mathbb{R}^2 \setminus \mathrm{cl}\Omega^o \text{ and } \mathbb{R}^2 \setminus \mathrm{cl}\Omega^i \text{ are connected}\\&\text{and such that the origin 0 of } \mathbb{R}^2 \text{ belongs both to } \Omega^o \text{ and } \Omega^i.\end{aligned} \tag{1}$$

Here and in the sequel cl denotes the closure. For the definition of functions and sets of the usual Schauder classes $C^{0,\alpha}$ and $C^{1,\alpha}$, we refer for example to Gilbarg and Trudinger [5, §6.2]. We note that condition (1) implies that Ω^o and Ω^i have no holes and that there exists a real number ϵ_0 such that

$$\epsilon_0 > 0 \quad \text{and} \quad \epsilon \,\mathrm{cl}\, \Omega^i \subseteq \Omega^o \quad \text{for all } \epsilon \in \,]-\epsilon_0, \epsilon_0[.$$

Then we denote by $\Omega(\epsilon)$ the perforated domain defined by

$$\Omega(\epsilon) \equiv \Omega^o \setminus \left(\epsilon \,\mathrm{cl}\, \Omega^i\right) \quad \forall \epsilon \in \,]-\epsilon_0, \epsilon_0[.$$

A simple topological argument shows that $\Omega(\epsilon)$ is an open bounded connected subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ for all $\epsilon \in \,]-\epsilon_0, \epsilon_0[\, \setminus \{0\}$. Moreover, the boundary $\partial\Omega(\epsilon)$ of $\Omega(\epsilon)$ has exactly the two connected components $\partial\Omega^o$ and $\epsilon\partial\Omega^i$, for all $\epsilon \in \,]-\epsilon_0, \epsilon_0[$. We also note that $\Omega(0) = \Omega^o \setminus \{0\}$.

Now let $g^o \in C^{1,\alpha}(\partial\Omega^o)$ and $g^i \in C^{1,\alpha}(\partial\Omega^i)$. For all $\epsilon \in \,]-\epsilon_0, \epsilon_0[\, \setminus \{0\}$, let u_ϵ be the unique function of $C^{1,\alpha}(\mathrm{cl}\, \Omega(\epsilon))$ such that

$$\begin{cases} \Delta u_\epsilon = 0 & \text{in } \Omega(\epsilon), \\ u_\epsilon(x) = g^o(x) & \text{for } x \in \partial\Omega^o, \\ u_\epsilon(x) = g^i(x/\epsilon) & \text{for } x \in \epsilon\partial\Omega^i. \end{cases} \tag{2}$$

Let u_0 be the unique function of $C^{1,\alpha}(\mathrm{cl}\, \Omega^o)$ such that

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega^o, \\ u_0(x) = g^o(x) & \text{for } x \in \partial\Omega^o. \end{cases} \tag{3}$$

We fix a point p in $\Omega^o \setminus \{0\}$ and take $\epsilon_p \in \,]0, \epsilon_0[$ such that $p \in \Omega(\epsilon)$ for all $\epsilon \in \,]-\epsilon_p, \epsilon_p[$. Then $u_\epsilon(p)$ is defined for all $\epsilon \in \,]-\epsilon_p, \epsilon_p[$ and we can ask, for example, the following question.

What can be said of the function from $]0, \epsilon_p[$ to $\mathbb{R}$ which takes ϵ to $u_\epsilon(p)$?

Questions of this type are typical in the frame of asymptotic analysis and are usually investigated by means of asymptotic expansion methods (see for example Maz'ya, Nazarov, and Plamenevskij [13, §2.4.1]). The techniques of asymptotic analysis usually aim at representing the behavior of $u_\epsilon(p)$ as $\epsilon \to 0^+$ in terms of regular functions of ϵ plus a remainder which is smaller than a known infinitesimal function of ϵ. In this paper, instead, we adopt the functional analytic approach proposed by Lanza de Cristoforis. By such an approach, one can prove that there exist $\epsilon_p \in \,]0, \epsilon_0]$, $\epsilon_p < 1$, and a real analytic function U_p from $]-\epsilon_p, \epsilon_p[\times]1/\log \epsilon_p, -1/\log \epsilon_p[$ to $\mathbb{R}$ such that

$$u_\epsilon(p) = U_p[\epsilon, 1/\log \epsilon] \quad \forall \epsilon \in \,]0, \epsilon_p[\tag{4}$$

and that $u_0(p) = U_p[0, 0]$ (cf., *e.g.*, Lanza de Cristoforis [10]). We observe that the logarithmic behavior displayed by u_ϵ for ϵ small only arises in dimension two and does not appear in higher dimensions (cf., *e.g.*, Lanza de Cristoforis [10]). Also, if instead of considering a Dirichlet boundary value problem we considered a mixed boundary value problem with a Dirichlet condition in the inner component of the boundary and a Neumann condition in the outer component, then one can prove that the logarithmic behavior appears only for Neumann data with non-zero integral (cf. Maz'ya, Nazarov, and Plamenevskij [13, §2.4.2]). Such a situation is convenient because we have a condition on the boundary data which ensures that u_ϵ will not display a logarithmic behavior. The first purpose of this paper is to find a similar condition also for the Dirichlet problem. Namely, we want to find a condition on g^o and g^i which ensures that for all $p \in \Omega^o \setminus \{0\}$ the function $u_\epsilon(p)$ can be expanded into powers of ϵ, *i.e.*, that

$$u_\epsilon(p) = V_p[\epsilon] \quad \forall \epsilon \in \,]0, \epsilon_p[\tag{5}$$

where V_p is a real analytic function from $]-\epsilon_p, \epsilon_p[$ to $\mathbb{R}$. In Theorem 3.6 we exhibit such a condition (see also condition (c) here below). Moreover, we show that the existence of at least one point p for which (5) holds is equivalent to the fact that it holds for all the points $p \in \Omega^o \setminus \{0\}$.

Then we observe that both the left hand side $u_\epsilon(p)$ and the right hand side $V_p[\epsilon]$ of equality (5) are defined for all $\epsilon \in \,]-\epsilon_p, \epsilon_p[$. However, the validity of the equality is stated only for ϵ positive. Thus it is natural to ask the following question.

What happens to equality $u_\epsilon(p) = V_p[\epsilon]$ for ϵ negative?

Moreover, one would like to understand if, for ϵ negative, $V_p[\epsilon]$ is related to the value attained at the point p of some harmonic function defined on the set $\Omega(\epsilon)$. In Theorem 3.6 we answer by proving that the validity of (5) for ϵ positive implies that

$$u_\epsilon(p) = V_p[\epsilon] \quad \forall \epsilon \in \,]-\epsilon_p, \epsilon_p[. \tag{6}$$

Also, the validity of (5) for at least one point p implies the validity of (6) for all the points $p \in \Omega^o \setminus \{0\}$. We stress that in order to prove (6) it is not sufficient to verify an analog of (5) for ϵ negative, namely it is not sufficient to show that there exists a real analytic function V_p^- from $]-\epsilon_p, \epsilon_p[$ to $\mathbb{R}$ such that $u_\epsilon(p) = V_p^-[\epsilon]$ for all $\epsilon \in \,]-\epsilon_p, 0[$. The reason is that the functions V_p^- and V_p may not coincide in a neighborhood of 0 and a gluing argument may fail to be applicable, as it actually occurs in dimension $n \geq 3$ odd when the boundary data are not trivial (cf. [1]). Furthermore, equality (6) together with some symmetry assumptions ensuring that $u_\epsilon = u_{-\epsilon}$ implies that $u_\epsilon(p)$ can be represented in terms of a convergent power series of ϵ^2. As pointed out in [1], this is also what happens when the dimension n is even and bigger than or equal to 4, in contrast to the case of odd dimension.

Our strategy is the following. First we apply a functional analytic approach which stems from that of Lanza de Cristoforis [10] to investigate equality (4). We consider also the case of boundary data which depend real analytically on $(\epsilon, 1/\log|\epsilon|)$. Moreover, we analyze what we call the 'macroscopic' behavior of the family $\{u_\epsilon\}_{\epsilon \in]-\epsilon_0, \epsilon_0[}$. Indeed, if $\Omega_M \subseteq \Omega^o$ is open, and $0 \notin \mathrm{cl}\,\Omega_M$, and $\epsilon_M \in \,]0, \epsilon_0]$, $\epsilon_M < 1$, is such that $\mathrm{cl}\,\Omega_M \cap (\epsilon\,\mathrm{cl}\,\Omega^i) = \emptyset$ for all $\epsilon \in \,]-\epsilon_M, \epsilon_M[$, then $\mathrm{cl}\,\Omega_M \subseteq \mathrm{cl}\,\Omega(\epsilon)$ for all $\epsilon \in \,]-\epsilon_M, \epsilon_M[$. Thus it makes sense to consider the restriction $u_{\epsilon|\mathrm{cl}\,\Omega_M}$ for all $\epsilon \in \,]-\epsilon_M, \epsilon_M[$. In particular, it makes sense to consider the map from $]-\epsilon_M, \epsilon_M[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ which takes ϵ to $u_{\epsilon|\mathrm{cl}\,\Omega_M}$. In Theorem 3.1 below we show that there exist an open neighborhood $\mathcal{U}_M$ of $\{(\epsilon, 1/\log|\epsilon|) : \epsilon \in \,]-\epsilon_M, \epsilon_M[\setminus \{0\}\} \cup \{(0, 0)\}$ in $\mathbb{R}^2$ and a real analytic map U_M from $\mathcal{U}_M$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ such that

$$u_{\epsilon|\mathrm{cl}\,\Omega_M} = U_M[\epsilon, 1/\log|\epsilon|] \quad \forall \epsilon \in \,]-\epsilon_M, \epsilon_M[\setminus \{0\} \tag{7}$$

(for the definition and properties of real analytic maps in Banach space see, *e.g.*, Deimling [3, §15]). Here the letter 'M' stands for 'macroscopic'.

It is worth noting that the real analytic map U_M from $\mathcal{U}_M$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ is univocally determined by the equality in (7) restricted to the positive interval $]0, \epsilon_M[$ (see Lemma 3.3 below). Moreover, for all fixed ϵ^* in the negative interval $]-\epsilon_M, 0[$, u_{ϵ^*} coincides with the unique real analytic extension of $U_M[\epsilon^*, 1/\log|\epsilon^*|]_{|\Omega_M}$

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to $\Omega(\epsilon^*)$. In this sense, the definition of u_ϵ for ϵ negative can be seen as a consequence of the analytic dependence on ϵ displayed by u_ϵ for ϵ positive.

Some further consequences of (7) are presented in Proposition 3.4, where we investigate the coefficients of the power series expansion of U_M around $(0,0)$ under certain symmetry assumptions.

Then we turn to consider the possibility of choosing boundary data g^o and g^i such that the following condition (a1) holds.

(a1) For all $\Omega_M \subseteq \Omega^o$ open and such that $0 \notin \mathrm{cl}\,\Omega_M$ and all $\epsilon_M \in]0,\epsilon_0[$ such that $\mathrm{cl}\,\Omega_M \cap \epsilon\,\mathrm{cl}\,\Omega^i = \emptyset$ for all $\epsilon \in]-\epsilon_M,\epsilon_M[$, there exists a real analytic map V_M from $]-\epsilon_M,\epsilon_M[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ such that

$$u_{\epsilon|\mathrm{cl}\,\Omega_M} = V_M[\epsilon] \quad \forall \epsilon \in]-\epsilon_M,\epsilon_M[.$$

Here we are asking to get rid of the logarithmic behavior displayed by u_ϵ for ϵ small. In Theorem 3.6 below we show that condition (a1) is equivalent to the following condition (b1).

(b1) There exist $x^o \in \Omega^o \setminus \{0\}$, $\epsilon^o \in]0,\epsilon_0[$, and a real analytic map V^o from $]-\epsilon^o,\epsilon^o[$ to $\mathbb{R}$ such that $x^o \in \Omega(\epsilon)$ for all $\epsilon \in]-\epsilon^o,\epsilon^o[$ and

$$u_\epsilon(x^o) = V^o[\epsilon] \quad \forall \epsilon \in]0,\epsilon^o[.$$

As a consequence, either $u_\epsilon(x^o)$ displays a logarithmic behavior for every point $x^o \in \Omega^o \setminus \{0\}$, or $u_\epsilon(x^o)$ does not display a logarithmic behavior for any point $x^o \in \Omega^o \setminus \{0\}$. Also, there exists a pair of functions $(\rho^o[\epsilon], \rho^i[\epsilon]) \in C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$ which depends only on ϵ, $\partial\Omega^o$, and $\partial\Omega^i$ (cf. Proposition 2.6), such that (a1) and (b1) are equivalent to the following condition (c).

(c) It holds $\int_{\partial\Omega^o} g^o\rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} g^i\rho^i[\epsilon]\,d\sigma = 0$ for all $\epsilon \in]-\epsilon_0,\epsilon_0[$.

The advantage of condition (c) with respect to (a1) and (b1) is that (c) can be verified on the boundary data (g^o, g^i) and does not require the knowledge of the solution u_ϵ of (2). In some simple cases, one can make such a condition much more explicit. For example, if g^o and g^i are both constant functions, then condition (c) is equivalent to the fact that g^o and g^i are identically equal to the same real number (cf. Example 3.7). If both Ω^o and Ω^i coincide with the unit ball $\mathbb{B}_2$ of $\mathbb{R}^2$, then condition (c) is equivalent to $\int_{\partial\mathbb{B}_2} g^o\,d\sigma = \int_{\partial\mathbb{B}_2} g^i\,d\sigma$ (cf. Example 3.8).

We observe that in Theorem 3.6 the case in which the boundary data are given by real analytic functions of ϵ is also investigated. Moreover, one can also consider the 'microscopic' behavior of the family $\{u_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ near the boundary of the hole. To do so, one denotes by $u_\epsilon(\epsilon\,\cdot)$ the rescaled function which takes $x \in (1/\epsilon)\,\mathrm{cl}\,\Omega(\epsilon)$ to $u_\epsilon(\epsilon x)$, for all $\epsilon \in]-\epsilon_0,\epsilon_0[\setminus \{0\}$. If $\Omega_m \subseteq \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega^i$ is open and bounded, and $\epsilon_m \in]0,\epsilon_0]$, $\epsilon_m < 1$, is such that $\epsilon\,\mathrm{cl}\,\Omega_m \subseteq \Omega^o$ for all $\epsilon \in]-\epsilon_m,\epsilon_m[$, then it makes sense to consider the map from $]-\epsilon_m,\epsilon_m[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_m)$ which takes ϵ to $u_\epsilon(\epsilon\,\cdot)_{|\mathrm{cl}\,\Omega_m}$. Here the letter '$m$' stands for 'microscopic'. Then, by the equivalence of (a1) and (b1) and by an argument based on the Kelvin transform one can deduce that the following conditions (a2) and (b2) are equivalent one to the other.

(a2) For all $\Omega_m \subseteq \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega^i$ open and bounded and all $\epsilon_m \in]0,\epsilon_0[$ such that $\epsilon\,\mathrm{cl}\,\Omega_m \subseteq \Omega^o$ for all $\epsilon \in]-\epsilon_m,\epsilon_m[$, there exists a real analytic map V_m from $]-\epsilon_m,\epsilon_m[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_m)$ such that

$$u_\epsilon(\epsilon\,\cdot)_{|\mathrm{cl}\,\Omega_m} = V_m[\epsilon] \quad \forall \epsilon \in]-\epsilon_m,\epsilon_m[\setminus \{0\}. \tag{8}$$

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(b2) There exist $x^i \in \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega^i$, $\epsilon^i \in]0, \epsilon_0[$, and a real analytic function V^i from $]-\epsilon^i, \epsilon^i[$ to $\mathbb{R}$ such that $\epsilon x^i \in \Omega^o$ for all $\epsilon \in]-\epsilon^i, \epsilon^i[$ and

$$u_\epsilon(\epsilon x^i) = V^i[\epsilon] \quad \forall \epsilon \in]0, \epsilon^i[.$$

We note that we do not require in condition (a2) that the equality in (8) holds for $\epsilon = 0$. In particular, $u_0(0\,\cdot)_{|\mathrm{cl}\,\Omega_m}$ is necessarily a constant function on $\mathrm{cl}\,\Omega_m$, while $V_m[0]$ may be non-constant (see (3)).

Now we can consider families $\{w_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ consisting of functions which are not required to be solutions of a particular boundary value problem in $\Omega(\epsilon)$, but which satisfy the following conditions (d0), (d1), and (d2).

(d0) $w_\epsilon \in C^{1,\alpha}(\mathrm{cl}\,\Omega(\epsilon))$ and $\Delta w_\epsilon = 0$ in $\Omega(\epsilon)$ for all $\epsilon \in]0, \epsilon_0[$.

(d1) For all $\Omega_M \subseteq \Omega^o$ open and such that $0 \notin \mathrm{cl}\,\Omega_M$ there exist $\epsilon'_M \in]0, \epsilon_0[$ and a real analytic map W_M from $]-\epsilon'_M, \epsilon'_M[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ such that $\mathrm{cl}\,\Omega_M \cap \epsilon\,\mathrm{cl}\,\Omega^i = \emptyset$ for all $\epsilon \in]0, \epsilon'_M[$ and such that

$$w_{\epsilon|\mathrm{cl}\,\Omega_M} = W_M[\epsilon] \quad \forall \epsilon \in]0, \epsilon'_M[.$$

(d2) For all $\Omega_m \subseteq \mathbb{R}^2 \setminus \mathrm{cl}\,\Omega^i$ open and bounded there exist $\epsilon'_m \in]0, \epsilon_0[$ and a real analytic map W_m from $]-\epsilon'_m, \epsilon'_m[$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_m)$ such that $\epsilon\,\mathrm{cl}\,\Omega_m \subseteq \Omega^o$ for all $\epsilon \in]0, \epsilon'_m[$ and such that

$$w_\epsilon(\epsilon\,\cdot)_{|\mathrm{cl}\,\Omega_m} = W_m[\epsilon] \quad \forall \epsilon \in]0, \epsilon'_m[.$$

We say that $\{w_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ as above is a *right real analytic family of harmonic functions on* $\Omega(\epsilon)$ (see also [1, §1], where the analogous definition is given for the n-dimensional case with $n \geq 3$). Then we say that $\{v_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ is a *real analytic family of harmonic functions on* $\Omega(\epsilon)$ if

(a0) $v_0 \in C^{1,\alpha}(\mathrm{cl}\,\Omega^o)$ and $\Delta v_0 = 0$ in Ω^o, $v_\epsilon \in C^{1,\alpha}(\mathrm{cl}\,\Omega(\epsilon))$ and $\Delta v_\epsilon = 0$ in $\Omega(\epsilon)$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$

and in addition $\{v_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ satisfies the conditions in (a1) and (a2) with u_ϵ replaced by v_ϵ (see also [1, §1], where the analogous definition is given for the n-dimensional case with $n \geq 3$). Then, by the equivalence of (a1) and (b1) and by the equivalence of (a2) and (b2) (which hold also for boundary data depending analytically on ϵ), we deduce the validity of the following statement.

(∗) If $n = 2$ and if $\{w_\epsilon\}_{\epsilon\in]0,\epsilon_0[}$ is a *right real analytic family of harmonic functions on* $\Omega(\epsilon)$, then there exists a *real analytic family of harmonic functions* $\{v_\epsilon\}_{\epsilon\in]-\epsilon_0,\epsilon_0[}$ *on* $\Omega(\epsilon)$ such that $w_\epsilon = v_\epsilon$ for all $\epsilon \in]0, \epsilon_0[$.

We note that an analog of statement (∗) has been proved in [1] in the case of families of harmonic functions in a perforated domain of $\mathbb{R}^n$, with $n \geq 4$ even. In this sense, one can say that statement (∗) here above extends the validity of the analogous statement (j) in [1, §1] to the two-dimensional case. The case of dimension $n \geq 3$ and odd is also studied in [1], but in this case an analog of statement (∗) does not hold and we have a completely different phenomenon (cf. [1, (jj) of §1 and Thm. 3.2]).

The paper is organized as follows. Section 2 is a section of preliminaries where we introduce some notions of potential theory (cf. Subsection 2.1) and we transform the boundary value problem in (2) into an equivalent system of integral equations on $\partial\Omega^o$ and $\partial\Omega^i$ which we analyze by exploiting the implicit function theorem (cf. Subsections 2.2–2.4). In Section 3 we derive our main results. First we consider in Subsection 3.1 the case in which the boundary data of problem (2) are given by real analytic functions evaluated at $(\epsilon, 1/\log|\epsilon|)$ and we prove Theorem 3.1, which in particular implies the validity of (7). Then, in Subsection 3.2 we consider the case in which the boundary data are given by analytic functions of ϵ and we prove Theorem 3.6, which in particular implies the equivalence of conditions (a1), (b1), and (c), and the validity of statement (∗).

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Finally, we observe that the results of this paper can be exploited in the computation of the power series expansions of the real analytic maps which describe u_ϵ for ϵ close to 0. In forthcoming papers we will show that the coefficients of such series can be obtained by a fully constructive method which is rigorously justified on the basis of the present paper (cf., *e.g.*, [2]).

2. Preliminaries

2.1. Classical notions of potential theory

Let S be the function from $\mathbb{R}^2 \setminus \{0\}$ to $\mathbb{R}$ defined by

$$S(x) \equiv \frac{1}{2\pi} \log|x| \quad \forall x \in \mathbb{R}^2 \setminus \{0\}.$$

As is well known, S is a fundamental solution of the Laplace operator on $\mathbb{R}^2$.

Let Ω be an open bounded subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$. Let $\phi \in C^{0,\alpha}(\partial\Omega)$. Then $v[\partial\Omega, \phi]$ denotes the single layer potential with density ϕ. Namely,

$$v[\partial\Omega, \phi](x) \equiv \int_{\partial\Omega} \phi(y) S(x-y)\, d\sigma_y \quad \forall x \in \mathbb{R}^2,$$

where $d\sigma$ denotes the arc length element on $\partial\Omega$. As is well known, $v[\partial\Omega, \phi]$ is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ and the restrictions $v^+[\partial\Omega, \phi] \equiv v[\partial\Omega, \phi]_{|\mathrm{cl}\,\Omega}$ and $v^-[\partial\Omega, \phi] \equiv v[\partial\Omega, \phi]_{|\mathbb{R}^2\setminus\Omega}$ belong to $C^{1,\alpha}(\mathrm{cl}\,\Omega)$ and $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Omega)$, respectively. Here $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Omega)$ denotes the space of functions on $\mathbb{R}^2 \setminus \Omega$ which restrict to a function of $C^{1,\alpha}(\mathrm{cl}\,\mathcal{O})$ for all open bounded subsets $\mathcal{O}$ of $\mathbb{R}^2 \setminus \Omega$.

Let $\psi \in C^{1,\alpha}(\partial\Omega)$. Then $w[\partial\Omega, \psi]$ denotes the double layer potential with density ψ. Namely,

$$w[\partial\Omega, \psi](x) \equiv -\int_{\partial\Omega} \psi(y) \nu_\Omega(y) \cdot \nabla S(x-y)\, d\sigma_y \quad \forall x \in \mathbb{R}^2,$$

where ν_Ω denotes the outer unit normal to $\partial\Omega$. As is well known, the restriction $w[\partial\Omega, \psi]_{|\Omega}$ extends to a function $w^+[\partial\Omega, \psi] \in C^{1,\alpha}(\mathrm{cl}\,\Omega)$ and the restriction $w[\partial\Omega, \psi]_{|\mathbb{R}^2\setminus\mathrm{cl}\,\Omega}$ extends to a function $w^-[\partial\Omega, \psi] \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \Omega)$.

Let

$$W_\Omega[\psi](x) \equiv -\int_{\partial\Omega} \psi(y) \nu_\Omega(y) \cdot \nabla S(x-y)\, d\sigma_y \quad \forall x \in \partial\Omega,$$

for all $\psi \in C^{1,\alpha}(\partial\Omega)$, and

$$W^*_\Omega[\phi](x) \equiv \int_{\partial\Omega} \phi(y) \nu_\Omega(x) \cdot \nabla S(x-y)\, d\sigma_y \quad \forall x \in \partial\Omega,$$

for all $\phi \in C^{0,\alpha}(\partial\Omega)$. Then W_Ω is a compact operator from $C^{1,\alpha}(\partial\Omega)$ to itself and W^*_Ω is a compact operator from $C^{0,\alpha}(\partial\Omega)$ to itself (cf., *e.g.*, Schauder [14,15]). The operators W_Ω and W^*_Ω are adjoint one to the other with respect to the duality on $C^{1,\alpha}(\partial\Omega) \times C^{0,\alpha}(\partial\Omega)$ induced by the inner product of the Lebesgue space $L^2(\partial\Omega)$ (cf., *e.g.*, Kress [7, Chap. 4]). Moreover,

$$w^{\pm}[\partial\Omega,\psi]_{|\partial\Omega} = \pm\frac{1}{2}\psi + W_{\Omega}[\psi] \ \forall\psi \in C^{1,\alpha}(\partial\Omega),$$
$$\nu_{\Omega}\cdot\nabla v^{\pm}[\partial\Omega,\phi]_{|\partial\Omega} = \mp\frac{1}{2}\phi + W^{*}_{\Omega}[\phi] \ \forall\phi \in C^{0,\alpha}(\partial\Omega). \tag{9}$$

We now introduce some more notation which we shall use in the sequel.

If Ω is an open bounded subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$ and $\mathcal{X}$ is a subspace of $L^1(\partial\Omega)$, then $\mathcal{X}_0$ denotes the subspace of $\mathcal{X}$ consisting of those functions f such that $\int_{\partial\Omega} f\,d\sigma = 0$.

If $\mathcal{O}$ is an open subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$, then $\mathbb{R}_{\mathcal{O}}$ denotes the set of the functions from $\mathcal{O}$ to $\mathbb{R}$ which are constant, $\mathbb{R}_{\mathcal{O},\mathrm{loc}}$ denotes the set of functions from $\mathcal{O}$ to $\mathbb{R}$ which are constant on each connected component of $\mathcal{O}$, $(\mathbb{R}_{\mathcal{O}})_{|\partial\mathcal{O}}$ denotes the set of the functions on $\partial\mathcal{O}$ which are traces on $\partial\mathcal{O}$ of functions of $\mathbb{R}_{\mathcal{O}}$, and $(\mathbb{R}_{\mathcal{O},\mathrm{loc}})_{|\partial\mathcal{O}}$ denotes the set of the functions on $\partial\mathcal{O}$ which are traces on $\partial\mathcal{O}$ of functions of $\mathbb{R}_{\mathcal{O},\mathrm{loc}}$.

Then one has the following classical lemma (cf., *e.g.*, Folland [4, Chap. 3]).

Lemma 2.1. *Let Ω be an open bounded subset of $\mathbb{R}^2$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) *The operator from $\mathrm{Ker}(\frac{1}{2}I + W^*_{\Omega})$ to $\mathrm{Ker}(\frac{1}{2}I + W_{\Omega})$ which takes μ to $v[\partial\Omega,\mu]_{|\partial\Omega}$ is a linear homeomorphism.*
(ii) *$\mathrm{Ker}(\frac{1}{2}I + W_{\Omega})$ consists of those functions of $(\mathbb{R}_{\mathbb{R}^2\setminus\mathrm{cl}\,\Omega,\mathrm{loc}})_{|\partial(\mathbb{R}^2\setminus\mathrm{cl}\,\Omega)}$ which vanish on the boundary of the unbounded connected component of $\mathbb{R}^2\setminus\mathrm{cl}\,\Omega$.*
(iii) *$\mathrm{Ker}(-\frac{1}{2}I + W_{\Omega}) = (\mathbb{R}_{\Omega,\mathrm{loc}})_{|\partial\Omega}$.*
(iv) *If $\phi \in \mathrm{Ker}(\frac{1}{2}I + W^*_{\Omega})$ and $\int_{\partial\Omega}\phi\psi\,d\sigma = 0$ for all $\psi \in \mathrm{Ker}(\frac{1}{2}I + W_{\Omega})$, then $\phi = 0$.*
(v) *If $\phi \in \mathrm{Ker}(-\frac{1}{2}I + W^*_{\Omega})$ and $\int_{\partial\Omega}\phi\psi\,d\sigma = 0$ for all $\psi \in \mathrm{Ker}(-\frac{1}{2}I + W_{\Omega})$, then $\phi = 0$.*

Definition 2.2. If $\epsilon \in \,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$, we denote by χ_ϵ the function from $\partial\Omega(\epsilon)$ to $\mathbb{R}$ defined by

$$\chi_\epsilon(x) \equiv \begin{cases} 0 & \text{if } x \in \partial\Omega^o, \\ 1 & \text{if } x \in \epsilon\partial\Omega^i. \end{cases}$$

We observe that χ_ϵ is a generator of $\mathrm{Ker}(\frac{1}{2}I + W_{\Omega(\epsilon)})$ (cf. Lemma 2.1 (ii)).

2.2. The map M and the pair of functions $(\rho^o[\epsilon],\rho^i[\epsilon])$

We introduce in this subsection the pair of functions $(\rho^o[\epsilon],\rho^i[\epsilon])$ which is related to the generator of the one dimensional space $\mathrm{Ker}(\frac{1}{2}I + W^*_{\Omega(\epsilon)})$ (see Lemma 2.1 (i), (ii) and Proposition 2.4 here below). To do so, we define the map $M \equiv (M^o, M^i, M^c)$ from $]-\epsilon_0,\epsilon_0[\,\times C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$ by setting

$$M^o[\epsilon,\rho^o,\rho^i](x) \equiv \frac{1}{2}\rho^o(x) + W^*_{\Omega^o}[\rho^o](x) + \int_{\partial\Omega^i}\rho^i(y)\nu_{\Omega^o}(x)\cdot\nabla S(x-\epsilon y)\,d\sigma_y \quad \forall x \in \partial\Omega^o,$$
$$M^i[\epsilon,\rho^o,\rho^i](x) \equiv \frac{1}{2}\rho^i(x) - W^*_{\Omega^i}[\rho^i](x) - \epsilon\int_{\partial\Omega^o}\rho^o(y)\nu_{\Omega^i}(x)\cdot\nabla S(\epsilon x - y)\,d\sigma_y \quad \forall x \in \partial\Omega^i,$$
$$M^c[\epsilon,\rho^o,\rho^i] \equiv \int_{\partial\Omega^i}\rho^i\,d\sigma - 1,$$

for all $(\epsilon,\rho^o,\rho^i) \in \,]-\epsilon_0,\epsilon_0[\,\times C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$.

Remark 2.3. Here one has to verify that M has values in the product space $C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)_0\times\mathbb{R}$. By standard properties of integral operators with real analytic kernels and with no singularity and by classical mapping properties of layer potentials, one deduces that M has values in $C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)\times\mathbb{R}$ (see also Subsection 2.1). To show that $\int_{\partial\Omega^i} M^i[\epsilon,\rho^o,\rho^i]\,d\sigma=0$ for all $(\epsilon,\rho^o,\rho^i)\in\,]-\epsilon_0,\epsilon_0[\,\times C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$ one observes that

$$\int_{\partial\Omega^i}\left(\frac{1}{2}\rho^i - W^*_{\Omega^i}\left[\rho^i\right]\right)d\sigma = 0$$

by the orthogonality of $\mathrm{Ker}(-\frac{1}{2}I+W_{\Omega^i})=(\mathbb{R}_{\Omega^i})_{|\partial\Omega^i}$ and of $\mathrm{Ran}(-\frac{1}{2}I+W^*_{\Omega^i})$ (cf. statement (iii) of Lemma 2.1). Moreover,

$$\int_{\partial\Omega^i}\epsilon\int_{\partial\Omega^o}\rho^o(y)\nu_{\Omega^i}(x)\cdot\nabla S(\epsilon x-y)\,d\sigma_y\,d\sigma_x=\int_{\partial\Omega^i}\nu_{\Omega^i}(x)\cdot\nabla_x\left(\int_{\partial\Omega^o}\rho^o(y)S(\epsilon x-y)\,d\sigma_y\right)d\sigma_x=0,$$

where the second equality is trivial for $\epsilon=0$ and follows by the divergence theorem for $\epsilon\in\,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$.

The following Propositions 2.4 and 2.5 concern the equation $M[\epsilon,\rho^o,\rho^i]=0$ for $\epsilon\in\,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$ and for $\epsilon=0$, respectively. A proof can be effected by exploiting the standard theorem on change of variables in integrals.

Proposition 2.4. *Let $\epsilon\in\,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$. Let $(\rho^o,\rho^i)\in C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$. Let $\tau\in C^{0,\alpha}(\partial\Omega(\epsilon))$ be defined by*

$$\tau(x)\equiv\begin{cases}\rho^o(x) & \text{if } x\in\partial\Omega^o,\\ |\epsilon|^{-1}\rho^i(x/\epsilon) & \text{if } x\in\epsilon\partial\Omega^i.\end{cases}$$

Then $M[\epsilon,\rho^o,\rho^i]=0$ if and only if

$$\frac{1}{2}\tau+W^*_{\Omega(\epsilon)}[\tau]=0\quad\text{and}\quad\int_{\epsilon\partial\Omega^i}\tau\,d\sigma=1. \tag{10}$$

Proposition 2.5. *Let $(\rho^o,\rho^i)\in C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$. Then $M[0,\rho^o,\rho^i]=0$ if and only if*

$$\frac{1}{2}\rho^o+W^*_{\Omega^o}\left[\rho^o\right]=-\nu_{\Omega^o}\cdot(\nabla S)_{|\partial\Omega^o},\qquad -\frac{1}{2}\rho^i+W^*_{\Omega^i}\left[\rho^i\right]=0\quad\text{and}\quad\int_{\partial\Omega^i}\rho^i\,d\sigma=1.$$

Then we have the following.

Proposition 2.6. *For all $\epsilon\in\,]-\epsilon_0,\epsilon_0[$ there exists a unique pair $(\rho^o[\epsilon],\rho^i[\epsilon])\in C^{0,\alpha}(\partial\Omega^o)\times C^{0,\alpha}(\partial\Omega^i)$ such that $M[\epsilon,\rho^o[\epsilon],\rho^i[\epsilon]]=0$.*

Proof. If $\epsilon\in\,]-\epsilon_0,\epsilon_0[\,\setminus\{0\}$, then the existence and uniqueness of $(\rho^o[\epsilon],\rho^i[\epsilon])$ is equivalent to the existence and uniqueness of the solution τ of Eq. (10) (cf. Proposition 2.4). By statement (ii) of Lemma 2.1 and by standard Fredholm theory, $\mathrm{Ker}(\frac{1}{2}I+W^*_{\Omega(\epsilon)})$ has dimension one. Hence, the first equation in (10) admits at least one non-zero solution. Then, statements (ii) and (iv) of Lemma 2.1 imply that the function which satisfies both the equations in (10) exists and is unique. If $\epsilon=0$, then the existence and uniqueness of $\rho^o[0]$ follows by Proposition 2.5, by statement (ii) of Lemma 2.1, and by standard Fredholm theory. By statement (iii) of Lemma 2.1 and by standard Fredholm theory, $\mathrm{Ker}(-\frac{1}{2}I+W^*_{\Omega^i})$ has dimension one. Hence, the existence and uniqueness of $\rho^i[0]$ follows by Proposition 2.5 and by statement (iii) and (v) of Lemma 2.1. □

In Lemma 2.7 below we show that M is real analytic and that the partial differential of M with respect to (ρ^o, ρ^i) is a linear homeomorphism.

Lemma 2.7. *The map M is real analytic. For all $(\epsilon, \rho^o, \rho^i) \in]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$ the partial differential $\partial_{(\rho^o,\rho^i)}M[\epsilon, \rho^o, \rho^i]$ is a linear homeomorphism from $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$ to $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$.*

Proof. By standard properties of integral operators with real analytic kernels and with no singularity and by classical mapping properties of layer potentials, one deduces that M is real analytic (cf. [12, §4]). The partial differential $\partial_{(\rho^o,\rho^i)}M[\epsilon, \rho^o, \rho^i]$ at $(\epsilon, \rho^o, \rho^i) \in]-\epsilon_0, \epsilon_0[\times C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$ is delivered by

$$\partial_{(\rho^o,\rho^i)}M[\epsilon, \rho^o, \rho^i](\bar{\rho}^o, \bar{\rho}^i) = \left(M^o[\epsilon, \bar{\rho}^o, \bar{\rho}^i], M^i[\epsilon, \bar{\rho}^o, \bar{\rho}^i], \int_{\partial\Omega^i} \bar{\rho}^i \, d\sigma\right)$$

for all $(\bar{\rho}^o, \bar{\rho}^i) \in C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$. Now fix $(f^o, f^i, c) \in C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)_0 \times \mathbb{R}$. We show that there exists unique $(\bar{\rho}^o, \bar{\rho}^i) \in C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$ such that

$$\partial_{(\rho^o,\rho^i)}M[\epsilon, \rho^o, \rho^i](\bar{\rho}^o, \bar{\rho}^i) = (f^o, f^i, c). \tag{11}$$

If $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$ then (11) is equivalent to

$$\frac{1}{2}\bar{\tau} + W^*_{\Omega(\epsilon)}[\bar{\tau}] = f \quad \text{and} \quad \int_{\epsilon\partial\Omega^i} \bar{\tau} \, d\sigma = c, \tag{12}$$

with $\bar{\tau}_{|\partial\Omega^o} \equiv \bar{\rho}^o$, $\bar{\tau}_{|\epsilon\partial\Omega^i} \equiv |\epsilon|^{-1}\bar{\rho}^i(\cdot/\epsilon)$, $f_{|\partial\Omega^o} \equiv f^o$, and $f_{|\epsilon\partial\Omega^i} \equiv f^i(\cdot/\epsilon)$. The first equation in (12) has at least a solution $\bar{\tau}$ because $\int_{\partial\Omega(\epsilon)} f\chi_\epsilon \, d\sigma = 0$ and χ_ϵ is a generator of $\mathrm{Ker}(\frac{1}{2}I + W_{\Omega(\epsilon)})$. Then, by statement (iv) of Lemma 2.1 one deduces the existence and uniqueness of the solution $\bar{\tau}$ of the equations in (12). If instead $\epsilon = 0$, then equality (11) is equivalent to

$$\frac{1}{2}\bar{\rho}^o + W^*_{\Omega^o}[\bar{\rho}^o] = -c\nu_{\Omega^o} \cdot (\nabla S)_{|\partial\Omega^o} + f^o,$$
$$-\frac{1}{2}\bar{\rho}^i + W^*_{\Omega^i}[\bar{\rho}^i] = -f^i \quad \text{and} \quad \int_{\partial\Omega^i} \bar{\rho}^i \, d\sigma = c.$$

Then the existence and uniqueness of $\bar{\rho}^o$ follows by statement (ii) of Lemma 2.1, and by standard Fredholm theory. The existence and uniqueness of $\bar{\rho}^i$ follows by the orthogonality of f^i and $\mathrm{Ker}(-\frac{1}{2}I + W_{\Omega^i}) = (\mathbb{R}^2_{\Omega^i})_{|\partial\Omega^i}$ (cf. statement (iii) of Lemma 2.1), by standard Fredholm theory, and by statement (v) of Lemma 2.1. Now the validity of the proposition follows by the open mapping theorem. □

Then by Proposition 2.6 and Lemma 2.7, and by the implicit function theorem for real analytic maps (cf., *e.g.*, Deimling [3, §15]) one deduces the validity of the following.

Proposition 2.8. *The map from $]-\epsilon_0, \epsilon_0[$ to $C^{0,\alpha}(\partial\Omega^o) \times C^{0,\alpha}(\partial\Omega^i)$ which takes ϵ to $(\rho^o[\epsilon], \rho^i[\epsilon])$ is real analytic.*

We introduce the following definition.

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Definition 2.9. If $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, then we denote by $\tau[\epsilon]$ the function of $C^{0,\alpha}(\partial\Omega(\epsilon))$ defined by

$$\tau[\epsilon](x) \equiv \begin{cases} \rho^o[\epsilon](x) & \text{if } x \in \partial\Omega^o, \\ |\epsilon|^{-1}\rho^i[\epsilon](x/\epsilon) & \text{if } x \in \epsilon\partial\Omega^i. \end{cases}$$

We observe that $\tau[\epsilon]$ is a generator of $\mathrm{Ker}(\frac{1}{2}I + W^*_{\Omega(\epsilon)})$ and $\int_{\epsilon\partial\Omega^i}\tau[\epsilon]\,d\sigma = 1$ and that the function $v[\partial\Omega(\epsilon),\tau[\epsilon]]_{|\partial\Omega(\epsilon)}$ is a generator of $\mathrm{Ker}(\frac{1}{2}I + W_{\Omega(\epsilon)})$ (cf. Propositions 2.4 and 2.6 and statement (i) of Lemma 2.1). Hence, there exists $c_\epsilon \in \mathbb{R}\setminus\{0\}$ such that

$$v\big[\partial\Omega(\epsilon),\tau[\epsilon]\big]_{|\partial\Omega(\epsilon)} = c_\epsilon\chi_\epsilon \tag{13}$$

(cf. statement (ii) of Lemma 2.1).

2.3. The map Λ and the pair of functions $(\theta^o[\epsilon,g^o,g^i],\theta^i[\epsilon,g^o,g^i])$

Now we introduce the map $\Lambda \equiv (\Lambda^o,\Lambda^i)$ related to the boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega(\epsilon), \\ w(x) = g^o(x) & \text{for all } x \in \partial\Omega^o, \\ w(x) = g^i(x/\epsilon) - \int_{\partial\Omega^o} g^o\rho^o[\epsilon]\,d\sigma - \int_{\partial\Omega^i} g^i\rho^i[\epsilon]\,d\sigma & \text{for all } x \in \epsilon\partial\Omega^i, \end{cases}$$

for $\epsilon \in]-\epsilon_0,\epsilon_0[\setminus\{0\}$. The first component Λ^o corresponds to the boundary condition on $\partial\Omega^o$ and the second component Λ^i corresponds to the boundary condition on $\epsilon\partial\Omega^i$. The map $\Lambda \equiv (\Lambda^o,\Lambda^i)$ is defined from $]-\epsilon_0,\epsilon_0[\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0$ to $C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$ by setting

$$\begin{aligned} \Lambda^o\big[\epsilon,g^o,g^i,\theta^o,\theta^i\big](x) \equiv{}& \frac{1}{2}\theta^o(x) + W_{\Omega^o}\big[\theta^o\big](x) \\ &+ \epsilon\int_{\partial\Omega^i}\theta^i(y)\nu_{\Omega^i}(y)\cdot\nabla S(x-\epsilon y)\,d\sigma_y - g^o(x) \quad \forall x \in \partial\Omega^o, \\ \Lambda^i\big[\epsilon,g^o,g^i,\theta^o,\theta^i\big](x) \equiv{}& \frac{1}{2}\theta^i(x) - W_{\Omega^i}\big[\theta^i\big](x) + w\big[\partial\Omega^o,\theta^o\big](\epsilon x) \\ &- g^i(x) + \int_{\partial\Omega^o} g^o\rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} g^i\rho^i[\epsilon]\,d\sigma \quad \forall x \in \partial\Omega^i, \end{aligned}$$

for all $(\epsilon,g^o,g^i,\theta^o,\theta^i)$ in $]-\epsilon_0,\epsilon_0[\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)\times C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0$. The following Propositions 2.10 and 2.11 concern the equation $\Lambda[\epsilon,g^o,g^i,\theta^o,\theta^i] = 0$ for $\epsilon \in]-\epsilon_0,\epsilon_0[\setminus\{0\}$ and for $\epsilon = 0$, respectively. A proof of Proposition 2.10 can be effected by exploiting the classical theorem on change of variables in integrals.

Proposition 2.10. *Let $\epsilon \in]-\epsilon_0,\epsilon_0[\setminus\{0\}$. Let $(g^o,g^i) \in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)$. Let $(\theta^o,\theta^i) \in C^{1,\alpha}(\partial\Omega^o)\times C^{1,\alpha}(\partial\Omega^i)_0$. Let $g \in C^{1,\alpha}(\partial\Omega(\epsilon))$ be defined by*

$$g(x) \equiv \begin{cases} g^o(x) & \text{if } x \in \partial\Omega^o, \\ g^i(x/\epsilon) & \text{if } x \in \epsilon\partial\Omega^i. \end{cases}$$

Let $\theta \in C^{1,\alpha}(\partial\Omega(\epsilon))$ be defined by

$$\theta(x) \equiv \begin{cases} \theta^o(x) & \text{if } x \in \partial\Omega^o, \\ \theta^i(x/\epsilon) & \text{if } x \in \epsilon\partial\Omega^i. \end{cases}$$

Then $\Lambda[\epsilon, g^o, g^i, \theta^o, \theta^i] = 0$ *if and only if*

$$\frac{1}{2}\theta + W_{\Omega(\epsilon)}[\theta] = g - \int_{\partial\Omega(\epsilon)} g\tau[\epsilon]\,d\sigma\,\chi_\epsilon.$$

Proposition 2.11. *Let* $(g^o, g^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$. *Let* $(\theta^o, \theta^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$. *Then* $\Lambda[0, g^o, g^i, \theta^o, \theta^i] = 0$ *if and only if*

$$\frac{1}{2}\theta^o + W_{\Omega^o}\big[\theta^o\big] = g^o \quad \text{and} \quad -\frac{1}{2}\theta^i + W_{\Omega^i}\big[\theta^i\big] = -g^i + \int_{\partial\Omega^i} g^i\rho^i[0]\,d\sigma. \qquad (14)$$

Proof. The equivalence of $\Lambda^o[0, g^o, g^i, \theta^o, \theta^i] = 0$ and of the first equation in (14) follows by the definition of Λ. Then one observes that $\Lambda^i[0, g^o, g^i, \theta^o, \theta^i] = 0$ if and only if

$$-\frac{1}{2}\theta^i + W_{\Omega^i}\big[\theta^i\big] = w\big[\partial\Omega^o, \theta^o\big](0) - g^i + \int_{\partial\Omega^o} g^o\rho^o[0]\,d\sigma + \int_{\partial\Omega^i} g^i\rho^i[0]\,d\sigma.$$

By Proposition 2.5, by standard properties of adjoint operators, and by the first equation in (14) one has

$$\begin{aligned} w\big[\partial\Omega^o, \theta^o\big](0) &= \int_{\partial\Omega^o} \theta^o\nu_{\Omega^o} \cdot \nabla S\,d\sigma \\ &= -\int_{\partial\Omega^o} \theta^o\left(\frac{1}{2}\rho^o[0] + W^*_{\Omega^o}\big[\rho^o[0]\big]\right)d\sigma \\ &= -\int_{\partial\Omega^o}\left(\frac{1}{2}\theta^o + W_{\Omega^o}\big[\theta^o\big]\right)\rho^o[0]\,d\sigma = -\int_{\partial\Omega^o} g^o\rho^o[0]\,d\sigma. \end{aligned}$$

Then the equivalence of $\Lambda^i[0, g^o, g^i, \theta^o, \theta^i] = 0$ and of the second equation in (14) follows by a straightforward calculation. □

Then we have the following.

Proposition 2.12. *For all* $(\epsilon, g^o, g^i) \in]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$ *there exists a unique pair* $(\theta^o[\epsilon, g^o, g^i], \theta^i[\epsilon, g^o, g^i]) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$ *such that* $\Lambda[\epsilon, g^o, g^i, \theta^o[\epsilon, g^o, g^i], \theta^i[\epsilon, g^o, g^i]] = 0$.

Proof. If $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$, then χ_ϵ generates $\mathrm{Ker}(\frac{1}{2}I + W_{\Omega(\epsilon)})$, $\tau(\epsilon)$ generates $\mathrm{Ker}(\frac{1}{2}I + W^*_{\Omega(\epsilon)})$, and $\int_{\epsilon\partial\Omega^i} \tau[\epsilon]\,d\sigma = 1$. Hence, the existence and uniqueness of $(\theta^o[\epsilon, g^o, g^i], \theta^i[\epsilon, g^o, g^i])$ follows by Proposition 2.10, by standard Fredholm theory, and by condition $\int_{\partial\Omega^i} \theta^i[\epsilon, g^o, g^i]\,d\sigma = 0$. If $\epsilon = 0$, then the existence and uniqueness of $\theta^o[0, g^o, g^i]$ follows by Proposition 2.11, by statement (ii) of Lemma 2.1, and by standard Fredholm theory. Then one observes that $\mathrm{Ker}(-\frac{1}{2}I + W^*_{\Omega^i})$ has dimension one by statement (iii) of Lemma 2.1 and by standard Fredholm theory, and that $\rho^i[0]$ is a generator of $\mathrm{Ker}(-\frac{1}{2}I + W^*_{\Omega^i})$ and $\int_{\partial\Omega^i} \rho^i[0]\,d\sigma = 1$ by Propositions 2.5 and 2.6. Hence, the existence and uniqueness of $\theta^i[0, g^o, g^i]$ follows by Proposition 2.11, by condition $\int_{\partial\Omega^i} \theta^i[0, g^o, g^i]\,d\sigma = 0$, by statement (iii) of Lemma 2.1, and by standard Fredholm theory. □

In Lemma 2.13 below we show that $\Lambda[\epsilon, g^o, g^i, \theta^o, \theta^i]$ is orthogonal to $(\rho^o[\epsilon], \rho^i[\epsilon])$ in $L^2(\partial\Omega^o) \times L^2(\partial\Omega^i)$.

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Lemma 2.13. *For all* $(\epsilon, g^o, g^i, \theta^o, \theta^i) \in]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$ *the following equality holds*

$$\int_{\partial\Omega^o} \Lambda^o\big[\epsilon, g^o, g^i, \theta^o, \theta^i\big]\rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} \Lambda^i\big[\epsilon, g^o, g^i, \theta^o, \theta^i\big]\rho^i[\epsilon]\,d\sigma = 0.$$

Proof. If $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus \{0\}$, then by definition of Λ and by the classical theorem on change of variables in integrals one has

$$\Lambda\big[\epsilon, g^o, g^i, \theta^o, \theta^i\big] = \frac{1}{2}\theta + W_{\Omega(\epsilon)}[\theta] - g + \int_{\partial\Omega(\epsilon)} g\tau[\epsilon]\,d\sigma\chi_\epsilon$$

(see also Proposition 2.10). Then the validity of the statement follows by the orthogonality of $\mathrm{Ran}(\frac{1}{2}I+W_{\Omega(\epsilon)})$ and $\mathrm{Ker}(\frac{1}{2}I + W^*_{\Omega(\epsilon)})$, by equality $\int_{\epsilon\partial\Omega^i} \tau[\epsilon]\,d\sigma = 1$, and by a straightforward calculation. If instead $\epsilon = 0$, then

$$\Lambda^o\big[0, g^o, g^i, \theta^o, \theta^i\big] = \frac{1}{2}\theta^o + W_{\Omega^o}\big[\theta^o\big] - g^o,$$
$$\Lambda^i\big[0, g^o, g^i, \theta^o, \theta^i\big] = \frac{1}{2}\theta^i - W_{\Omega^i}\big[\theta^i\big] + w\big[\partial\Omega^o, \theta^o\big](0) - g^i + \int_{\partial\Omega^o} g^o\rho^o[0]\,d\sigma + \int_{\partial\Omega^i} g^i\rho^i[0]\,d\sigma.$$

So that

$$\begin{aligned}\int_{\partial\Omega^o} \Lambda^o\big[0, g^o, g^i, \theta^o, \theta^i\big]\rho^o[0]\,d\sigma &= \int_{\partial\Omega^o}\Big(\frac{1}{2}\theta^o + W_{\Omega^o}\big[\theta^o\big]\Big)\rho[0]\,d\sigma - \int_{\partial\Omega^o} g^o\rho[0]\,d\sigma \\ &= \int_{\partial\Omega^o}\theta^o\Big(\frac{1}{2}\rho^o[0] + W^*_{\Omega^o}\big[\rho^o[0]\big]\Big)\,d\sigma - \int_{\partial\Omega^o} g^o\rho^o[0]\,d\sigma \\ &= -\int_{\partial\Omega^o}\theta^o\nu_{\Omega^o}\cdot\nabla S\,d\sigma - \int_{\partial\Omega^o} g^o\rho^o[0]\,d\sigma \\ &= -w\big[\partial\Omega^o, \theta^o\big](0) - \int_{\partial\Omega^o} g^o\rho^o[0]\,d\sigma\end{aligned}$$

(see also Proposition 2.5). By the orthogonality of $\mathrm{Ran}(-\frac{1}{2}I + W_{\Omega^i})$ and $\mathrm{Ker}(-\frac{1}{2}I + W^*_{\Omega^i})$ and by equality $\int_{\partial\Omega^i} \rho^i[0]\,d\sigma = 1$ one has

$$\int_{\partial\Omega^i} \Lambda^i\big[0, g^o, g^i, \theta^o, \theta^i\big]\rho^i[0]\,d\sigma = w\big[\partial\Omega^o, \theta^o\big](0) + \int_{\partial\Omega^o} g^o\rho^o[0]\,d\sigma.$$

Then the validity of the statement follows by a straightforward calculation. □

In Lemma 2.14 below we show that Λ is real analytic and that the partial differential of Λ with respect to (θ^o, θ^i) is a linear homeomorphism from $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$ onto a suitable subspace of $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$.

Lemma 2.14. *The map* Λ *is real analytic. For all* $(\epsilon, g^o, g^i, \theta^o, \theta^i) \in]-\epsilon_0, \epsilon_0[\times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$ *the partial differential* $\partial_{(\theta^o,\theta^i)}\Lambda[\epsilon, g^o, g^i, \theta^o, \theta^i]$ *is a linear homeomorphism from*

$C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$ to the subspace of $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$ consisting of those pairs (ψ^o, ψ^i) such that

$$\int_{\partial\Omega^o} \psi^o \rho^o[\epsilon]\, d\sigma + \int_{\partial\Omega^i} \psi^i \rho^i[\epsilon]\, d\sigma = 0. \tag{15}$$

Proof. By standard properties of integral operators with real analytic kernels and with no singularity and by classical mapping properties of layer potentials, it follows that Λ is real analytic (cf. [12, §4]). Then the partial differential $\partial_{(\theta^o,\theta^i)}\Lambda[\epsilon, g^o, g^i, \theta^o, \theta^i]$ is delivered by

$$\partial_{(\theta^o,\theta^i)}\Lambda\big[\epsilon, g^o, g^i, \theta^o, \theta^i\big]\big(\bar\theta^o, \bar\theta^i\big) = \Lambda\big[\epsilon, 0, 0, \bar\theta^o, \bar\theta^i\big]$$

for all $(\bar\theta^o, \bar\theta^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$. Then $\partial_{(\theta^o,\theta^i)}\Lambda[\epsilon, g^o, g^i, \theta^o, \theta^i](\bar\theta^o, \bar\theta^i) = (\psi^o, \psi^i)$ is equivalent to $\Lambda[\epsilon, \psi^o, \psi^i, \bar\theta^o, \bar\theta^i] = 0$ for all $(\psi^o, \psi^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$ such that (15) holds. Hence the validity of the lemma follows by the open mapping theorem and by Proposition 2.12. □

We now introduce in the following Lemma 2.15 a technical corollary of the implicit function theorem for real analytic maps. For a proof we refer to Lanza de Cristoforis [8, Thm. 13].

Lemma 2.15. *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, $\mathcal{Z}_1$ be Banach spaces. Let $\mathcal{O}$ be an open set of $\mathcal{X} \times \mathcal{Y}$ such that $(x_*, y_*) \in \mathcal{O}$. Let F be a real analytic map from $\mathcal{O}$ to $\mathcal{Z}$ such that $F(x_*, y_*) = 0$. Let the partial differential $\partial_y F(x_*, y_*)$ with respect to the variable y be a homeomorphism from $\mathcal{Y}$ onto its image $V \equiv \mathrm{Ran}(\partial_y F(x_*, y_*))$. Assume that there exists a closed subspace V_1 of $\mathcal{Z}$ such that $\mathcal{Z} = V \oplus V_1$ algebraically. Let $\mathcal{O}_1$ be an open subset of $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$ containing $(x_*, y_*, 0)$ and such that $(x, y, F(x, y))$ and $(x, y, 0)$ belong to $\mathcal{O}_1$ for all $(x, y) \in \mathcal{O}$. Let G be a real analytic map from $\mathcal{O}_1$ to $\mathcal{Z}_1$ such that $G(x, y, F(x, y)) = 0$ for all $(x, y) \in \mathcal{O}$, $G(x, y, 0) = 0$ for all $(x, y) \in \mathcal{O}$, and such that the partial differential $\partial_z G(x_*, y_*, 0)$ is surjective onto $\mathcal{Z}_1$ and has kernel equal to V. Then there exist an open neighborhood $\mathcal{U}$ of x_* in $\mathcal{X}$, an open neighborhood $\mathcal{V}$ of y_* in $\mathcal{Y}$ with $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{O}$, and a real analytic map T_* from $\mathcal{U}$ to $\mathcal{V}$ such that the set of zeros of F in $\mathcal{U} \times \mathcal{V}$ coincides with the graph of T_*.*

Then we have the following.

Proposition 2.16. *The function from $]-\epsilon_0, \epsilon_0[\, \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$ which takes (ϵ, g^o, g^i) to $(\theta^o[\epsilon, g^o, g^i], \theta^i[\epsilon, g^o, g^i])$ is real analytic.*

Proof. Let $(\epsilon_*, g^o_*, g^i_*, \theta^o_*, \theta^i_*) \in\,]-\epsilon_0, \epsilon_0[\, \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$ be such that $\Lambda[\epsilon_*, g^o_*, g^i_*, \theta^o_*, \theta^i_*] = 0$. Let $\mathcal{X} \equiv \mathbb{R} \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$, $\mathcal{Y} \equiv C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$, $\mathcal{Z} \equiv C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$, $\mathcal{Z}_1 \equiv \mathbb{R}$, $\mathcal{O} \equiv\,]-\epsilon_0, \epsilon_0[\, \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$. Let $F \equiv \Lambda$. Let $x_* \equiv (\epsilon_*, g^o_*, g^i_*)$ and $y_* \equiv (\theta^o_*, \theta^i_*)$. Let V be the subspace of $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$ consisting of those pairs (ψ^o, ψ^i) which satisfy the condition in (15) with $\epsilon = \epsilon_*$, let V_1 be the 1-dimensional subspace of $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$ generated by $(\rho^o[\epsilon_*], \rho^i[\epsilon_*])$. Let $\mathcal{O}_1 \equiv\,]-\epsilon_0, \epsilon_0[\, \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i) \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0 \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$. Let

$$G\big(\epsilon, g^o, g^i, \theta^o, \theta^i, \psi^o, \psi^i\big) \equiv \int_{\partial\Omega^o} \psi^o \rho^o[\epsilon]\, d\sigma + \int_{\partial\Omega^i} \psi^i \rho^i[\epsilon]\, d\sigma$$

for all $(\epsilon, g^o, g^i, \theta^o, \theta^i, \psi^o, \psi^i) \in \mathcal{O}_1$. Then Lemma 2.15 implies that there exist an open neighborhood of $\mathcal{U}$ of $(\epsilon_*, g^o_*, g^i_*)$ in $]-\epsilon_0, \epsilon_0[\, \times C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$, an open neighborhood $\mathcal{V}$ of (θ^o_*, θ^i_*) in $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)_0$, and a real analytic map $T_* \equiv (T^o_*, T^i_*)$ from $\mathcal{U}$ to $\mathcal{V}$ such that the set of zeros of Λ in $\mathcal{U} \times \mathcal{V}$

coincides with the graph of T_*. Then Proposition 2.12 implies that $(\theta^o[\epsilon, g^o, g^i], \theta^i[\epsilon, g^o, g^i]) = T_*[\epsilon, g^o, g^i]$ for all $(\epsilon, g^o, g^i) \in \mathcal{U}$ and the validity of the proposition follows. □

2.4. Solution of the Dirichlet boundary value problem

In the following Proposition 2.17 we write a representation formula for $u_\epsilon(x)$ in terms of the functions $\rho^o[\epsilon]$, $\rho^i[\epsilon]$, $\theta^o[\epsilon, g^o, g^i]$, and $\theta^i[\epsilon, g^o, g^i]$ introduced in the previous Subsections 2.2 and 2.3. The validity of Proposition 2.17 follows by equalities (9) and (13), by Propositions 2.4, 2.6, 2.10, and 2.12, and by a straightforward calculation.

Proposition 2.17. *Let $\epsilon \in \left]-\epsilon_0, \epsilon_0\right[\setminus \{0\}$. Let $(g^o, g^i) \in C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$. Let $u_\epsilon \in C^{1,\alpha}(\mathrm{cl}\,\Omega(\epsilon))$ be the unique solution of (2). Then*

$$\begin{aligned} u_\epsilon(x) = {} & w^+\big[\partial\Omega^o, \theta^o\big[\epsilon, g^o, g^i\big]\big](x) + \epsilon \int_{\partial\Omega^i} \theta^i\big[\epsilon, g^o, g^i\big](y)\nu_{\Omega^i}(y) \cdot \nabla S(x - \epsilon y)\, d\sigma_y \\ & + \bigg(\int_{\partial\Omega^o} g^o \rho^o[\epsilon]\, d\sigma + \int_{\partial\Omega^i} g^i \rho^i[\epsilon]\, d\sigma\bigg) \\ & \times \bigg(v^+\big[\partial\Omega^o, \rho^o[\epsilon]\big](x) + \int_{\partial\Omega^i} \rho^i[\epsilon](y) S(x - \epsilon y)\, d\sigma_y\bigg) \\ & \times \bigg(\frac{1}{\int_{\partial\Omega^i} d\sigma} \int_{\partial\Omega^i} v\big[\partial\Omega^o, \rho^o[\epsilon]\big](\epsilon y) + v\big[\partial\Omega^i, \rho^i[\epsilon]\big](y)\, d\sigma_y + \frac{\log|\epsilon|}{2\pi}\bigg)^{-1} \end{aligned}$$

for all $x \in \mathrm{cl}\,\Omega^o \setminus \epsilon\, \mathrm{cl}\,\Omega^i$.

Remark 2.18. Under the assumptions of Proposition 2.17, one has

$$\begin{aligned} & \frac{1}{\int_{\partial\Omega^i} d\sigma} \int_{\partial\Omega^i} v\big[\partial\Omega^o, \rho^o[\epsilon]\big](\epsilon y) + v\big[\partial\Omega^i, \rho^i[\epsilon]\big](y)\, d\sigma_y + \frac{\log|\epsilon|}{2\pi} \\ & = \frac{1}{\int_{\epsilon\partial\Omega^i} d\sigma} \int_{\epsilon\partial\Omega^i} v\big[\partial\Omega(\epsilon), \tau[\epsilon]\big]\, d\sigma = c_\epsilon \neq 0 \quad \forall \epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\} \end{aligned}$$

(cf. equality (13)).

3. Real analytic families of harmonic functions

3.1. Harmonic functions depending analytically on $(\epsilon, 1/\log|\epsilon|)$

In this subsection we prove Theorem 3.1, where we examine a family of solutions of a Dirichlet problem with boundary data depending analytically on $(\epsilon, 1/\log|\epsilon|)$.

Theorem 3.1. *Let $\epsilon_0^* \equiv \min\{\epsilon_0, 1\}$. Let $\epsilon_1 \equiv -1/\log|\epsilon_0^*|$ if $\epsilon_0^* < 1$ and $\epsilon_1 \equiv +\infty$ if $\epsilon_0^* = 1$. Let $G \equiv (G^o, G^i)$ be a real analytic map from $\left]-\epsilon_0^*, \epsilon_0^*\right[\times \left]-\epsilon_1, \epsilon_1\right[$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$. For all $\epsilon \in \left]-\epsilon_0^*, \epsilon_0^*\right[\setminus \{0\}$, let $u[\epsilon]$ be the unique function in $C^{1,\alpha}(\mathrm{cl}\,\Omega(\epsilon))$ such that*

$$\Delta u[\epsilon] = 0 \quad \text{in } \Omega(\epsilon), \qquad u[\epsilon]_{|\partial\Omega^o} = G^o\big[\epsilon, 1/\log|\epsilon|\big], \qquad u[\epsilon]_{|\epsilon\partial\Omega^i} = G^i\big[\epsilon, 1/\log|\epsilon|\big](\cdot/\epsilon).$$

Let $\Omega_M \subseteq \Omega^o$ be open and $0 \notin \mathrm{cl}\,\Omega_M$. Let $\epsilon_M \in]0, \epsilon_0^*[$ be such that $\mathrm{cl}\,\Omega_M \cap \epsilon\,\mathrm{cl}\,\Omega^i = \emptyset$ for all $\epsilon \in]-\epsilon_M, \epsilon_M[$. Then there exist an open neighborhood $\mathcal{U}_M$ of $\{(\epsilon, 1/\log|\epsilon|) : \epsilon \in]-\epsilon_M, \epsilon_M[\setminus \{0\}\} \cup \{(0,0)\}$ in $]-\epsilon_M, \epsilon_M[\times]-\epsilon_1, \epsilon_1[$ and a real analytic map U_M from $\mathcal{U}_M$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ such that

$$u[\epsilon]_{|\mathrm{cl}\,\Omega_M} = U_M\big[\epsilon, 1/\log|\epsilon|\big] \quad \forall \epsilon \in]-\epsilon_M, \epsilon_M[\setminus \{0\}. \tag{16}$$

Proof. We set

$$C\big[\epsilon, \epsilon'\big] \equiv \frac{\epsilon'}{\int_{\partial\Omega^i} d\sigma} \int_{\partial\Omega^i} v\big[\partial\Omega^o, \rho^o[\epsilon]\big](\epsilon y) + v\big[\partial\Omega^i, \rho^i[\epsilon]\big](y)\, d\sigma_y + \frac{1}{2\pi}$$
$$\forall \big(\epsilon, \epsilon'\big) \in]-\epsilon_0, \epsilon_0[\times\mathbb{R},$$

and we note that $c_\epsilon = \log|\epsilon| C[\epsilon, 1/\log|\epsilon|]$ for all $\epsilon \in]-\epsilon_0^*, \epsilon_0^*[\setminus \{0\}$ (cf. equality (13)). By Remark 2.18, by Proposition 2.8, by standard properties of integral operators with real analytic kernels and with no singularity, and by classical mapping properties of layer potentials, we deduce that C is a real analytic function from $]-\epsilon_0, \epsilon_0[\times\mathbb{R}$ to $\mathbb{R}$ and that there exists an open neighborhood $\mathcal{U}_C$ of $\{(\epsilon, 1/\log|\epsilon|) : \epsilon \in]-\epsilon_0^*, \epsilon_0^*[\setminus \{0\}\} \cup \{(0,0)\}$ in $]-\epsilon_0^*, \epsilon_0^*[\times]-\epsilon_1, \epsilon_1[$ where C does not vanish (cf. [12, §4]). Now let $\mathcal{U}_M$ be an open neighborhood of $\{(\epsilon, 1/\log|\epsilon|) : \epsilon \in]-\epsilon_M, \epsilon_M[\setminus \{0\}\} \cup \{(0,0)\}$ in $\mathcal{U}_C$. We introduce the map U_M by setting

$$\begin{aligned} U_M\big[\epsilon, \epsilon'\big](x) \equiv\ & w^+\big[\partial\Omega^o, \theta^o\big[\epsilon, G^o\big[\epsilon, \epsilon'\big], G^i\big[\epsilon, \epsilon'\big]\big]\big](x) \\ & + \epsilon \int_{\partial\Omega^i} \theta^i\big[\epsilon, G^o\big[\epsilon, \epsilon'\big], G^i\big[\epsilon, \epsilon'\big]\big](y)\nu_{\Omega^i}(y) \cdot \nabla S(x - \epsilon y)\, d\sigma_y \\ & + \epsilon'\bigg(\int_{\partial\Omega^o} G^o\big[\epsilon, \epsilon'\big]\rho^o[\epsilon]\, d\sigma + \int_{\partial\Omega^i} G^i\big[\epsilon, \epsilon'\big]\rho^i[\epsilon]\, d\sigma\bigg) \\ & \times \bigg(v^+\big[\partial\Omega^o, \rho^o[\epsilon]\big](x) + \int_{\partial\Omega^i} \rho^i[\epsilon] S(x - \epsilon y)\, d\sigma_y\bigg) C\big[\epsilon, \epsilon'\big]^{-1} \end{aligned} \tag{17}$$

for all $x \in \mathrm{cl}\,\Omega_M$ and all $(\epsilon, \epsilon') \in \mathcal{U}_M$. By the definition of U_M and Proposition 2.17, we have $u[\epsilon]_{|\,\mathrm{cl}\,\Omega_M} = U_M[\epsilon, 1/\log|\epsilon|]$ for all $\epsilon \in]-\epsilon_M, \epsilon_M[\setminus \{0\}$. Moreover, by classical mapping properties of layer potentials, by Propositions 2.8 and 2.16, by standard calculus in Banach spaces, and by standard properties of functions in Schauder spaces, we verify that U_M is a real analytic map from $\mathcal{U}_M$ to $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ (cf. [12, §3] and [1, proof of Thm. 3.1]). Thus the validity of the theorem is proved. □

Remark 3.2. By (17), by Propositions 2.11 and 2.12, and by (9), one has $U_M[0,0] = u[0]_{|\mathrm{cl}\,\Omega_M}$, where $u[0]$ is the unique function of $C^{1,\alpha}(\mathrm{cl}\,\Omega^o)$ such that $\Delta u[0] = 0$ in Ω^o and $u[0]_{|\partial\Omega^o} = G^o[0,0]$.

By Lemma 3.3 below, one readily deduces that equality (16) univocally identifies the map U_M. The validity of Lemma 3.3 is known and is related to the non-subanalyticity of the curve $(\epsilon, 1/\log\epsilon)$ for $\epsilon \in]0,1[$ (cf., *e.g.*, Krantz and Parks [6, Chap. 5]). For the sake of completeness we present here an elementary proof based on standard properties of real analytic functions.

Lemma 3.3. *Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open connected neighborhood of $(0,0)$. Let U be a real analytic function from $\mathcal{U}$ to $\mathbb{R}$. Assume that $U(\epsilon, 1/\log\epsilon) = 0$ for all $(\epsilon, 1/\log\epsilon) \in \mathcal{U}$ with $\epsilon > 0$. Then $U(\epsilon, \epsilon') = 0$ for all $(\epsilon, \epsilon') \in \mathcal{U}$.*

Proof. Since U is real analytic, there exist $\delta, \delta' > 0$ and a family of real numbers $\{a_{(i,j)}\}_{(i,j)\in\mathbb{N}^2}$ such that $]-\delta,\delta[\times]-\delta',\delta'[\subseteq \mathcal{U}$ and that $U(\epsilon,\epsilon') = \sum_{(i,j)\in\mathbb{N}^2} a_{(i,j)}\epsilon^i\epsilon'^j$ for all $(\epsilon,\epsilon') \in]-\delta,\delta[\times]-\delta',\delta'[$, where the series converges absolutely and uniformly. We now prove that $a_{(i,j)} = 0$ for all $(i,j) \in \mathbb{N}^2$. Then the validity of the lemma follows by the Identity Principle for real analytic functions. Possibly shrinking δ, one can assume that $1/\log\epsilon \in]-\delta', 0[$ for all $\epsilon \in]0,\delta[$. Hence, $\sum_{(i,j)\in\mathbb{N}^2} a_{(i,j)}\epsilon^i(1/\log\epsilon)^j = U(\epsilon, 1/\log\epsilon) = 0$ for all $\epsilon \in]0,\delta[$. It follows that $a_{(0,0)} = \lim_{\epsilon\to 0^+} U(\epsilon, 1/\log\epsilon) = 0$. Then, an induction argument on j shows that $a_{(0,j)} = 0$ for all $j \in \mathbb{N}$. Indeed, if $k \in \mathbb{N}$ and $a_{(0,j)} = 0$ for all $j \le k$, then $(\log\epsilon)^{k+1}U(\epsilon, 1/\log\epsilon) = \sum_{(i,j)\in\mathbb{N}^2} a_{(i,j)}\epsilon^i(1/\log\epsilon)^{j-k-1} = 0$ for all $\epsilon \in]0,\delta[$ and thus $a_{(0,k+1)} = \lim_{\epsilon\to 0^+}(\log\epsilon)^{k+1}U(\epsilon, 1/\log\epsilon) = 0$. Now we argue by induction on i. Let $k \in \mathbb{N}$ and assume that $a_{(i,j)} = 0$ for all $(i,j) \in \mathbb{N}^2$ with $i \le k$. Let U_{k+1} denote the function from $]-\delta,\delta[\times]-\delta',\delta'[$ to $\mathbb{R}$ defined by $U_{k+1}(\epsilon,\epsilon') = \epsilon^{-(k+1)}U(\epsilon,\epsilon')$ for all $(\epsilon,\epsilon') \in]-\delta,\delta[\times]-\delta',\delta'[$. Then $U_{k+1}(\epsilon,\epsilon') = \sum_{(i,j)\in\mathbb{N}^2} b_{(i,j)}\epsilon^i\epsilon'^j$ for all $(\epsilon,\epsilon') \in]-\delta,\delta[\times]-\delta',\delta'[$, where $b_{(i,j)} = a_{(i+k+1,j)}$ for all $(i,j) \in \mathbb{N}^2$ and where the series converges absolutely and uniformly. Moreover, $U_{k+1}(\epsilon, 1/\log\epsilon) = 0$ for all $\epsilon \in]0,\delta[$. Then, by arguing as above for U, one verifies that $b_{(0,j)} = a_{(k+1,j)} = 0$ for all $j \in \mathbb{N}$. Hence $a_{(i,j)} = 0$ for all $(i,j) \in \mathbb{N}^2$ and the proof is completed. □

Then we have the following Proposition 3.4 whose validity can be deduced by a slight modification of the proof of [1, Prop. 4.2] and by exploiting Lemma 3.3.

Proposition 3.4. *Let the assumptions and notation of Theorem 3.1 hold. Let $\zeta \in \{-1,1\}$. If $\Omega^i = -\Omega^i$ and*

$$G^o\big[\epsilon, 1/\log|\epsilon|\big](x) = \zeta G^o\big[-\epsilon, 1/\log|\epsilon|\big](x) \quad \forall x \in \partial\Omega^o,$$
$$G^i\big[\epsilon, 1/\log|\epsilon|\big](x) = \zeta G^i\big[-\epsilon, 1/\log|\epsilon|\big](-x) \quad \forall x \in \partial\Omega^i,$$

for all $\epsilon \in]-\epsilon_0^*, \epsilon_0^*[\setminus\{0\}$, then there exist $\tilde{\epsilon}_M \in]0, \epsilon_M[$ and a family $\{u_{M,(i,j)}\}_{(i,j)\in\mathbb{N}^2}$ in $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ such that

$$u[\epsilon]_{|\mathrm{cl}\,\Omega_M} = \epsilon^{(1-\zeta)/2}\sum_{(i,j)\in\mathbb{N}^2} u_{M,(i,j)}\epsilon^{2i}\big(1/\log|\epsilon|\big)^j \quad \forall \epsilon \in]-\tilde{\epsilon}_M, \tilde{\epsilon}_M[\setminus\{0\},$$

where the series converges absolutely and uniformly in $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$.

3.2. Harmonic functions depending analytically on ϵ

In this subsection we prove Theorem 3.6, where we investigate a family of solution of a Dirichlet problem with boundary data depending analytically on ϵ. We first introduce in the following Lemma 3.5 an elementary consequence of the asymptotic behavior of $\log\epsilon$ and of standard properties of real analytic functions.

Lemma 3.5. *Let $\epsilon^* > 0$. Let A and B be real analytic functions from $]-\epsilon^*, \epsilon^*[$ to $\mathbb{R}$ such that $A[\epsilon]\log\epsilon = B[\epsilon]$ for all $\epsilon \in]0, \epsilon^*[$. Then $A = B = 0$.*

Theorem 3.6. *Let $F \equiv (F^o, F^i)$ be a real analytic map from $]-\epsilon_0, \epsilon_0[$ to $C^{1,\alpha}(\partial\Omega^o) \times C^{1,\alpha}(\partial\Omega^i)$. For all $\epsilon \in]-\epsilon_0, \epsilon_0[\setminus\{0\}$, let $v[\epsilon]$ be the unique function in $C^{1,\alpha}(\mathrm{cl}\,\Omega(\epsilon))$ such that*

$$\Delta v[\epsilon] = 0 \quad \text{in } \Omega(\epsilon), \qquad v[\epsilon]_{|\partial\Omega^o} = F^o[\epsilon], \qquad v[\epsilon]_{|\epsilon\partial\Omega^i} = F^i[\epsilon](\cdot/\epsilon).$$

Let $v[0] \in C^{1,\alpha}(\mathrm{cl}\,\Omega^o)$ be such that $\Delta v[0] = 0$ in Ω^o and $v[0]_{|\partial\Omega^o} = F^o[0]$. Then the following statements are equivalent.

(i) *For all* $\Omega_M \subseteq \Omega^o$ *open and such that* $0 \notin \mathrm{cl}\,\Omega_M$ *and all* $\epsilon_M \in]0, \epsilon_0[$ *such that* $\mathrm{cl}\,\Omega_M \cap \epsilon\,\mathrm{cl}\,\Omega^i = \emptyset$ *for all* $\epsilon \in]-\epsilon_M, \epsilon_M[$*, there exists a real analytic map* V_M *from* $]-\epsilon_M, \epsilon_M[$ *to* $C^{1,\alpha}(\mathrm{cl}\,\Omega_M)$ *such that*

$$v[\epsilon]_{|\mathrm{cl}\,\Omega_M} = V_M[\epsilon] \quad \forall \epsilon \in]-\epsilon_M, \epsilon_M[.$$

(ii) *There exist* $x^o \in \Omega^o \setminus \{0\}$, $\epsilon^o \in]0, \epsilon_0[$*, and a real analytic function* V^o *from* $]-\epsilon^o, \epsilon^o[$ *to* $\mathbb{R}$ *such that* $x^o \in \Omega(\epsilon)$ *for all* $\epsilon \in]-\epsilon^o, \epsilon^o[$ *and*

$$v[\epsilon](x^o) = V^o[\epsilon] \quad \forall \epsilon \in]0, \epsilon^o[.$$

(iii) $\int_{\partial\Omega^o} F^o[\epsilon]\rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} F^i[\epsilon]\rho^i[\epsilon]\,d\sigma = 0$ *for all* $\epsilon \in]-\epsilon_0, \epsilon_0[$.

Proof. Clearly (i) implies (ii). The proof that (iii) implies (i) can be effected by arguing as in the proof of Theorem 3.1 with G^o and G^i replaced by F^o and F^i and by noting that the last term in the right hand side of the equality corresponding to (17) is identically zero by condition (iii) (see Remark 3.2). To complete the proof we show that (ii) implies (iii).

Assume that (ii) holds. Set

$$A^o[\epsilon] \equiv \int_{\partial\Omega^o} F^o[\epsilon]\rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} F^i[\epsilon]\rho^i[\epsilon]\,d\sigma,$$

$$B^o[\epsilon] \equiv v\big[\partial\Omega^o, \rho^o[\epsilon]\big](x^o) + \int_{\partial\Omega^i} \rho^i[\epsilon] S(x^o - \epsilon y)\,d\sigma_y,$$

$$C^o[\epsilon] \equiv V^o[\epsilon] - w\big[\partial\Omega^o, \theta^o\big[\epsilon, F^o[\epsilon], F^i[\epsilon]\big]\big](x^o) - \epsilon \int_{\partial\Omega^i} \theta^i\big[\epsilon, F^o[\epsilon], F^i[\epsilon]\big](y)\nu_{\Omega^i}(y)\cdot\nabla S(x^o - \epsilon y)\,d\sigma_y,$$

$$D^o[\epsilon] \equiv \frac{1}{\int_{\partial\Omega^i} d\sigma} \int_{\partial\Omega^i} v\big[\partial\Omega^o, \rho^o[\epsilon]\big](\epsilon y) + v\big[\partial\Omega^i, \rho^i[\epsilon]\big](y)\,d\sigma_y, \quad \forall \epsilon \in]-\epsilon^o, \epsilon^o[.$$

Then, by classical mapping properties of layer potentials and by Propositions 2.8 and 2.16, A^o, B^o, C^o, and D^o are real analytic functions from $]-\epsilon^o, \epsilon^o[$ to $\mathbb{R}$ (cf. [12, §4]). Also, a straightforward calculation based on statement (ii) and Proposition 2.17 shows that

$$A^o[\epsilon]B^o[\epsilon] - C^o[\epsilon]D^o[\epsilon] = C^o[\epsilon]\frac{\log\epsilon}{2\pi} \quad \forall \epsilon \in]0, \epsilon^o[$$

(see also Remark 2.18). Thus, Lemma 3.5 implies that

$$A^o[\epsilon]B^o[\epsilon] - C^o[\epsilon]D^o[\epsilon] = 0 \quad \text{and} \quad C^o[\epsilon] = 0 \quad \forall \epsilon \in]-\epsilon^o, \epsilon^o[.$$

Hence $A^o[\epsilon]B^o[\epsilon] = 0$ and one deduces that

$$\left(\int_{\partial\Omega^o} F^o[\epsilon]\rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} F^i[\epsilon]\rho^i[\epsilon]\,d\sigma\right) v\big[\partial\Omega(\epsilon), \tau[\epsilon]\big](x^o) = 0$$

for all $\epsilon \in]-\epsilon^o, \epsilon^o[\setminus\{0\}$ (see also Definition 2.9). Let now $\epsilon \in]-\epsilon^o, \epsilon^o[\setminus\{0\}$. Since $v[\partial\Omega(\epsilon), \tau[\epsilon]]_{|\partial\Omega(\epsilon)} = c_\epsilon \chi_\epsilon$ with $c_\epsilon \in \mathbb{R}\setminus\{0\}$ (cf. Definition 2.2 and (13)), then the maximum principle implies that 0 is a maximum or minimum value for $v[\partial\Omega(\epsilon), \tau[\epsilon]]$ in $\mathrm{cl}\,\Omega(\epsilon)$ and $v[\partial\Omega(\epsilon), \tau[\epsilon]]$ can attain the value 0 only on the boundary of $\Omega(\epsilon)$. Hence $v[\partial\Omega(\epsilon), \tau[\epsilon]](x^o) \neq 0$ because x^o belongs to $\Omega(\epsilon)$. It follows that

$$\int_{\partial\Omega^o} F^o[\epsilon]\rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} F^i[\epsilon]\rho^i[\epsilon]\,d\sigma = 0 \quad \forall\epsilon \in \left]-\epsilon^o, \epsilon^o\right[\setminus \{0\}$$

and by continuity one deduces the validity of (iii). □

In the following Examples 3.7 and 3.8 we consider some simple cases for which we can obtain more explicit equivalent conditions for (i)–(iii) of Theorem 3.6. The following Example 3.7 concerns the case of ϵ-dependent boundary data which are constant on $\partial\Omega(\epsilon)$.

Example 3.7. Let the notation of Theorem 3.6 hold. Assume that there exist two real analytic functions c^o and c^i from $]-\epsilon_0, \epsilon_0[$ to $\mathbb{R}$ such that

$$F^o[\epsilon](x) = c^o[\epsilon] \quad \text{and} \quad F^i[\epsilon](y) = c^i[\epsilon] \quad \forall\epsilon \in \left]-\epsilon_0, \epsilon_0\right[,\ x \in \partial\Omega^o,\ y \in \partial\Omega^i.$$

Then $v[\epsilon]$ satisfies the (equivalent) conditions in (i)–(iii) if and only if $c^o[\epsilon] = c^i[\epsilon]$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$.

Proof. If $c^o = c^i$ then $v[\epsilon](x) = c^o[\epsilon]$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$ and all $x \in \operatorname{cl}\Omega(\epsilon)$. Then one immediately verifies the validity of (i), and accordingly of (ii),(iii) by Theorem 3.6. In particular, if $c^o[\epsilon] = c^i[\epsilon] = 1$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$, then

$$\int_{\partial\Omega^o} \rho^o[\epsilon]\,d\sigma + \int_{\partial\Omega^i} \rho^i[\epsilon]\,d\sigma = 0 \quad \forall\epsilon \in \left]-\epsilon_0, \epsilon_0\right[\tag{18}$$

by (iii). Now we prove that (i)–(iii) imply that $c^o = c^i$. Assume by contradiction that $c^o \neq c^i$ and $v[\epsilon]$ satisfies the condition in (iii). Then there exists $\epsilon_* \in]-\epsilon_0, \epsilon_0[$ such that $c^o[\epsilon_*] \neq c^i[\epsilon_*]$ and

$$c^o[\epsilon_*] \int_{\partial\Omega^o} \rho^o[\epsilon_*]\,d\sigma + c^i[\epsilon_*] \int_{\partial\Omega^i} \rho^i[\epsilon_*]\,d\sigma = 0. \tag{19}$$

But then, equalities (18) (which does not depend on c^o, c^i) and (19) imply that $\int_{\partial\Omega^i} \rho^i[\epsilon_*]\,d\sigma = 0$. A contradiction, because $\int_{\partial\Omega^i} \rho^i[\epsilon]\,d\sigma = 1$ for all $\epsilon \in]-\epsilon_0, \epsilon_0[$ (cf. Propositions 2.4, 2.5, and 2.6). □

In Example 3.8 here below we consider the case where the domain $\Omega(\epsilon)$ is a circular annulus.

Example 3.8. Let the notation of Theorem 3.6 hold. Assume that $\Omega^o = \Omega^i = \mathbb{B}_2$. Let $\Omega(\epsilon) = \mathbb{B}_2 \setminus \epsilon \operatorname{cl}\mathbb{B}_2$ for all $\epsilon \in]-1, 1[$. Then $v[\epsilon]$ satisfies the (equivalent) conditions in (i)–(iii) if and only if

$$\int_{\partial\mathbb{B}_2} F^o[\epsilon]\,d\sigma = \int_{\partial\mathbb{B}_2} F^i[\epsilon]\,d\sigma \quad \forall\epsilon \in \left]-\epsilon_0, \epsilon_0\right[. \tag{20}$$

Proof. By Propositions 2.4–2.6 and by a standard symmetry argument one verifies that $\rho^o[\epsilon](x) = \rho^o[\epsilon](Tx)$ and $\rho^o[\epsilon](x) = \rho^o[\epsilon](Tx)$ for all $x \in \partial\Omega^o = \partial\Omega^o = \partial\mathbb{B}_2$ and for all orthogonal transformation T on $\mathbb{R}^2$. It follows that $\rho^o[\epsilon]$ and $\rho^o[\epsilon]$ are constant functions on $\partial\mathbb{B}_2$. Then, by equalities $\int_{\partial\mathbb{B}_2} \rho^i[\epsilon]\,d\sigma = 1$ (cf. Propositions 2.4–2.6) and (18) one deduces that

$$\rho^o[\epsilon](x) = -\frac{1}{2\pi} \quad \text{and} \quad \rho^i[\epsilon](x) = \frac{1}{2\pi} \quad \forall x \in \partial\mathbb{B}_2,\ \epsilon \in \left]-1, 1\right[.$$

Hence, the conditions in (iii) of Theorem 3.6 and in (20) are equivalent. □

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Uncited references

[9] [11]

References

[1] M. Dalla Riva, P. Musolino, Real analytic families of harmonic functions in a domain with a small hole, J. Differential Equations 252 (2012) 6337–6355.
[2] M. Dalla Riva, P. Musolino, S.V. Rogosin, Series expansions for the solution of the Dirichlet problem in a planar domain with a small hole, submitted for publication.
[3] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.
[4] G.B. Folland, Introduction to Partial Differential Equations, Princeton University Press, Princeton, NJ, 1995.
[5] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 2001.
[6] S.G. Krantz, H.R. Parks, A Primer of Real Analytic Functions, Birkhäuser Verlag, Basel, 1992.
[7] R. Kress, Linear Integral Equations, Springer-Verlag, New York, 1999.
[8] M. Lanza de Cristoforis, Perturbation problems in potential theory, a functional analytic approach, J. Appl. Funct. Anal. 2 (2007) 197–222.
[9] M. Lanza de Cristoforis, Asymptotic behavior of the solutions of a nonlinear Robin problem for the Laplace operator in a domain with a small hole: a functional analytic approach, Complex Var. Elliptic Equ. 52 (2007) 945–977.
[10] M. Lanza de Cristoforis, Asymptotic behavior of the solutions of the Dirichlet problem for the Laplace operator in a domain with a small hole. A functional analytic approach, Analysis (Munich) 28 (2008) 63–93.
[11] M. Lanza de Cristoforis, Asymptotic behaviour of the solutions of a non-linear transmission problem for the Laplace operator in a domain with a small hole. A functional analytic approach, Complex Var. Elliptic Equ. 55 (2010) 269–303.
[12] M. Lanza de Cristoforis, P. Musolino, A real analyticity result for a nonlinear integral operator, J. Integral Equations Appl. 25 (2013) 21–46.
[13] V. Maz'ya, S. Nazarov, B. Plamenevskij, Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains, vol. I, Birkhäuser Verlag, Basel, 2000.
[14] J. Schauder, Potentialtheoretische Untersuchungen, Math. Z. 33 (1931) 602–640.
[15] J. Schauder, Bemerkung zu meiner Arbeit "Potentialtheoretische Untersuchungen I (Anhang)", Math. Z. 35 (1932) 536–538.

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