Alternating Projections Methods for Discrete-time Stabilization of Quantum States

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***Abstract*—We study sequences (both cyclic and randomized) of idempotent completely-positive trace-preserving quantum maps, and show how they asymptotically converge to the intersection of their fixed point sets via alternating projection methods, highlighting the robustness features of the protocol against randomization. The general results are then specialized to stabilizing entangled states in finite-dimensional multipartite quantum systems subject to locality constraints, a problem of key interest for quantum information applications.**

I. Introduction

Driving a quantum system to a desired state is a prerequisite for quantum control applications ranging from quantum chemistry to quantum computation [1]. Many methods for *state preparation*, a control task where an all-to-one transition towards the target state is required, rely on a "fixed" dissipative mechanism to first prepare a known (but not yet the target) pure state independently of the initial condition, followed by unitary control implementing a one-to-one transition [2]. In this spirit, in the circuit model of quantum computation [3], preparation of arbitrary pure states is attained by initializing the quantum register in a known factorized pure state, and then implementing a sequence of unitary transformations ("quantum gates") drawn from a universal set. Additional possibilities for state preparation arise if the target system is allowed to couple to an auxiliary quantum system, so that the pair can be jointly initialized and controlled, and the ancilla reset or traced over [4]. For example, sequential unitary coupling to an ancilla may be used to design a sequence of non-unitary transformations ("quantum channels") on a multi-qubit system, that dissipatively prepare it in a matrix product state [5].

A more powerful setting is to allow *dissipative control design* from the outset [6], [4]. This opens up the possibility to synthesize all-to-one open-system dynamics that not only prepare the target state of interest but, additionally, leave it *invariant* throughout – that is, achieves *stabilization*, which is the task we focus on in this work. Quantum state stabilization has been theoretically investigated from different perspectives, including feedback design with classical [7], [8], [9], [10], [11], [12] and quantum [13], [14], [15], [16] controllers, as well as open-loop reservoir engineering techniques with both time-independent dynamics and switching control [17], [18], [19], [20], [21], [22], [23]. Most of this research effort, however, has focused on continuous-time models, with fewer studies addressing *discrete-time* quantum dynamics. With "digital" open-system quantum simulators being now experimentally accessible [24], [25], investigating quantum stabilization problems in discrete time becomes both natural and important. Thanks to the invariance requirement, stabilizing pure or mixed target states using "dissipative quantum circuits" brings distinctive advantages for *on-demand* state preparation: (i) repeating a stabilizing protocol or even portions of it, will further maintain the system in the target state (if so desired), without disruption; (ii) The order of the applied control operations need no longer be crucial, allowing for the target state to still be reached probabilistically (in a suitable sense); and, (iii) if at a certain instant a wrong map is implemented, or some transient noise perturbs the dynamics, these unwanted effects can be reabsorbed without requiring active intervention or the whole preparation protocol having to be re-implemented correctly.

Discrete-time quantum Markov dynamics are described by sequences of quantum channels, namely, completely-positive, trace-preserving (CPTP) maps [26]. This give rise to a rich stability theory that can be seen as the non-commutative generalization of the asymptotic analysis of classical Markov chains, and that thus far has being studied in depth only in the time-homogeneous case [27], including elementary feedback stabilizability and reachability problems [28], [29].

In this work, we show that *time-dependent* sequences of CPTP maps can be used to make their common fixed states the minimal asymptotically stable sets, which are reached by iterating cyclically a finite subsequence. The methods we introduce employ a finite number of idempotent CPTP maps, which we call *CPTP projections*, and can be considered a quantum version of *alternated projections methods*. The latter, stemming from seminal results by von Neumann [30] and extended by Halperin [31] and others [32], [33], are a family of (classical) algorithms that, loosely speaking, aim to select an element in the intersection of a number of sets that minimizes a natural (quadratic) distance with respect to the input. The numerous applications of such classical algorithms include estimation [34] and control [35] and, recently, specific tasks in quantum information, such as quantum channel construction [36]. In the context of quantum stabilization, we show that instead of working with the standard (Hilbert-Schmidt) inner

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product, it is natural to resort to a different inner product, a *weighted inner product* for which the CPTP projections become orthogonal, and the original results apply. When, depending on the structure of the fixed-point set, this strategy is not viable, we establish convergence by a different proof that does not directly build on existing alternating projection theorems. For all the proposed sequences, the order of implementation is not crucial, and convergence in probability is guaranteed even when the sequence is randomized, under very mild hypotheses on the distribution. As an application, we specialize these results to distributed stabilization of entangled states on multipartite quantum systems, where the robustness properties imply that the target can be reached by *unsupervised randomized applications* of dissipative quantum maps.

II. Preliminary material

A. Models and stability notions

We consider a finite-dimensional quantum system, associated to a Hilbert space $\mathcal{H} \approx \mathbb{C}^d$. Let $\mathcal{B}(\mathcal{H})$ denote the space of linear bounded operators on $\mathcal{H}$, with $\dagger$ being the adjoint operation. We are concerned with discrete-time evolution, indexed by $t \in \mathbb{N}_+$. The state of the system at each time $t \geq 0$ is a *density matrix* in $\mathfrak{D}(\mathcal{H})$, namely a positive-semidefinite, trace one matrix. Let ρ_0 be the initial state. We consider *time-inhomogenous* Markov dynamics, namely, sequences of CPTP maps $\{\mathcal{E}_t\}$, defining the state evolution through the dynamical equation: $\rho_{t+1} = \mathcal{E}_t(\rho_t), \quad t \geq 0.$

Recall that a linear map $\mathcal{E}$ is CPTP if and only if it admits an operator-sum representation (OSR) [26]: $\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger$, where the (Hellwig-Kraus) operators $\{M_k\} \subset \mathcal{B}(\mathcal{H})$ satisfy $\sum_k M_k^\dagger M_k = I$. We shall assume that for all $t > 0$ the map $\mathcal{E}_t = \mathcal{E}_{j(t)}$ is chosen from a set of "available" maps, to be designed within the available control capabilities. In particular, in Section IV we will focus on locality-constrained dynamics. For any $t \geq s \geq 0$, we shall denote by

$$\mathcal{E}_{t,s} \equiv \mathcal{E}_{t-1} \circ \mathcal{E}_{t-2} \circ \ldots \circ \mathcal{E}_s, \quad (\mathcal{E}_{t,t} = \mathcal{I}),$$

the evolution map, or "propagator", from s to t. Define the distance of an operator ρ from a set $\mathcal{S}$ as $d(\rho, \mathcal{S}) \equiv \inf_{\tau \in \mathcal{S}} \|\rho - \tau\|_1$, with $\|\cdot\|_1$ being the trace norm. A set $\mathcal{S}$ is *invariant* for the dynamics if $\mathcal{E}_{t,s}(\tau) \in \mathcal{S}$ for all $\tau \in \mathcal{S}$. An invariant set $\mathcal{S}$ is (uniformly) *simply stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\tau, \mathcal{S}) < \delta$ ensures $d(\mathcal{E}_{t,s}(\tau), \mathcal{S}) < \varepsilon$ for all $t \geq s \geq 0$. An invariant set $\mathcal{S}$ is *globally asymptotically stable* (GAS) if it is simply stable and

$$\lim_{t \to \infty} d(\mathcal{E}_{t,s}(\rho), \mathcal{S}) = 0, \quad \forall \rho, \ s \geq 0. \tag{1}$$

Notice that, since we are dealing with finite-dimensional systems, convergence in any matrix norm is equivalent. Furthermore, since CPTP maps are trace-norm contractions [3], we have that simple stability is always guaranteed (and actually the distance is monotonically non-increasing):

Proposition 1: If a set $\mathcal{S}$ is invariant for the dynamics $\{\mathcal{E}_{t,s}\}_{t,s \geq 0}$, then it is simply stable.

Proof: We have, for all $t, s \geq 0$:

$$\begin{aligned} d(\mathcal{E}_{t,s}(\rho), \mathcal{S}) &\leq d(\mathcal{E}_{t,s}(\rho), \mathcal{E}_{t,s}(\tau^*_{t,s})) \\ &\leq d(\rho, \tau^*_{t,s}) = d(\rho, \mathcal{S}). \end{aligned}$$

The first inequality is true, by definition, for all $\tau_{t,s} \in \mathcal{S}$, and also on the closure $\bar{\mathcal{S}}$, thanks to continuity of $\mathcal{E}_{t,s}$; the second holds due to contractivity of $\mathcal{E}$, and the last equality follows by letting $\tau^*_{t,s} \equiv \arg\min_{\tau \in \bar{\mathcal{S}}} \|\rho - \tau\|_1$, where we can take the min since $\bar{\mathcal{S}}$ is closed and compact. □

B. Fixed points of CP maps

We collect in this section some relevant facts on the structure of fixed-point sets $\mathrm{fix}(\mathcal{E})$ for a CP map $\mathcal{E}$. More details can be found e.g. in [37], [38], [39].

Let $\mathrm{alg}(\mathcal{E})$ denote the $\dagger$-closed algebra generated by the operators in the OSR of $\mathcal{E}$, and $\mathcal{A}'$ denote the commutant of $\mathcal{A}$, namely the set of operators which commute with all the elements of $\mathcal{A}$. For *unital* CP maps, $\mathrm{fix}(\mathcal{E})$ is a $\dagger$-closed algebra, $\mathrm{fix}(\mathcal{E}) = \mathrm{alg}(\mathcal{E})' = \mathrm{fix}(\mathcal{E}^\dagger)$ [39]. This implies that it admits a (Wedderburn) block decomposition [40]:

$$\mathrm{fix}(\mathcal{E}) = \bigoplus_\ell \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes I_{F,\ell}, \tag{2}$$

with respect to a Hilbert space decomposition:

$$\mathcal{H} = \bigoplus_\ell \mathcal{H}_{S,\ell} \otimes \mathcal{H}_{F,\ell}.$$

For a *general* (not necessarily unital) CPTP map, it is possible to show [39], [37] that the fixed-point set has a related structure. Given a CPTP map $\mathcal{E}$, and a maximal-rank fixed point ρ with $\tilde{\mathcal{H}} \equiv \mathrm{supp}(\rho)$, let $\tilde{\mathcal{E}}$ denote the reduction of $\mathcal{E}$ to $\mathcal{B}(\tilde{\mathcal{H}})$. Then, $\tilde{\mathcal{E}}$ is CPTP on its support, $\tilde{\mathcal{E}}^\dagger$ is unital and:

$$\mathrm{fix}(\mathcal{E}) = \rho^{\frac{1}{2}} \left(\ker(\tilde{\mathcal{E}}^\dagger) \oplus \mathbb{O}\right) \rho^{\frac{1}{2}}, \tag{3}$$

where $\mathbb{O}$ is the zero operator on the complement of $\tilde{\mathcal{H}}$. Moreover, with respect to the decomposition of $\mathrm{fix}(\tilde{\mathcal{E}}^\dagger) = \bigoplus_\ell \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes I_{F,\ell}$, any maximal-rank fixed state has the structure:

$$\rho = \bigoplus_\ell p_\ell \rho_{S,\ell} \otimes \tau_{F,\ell}, \tag{4}$$

where $\rho_{S,\ell}$ and $\tau_{F,\ell}$ are *full-rank* density operators of appropriate dimension, and p_ℓ a set of convex weights.

Given a CPTP map admitting a full-rank invariant state ρ, by using (4) and (2) in (3), the fixed-point sets $\mathrm{fix}(\mathcal{E})$ is a ρ-*distorted algebra*, namely, an associative algebra with respect to a modified product (i.e. $X \times_\rho Y = X\rho^{-1}Y$), with structure

$$\mathcal{A}_\rho = \bigoplus_\ell \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_{F,\ell}, \tag{5}$$

where $\tau_{F,\ell}$ are a set of density operators of appropriate dimension (the same for every element in $\mathrm{fix}(\mathcal{E})$).

In addition, since ρ has the same block structure (4), $\mathrm{fix}(\mathcal{E})$ is clearly invariant with respect to the action of the linear map $\mathcal{M}_{\rho,\lambda}(X) \equiv \rho^\lambda X \rho^{-\lambda}$ for any $\lambda \in \mathbb{C}$. The same holds for the fixed points of the dual dynamics. In fact, using a finite-dimensional version of Takesaki's theorem [38], it has been proved in [37] that commutativity with $\mathcal{M}_{\rho,1/2}$ is actually *sufficient* to ensure that a distorted algebra is a valid fixed-point set. More precisely:

Theorem 1 (Existence of ρ-preserving dynamics): Let ρ be a full-rank density operator and $\mathcal{A}_\rho$ a distorted algebra such that $\rho \in \mathcal{A}_\rho$. Then there exists a CPTP map $\mathcal{E}$ such that $\mathrm{fix}(\mathcal{E}) = \mathcal{A}_\rho$ if and only if $\mathcal{A}_\rho$ is invariant for $\mathcal{M}_{\rho,\frac{1}{2}}$.

III. Alternating projection methods

A. von Neumann-Halperin Theorem

Many of the ideas we use in this paper are inspired by a classical result originally due to von Neumann [30], and later extended by Halperin to multiple projectors:

Theorem 2 (von Neumann-Halperin alternating projections): If $\mathcal{M}_1,\ldots,\mathcal{M}_r$ are closed subspaces in a Hilbert space $\mathcal{H}$, and $P_{\mathcal{M}_j}$ are the corresponding orthogonal projections, then

$$\lim_{n\to\infty}(P_{\mathcal{M}_1}...P_{\mathcal{M}_r})^n x = Px, \quad \forall x \in \mathcal{H},$$

where P is the orthogonal projection onto $\bigcap_{i=1}^r \mathcal{M}_i$.

A proof for this theorem can be found in Halperin's original work [31]. Since then, the result has been refined in many ways, has inspired similar convergence results that use information projections [41] and, in full generality, projections in the sense of Bregman divergences [42], [32]. The applications of the results are manifold, especially in algorithms: while it is beyond the scope of this work to attempt a review, a good collection is presented in [33]. Some bounds on the convergence rate for the alternating projection methods can be derived by looking at the *angles between the subspaces* we are projecting on, see again [33] for more details.

B. CPTP projections and orthogonality

We call an idempotent CPTP map, namely, one that satisfies $\mathcal{E}^2 = \mathcal{E}$, a *CPTP projection.* As any linear idempotent map, $\mathcal{E}$ has only $0, 1$ eigenvalues and maps any operator X onto the set of its fixed points, $\mathrm{fix}(\mathcal{E})$. Recall that

$$\mathrm{fix}(\mathcal{E}) = \bigoplus_\ell [\mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_{F,\ell}] \oplus \mathbb{0}_R, \tag{6}$$

for some Hilbert-space decomposition:

$$\mathcal{H} = \bigoplus_\ell (\mathcal{H}_{S,\ell} \otimes \mathcal{H}_{F,\ell}) \oplus \mathcal{H}_R, \tag{7}$$

where the last zero-block is not present if there exists a $\rho > 0$ in $\mathrm{fix}(\mathcal{E})$. We next give the structure of the CPTP projection associated to $\mathrm{fix}(\mathcal{E})$: it is known (see e.g. [38]) that given a CPTP map $\mathcal{E}$ with ρ a fixed point of maximal rank, a CPTP projection onto $\mathcal{A}_\rho = \mathrm{fix}(\mathcal{E})$ exists and is given by

$$\mathcal{E}_{\mathcal{A}_\rho}(X) = \lim_{T\to+\infty} \frac{1}{T} \sum_{i=0}^{T-1} \mathcal{E}^i(X). \tag{8}$$

If the fixed point ρ is full rank, then the CPTP projection onto $\mathcal{A}_\rho = \oplus_\ell \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_{F,\ell}$ is equivalently given by

$$\mathcal{E}_{\mathcal{A}_\rho}(X) = \bigoplus_\ell \mathrm{Tr}_{F,\ell}(\Pi_{SF,\ell} X \,\Pi_{SF,\ell}) \otimes \tau_{F,\ell}, \tag{9}$$

where $\Pi_{SF,\ell}$ is the orthogonal projection from $\mathcal{H}$ onto the subspace $\mathcal{H}_{S,\ell} \otimes \mathcal{H}_{F,\ell}$.

For a full-rank fixed-point set, CPTP projections are *not* orthogonal projections onto $\mathrm{fix}(\mathcal{E})$, at least with respect to the Hilbert-Schmidt inner product, unless they are unital – a proof is provided in the Appendix.

We are nonetheless going to show that $\mathcal{E}_\mathcal{A}$ *is* an orthogonal projection with respect to a different inner product. This proves that the map in Eq. (9) is the unique CPTP projection onto $\mathcal{A}_\rho$. If the fixed-point set does not contains a full-rank state, Eq. (8) still defines a valid CPTP projection onto $\mathrm{fix}(\mathcal{E})$; however, this need not be unique. We will exploit this fact in the proof of Theorem 3, where we choose a particular one.

Definition 1: Let ξ be a positive-definite operator. Define: (i) the *ξ-inner product* as

$$\langle X, Y\rangle_\xi \equiv \mathrm{Tr}(X\xi Y); \tag{10}$$

(ii) the *symmetric ξ-inner product* as

$$\langle X, Y\rangle_{\xi,s} \equiv \mathrm{Tr}(X\xi^{\frac{1}{2}} Y \xi^{\frac{1}{2}}). \tag{11}$$

It is straightforward to verify that both (10) and (11) are valid inner products.

We next show that $\mathcal{E}_\mathcal{A}$ is an orthogonal projection with respect to (10) and (11), when $\xi = \rho^{-1}$ for a full rank fixed point ρ. We will need a preliminary lemma. With $W \equiv \bigoplus W_i$ we will denote an operator that acts as W_i on $\mathcal{H}_i$, for a direct-sum decomposition of $\mathcal{H} = \bigoplus_i \mathcal{H}_i$.

Lemma 1: Consider $Y, W \in \mathcal{B}(\mathcal{H})$, where W admits an orthogonal block-diagonal representation $W = \bigoplus_\ell W_\ell$. Then $\mathrm{Tr}(WY) = \sum_\ell \mathrm{Tr}(W_\ell Y_\ell)$, where $Y_\ell = \Pi_\ell Y \Pi_\ell$.

Proof: Let Π_ℓ be the projector onto $\mathcal{H}_\ell$. Remembering that $\sum_\ell \Pi_\ell = I$ and $\Pi_\ell = \Pi_\ell^2$, it follows that

$$\mathrm{Tr}(X) = \sum_\ell \mathrm{Tr}(\Pi_\ell X) = \sum_\ell \mathrm{Tr}(\Pi_\ell X \Pi_\ell).$$

Therefore, we obtain:

$$\begin{aligned}\mathrm{Tr}(WY) &= \mathrm{Tr}(\sum_\ell \Pi_\ell \bigoplus_j W_j Y) = \sum_\ell \mathrm{Tr}(\Pi_\ell W_\ell Y) \\ &= \sum_\ell \mathrm{Tr}(\Pi_\ell W_\ell \Pi_\ell Y) = \sum_\ell \mathrm{Tr}(W_\ell \Pi_\ell Y \Pi_\ell) \\ &= \sum_\ell \mathrm{Tr}(W_\ell Y_\ell). \end{aligned}$$

□

Proposition 2: Let $\xi = \rho^{-1}$, where ρ is a full-rank fixed state in $\mathcal{A}_\rho$, which is invariant for $\mathcal{M}_{\rho,\frac{1}{2}}$. Then $\mathcal{E}_{\mathcal{A}_\rho}$ is an orthogonal projection with respect to the inner products in (10) and (11).

Proof: We already know that $\mathcal{E}$ is linear and idempotent. In order to show that $\mathcal{E}$ is an orthogonal projection, we need to show that it is self-adjoint relative to the relevant inner product. Let us consider $\rho = \bigoplus \rho_\ell \otimes \tau_\ell$ and, as above:

$$\begin{aligned} X_\ell = &\ \Pi_{SF,\ell} X \,\Pi_{SF,\ell} = \textstyle\sum_k A_{k,\ell} \otimes B_{k,\ell}, \\ Y_\ell = &\ \Pi_{SF,\ell} Y \,\Pi_{SF,\ell} = \textstyle\sum_j C_{j,\ell} \otimes D_{j,\ell}. \end{aligned}$$

If we apply Lemma 1 to the operator

$$W = \mathcal{E}_{\mathcal{A}_\rho}(X)\rho^{-1} = \bigoplus_\ell ([\mathrm{Tr}_{F,\ell}(X_\ell) \otimes \tau_\ell](\rho_\ell^{-1} \otimes \tau_\ell^{-1})),$$

we obtain:

$$\begin{aligned} \langle \mathcal{E}(X), Y\rangle_\xi &= \mathrm{Tr}(\mathcal{E}_{\mathcal{A}_\rho}(X)\rho^{-1} Y) \\ &= \mathrm{Tr}(\bigoplus_\ell \mathrm{Tr}_{F,\ell}(X_\ell) \otimes \tau_\ell(\rho_\ell^{-1} \otimes \tau_\ell^{-1}) Y_\ell) \\ &= \sum_{\ell,k,j} \mathrm{Tr}([A_{k,\ell}\mathrm{Tr}(B_{k,\ell})\rho_\ell^{-1} \otimes I][C_{j,\ell} \otimes D_{j,\ell}]) \\ &= \sum_{\ell,k,j} \mathrm{Tr}(B_{k,\ell})\mathrm{Tr}(A_{k,\ell}\rho_\ell^{-1} C_{j,\ell})\mathrm{Tr}(D_{j,\ell}). \end{aligned}$$

By similar calculation,

$$\begin{aligned}\langle X, \mathcal{E}(Y)\rangle_\xi &= \mathrm{Tr}(X\rho^{-1}\mathcal{E}(Y))\\ &= \sum_{\ell,k,j} \mathrm{Tr}(B_{k,\ell})\mathrm{Tr}(A_{k,\ell}\rho_\ell^{-1}C_{j,\ell})\mathrm{Tr}(D_{j,\ell}).\end{aligned}$$

By comparison, we infer that $\langle \mathcal{E}(X), Y\rangle_\xi = \langle X, \mathcal{E}(Y)\rangle_\xi$. A similar proof can be carried over using the symmetric ξ-inner product of Eq. (11). □

We are now ready to prove the main results of this section. The first shows that the set of states with support on a target subspace can be made GAS by sequences of CPTP projections on larger subspaces that have the target as intersection.

Theorem 3 (Subspace stabilization): Let $\mathcal{H}_j$, $j = 1, \dots, r$, be subspaces such that $\bigcap_j \mathcal{H}_j \equiv \hat{\mathcal{H}}$. Then there exists CPTP projections $\mathcal{E}_1, \dots, \mathcal{E}_r$ onto $\mathcal{B}(\mathcal{H}_j)$, $j = 1, \dots, r$, such that $\forall \tau \in \mathcal{D}(\mathcal{H})$:

$$\lim_{n\to\infty} (\mathcal{E}_r \dots \mathcal{E}_1)^n(\tau) = \mathcal{E}_{\mathcal{B}(\hat{\mathcal{H}})}(\tau), \tag{12}$$

where $\mathcal{E}_{\mathcal{B}(\hat{\mathcal{H}})}$ is a CPTP projection onto $\mathcal{B}(\hat{\mathcal{H}})$.

Proof: We shall explicitly construct CPTP maps whose cyclic application ensures stabilization. Define P_j to be the projector onto $\mathcal{H}_j$, and the map:

$$\mathcal{E}_j(\cdot) \equiv P_j(\cdot)P_j + \frac{P_j \,\mathrm{Tr}\,(P_j^\perp(\cdot)P_j^\perp)P_j}{\mathrm{Tr}(P_j)}. \tag{13}$$

The latter is CP as it is obtained as sum and concatenation of CP maps, as the trace $\mathrm{Tr}(\cdot) = \sum_k \langle k|\cdot|k\rangle$, $\{|k\rangle\}$ being an orthonormal basis, is CP, and it can be verified to be TP by simple calculations. Consider $\hat{P}$ the orthogonal projection onto $\hat{\mathcal{H}}$ and the positive-semidefinite function $V(\tau) = 1 - \mathrm{Tr}(\hat{P}\,\tau)$, $\tau \in \mathcal{B}(\mathcal{H})$. The variation of V, when a $\mathcal{E}_j$ is applied, is

$$\Delta V(\tau) \equiv V(\mathcal{E}_j(\tau)) - V(\tau) = -\mathrm{Tr}[\hat{P}\,(\mathcal{E}_j(\tau) - \tau)] \equiv \Delta V_j(\tau).$$

If we show that this function is non-increasing along the trajectories generated by repetitions of the cycle of all maps, namely, $\mathcal{E}_{\rm cycle} \equiv \mathcal{E}_r \circ \dots \circ \mathcal{E}_1$, the system is periodic thus its stability can be studied as a time-invariant one. Hence, by LaSalle-Krasowskii theorem [43], the trajectories (being all bounded) will converge to the largest invariant set contained in the set of τ such that on a cycle $\Delta V_{\rm cycle}(\tau) = 0$. We next show that this set must have support *only* on $\hat{\mathcal{H}}$. If an operator ρ has support on $\hat{\mathcal{H}}$, it is clearly invariant and $\Delta V(\rho) = 0$. Assume now that $\mathrm{supp}(\tau) \nsubseteq \mathcal{H}_j$ for some j, that is, $\mathrm{Tr}(\tau P_j^\perp) > 0$. By using the form of the map $\mathcal{E}_j$ given in Eq. (13), we have

$$\Delta V_j(\tau) = -\mathrm{Tr}(\hat{P}(P_j\tau P_j)) - \mathrm{Tr}(\tau P_j^\perp)\frac{\mathrm{Tr}(\hat{P}(P_j))}{\mathrm{Tr}\,(P_j)} + \mathrm{Tr}(\hat{P}\tau)$$

The sum of the first and the third term in the above equation is zero since $\hat{P} \leq P_j$,. The second term, on the other hand, is *strictly negative*. This is because: (i) we assumed that $\mathrm{Tr}(\tau P_j^\perp) > 0$; (ii) with $\hat{P} \leq P_j$, and $\mathcal{E}_j(P_j)$ having the same support of P_j by construction, it also follows that $\mathrm{Tr}\,(\Pi\mathcal{E}_j(P_j)) > 0$. This implies that $\mathcal{E}_j$ either leaves τ (and hence $V(\tau)$) invariant, or $\Delta V_j(\rho) < 0$. Hence, each cycle $\mathcal{E}_{\rm cycle}$ is such that $\Delta V_{\rm cycle}(\tau) = \sum_{j=1}^r \Delta V_j(\tau) < 0$ for all $\tau \notin \mathfrak{D}(\hat{\mathcal{H}})$. We thus showed that no state τ with support outside of $\hat{\mathcal{H}}$ can be in the attractive set for the dynamics. Hence, the dynamics asymptotically converges onto $\mathfrak{D}(\hat{\mathcal{H}})$ which is the only invariant set for all the $\mathcal{E}_j$. □

The second result shows that a similar property holds for more general fixed-point sets, as long as they contain a full-rank state:

Theorem 4 (Full-rank fixed-set stabilization): Let the maps $\mathcal{E}_1, \dots, \mathcal{E}_r$ be CPTP projections onto $\mathcal{A}_i$, $i = 1, \dots, r$, and assume that $\hat{\mathcal{A}} \equiv \bigcap_{i=1}^r \mathcal{A}_i$ contains a full-rank state ρ. Then $\forall \tau \in \mathcal{D}(\mathcal{H})$:

$$\lim_{n\to\infty} (\mathcal{E}_r \dots \mathcal{E}_1)^n(\tau) = \mathcal{E}_{\hat{\mathcal{A}}}(\tau), \tag{14}$$

where $\mathcal{E}_{\hat{\mathcal{A}}}$ is the CPTP projection onto $\hat{\mathcal{A}}$.

Proof: Let us consider $\xi = \rho^{-1}$; then $\rho \in \hat{\mathcal{A}}$ implies that the maps $\hat{\mathcal{E}}_i$ are all orthogonal projections with respect to the *same* ρ^{-1}-modified inner product. Hence, it suffice to apply von Neumann-Halperin, Theorem 2: asymptotically, the cyclic application of orthogonal projections onto subsets converges to the projection onto the intersection of the subsets; in our case, the latter is $\hat{\mathcal{A}}$. □

Together with Theorem 1, the above result implies that the intersection of fixed-point sets is still a fixed-point set of *some* map, as long as it contains a full-rank state:

Corollary 1: If $\mathcal{A}_i$, $i = 1, \dots, r$, are ρ-distorted algebras, with ρ full rank, and are invariant for $\mathcal{M}_{\rho,\frac{1}{2}}$, then $\hat{\mathcal{A}} = \bigcap_{i=1}^r \mathcal{A}_i$ is also a ρ-distorted algebra, invariant for $\mathcal{M}_{\rho,\frac{1}{2}}$.

Proof: $\hat{\mathcal{A}}$ contains ρ and the previous Theorem ensures that a CPTP projection onto it exists. Then by Theorem 1 it is invariant for $\mathcal{M}_{\rho,\frac{1}{2}}$. □

Lastly, combining the ideas of the proof of Theorem 3 and 4, we obtain *sufficient* conditions for general fixed-point sets.

Theorem 5 (General fixed-point set stabilization): Assume that the CPTP fixed-point sets $\mathcal{A}_i$, $i = 1, \dots, r$, are such that $\hat{\mathcal{A}} \equiv \bigcap_{i=1}^r \mathcal{A}_i$ satisfies

$$\mathrm{supp}(\hat{\mathcal{A}}) = \bigcap_{i=1}^r \mathrm{supp}(\mathcal{A}_i).$$

Then there exist CPTP projections $\mathcal{E}_1, \dots, \mathcal{E}_r$ onto $\mathcal{A}_i$, $i = 1, \dots, r$, such that $\forall \tau \in \mathcal{D}(\mathcal{H})$:

$$\lim_{n\to\infty} (\mathcal{E}_r \dots \mathcal{E}_1)^n(\tau) = \mathcal{E}_{\hat{\mathcal{A}}}(\tau), \tag{15}$$

where $\mathcal{E}$ is a CPTP projection onto $\hat{\mathcal{A}}$.

Proof: To prove the claim, we explicitly construct the maps combining the ideas from the two previous theorems. Define P_j to be the projector onto $\mathrm{supp}(\mathcal{A}_j)$, and the maps

$$\mathcal{E}_j^0(\cdot) \equiv P_j(\cdot)P_j + \frac{P_j \mathrm{Tr}\,(P_j^\perp \cdot P_j^\perp)P_j}{\mathrm{Tr}(P_j)}, \quad \mathcal{E}_j^1 \equiv \mathcal{E}_{\mathcal{A}_j} \oplus \mathcal{I}_{\mathcal{A}_j^\perp}, \tag{16}$$

where the first is a CPTP map similar to (13), and $\mathcal{E}_{\mathcal{A}_j} : \mathcal{B}(\mathrm{supp}(\mathcal{A}_j)) \to \mathcal{B}(\mathrm{supp}(\mathcal{A}_j))$ is the unique CPTP projection onto $\mathcal{A}_j$ (notice that on its own support $\mathcal{A}_j$ includes a full-rank state), and $\mathcal{I}_{\mathcal{A}_j^\perp}$ denotes the identity map on operators on $\mathrm{supp}(\mathcal{A}_j)^\perp$. Now construct $\mathcal{E}_j(\cdot) \equiv \mathcal{E}_j^1 \circ \mathcal{E}_j^0(\cdot)$. Since each map $\mathcal{E}_j^1$ leaves the support of P_j invariant, the same Lyapunov argument of Theorem 3 shows that:

$$\text{supp}(\lim_{n\to\infty}(\mathcal{E}_r \dots \mathcal{E}_1)^n(\rho)) \subseteq \text{supp}(\mathcal{E}_{\mathcal{B}(\hat{\mathcal{H}})}(\tau)). \tag{17}$$

We thus have that the largest invariant set for a cycle of maps $\mathcal{E}_r \dots \mathcal{E}_1$ has support equal to $\hat{\mathcal{A}}$, and by the discrete-time invariance principle [43], the dynamics converge to that.

Now notice that, since $\hat{\mathcal{A}}$ is contained in each of the $\mathcal{A}_j = \text{fix}(\mathcal{E}_j)$, such is any maximum-rank operator in $\hat{\mathcal{A}}$, which implies (see e.g. Lemma 1 in [27]) that $\text{supp}(\hat{\mathcal{A}})$ is an invariant subspace for each $\mathcal{E}_j$. Hence, $\mathcal{E}_j$ restricted to $\mathcal{B}(\text{supp}(\hat{\mathcal{A}}))$ is still CPTP, and by construction projects onto the elements of $\mathcal{A}_j$ that have support contained in $\text{supp}(\hat{\mathcal{A}})$. Such a set, call it $\hat{\mathcal{A}}_j$, is thus a valid fixed-point set. By Theorem 4, we have that on the support of $\hat{\mathcal{A}}$ the limit in Eq. (15) converges to $\hat{\mathcal{A}}$. This shows that the largest invariant set for the cycle is exactly $\hat{\mathcal{A}}$, hence the claim is proved. □

Remark: In order for the proposed quantum alternating projection methods to be effective, it is important that the relevant CPTP maps be sufficiently simple to evaluate and implement. Assuming that the map $\mathcal{E}$ is easily achievable, it is useful to note that the projection map $\mathcal{E}_{\mathcal{A}_\rho}$ defined in Eq. (8) may be approximated through iteration of a map $\tilde{\mathcal{E}}_\lambda \equiv (1-\lambda)\mathcal{E} + \lambda\mathcal{I}$, where $\lambda \in (0,1)$. Since $\tilde{\mathcal{E}}_\lambda$ has 1 as the only eigenvalue on the unit circle, it is easy to show that $\lim_{n\to\infty} \tilde{\mathcal{E}}_\lambda^n = \mathcal{E}_{\mathcal{A}_\rho}$, $\mathcal{E}_{\mathcal{A}_\rho} \approx \tilde{\mathcal{E}}_\lambda^n$ for a sufficiently large number of iterations.

C. Robustness with respect to randomization

While Theorem 3 and Theorem 4 require *deterministic* cyclic repetition of the CPTP projections, the order is not critical for convergence. Randomizing the order of the maps still leads to asymptotic convergence, albeit in probability. We say that an operator-valued process $X(t)$ *converges in probability* to X^* if, for any $\delta, \varepsilon > 0$, there exists a time $T > 0$ such that $\mathbb{P}[\,\text{Tr}((X(T) - X_*)^2) > \varepsilon\,] < \delta$. Likewise, $X(t)$ *converges in expectation* if $\mathbb{E}(\rho(t)) \to \rho^*$ when $t \to +\infty$. Establishing convergence in probability uses the following Borel-Cantelli-type lemma, adapted from [44]:

Lemma 2 (Convergence in probability): Consider a finite number of CPTP maps $\{\mathcal{E}_j\}_{j=1}^M$, and a (Lyapunov) function $V(\rho)$, such that $V(\rho) \geq 0$ and $V(\rho) = 0$ if and only if $\rho \in \mathcal{S}$, with $\mathcal{S} \subset \mathcal{D}(\mathcal{H})$ some set of density operators. Assume, furthermore that:

(i) For each j and state ρ, $V(\mathcal{E}_j(\rho)) \leq V(\rho)$.

(ii) For each $\varepsilon > 0$ there exists a finite sequence of maps

$$\mathcal{E}_\varepsilon = \mathcal{E}_{j_K} \circ \dots \circ \mathcal{E}_{j_1}, \tag{18}$$

with $j_\ell \in \{1, \dots, M\}$ for all ℓ, such that $V(\mathcal{E}_\varepsilon(\rho)) < \varepsilon$ for all $\rho \neq \mathcal{S}$.

Assume that the maps are selected at random, with independent probability distribution $\mathbb{P}_t[\mathcal{E}_j]$ at each time t, and that there exists $\varepsilon > 0$ for which $\mathbb{P}_t[\mathcal{E}_j] > \varepsilon$ for all t. Then, for any $\gamma > 0$, the probability of having $V(\rho(t)) < \gamma$ converges to 1 as $t \to +\infty$.

Using the above result, we can prove the following:

Corollary 2: Let $\mathcal{E}_1, \dots, \mathcal{E}_r$ CPTP projections onto $\mathcal{A}_i = \mathfrak{B}(\mathcal{H}_i)$, $i = 1, \dots, r$. Assume that at each step $t \geq 0$ the map $\mathcal{E}_{j(t)}$ is selected randomly from a probability distribution

$$\Big\{q_j(t) = \mathbb{P}[\mathcal{E}_{j(t)}] > 0 | \sum_j q_j(t) = 1\Big\},$$

and that $q_j(t) > \epsilon > 0$ for all j and $t \geq 0$. For all $\tau \in \mathfrak{D}(\mathcal{H})$, let $\tau(t) \equiv \mathcal{E}_{j(t)} \circ \dots \circ \mathcal{E}_{j(1)}(\tau)$. Then $\tau(t)$ converges in probability and in expectation to $\tau^* = \mathcal{E}_{\hat{\mathcal{A}}}(\tau)$, where $\mathcal{E}$ is the CPTP projection onto $\hat{\mathcal{A}}$.

Proof: Given Lemma 2, it suffices to consider $V(\tau) \equiv 1 - \text{Tr}(\hat{\text{P}}\,\tau)$. It is non-increasing, and Theorem 4 also ensures that for every $\varepsilon > 0$, there exists a finite number of cycles of the maps that makes $V(\tau) < \varepsilon$. □

A similar result holds for the full-rank case:

Corollary 3: Let $\mathcal{E}_1, \dots, \mathcal{E}_r$ CPTP projections onto $\mathcal{A}_i$, $i = 1, \dots, r$, and assume that $\hat{\mathcal{A}} = \bigcap_{i=1}^r \mathcal{A}_i$ contains a full-rank state ρ. Assume that at each step $t \geq 0$ the map $\mathcal{E}_{j(t)}$ is selected randomly from a probability distribution

$$\Big\{q_j(t) = \mathbb{P}[\mathcal{E}_{j(t)}] > 0 | \sum_j q_j(t) = 1\Big\},$$

and that $q_j(t) > \epsilon > 0$ for all j and $t \geq 0$. For all $\tau \in \mathfrak{D}(\mathcal{H})$, let $\tau(t) \equiv \mathcal{E}_j(t) \circ \dots \circ \mathcal{E}_j(1)(\tau)$. Then $\tau(t)$ converges in probability and in expectation to $\tau^* = \mathcal{E}_{\hat{\mathcal{A}}}(\tau)$, where $\mathcal{E}$ is the CPTP projection onto $\hat{\mathcal{A}}$.

Proof: Given the Lemma 2, it suffices to consider $V(\tau) \equiv \langle(\tau - \tau^*), (\tau - \tau^*)\rangle_{\rho^{-1}}$. It is non-increasing, and Theorem 4 ensures that for every $\varepsilon > 0$ there exists a finite number of cycles of the maps that makes $V(\tau) < \varepsilon$. □

IV. Quasi-Local State Stabilization

A. Locality notion and stabilizability

In this section we specialize to a multipartite quantum system consisting of n (distinguishable) subsystems, or "qudits", defined on a tensor-product Hilbert space

$$\mathcal{H} \equiv \bigotimes_{a=1}^n \mathcal{H}_a, \quad a = 1, \dots, n, \ \dim(\mathcal{H}_a) = d_a, \ \dim(\mathcal{H}) = d.$$

In order to impose *quasi-locality constraints* on operators and dynamics on $\mathcal{H}$, we introduce *neighborhoods*. Following [19], [20], [37], neighborhoods $\{\mathcal{N}_j\}$ are subsets of indexes labeling the subsystems, that is, $\mathcal{N}_j \subsetneq \{1, \dots, n\}, \quad j = 1, \dots, K$. A *neighborhood operator* M is an operator on $\mathcal{H}$ such that there exists a neighborhood $\mathcal{N}_j$ for which we may write

$$M \equiv M_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}}_j},$$

where $M_{\mathcal{N}_j}$ accounts for the action of M on subsystems in $\mathcal{N}_j$, and $I_{\overline{\mathcal{N}}_j} \equiv \bigotimes_{a \notin \mathcal{N}_j} I_a$ is the identity on the remaining ones. Once a state $\rho \in \mathcal{D}(\mathcal{H})$ and a neighborhood structure are assigned on $\mathcal{H}$, *reduced neighborhood states* may be computed as $\rho_{\mathcal{N}_j} \equiv \text{Tr}_{\overline{\mathcal{N}}_j}(\rho)$, where $\text{Tr}_{\overline{\mathcal{N}}_j}$ indicates the partial trace over the tensor complement of the neighborhood $\mathcal{N}_j$, namely, $\mathcal{H}_{\overline{\mathcal{N}}_j} \equiv \bigotimes_{a \notin \mathcal{N}_j} \mathcal{H}_a$. A strictly "local" setting corresponds to the case where $\mathcal{N}_j \equiv \{j\}$, that is, each subsystem forms a distinct neighborhood.

Assume that some quasi-locality notion is *fixed* by specifying a set of neighborhoods, $\mathcal{N} \equiv \{\mathcal{N}_j\}$. A *CP map $\mathcal{E}$ is a neighborhood map* relative to $\mathcal{N}$ if, for some j, $\mathcal{E} = \mathcal{E}_{\mathcal{N}_j} \otimes \mathcal{I}_{\overline{\mathcal{N}}_j}$, where $\mathcal{E}_{\mathcal{N}_j}$ is the restriction of $\mathcal{E}$ to operators on the subsystems in $\mathcal{N}_j$ and $\mathcal{I}_{\overline{\mathcal{N}}_j}$ is the identity map for operators on $\mathcal{H}_{\overline{\mathcal{N}}_j}$. An equivalent formulation can be given in terms of the OSR: that is, $\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger$ is a neighborhood map relative to $\mathcal{N}$ if there exists a neighborhood $\mathcal{N}_j$ such that, *for all* k, $M_k = M_{\mathcal{N}_j,k} \otimes I_{\overline{\mathcal{N}}_j}$. The reduced map on the neighborhood is then $\mathcal{E}_{\mathcal{N}_j}(\cdot) = \sum_k M_{\mathcal{N}_j,k} \cdot M_{\mathcal{N}_j,k}^\dagger$. Since the identity factor is preserved by sums (and products) of the M_k, it is immediate to verify that the property of $\mathcal{E}$ being a neighborhood map is well-defined with respect to the freedom in the OSR [3].

Definition 2: A state ρ is discrete-time *Quasi-Locally Stabilizable* (QLS) if there exists a sequence $\{\mathcal{E}_t\}_{t\geq 0}$ of neighborhood maps such that ρ is GAS for the associated propagator $\mathcal{E}_{t,s} = \mathcal{E}_{t-1} \circ \ldots \circ \mathcal{E}_s$, namely:

$$\mathcal{E}_{t,s}(\rho) = \rho, \quad \forall t \geq s \geq 0; \tag{19}$$

$$\lim_{t\to\infty} \|\mathcal{E}_{t,s}(\sigma), \rho\|_1 = 0, \quad \forall \sigma \in \mathfrak{D}(\mathcal{H}),\, \forall s \geq 0. \tag{20}$$

Remark: With respect to the definition of quasi-locality that naturally emerges for continuous-time Markov dynamics [19], [20], [37], it is important to appreciate that constraining discrete-time dynamics to be QL in the above sense is more restrictive. In fact, even if a generator $\mathcal{L}$ of a continuous-time (homogeneous) semigroup can be written as a sum of neighborhood generators, namely, $\mathcal{L} = \sum_k \mathcal{L}_k$, the generated semigroup $\mathcal{E}_t \equiv e^{\mathcal{L}t}, t \geq 0$, is *not*, in general, QL at any time. In some sense, one may think of the different noise components $\mathcal{L}_1, \ldots, \mathcal{L}_k$ of the continuous-time generator as acting "in parallel". On the other hand, were the maps $\mathcal{E}_j$ we consider in this paper each generated by some corresponding neighborhood generator $\mathcal{L}_j$, then by QL discrete-time dynamics we would be requesting that, on each time interval, a *single* noise operator is active, thus obtaining global switching dynamics [23] of the form $e^{\mathcal{L}_k T_k} \circ e^{\mathcal{L}_{k-1} T_{k-1}} \circ \ldots \circ e^{\mathcal{L}_1 T_1}$.

We could have requested each $\mathcal{E}_t$ to be a convex combination of neighborhood maps acting on different neighborhoods, however it is not difficult to see that this case can be studied as the convergence in expectation for a randomized sequence. Hence, we are focusing on the *most restrictive definition of QL constraint* for discrete-time Markov dynamics. With respect to the continuous dynamics, however, we allow for the evolution to be time-inhomogeneous. Remarkably, we shall find a characterization of QLS pure states that is equivalent to the continuous-time case, when the latter dynamics are required to be *frustration-free* (FF) [37].

B. Invariance conditions and minimal fixed point sets

In this section, we build on the invariance requirement of Eq. (19) to find *necessary* conditions that the discrete-time dynamics must satisfy in order to have a given state ρ as its unique and attracting equilibrium. These impose a certain minimal fixed-point set, and hence suggest a structure for the stabilizing dynamics.

Following [37], given an operator $X \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$, with corresponding (operator) Schmidt decomposition $X = \sum_j A_j \otimes B_j$, we define its *Schmidt span* as $\Sigma_A(X) \equiv \mathrm{span}(\{A_j\})$. The Schmidt span is important because, if we want to leave an operator invariant with a neighborhood map, this also imposes the invariance of its Schmidt span [37].

In our case, this specifically means that, given a $\rho \in \mathcal{D}(\mathcal{H}_{\mathcal{N}_j} \otimes \mathcal{H}_{\overline{\mathcal{N}}_j})$ and a neighborhood $\mathcal{E} = \mathcal{E}_{\mathcal{N}_j} \otimes I_{\overline{\mathcal{N}}_j}$, then $\mathrm{span}(\rho) \subseteq \mathrm{fix}(\mathcal{E})$ implies

$$\Sigma_{\mathcal{N}_j}(\rho) \otimes \mathcal{B}(\mathcal{H}_{\overline{\mathcal{N}}_j}) \subseteq \mathrm{fix}(\mathcal{E}_{\mathcal{N}_j}).$$

However, a Schmidt span need *not* be a valid fixed-point set, namely, a ρ-distorted algebra that is invariant for $\mathcal{M}_{\rho,\frac{1}{2}}$. In general, we need to further enlarge the QL fixed-point sets from the Schmidt span to suitable algebras. We discuss separately two relevant cases.

• *Pure states.—* Let $\rho = |\psi\rangle\langle\psi|$ be a pure state and assume that, with respect to the factorization $\mathcal{H}_{\mathcal{N}_j} \otimes \mathcal{H}_{\overline{\mathcal{N}}_j}$, its Schmidt decomposition $|\psi\rangle = \sum_k c_k |\psi_k\rangle \otimes |\phi_k\rangle$. Let $\mathcal{H}^0_{\mathcal{N}_j} \equiv \mathrm{span}\{|\psi_k\rangle\} = \mathrm{supp}(\rho_{\mathcal{N}_j})$. Then we have [37]:

$$\Sigma_{\mathcal{N}_j}(\rho) = \mathcal{B}(\mathcal{H}^0_{\mathcal{N}_j}). \tag{21}$$

In this case the Schmidt span is indeed a valid fixed-point set, and no further enlargement is needed. The *minimal fixed-point set* for neighborhood maps required to preserve ρ is thus $\mathcal{F}_j \equiv \mathcal{B}(\mathcal{H}^0_{\mathcal{N}_j}) \otimes \mathcal{B}(\mathcal{H}_{\overline{\mathcal{N}}_j})$. By construction, each $\mathcal{F}_j$ contains ρ. Notice that their intersection is just ρ if and only if

$$\mathrm{span}\{|\psi\rangle\} = \bigcap_j \mathcal{H}^0_{\mathcal{N}_j} \otimes \mathcal{H}_{\overline{\mathcal{N}}_j} = \bigcap_j \mathcal{H}^0_j, \tag{22}$$

where we have defined $\mathcal{H}^0_j \equiv \mathcal{H}^0_{\mathcal{N}_j} \otimes \mathcal{H}_{\overline{\mathcal{N}}_j}$.

• *Full rank states.—* If ρ is a full-rank state, and W a set of operators, the *minimal fixed-point set generated by ρ and W*, by Theorem 1, is the smallest ρ-distorted algebra generated by W which is invariant with respect to $\mathcal{M}_{\rho,\frac{1}{2}}$. Notice that, since ρ is full rank, its reduced states $\rho_{\mathcal{N}_j}$ are also full rank. Denote by $\mathrm{alg}_\rho(W)$ the †-closed ρ-distorted algebra generated by W. Call $W_j \equiv \Sigma_{\mathcal{N}_j}(\rho)$. The minimal fixed-point sets $\mathcal{F}_{\rho_{\mathcal{N}_j}}(W_j)$ can then be constructed iteratively from $\mathcal{F}_j^{(0)} \equiv \mathrm{alg}_{\rho_{\mathcal{N}_j}}(W_j)$, with the k-th step given by [37]:

$$\mathcal{F}_j^{(k+1)} \equiv \mathrm{alg}_{\rho_{\mathcal{N}_j}}(\mathcal{M}_{\rho_{\mathcal{N}_j},\frac{1}{2}}(\mathcal{F}_j^{(k)}), \mathcal{F}_j^{(k)}).$$

We keep iterating until $\mathcal{F}_j^{(k+1)} = \mathcal{F}_j^{(k)} = \mathcal{F}_{\rho_{\mathcal{N}_j}}(W_j)$. When that happens, define

$$\mathcal{F}_j \equiv \mathcal{F}_{\rho_{\mathcal{N}_j}}(\Sigma_{\mathcal{N}_j}(\rho)) \otimes \mathcal{B}(\mathcal{H}_{\overline{\mathcal{N}}_j}). \tag{23}$$

Since the $\mathcal{F}_j$ are constructed to be the minimal sets for neighborhood maps that contain the given state and its corresponding Schmidt span, then clearly: $\mathrm{span}(\rho) \subset \bigcap_j \mathcal{F}_j$.

C. Stabilizability under quasi-locality constraints

In the case of a pure target state, we can prove the following:

Theorem 6 (QLS pure states): A pure state $\rho = |\psi\rangle\langle\psi|$ is QLS by discrete-time dynamics if and only if

$$\mathrm{supp}(\rho) = \bigcap_j \mathcal{H}^0_j. \tag{24}$$

Proof: Given the discussion of Section IV-B, any dynamics that make ρ QLS (and hence leaves it invariant) must consist of neighborhood maps $\{\mathcal{E}_j\}$ with corresponding fixed points such that $\mathcal{F}_k \subseteq \text{fix}(\mathcal{E}_j)$, whenever $\mathcal{E}_j$ is a $\mathcal{N}_k$-neighborhood map. If the intersection of the fixed-point sets is not unique, ρ cannot be GAS, since there would be another state that is not attracted to it. Given Eq. (22), we have

$$\text{span}(\rho) = \bigcap_k \mathcal{F}_k \iff \text{supp}(\rho) = \bigcap_j \mathcal{H}_j^0.,$$

which proves necessity. For sufficiency, we explicitly construct neighborhood maps whose cyclic application ensures stabilization. Define$P_{\mathcal{N}_j}$ to be the projector onto $\text{supp}(\rho_{\mathcal{N}_j})$, and the CPTP maps: $\mathcal{E}_{\mathcal{N}_j}(\cdot) \equiv P_{\mathcal{N}_j}(\cdot)P_{\mathcal{N}_j} + \frac{P_{\mathcal{N}_j}}{\text{Tr}(P_{\mathcal{N}_j})}\,\text{Tr}\,(P_{\mathcal{N}_j}^{\perp}\cdot)$, with $\mathcal{E}_j \equiv \mathcal{E}_{\mathcal{N}_j} \otimes \mathcal{I}_{\overline{\mathcal{N}}_j}$. Consider the positive-semidefinite function $V(\tau) = 1 - \text{Tr}\,(\rho\,\tau)\,,\, \tau \in \mathcal{B}(\mathcal{H})$. The result then follows from Theorem 3. □

For full-rank states we have the following characterization:

Theorem 7 (QLS full-rank states): A full-rank state $\rho \in \mathcal{D}(\mathcal{H})$ is QLS by discrete-time dynamics if and only if

$$\text{span}(\rho) = \bigcap_k \mathcal{F}_k \tag{25}$$

Proof: As before, by contradiction, suppose that $\rho_2 \in \bigcap_k \mathcal{F}_k$ exists, such that $\rho_2 \neq \rho$. This clearly implies that ρ cannot be GAS because there would exist another invariant state, which is not attracted to ρ. This proves necessity. Sufficiency derives from the alternating CPTP projection theorem. Specifically, let $\mathcal{E}_{\mathcal{N}_k}$ be the CPTP projection onto $\mathcal{F}_k$, and $\mathcal{E}_k \equiv \mathcal{E}_{\mathcal{N}_k} \otimes \text{Id}_{\overline{\mathcal{N}}_k}$. By Theorem 4, we already know that for every ρ, $(\mathcal{E}_M \ldots \mathcal{E}_1)^k(\rho) \to \bigcap_k \mathcal{F}_k$ for $k \to \infty$. Now, by hypothesis, $\bigcap_k \mathcal{F}_k = \text{span}(\rho)$ and, being ρ the only (trace one) state in his own span, ρ is GAS. □

A set of *sufficient conditions*, stemming from Theorem 5, can be also derived in an analogous way for a general target state.

Remark: The conditions that guarantee either a pure or a full-rank state to be QLS in discrete time are the same that guarantee existence of a *QL FF stabilizing generator in continuous time* [37]. Hence, all the examples of stabilizable states and classes of states, as well as the non stabilizable ones, carry over from that setting. We stress that if more general continuous-time generators are allowed, namely, frustration is permitted as in Hamiltonian-assisted stabilization [20], then the continuous-time setting can be more powerful. On the one hand, considering the stricter nature of the QL constraint for the discrete-time setting, this is not surprising. On the other hand, if Liouvillian is no longer FF, then the target is globally invariant for $\mathcal{L}$ but no longer invariant for individual QL components $\mathcal{L}_j$, suggesting that a weaker ("stroboscopic") invariance requirement could be more appropriate to "mimic" the effect of frustration in the discrete-time QL setting.

V. Conclusions

We have introduced alternating projection methods based on sequences of CPTP projections, and used them in designing discrete-time stabilizing dynamics for entangled states in multipartite quantum systems subject to realistic quasi-locality constraints. We show that the proposed methods are also suitable for distributed, randomized and unsupervised implementations on large networks. While the locality constraints we impose on the discrete-time dynamics are stricter, the stabilizable states are, remarkably, the same that are stabilizable for continuous-time frustration-free generators. From a methodological standpoint, our results shed further light on the structure and *intersection of fixed-point sets* of CPTP maps In particular, we show that the intersection of fixed-point sets is *still* a fixed-point set, as long as it contains a full-rank state.

Towards applications, the proposed alternating projection methods are in principle suitable for implementation in digital open-quantum system simulators, such as demonstrated in trapped-ion experiments [24]. Beside providing protocols for stabilizing relevant classes of entangles states, including *graph product states and commuting Gibbs states* [37], our methods point to an alternative approach for constructing quantum samplers using quasi-local resources [45].

Some possible developments are worth highlighting. First, in order to extend the applicability of the proposed methods to more general classes of states, as well as to establish a tighter link to quantum error correction and dissipative code preparation [46], [47], it is natural to look at discrete-time *conditional stabilization*, in the spirit of [20]. Second, while it is possible to use basic classical bounds on the convergence speed of the proposed alternating-projection methods [33], their geometric nature makes it hard to obtain physical insight from them. A more intuitive approach, following [18], [27], [48], may offer a promising alternative venue in that respect. It would be interesting to extend the analysis to the non-homogeneous, discrete-time cases considered in this work. Lastly, the characterization of scenarios in which *finite-time stabilization* is possible under QL constraints is a challenging open problem, which we plan to address elsewhere [49].

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Appendix

A. Non-orthogonality of $\mathcal{E}_{\mathcal{A}}$ with respect to the Hilbert-Schmidt inner product

Let us decompose a full-rank fixed point set $\mathcal{A}_\rho = \bigoplus_\ell \mathcal{A}_\ell = \bigoplus_\ell \mathcal{B}(\mathcal{H}_{S,\ell}) \otimes \tau_\ell$, (where $\tau_\ell \equiv \tau_{F,\ell}$). By definition, the *orthogonal* projection of X onto $\mathcal{A}_i$ is given by

$$P_{\mathcal{A}}(X) \equiv \sum_{\ell,i} \langle \sigma_{\ell,i} \otimes \tau_\ell, X\rangle_{HS}\, \sigma_{\ell,i} \otimes \tau_\ell,$$

where $\sigma_{\ell,i} \otimes \tau_\ell$ is an orthonormal basis for $\mathcal{A}_\ell$. Note that the outcome only depends on the restrictions of X to the supports of the $\mathcal{A}_\ell$. Hence, let $X \equiv \sum_\ell X_\ell + \Delta X$, where $X_\ell =$

$\Pi_{SF,\ell} X \Pi_{SF,\ell}$, and further decompose $X_\ell \equiv \sum_k A_{\ell,k} \otimes B_{\ell,k}$, so we can write:

$$P_{\mathcal{A}}(X) = \bigoplus_i \sum_{j,\ell} \Big(\sum_k \mathrm{Tr}[(\sigma_j \otimes \tau_\ell)(A_{\ell,k} \otimes B_{\ell,k})] \sigma_j \otimes \tau_\ell \Big)$$
$$= \bigoplus_\ell \sum_{j,\ell} \Big(\mathrm{Tr}[\sigma_j \sum_k (A_{\ell,k} \mathrm{Tr}(\tau_\ell B_{\ell,k}))] \sigma_j \otimes \tau_\ell) \Big).$$

By comparing to Eq. (9), we have that $P_{\mathcal{A}} = \mathcal{E}_{\mathcal{A}}$ if and only if $\sum_k (A_k \mathrm{Tr}(\tau_j B_k)) = \mathrm{Tr}_{F,\ell}(X_\ell)$, which is equivalent to request that $\tau_j = \lambda_\ell I$. Thus, unless $\mathcal{A}_\rho$ contains the completely mixed state, $\mathcal{E}_{\mathcal{A}}$ in Eq. (9) is not an orthogonal projection with respect to the Hilbert-Schmidt inner product. □

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