Spectral analysis of the biharmonic operator subject to Neumann boundary conditions on dumbbell domains

José M. Arrieta[*]   Francesco Ferraresso[†]   Pier Domenico Lamberti[‡]

December 26, 2017

Abstract

We consider the biharmonic operator subject to homogeneous boundary conditions of Neumann type on a planar dumbbell domain which consists of two disjoint domains connected by a thin channel. We analyse the spectral behaviour of the operator, characterizing the limit of the eigenvalues and of the eigenprojections as the thickness of the channel goes to zero. In applications to linear elasticity, the fourth order operator under consideration is related to the deformation of a free elastic plate, a part of which shrinks to a segment. In contrast to what happens with the classical second order case, it turns out that the limiting equation is here distorted by a strange factor depending on a parameter which plays the role of the Poisson coefficient of the represented plate.

1   Introduction

This paper is devoted to a spectral analysis of the biharmonic operator subject to Neumann boundary conditions on a domain which undergoes a singular perturbation. The focus is on planar dumbbell-shaped domains Ω_ϵ, with $\epsilon > 0$, described in Figure 1. Namely, given two bounded smooth domains Ω_L, Ω_R in $\mathbb{R}^2$ with $\Omega_L \cap \Omega_R = \emptyset$ such that $\partial\Omega_L \supset \{(0, y) \in \mathbb{R}^2 : -1 < y < 1\}, \partial\Omega_R \supset \{(1, y) \in \mathbb{R}^2 : -1 < y < 1\}$, and $(\Omega_R \cup \Omega_L) \cap ([0, 1] \times [-1, 1]) = \emptyset$, we set

$$\Omega = \Omega_L \cup \Omega_R, \ \ \text{and} \ \ \Omega_\epsilon = \Omega \cup R_\epsilon \cup L_\epsilon,$$

for all $\epsilon > 0$ small enough. Here $R_\epsilon \cup L_\epsilon$ is a thin channel connecting Ω_L and Ω_R defined by

$$R_\epsilon = \{(x, y) \in \mathbb{R}^2 : x \in (0, 1), 0 < y < \epsilon g(x)\}, \tag{1.1}$$

$$L_\epsilon = (\{0\} \times (0, \epsilon g(0)) \cup (\{1\} \times (0, \epsilon g(1)))),$$

where $g \in C^2[0, 1]$ is a positive real-valued function. Note that Ω_ϵ collapses to the limit set $\Omega_0 = \Omega \cup ([0, 1] \times \{0\})$ as $\epsilon \to 0$.

---

[*]Departamento de Matemática Aplicada, Universidad Complutense de Madrid, arrieta@mat.ucm.es

[†]Dipartimento di matematica "Tullio Levi-Civita", Università degli Studi di Padova, fferrare@math.unipd.it

[‡]Dipartimento di matematica "Tullio Levi-Civita", Università degli Studi di Padova, lamberti@math.unipd.it
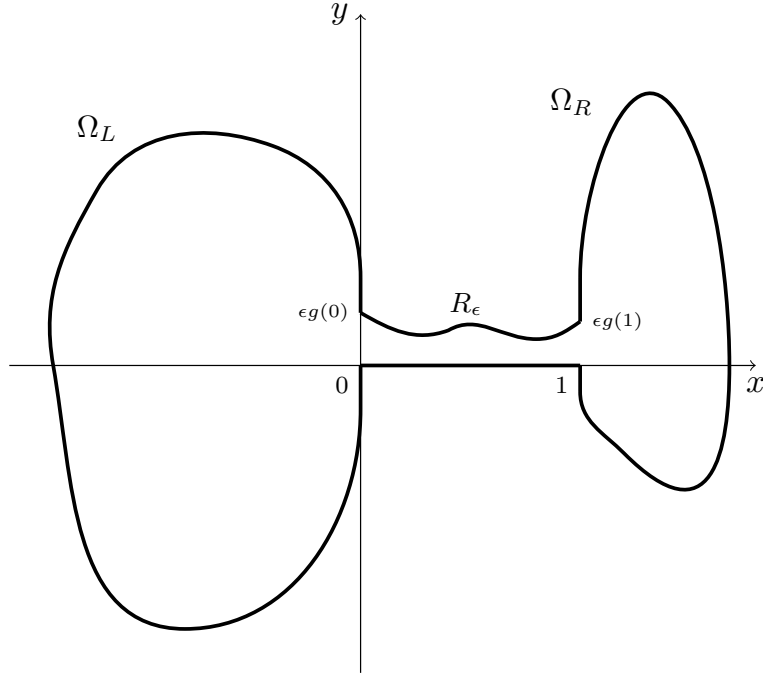
Figure 1: The dumbbell domain Ω_ϵ.

We consider the eigenvalue problem

$$
\begin{cases}
\Delta^2 u - \tau \Delta u + u = \lambda u, & \text{in } \Omega_\epsilon, \\
(1-\sigma)\frac{\partial^2 u}{\partial n^2} + \sigma \Delta u = 0, & \text{on } \partial\Omega_\epsilon, \\
\tau \frac{\partial u}{\partial n} - (1-\sigma)\, \mathrm{div}_{\partial\Omega_\epsilon}(D^2 u \cdot n)_{\partial\Omega_\epsilon} - \frac{\partial(\Delta u)}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon,
\end{cases}
\tag{1.2}
$$

where $\tau \geq 0$, $\sigma \in (-1,1)$ are fixed parameters, and we analyse the behaviour of the eigenvalues and of the corresponding eigenfunctions as $\epsilon \to 0$. Here $\mathrm{div}_{\partial\Omega_\epsilon}$ is the tangential divergence operator, and $(\cdot)_{\partial\Omega_\epsilon}$ is the projection on the tangent line to $\partial\Omega_\epsilon$. The corresponding Poisson problem reads

$$
\begin{cases}
\Delta^2 u - \tau \Delta u + u = f, & \text{in } \Omega_\epsilon, \\
(1-\sigma)\frac{\partial^2 u}{\partial n^2} + \sigma \Delta u = 0, & \text{on } \partial\Omega_\epsilon, \\
\tau \frac{\partial u}{\partial n} - (1-\sigma)\, \mathrm{div}_{\partial\Omega_\epsilon}(D^2 u \cdot n)_{\partial\Omega_\epsilon} - \frac{\partial(\Delta u)}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon,
\end{cases}
\tag{1.3}
$$

with datum $f \in L^2(\Omega_\epsilon)$.

Since $\partial\Omega_\epsilon$ has corner singularities at the junctions $(0,0)$, $(0, \epsilon g(0))$, $(1,0)$, $(1, \epsilon g(1))$ and H^4 regularity does not hold around those points, we shall always understand problems (1.2), (1.3), (as well as analogous problems) in a weak (variational) sense, in which case only H^2 regularity is required.

Namely, the variational formulation of problem (1.3) is the following: find $u \in H^2(\Omega_\epsilon)$ such that

$$
\int_{\Omega_\epsilon} (1-\sigma)D^2 u : D^2\varphi + \sigma\Delta u\Delta\varphi + \tau\nabla u \cdot \nabla\varphi + u\varphi \, dx = \int_{\Omega_\epsilon} f\varphi \, dx,
\tag{1.4}
$$

for all $\varphi \in H^2(\Omega_\epsilon)$. The quadratic form associated with the left-hand side of (1.4) - call it $B_{\Omega_\epsilon}(u, \varphi)$ - is coercive for all $\tau \geq 0$ and $\sigma \in (-1,1)$, see e.g. [13], [14]. In particular, by standard spectral theory this quadratic form allows to define a non-negative self-adjoint

2

operator $T = (\Delta^2 - \tau\Delta + I)_{N(\sigma)}$ in $L^2(\Omega_\epsilon)$ which plays the role of the classical operator $\Delta^2 - \tau\Delta + I$ subject to the boundary conditions above. More precisely, T is uniquely defined by the relation

$$B_{\Omega_\epsilon}(u, \varphi) = < T^{1/2}u, T^{1/2}\varphi >_{L^2(\Omega_\epsilon)},$$

for all $u, \varphi \in H^2(\Omega_\epsilon)$. In particular the domain of the square root $T^{1/2}$ of T is $H^2(\Omega_\epsilon)$ and a function u belongs to the domain of T if and only if $u \in H^2(\Omega_\epsilon)$ and there exists $f \in L^2(\Omega_\epsilon)$ such that $B_{\Omega_\epsilon}(u, \varphi) = < f, \varphi >_{L^2(\Omega_\epsilon)}$ for all $\varphi \in H^2(\Omega_\epsilon)$, in which case $Tu = f$. We refer to [16, Chp. 4] for a general introduction to the variational approach to the spectral analysis of partial differential operators on non-smooth domains.

The operator T is densely defined and its eigenvalues and eigenfunctions are exactly those of problem (1.2). Moreover, since the embedding $H^2(\Omega_\epsilon) \subset L^2(\Omega_\epsilon)$ is compact, $(\Delta^2 - \tau\Delta + I)_{N(\sigma)}$ has compact resolvent, hence the spectrum is discrete and consists of a divergent increasing sequence of positive eigenvalues $\lambda_n(\Omega_\epsilon)$, $n \in \mathbb{N}$, with finite multiplicity (here each eigenvalue is repeated as many times as its multiplicity).

Problem (1.2) arises in linear elasticity in connection with the Kirchhoff-Love model for the study of vibrations and deformations of free plates, in which case σ represents the Poisson ratio of the material and τ the lateral tension. In this sense, the dumbbell domain Ω_ϵ could represent a plate and R_ϵ a part of it which degenerates to the segment $[0, 1] \times \{0\}$.

We note that problem (1.2) can be considered as a natural fourth order version of the corresponding eigenvalue problem for the Neumann Laplacian $-\Delta_N$, namely

$$\begin{cases} -\Delta u + u = \lambda u, & \text{in } \Omega_\epsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon, \end{cases} \tag{1.5}$$

the variational formulation of which reads

$$\int_{\Omega_\epsilon} Du \cdot D\varphi + u\varphi \, dx = \lambda \int_{\Omega_\epsilon} u\varphi \, dx, \tag{1.6}$$

where the test functions φ and the unknown u are considered in $H^1(\Omega_\epsilon)$.

Although the terminology used in the literature to refer to boundary value problems for fourth order operators is sometimes a bit misleading, we emphasise that the formulation of problems (1.2), (1.3) is rather classical, see e.g. [28, Example 2.15] where problem (1.3) with $\tau = 0$ is referred to as the Neumann problem for the biharmonic operator. Moreover, we point out that a number of recent papers devoted to the analysis of (1.2) have confirmed that problem (1.2) can be considered as the natural Neumann problem for the biharmonic operator, see [7], [8], [10], [11], [12], [13], [14], [29]. We also refer to [22] for an extensive discussion on boundary value problems for higher order elliptic operators.

It is known that the eigenelements of the Neumann Laplacian on a typical dumbbell domain as above have a singular behaviour, see [1], [2], [3], [4], [5], [6], and the references therein. For example, it is known that not all the eigenvalues of $-\Delta_N$ on Ω_ϵ converge to the eigenvalues of $-\Delta_N$ in Ω; indeed, some of the eigenvalues of the dumbbell domain are asymptotically close to the eigenvalues of a boundary value problem defined in the channel R_ϵ. This allows the appearance in the limit of extra eigenvalues associated with an ordinary differential equation in the segment $(0, 1)$, which are generally different from the eigenvalues of $-\Delta_N$ in Ω. Such singular behaviour reflects a general characteristic of boundary value problems with Neumann boundary conditions, the stability of which

requires rather strong assumptions on the admissible domain perturbations, see e.g., [4], [8], [27]. We refer to [15, p. 420] for a classical counterexample.

The aim of the present paper is to clarify how Neumann boundary conditions affect the spectral behaviour of the operator $\Delta^2 - \tau\Delta$ on dumbbell domains, by extending the validity of some results known for the second order operator $-\Delta_N$ to the fourth-order operator $(\Delta^2 - \tau\Delta)_{N(\sigma)}$.

First of all, we prove that the eigenvalues of problem (1.2) can be asymptotically decomposed into two families of eigenvalues as

$$(\lambda_n(\Omega_\epsilon))_{n \geq 1} \approx (\omega_k)_{k \geq 1} \cup (\theta_l^\epsilon)_{l \geq 1}, \quad \text{as } \epsilon \to 0, \tag{1.7}$$

where $(\omega_k)_{k \geq 1}$ are the eigenvalues of problem

$$\begin{cases} \Delta^2 w - \tau\Delta w + w = \omega_k\, w, & \text{in } \Omega, \\ (1-\sigma)\frac{\partial^2 w}{\partial n^2} + \sigma\Delta w = 0, & \text{on } \partial\Omega, \\ \tau\frac{\partial w}{\partial n} - (1-\sigma)\operatorname{div}_{\partial\Omega}(D^2 w \cdot n)_{\partial\Omega} - \frac{\partial(\Delta w)}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

and $(\theta_l^\epsilon)_{l \geq 1}$ are the eigenvalues of problem

$$\begin{cases} \Delta^2 v - \tau\Delta v + v = \theta_l^\epsilon\, v, & \text{in } R_\epsilon, \\ (1-\sigma)\frac{\partial^2 v}{\partial n^2} + \sigma\Delta v = 0, & \text{on } \Gamma_\epsilon, \\ \tau\frac{\partial v}{\partial n} - (1-\sigma)\operatorname{div}_{\Gamma_\epsilon}(D^2 v \cdot n)_{\Gamma_\epsilon} - \frac{\partial(\Delta v)}{\partial n} = 0, & \text{on } \Gamma_\epsilon, \\ v = 0 = \frac{\partial v}{\partial n}, & \text{on } L_\epsilon. \end{cases} \tag{1.9}$$

The decomposition (1.7) is proved under the assumption that a certain condition on R_ϵ, called H-Condition, is satisfied. We provide in particular a simple condition on the profile function g which guarantees the validity of the H-Condition.

Thus, in order to analyse the behaviour of $\lambda_n(\Omega_\epsilon)$ as $\epsilon \to 0$, it suffices to study θ_l^ϵ as $\epsilon \to 0$. To do so, we need to pass to the limit in the variational formulation of problem (1.9). Since the domain R_ϵ collapses to a segment as $\epsilon \to 0$, we use thin domain techniques in order to find the appropriate limiting problem. As in the case of the Laplace operator, the limiting problem depends on the shape of the channel R_ϵ via the profile function $g(x)$. More precisely it can be written as follows

$$\begin{cases} \frac{1-\sigma^2}{g}(gh'')'' - \frac{\tau}{g}(gh')' + h = \theta h, & \text{in } (0,1) \\ h(0) = h(1) = 0, \\ h'(0) = h'(1) = 0. \end{cases} \tag{1.10}$$

This allows to prove convergence results for the eigenvalues and eigenfunctions of problem (1.3). The precise statement can be found in Theorem 7. Roughly speaking, Theorem 7 establishes the following alternative:

(A) either $\lambda_n(\Omega_\epsilon) \to \omega_k$, for some $k \geq 1$ in which case the corresponding eigenfunctions converge in Ω to the eigenfunctions associated with ω_k.

(B) or $\lambda_n(\Omega_\epsilon) \to \theta_l$ as $\epsilon \to 0$ for some $l \in \mathbb{N}$ in which case the corresponding eigenfunctions behave in R_ϵ like the eigenfunctions associated with θ_l.

Moreover, all eigenvalues ω_k and θ_l are reached in the limit by the eigenvalues $\lambda_n(\Omega_\epsilon)$.

We find it remarkable that for $\sigma \neq 0$ the limiting equation in (1.10) is distorted by the coefficient $1 - \sigma^2 \neq 1$. This phenomenon shows that the dumbbell problem for our fourth

order problem (1.2) with $\sigma \neq 0$ is significantly different from the second order problem (1.5) considered in the literature.

We also note that the Dirichlet problem for the operator $\Delta^2 u - \tau \Delta u + u$, namely

$$\begin{cases} \Delta^2 u - \tau \Delta u + u = \lambda u, & \text{in } \Omega_\epsilon, \\ u = 0, & \text{on } \partial\Omega_\epsilon, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_\epsilon \end{cases} \qquad (1.11)$$

is stable in the sense that its eigenelements converge to those of the operator $\Delta^2 - \tau\Delta + I$ in Ω as $\epsilon \to 0$. In other words, as for the Laplace operator, in the case of Dirichlet boundary conditions, no eigenvalues from the channel R_ϵ appear in the limit as $\epsilon \to 0$. In fact, it is well known that Dirichlet eigenvalues on thin domains diverge to $+\infty$ as $\epsilon \to 0$, because of the Poincaré inequality.

In order to prove our results, we study the convergence of the resolvent operators $(\Delta^2 - \tau\Delta + I)^{-1}_{N(\sigma,\tau)}$ and this is done by using the notion of $\mathcal{E}$-convergence, which is a useful tool in the analysis of boundary value problems defined on variable domains, see e.g., [5], [7], [8].

We point out that, although many papers in the literature have been devoted to the spectral analysis of second order operators with either Neumann or Dirichlet boundary conditions on dumbbell domains, see [2], [3], [23], [24] and references therein, very little seems to be known about these problems for higher order operators. We refer to [30] for a recent analysis of the dumbbell problem in the case of elliptic systems subject to Dirichlet boundary conditions.

Finally, we observe that it would be interesting to provide precise rates of convergence for the eigenvalues $\lambda_n(\Omega_\epsilon)$ and the corresponding eigenfunctions as $\epsilon \to 0$ in the spirit of the asymptotic analysis performed e.g., in [3], [18], [19], [20], [21], [23], [24] for second order operators. However, in case of higher order operators, this seems a challenging problem and is not addressed here.

The paper is organized as follows. In Section 2 we prove the asymptotic decomposition (1.7) of the eigenvalues $\lambda_n(\Omega_\epsilon)$. This is achieved in several steps. In Theorem 1 we provide a suitable upper bound for the eigenvalue $\lambda_n(\Omega_\epsilon)$. Then, in Definition 2 we introduce an assumption on the shape of the channel R_ϵ, called H-Condition, which is needed to prove a lower bound for $\lambda_n(\Omega_\epsilon)$ as $\epsilon \to 0$, see Theorem 2. Finally, we collect the results of the section in Theorem 3 to deduce a convergence result for the eigenvalues and the eigenfunctions of problem (1.2) under the assumption that the H-Condition holds. In Section 3 we show that a wide class of regular dumbbell domains satisfy the H-Condition. In Section 4 we study the convergence of the solutions of problem (1.9) as $\epsilon \to 0$, we identify the limiting problem in $(0, 1)$, and we prove the spectral convergence of problem (1.9) to problem (1.10). Finally, in Section 5 we combine the results of the previous sections and prove Theorem 7.

2 Decomposition of the eigenvalues

The main goal of this section is to prove the decomposition of the eigenvalues of problem (1.2) into the two families of eigenvalues coming from (1.8) and (1.9). First of all we note that, since Ω_ϵ, Ω and R_ϵ are sufficiently regular, by standard spectral theory for differential operators it follows that the operators associated with the quadratic forms appearing in the weak formulation of problems (1.2), (1.8), (1.9) have compact resolvents. Thus, the spectra of such problems are discrete and consist of positive eigenvalues of finite multiplicity. The

eigenpairs of problems (1.2), (1.8), (1.9) will be denoted by $(\lambda_n(\Omega_\epsilon), \varphi_n^\epsilon)_{n \geq 1}$, $(\omega_n, \varphi_n^\Omega)_{n \geq 1}$, $(\theta_n^\epsilon, \gamma_n^\epsilon)_{n \geq 1}$ respectively, where the three families of eigenfunctions $\varphi_n^\epsilon, \varphi_n^\Omega, \gamma_n^\epsilon$ are complete orthonormal bases of the spaces $L^2(\Omega_\epsilon), L^2(\Omega), L^2(R_\epsilon)$ respectively. Moreover we set $(\lambda_n^\epsilon)_{n \geq 1} = (\omega_k)_{k \geq 1} \cup (\theta_l^\epsilon)_{l \geq 1}$, where it is understood that the eigenvalues are arranged in increasing order and repeated according to their multiplicity. In particular if $\omega_k = \theta_l^\epsilon$ for some $k, l \in \mathbb{N}$, then such an eigenvalue is repeated in the sequence $(\lambda_n^\epsilon)_{n \geq 1}$ as many times as the sum of the multiplicities of ω_k and θ_l^ϵ. Let us note explicitly that the order in the sequence $(\lambda_n^\epsilon)_{n \geq 1}$ depends on ϵ. For each λ_n^ϵ we define the function $\phi_n^\epsilon \in H^2(\Omega) \oplus H^2(R_\epsilon)$ in the following way:

$$
\phi_n^\epsilon = \begin{cases} \varphi_k^\Omega, & \text{in } \Omega, \\ 0, & \text{in } R_\epsilon, \end{cases} \tag{2.1}
$$

if $\lambda_n^\epsilon = \omega_k$, for some $k \in \mathbb{N}$; otherwise

$$
\phi_n^\epsilon = \begin{cases} 0, & \text{in } \Omega, \\ \gamma_l^\epsilon, & \text{in } R_\epsilon, \end{cases} \tag{2.2}
$$

if $\lambda_n^\epsilon = \theta_l^\epsilon$, for some $l \in \mathbb{N}$. We observe that in the case $\lambda_n^\epsilon = \omega_k = \theta_l^\epsilon$ for some $k, l \in \mathbb{N}$, with ω_k of multiplicity m_1 and θ_l^ϵ of multiplicity m_2 we agree to order the eigenvalues (and the corresponding functions ϕ_n^ϵ) by listing first the m_1 eigenvalues ω_k, then the remaining m_2 eigenvalues θ_l^ϵ.

Note that $(\phi_i^\epsilon, \phi_j^\epsilon)_{L^2(\Omega_\epsilon)} = \delta_{ij}$ where δ_{ij} is the Kronecker symbol, that is $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Note also that although ϕ_n^ϵ defined by (2.2) are in $H^2(\Omega_\epsilon)$ (due to the Dirichlet boundary condition imposed in L_ϵ), the function ϕ_n^ϵ defined by (2.1) do not lie in $H^2(\Omega_\epsilon)$. To bypass this problem we define a sequence of functions in $H^2(\Omega_\epsilon)$ by setting

$$
\xi_n^\epsilon = \begin{cases} E\varphi_k^\Omega, & \text{if } \lambda_n^\epsilon = \omega_k, \\ \phi_n^\epsilon, & \text{if } \lambda_n^\epsilon = \theta_l^\epsilon, \end{cases}
$$

where E is a linear continuous extension operator mapping $H^2(\Omega)$ to $H^2(\mathbb{R}^N)$. Then it is easy to verify that for fixed i, j, we have $(\xi_i^\epsilon, \xi_j^\epsilon)_{L^2(\Omega_\epsilon)} = \delta_{ij} + o(1)$ as $\epsilon \to 0$. Then for fixed n and for ϵ small enough, $\xi_1^\epsilon, \ldots, \xi_n^\epsilon$ are linearly independent.

Now we prove an upper bound for the eigenvalues $\lambda_n(\Omega_\epsilon)$.

Theorem 1 (Upper bound). *Let $n \geq 1$ be fixed. The eigenvalues λ_n^ϵ are uniformly bounded in ϵ and*

$$
\lambda_n(\Omega_\epsilon) \leq \lambda_n^\epsilon + o(1), \quad \text{as } \epsilon \to 0. \tag{2.3}
$$

Proof. The fact that λ_n^ϵ remains bounded as $\epsilon \to 0$ is an easy consequence of the inequality

$$
\lambda_n^\epsilon \leq \omega_n < \infty, \tag{2.4}
$$

which holds by definition of λ_n^ϵ. In the sequel we write $\perp$ to denote the orthogonality in L^2, and $[f_1, \ldots, f_m]$ for the linear span of the functions $f_1, \ldots, f_m$.

By the variational characterization of the eigenvalues $\lambda_n(\Omega_\epsilon)$ we have

$$
\lambda_n(\Omega_\epsilon) = \min \left\{ \frac{\displaystyle\int_{\Omega_\epsilon} (1-\sigma)|D^2\psi|^2 + \sigma|\Delta\psi|^2 + \tau|\nabla\psi|^2 + |\psi|^2}{\displaystyle\int_{\Omega_\epsilon} |\psi|^2} \right.
$$
$$
\left. : \psi \in H^2(\Omega_\epsilon), \psi \not\equiv 0 \text{ and } \psi \perp \varphi_1^\epsilon, \ldots, \varphi_{n-1}^\epsilon \right\}. \tag{2.5}
$$

Since the functions $\xi_1^\epsilon, \dots, \xi_n^\epsilon$ are linearly independent, by a dimension argument there exists $\xi^\epsilon \in [\xi_1^\epsilon, \dots, \xi_n^\epsilon]$ such that $\|\xi^\epsilon\|_{L^2(\Omega_\epsilon)} = 1$, and $\xi^\epsilon \perp \varphi_1^\epsilon, \dots, \varphi_{n-1}^\epsilon$.

We can write $\xi^\epsilon = \sum_{i=1}^n \alpha_i \xi_i^\epsilon$, for some $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ depending on ϵ such that $\sum_{i=1}^n \alpha_i^2 = 1 + o(1)$ as $\epsilon \to 0$. By using ξ^ϵ as a test function in (2.5) we get

$$
\begin{aligned}
\lambda_n(\Omega_\epsilon) &\leq \int_{\Omega_\epsilon} (1-\sigma)|D^2\xi^\epsilon|^2 + \sigma|\Delta\xi^\epsilon|^2 + \tau|\nabla\xi^\epsilon|^2 + |\xi^\epsilon|^2 \\
&= \sum_{i=1}^n \alpha_i^2 \left(\int_{\Omega_\epsilon} (1-\sigma)|D^2\xi_i^\epsilon|^2 + \sigma|\Delta\xi_i^\epsilon|^2 + \tau|\nabla\xi_i^\epsilon|^2 + |\xi_i^\epsilon|^2 \right) \\
&\quad + \sum_{i \neq j} \alpha_i \alpha_j \left(\int_{\Omega_\epsilon} (1-\sigma)(D^2\xi_i^\epsilon : D^2\xi_j^\epsilon) + \sigma\Delta\xi_i^\epsilon \Delta\xi_j^\epsilon + \tau\nabla\xi_i^\epsilon \cdot \nabla\xi_j^\epsilon + \xi_i^\epsilon \xi_j^\epsilon \right).
\end{aligned}
\tag{2.6}
$$

By definition of ξ_i^ϵ and the absolute continuity of the Lebesgue integral, we have

$$
\int_{\Omega_\epsilon} (1-\sigma)|D^2\xi_i^\epsilon|^2 + \sigma|\Delta\xi_i^\epsilon|^2 + \tau|\nabla\xi_i^\epsilon|^2 + |\xi_i^\epsilon|^2 = \begin{cases} \omega_k + o(1), & \text{if } \exists k \text{ s.t. } \lambda_i^\epsilon = \omega_k, \\ \theta_\epsilon^l, & \text{if } \exists l \text{ s.t. } \lambda_i^\epsilon = \theta_\epsilon^l, \end{cases}
$$

which implies that $\int_{\Omega_\epsilon} (1-\sigma)|D^2\xi_i^\epsilon|^2 + \sigma|\Delta\xi_i^\epsilon|^2 + \tau|\nabla\xi_i^\epsilon|^2 + |\xi_i^\epsilon|^2 \leq \lambda_n^\epsilon + o(1)$.

Note that

$$
\sum_{i \neq j} \alpha_i \alpha_j \left(\int_{\Omega_\epsilon} (1-\sigma)(D^2\xi_i^\epsilon : D^2\xi_j^\epsilon) + \sigma\Delta\xi_i^\epsilon \Delta\xi_j^\epsilon + \tau\nabla\xi_i^\epsilon \cdot \nabla\xi_j^\epsilon + \xi_i^\epsilon \xi_j^\epsilon \right) = o(1).
$$

Hence, $\lambda_n(\Omega_\epsilon) \leq \sum_{i=1}^n \alpha_i^2(\lambda_n^\epsilon + o(1)) + o(1) \leq \lambda_n^\epsilon + o(1)$ which concludes the proof of (2.3). $\qquad\square$

Remark 1. *Note that the shape of the channel R_ϵ does not play any role in establishing the upper bound. The only fact needed is that the measure of R_ϵ tends to 0 as $\epsilon \to 0$.*

In the sequel we shall provide a lower bound for the eigenvalues $\lambda_n(\Omega_\epsilon)$. Before doing so, let us introduce some notation.

Definition 1. *Let $\sigma \in (-1, 1)$, $\tau \geq 0$. We denote by $H^2_{L_\epsilon}(R_\epsilon)$ the space obtained as the closure in $H^2(R_\epsilon)$ of $C^\infty(\overline{R_\epsilon})$ functions which vanish in a neighbourhood of L_ϵ. Furthermore, for any Lipschitz bounded open set U we define*

$$
[f]_{H^2_{\sigma,\tau}(U)} = \left| (1-\sigma)\|D^2 f\|^2_{L^2(U)} + \sigma\|\Delta f\|^2_{L^2(U)} + \tau\|\nabla f\|^2_{L^2(U)} + \|f\|^2_{L^2(U)} \right|^{1/2},
$$

for all $f \in H^2(U)$.

Note the functions u in $H^2_{L_\epsilon}(R_\epsilon)$ satisfy the conditions $u = 0$ and $\nabla u = 0$ on L_ϵ in the sense of traces.

Proposition 1. *Let $n \in \mathbb{N}$ be such that the following two conditions are satisfied:*

(i) For all $i = 1, \dots, n$,

$$
|\lambda_i^\epsilon - \lambda_i(\Omega_\epsilon)| \to 0 \qquad \text{as } \epsilon \to 0, \tag{2.7}
$$

(ii) There exists $\delta > 0$ such that

$$
\lambda_n^\epsilon \leq \lambda_{n+1}(\Omega_\epsilon) - \delta \tag{2.8}
$$

for any $\epsilon > 0$ small enough.

Let P_n be the projector from $L^2(\Omega_\epsilon)$ onto the linear span $[\phi_1^\epsilon, \ldots, \phi_n^\epsilon]$ defined by

$$P_n g = \sum_{i=1}^{n} (g, \phi_i^\epsilon)_{L^2(\Omega_\epsilon)} \phi_i^\epsilon , \qquad (2.9)$$

for all $g \in L^2(\Omega_\epsilon)$, where ϕ_i^ϵ is defined in (2.1), (2.2). Then

$$\|\varphi_i^\epsilon - P_n \varphi_i^\epsilon\|_{H^2(\Omega) \oplus H^2(R_\epsilon)} \to 0, \qquad (2.10)$$

as $\epsilon \to 0$, for all $i = 1, \ldots, n$.

Proof. By (2.3) and (2.4) we can extract a subsequence from both the sequences $(\lambda_i^\epsilon)_{\epsilon > 0}$ and $(\lambda_i(\Omega_\epsilon))_{\epsilon > 0}$ such that

$$\lambda_i^{\epsilon_k} \to \lambda_i, \quad \text{and} \quad \lambda_i(\Omega_{\epsilon_k}) \to \widehat{\lambda}_i,$$

as $k \to \infty$, for all $i = 1, \ldots, n + 1$.
By assumption we have $\lambda_i = \widehat{\lambda}_i$ for all $i = 1, \ldots, n$. Thus, by passing to the limit as $\epsilon \to 0$ in (2.3) (with n replaced by $n + 1$) and in (2.8), we get

$$\lambda_n \leq \widehat{\lambda}_{n+1} - \delta \leq \lambda_{n+1} - \delta.$$

We rewrite $\lambda_1, \ldots, \lambda_n$ without repetitions due to multiplicity in order to get a new sequence

$$\widetilde{\lambda}_1 < \widetilde{\lambda}_2 < \cdots < \widetilde{\lambda}_s = \lambda_n \qquad (2.11)$$

and set $\widetilde{\lambda}_{s+1} := \widehat{\lambda}_{n+1} \leq \lambda_{n+1}$. Thus, by assumption (2.8) we have that

$$\widetilde{\lambda}_s < \widetilde{\lambda}_{s+1}. \qquad (2.12)$$

For each $r = 1, \ldots, s$, let $\widetilde{\lambda}_r = \lambda_{i_r} = \cdots = \lambda_{j_r}$, for some $i_r \leq j_r$, $i_r, j_r \in \{1, \ldots, n\}$, where it is understood that $j_r - i_r + 1$ is the multiplicity of $\widetilde{\lambda}_r$. Furthermore, we define the eigenprojector Q_r from $L^2(\Omega_\epsilon)$ onto the linear span $[\varphi_{i_r}^\epsilon, \ldots, \varphi_{j_r}^\epsilon]$ by

$$Q_r g = \sum_{i=i_r}^{j_r} (g, \varphi_{i_r}^\epsilon)_{L^2(\Omega_\epsilon)} \varphi_{i_r}^\epsilon. \qquad (2.13)$$

We now proceed to prove the following

Claim: $\|\xi_i^{\epsilon_k} - Q_r \xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \to 0$ as $\epsilon \to 0$, for all $i_r \leq i \leq j_r$ and $r \leq s$.
Let us prove it by induction on $1 \leq r \leq s$.
If $r = 1$, we define the function

$$\chi_{\epsilon_k} = \xi_i^{\epsilon_k} - Q_1 \xi_i^{\epsilon_k} = \xi_i^{\epsilon_k} - \sum_{l=1}^{j_1} (\xi_i^{\epsilon_k}, \varphi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \varphi_l^{\epsilon_k}.$$

Then $\chi_{\epsilon_k} \in H^2(\Omega_{\epsilon_k})$, $(\chi_{\epsilon_k}, \varphi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0$ for all $l = 1, \ldots, j_1$ and by the min-max representation of $\lambda_2(\Omega_{\epsilon_k})$ we have that

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \geq \lambda_2(\Omega_{\epsilon_k}) \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} \geq \widetilde{\lambda}_2 \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} - o(1). \qquad (2.14)$$

On the other hand, it is easy to prove by definition of χ_{ϵ_k} that

$$\int_{\Omega_{\epsilon_k}} (1-\sigma)\big(D^2\chi_{\epsilon_k} : D^2\psi\big) + \sigma\Delta\chi_{\epsilon_k}\Delta\psi + \tau\nabla\chi_{\epsilon_k}\cdot\nabla\psi + \chi_{\epsilon_k}\psi\, dx$$
$$= \lambda_1(\Omega_{\epsilon_k})\int_{\Omega_{\epsilon_k}} \chi_{\epsilon_k}\psi\, dx + o(1) \quad (2.15)$$

for all $\psi \in H^2(\Omega_{\epsilon_k})$. This in particular implies that

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} = \lambda_1(\Omega_{\epsilon_k})\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1) \qquad (2.16)$$

and consequently,

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \le \widetilde{\lambda}_1\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1). \qquad (2.17)$$

Hence, inequalities (2.14), (2.17) imply that

$$\widetilde{\lambda}_2\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} - o(1) \le \widetilde{\lambda}_1\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1),$$

which implies that $\|\chi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k})} = o(1)$ (otherwise we would have $\widetilde{\lambda}_2 \le \widetilde{\lambda}_1 + o(1)$, against (2.11)). Finally, equation (2.16) implies that $[\chi_{\epsilon_k}]_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} = o(1)$, so that also $\|\chi_{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} = o(1)$.

Let $r > 1$ and assume by induction hypothesis that

$$\|\xi_i^{\epsilon_k} - Q_t\xi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \to 0 \qquad (2.18)$$

as $k \to \infty$, for all $i_t \le i \le j_t$ and for all $t = 1, \ldots, r-1$. We have to prove that (2.18) holds also for $t = r$. Let $i_r \le i \le j_r$ and let $\chi_{\epsilon_k} = \xi_i^{\epsilon_k} - Q_r\xi_i^{\epsilon_k}$. Then

$$(\chi_{\epsilon_k}, \varphi_h^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \to 0 \quad \text{as } k \to \infty, \text{ for all } h = 1, \ldots, j_r. \qquad (2.19)$$

Indeed, if $h \in \{i_r, \ldots, j_r\}$ then by definition of χ_{ϵ_k}, $(\chi_{\epsilon_k}, \varphi_h^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0$. Otherwise, if $h < i_r$, note that the function $\varphi_h^{\epsilon_k}$ satisfies

$$\int_{\Omega_{\epsilon_k}} (1-\sigma)\big(D^2\varphi_h^{\epsilon_k} : D^2\psi\big) + \sigma\Delta\varphi_h^{\epsilon_k}\Delta\psi + \tau\nabla\varphi_h^{\epsilon_k}\nabla\psi + \varphi_h^{\epsilon_k}\psi\, dx$$
$$= \lambda_h(\Omega_{\epsilon_k})\int_{\Omega_{\epsilon_k}} \varphi_h^{\epsilon_k}\psi\, dx\,,$$

for all $\psi \in H^2(\Omega_{\epsilon_k})$, briefly $B_{\Omega_{\epsilon_k}}(\varphi_h^{\epsilon_k}, \psi) = \lambda_h(\Omega_{\epsilon_k})(\varphi_h^{\epsilon_k}, \psi)_{L^2(\Omega_{\epsilon_k})}$, for all $\psi \in H^2(\Omega_{\epsilon_k})$, where B_U denotes the quadratic form associated with the operator $\Delta^2 - \tau\Delta + I$ on an open set U. Similarly, $B_{\Omega_{\epsilon_k}}(\xi_i^{\epsilon_k}, \psi) = \lambda_i^{\epsilon_k}(\xi_i^{\epsilon_k}, \psi)_{L^2(\Omega_{\epsilon_k})} + o(1)$ for all $\psi \in H^2(\Omega_{\epsilon_k})$. Thus, $\lambda_h(\Omega_{\epsilon_k})(\varphi_h^{\epsilon_k}, \xi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = \lambda_i^{\epsilon_k}(\xi_i^{\epsilon_k}, \varphi_h^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} + o(1)$ which implies

$$(\lambda_h(\Omega_{\epsilon_k}) - \lambda_i^{\epsilon_k})(\varphi_h^{\epsilon_k}, \xi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = o(1) \qquad (2.20)$$

and since $(\lambda_h(\Omega_{\epsilon_k}) - \lambda_i^{\epsilon_k}) \to (\widetilde{\lambda}_h - \widetilde{\lambda}_i) \ne 0$ by assumption, by (2.20) we deduce that $(\varphi_h^{\epsilon_k}, \xi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = o(1)$ as $\epsilon_k \to 0$, for all $h = 1, \ldots, j_r$, which implies (2.19).

As in the case $r = 1$ we may deduce that

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \ge \widetilde{\lambda}_{r+1}\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} - o(1). \qquad (2.21)$$

On the other hand, by definition of χ_{ϵ_k} we have

$$[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} \leq \widetilde{\lambda}_r \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1). \tag{2.22}$$

By (2.21), (2.22) and (2.11) it must be $\|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} = o(1)$ and by (2.22) we deduce that $[\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} = o(1)$, hence $\|\chi_{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \to 0$, as $k \to \infty$. This concludes the proof of the Claim.

Now define the projector $\widetilde{Q}_n$ from $L^2(\Omega_\epsilon)$ into the linear span $[\varphi^\epsilon_1, \ldots, \varphi^\epsilon_n]$ by

$$\widetilde{Q}_n g = \sum_{i=1}^n (g, \varphi^\epsilon_i)_{L^2(\Omega_\epsilon)} \varphi^\epsilon_i.$$

Then, as a consequence of the Claim we have that

$$\|\xi^{\epsilon_k}_i - \widetilde{Q}_n \xi^{\epsilon_k}_i\|_{H^2(\Omega_{\epsilon_k})} \to 0 \tag{2.23}$$

as $k \to \infty$, for all $i = 1, \ldots, n$. Indeed for all indexes $i = 1, \ldots, n$ there exists $1 \leq r \leq s$ such that $i_r \leq i \leq j_r$; let assume for simplicity that $r = 1$. Then we have $\|\xi^{\epsilon_k}_i - Q_1 \xi^{\epsilon_k}_i\|_{H^2(\Omega_{\epsilon_k})} \to 0$ as $k \to \infty$; and also

$$\|\xi^{\epsilon_k}_i - \widetilde{Q}_n \xi^{\epsilon_k}_i\|_{H^2(\Omega_{\epsilon_k})} \leq \|\xi^{\epsilon_k}_i - Q_1 \xi^{\epsilon_k}_i\|_{H^2(\Omega_{\epsilon_k})}$$
$$+ \sum_{l>j_1}^n \left|(\xi^{\epsilon_k}_i, \varphi^{\epsilon_k}_l)_{L^2(\Omega_{\epsilon_k})}\right| \|\varphi^{\epsilon_k}_l\|_{H^2(\Omega_{\epsilon_k})}$$

and the right-hand side tends to 0 as $k \to \infty$ because $\|\varphi^{\epsilon_k}_l\|_{H^2(\Omega_{\epsilon_k})}$ is uniformly bounded in k and $(\xi^{\epsilon_k}_i, \varphi^{\epsilon_k}_l)_{L^2(\Omega_{\epsilon_k})} \to 0$ as $k \to \infty$ (to see this it is sufficient to argue as in the proof of (2.20)). Moreover, since $\|\xi^{\epsilon_k}_i - \phi^{\epsilon_k}_i\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})} \to 0$ as $k \to \infty$ for all $i = 1, \ldots, n$, we also have $\|\phi^{\epsilon_k}_i - \widetilde{Q}_n \phi^{\epsilon_k}_i\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})} \to 0$ as $k \to \infty$, for all $i = 1, \ldots, n$. Thus $(\widetilde{Q}_n \phi^{\epsilon_k}_1, \ldots, \widetilde{Q}_n \phi^{\epsilon_k}_n)$ is a basis in $(L^2(\Omega_{\epsilon_k}))^n)$ for $[\varphi^{\epsilon_k}_1, \ldots, \varphi^{\epsilon_k}_n]$. Hence, $\varphi^{\epsilon_k}_i = \sum_{l=1}^n a^{\epsilon_k}_{li} \widetilde{Q}_n \phi^{\epsilon_k}_l$ for some coefficients $a^{\epsilon_k}_{li} = (\varphi^{\epsilon_k}_i, \phi^{\epsilon_k}_l)_{L^2(\Omega_{\epsilon_k})} + o(1)$ as $k \to \infty$. Then for all $i = 1, \ldots, n$ we have

$$\|\varphi^{\epsilon_k}_i - P_n \varphi^{\epsilon_k}_i\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})}$$
$$= \left\| \sum_{l=1}^n (\varphi^{\epsilon_k}_i, \phi^{\epsilon_k}_l)_{L^2} [\phi^{\epsilon_k}_l - \widetilde{Q}_n \phi^{\epsilon_k}_l] + o(1) \sum_{l=1}^n \widetilde{Q}_n \phi^{\epsilon_k}_l \right\|_{H^2(\Omega)\oplus H^2(R_{\epsilon_k})}$$

and the right-hand side tends to 0 as $k \to \infty$. $\qquad\square$

Remark 2. *In the previous proof one could prove that the matrix $A = (a^{\epsilon_k}_{li})_{l,i=1,\ldots,n}$ is almost orthogonal, in the sense that $AA^t = A^t A = \mathbb{I} + o(1)$ as $k \to \infty$. To prove this it is sufficient to show that the matrix $\tilde{A} = \left((\phi^{\epsilon_k}_l, \varphi^{\epsilon_k}_m)_{L^2(\Omega_{\epsilon_k})}\right)_{l,m=1,\ldots,n}$ is almost orthogonal. Let l be fixed and note that $\phi^{\epsilon_k}_l = \sum_{m=1}^n (\phi^{\epsilon_k}_l, \varphi^{\epsilon_k}_m)_{L^2(\Omega_{\epsilon_k})} \varphi^{\epsilon_k}_m + (\mathbb{I} - \widetilde{Q}_m)\phi^{\epsilon_k}_l$, hence, by (2.23) we deduce that*

$$\delta_{li} = (\phi^{\epsilon_k}_l, \phi^{\epsilon_k}_i)_{L^2(\Omega_{\epsilon_k})} = \sum_{m=1}^n (\phi^{\epsilon_k}_l, \varphi^{\epsilon_k}_m)_{L^2(\Omega_{\epsilon_k})} (\varphi^{\epsilon_k}_m, \phi^{\epsilon_k}_i)_{L^2(\Omega_{\epsilon_k})} + o(1), \tag{2.24}$$

as $k \to \infty$. Note that we can rewrite (2.24) as $\tilde{A}\tilde{A}^t = \mathbb{I} + o(1)$, and in a similar way we also get that $\tilde{A}^t \tilde{A} = \mathbb{I} + o(1)$, concluding the proof.

In the sequel we shall need the following lemma.

Lemma 1. *Let $1 \leq i \leq j \leq n$. Assume that $\widehat{\lambda} \in \mathbb{R}$ is such that, possibly passing to a subsequence, $\lambda_m(\Omega_\epsilon) \to \widehat{\lambda}$ as $\epsilon \to 0$ for all $m \in \{i, \ldots, j\}$. If $\chi_\epsilon \in [\varphi_i^\epsilon, \ldots, \varphi_j^\epsilon]$, $\|\chi_\epsilon\|_{L^2(\Omega_\epsilon)} = 1$ and $\chi_\epsilon|_\Omega \rightharpoonup \chi$ in $H^2(\Omega)$ then*

$$\int_\Omega (1 - \sigma)(D^2\chi : D^2\psi) + \sigma\Delta\chi\Delta\psi + \tau\nabla\chi \cdot \nabla\psi + \chi\psi \, dx = \widehat{\lambda} \int_\Omega \chi\psi \, dx, \qquad (2.25)$$

for all $\psi \in H^2(\Omega)$.

Proof. Since $\chi_\epsilon \in [\varphi_i^\epsilon, \ldots, \varphi_j^\epsilon]$ and $\|\chi_\epsilon\|_{L^2(\Omega_\epsilon)} = 1$ there exist coefficients $(a_l(\epsilon))_{l=i}^j$ such that $\chi_\epsilon = \sum_{l=i}^j a_l(\epsilon)\varphi_l^\epsilon$ and $\sum_{l=i}^j a_l^2(\epsilon) = 1$. Note that for all $m \in \{i, \ldots, j\}$, possibly passing to a subsequence, there exists $\widehat{\varphi}_m \in H^2(\Omega)$ such that $\varphi_m^\epsilon|_\Omega \rightharpoonup \widehat{\varphi}_m$ in $H^2(\Omega)$. Since $\chi_\epsilon|_\Omega \rightharpoonup \chi$ in $H^2(\Omega)$ by assumption, we get that $\chi = \sum_{l=i}^j a_l\widehat{\varphi}_l$ in Ω for some coefficients $(a_l)_{l=i}^j$. Let $\psi \in H^2(\Omega)$ be fixed and consider an extension $\widetilde{\psi} = E\psi \in H^2(\mathbb{R}^N)$. Then

$$\begin{aligned}
\int_{\Omega_\epsilon} &(1 - \sigma)\left(D^2\chi_\epsilon : D^2\widetilde{\psi}\right) + \sigma\Delta\chi_\epsilon\Delta\widetilde{\psi} + \tau\nabla\chi_\epsilon\nabla\widetilde{\psi} + \chi_\epsilon\widetilde{\psi} \\
&= \sum_{l=i}^j a_l(\epsilon)\left[\int_{\Omega_\epsilon} (1 - \sigma)\left(D^2\varphi_l^\epsilon : D^2\widetilde{\psi}\right) + \sigma\Delta\varphi_l^\epsilon\Delta\widetilde{\psi} + \tau\nabla\varphi_l^\epsilon\nabla\widetilde{\psi} + \varphi_l^\epsilon\widetilde{\psi}\right] \qquad (2.26) \\
&= \sum_{l=i}^j a_l(\epsilon)\lambda_l(\Omega_\epsilon)\int_{\Omega_\epsilon} \varphi_l^\epsilon\widetilde{\psi}.
\end{aligned}$$

Then it is possible to pass to the limit in both sides of (2.26) by splitting the integrals over Ω_ϵ into an integral over R_ϵ (that tends to 0 as $\epsilon \to 0$) and an integral over Ω. Moreover, the integrals over Ω will converge to the corresponding integrals in (2.25) as $\epsilon \to 0$, because of the weak convergence of χ_ϵ in $H^2(\Omega)$ and the strong convergence of $E\psi$ to ψ in $H^2(\Omega)$. $\qquad\square$

We proceed to prove the lower bound for $\lambda_n(\Omega_\epsilon)$. To do so, we need to add an extra assumption on the shape of Ω_ϵ. Hence, we introduce the following condition in the spirit of what is known for the Neumann Laplacian (see e.g., [1], [2], [6]).

Definition 2 (H-Condition). *We say that the family of dumbbell domains Ω_ϵ, $\epsilon > 0$, satisfies the H-Condition if, given functions $u_\epsilon \in H^2(\Omega_\epsilon)$ such that $\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \leq R$ for all $\epsilon > 0$, there exist functions $\bar{u}_\epsilon \in H^2_{L_\epsilon}(R_\epsilon)$ such that*

(i) $\|u_\epsilon - \bar{u}_\epsilon\|_{L^2(R_\epsilon)} \to 0$ as $\epsilon \to 0$,

(ii) $[\bar{u}_\epsilon]^2_{H^2_{\sigma,\tau}(R_\epsilon)} \leq [u_\epsilon]^2_{H^2_{\sigma,\tau}(\Omega_\epsilon)} + o(1)$ as $\epsilon \to 0$.

Recall that $[\cdot]_{H^2_{\sigma,\tau}}$ is defined above in Definition 1. We will show in Section 3 that a wide class of channels R_ϵ satisfies the H-Condition.

Theorem 2 (Lower bound). *Assume that the family of dumbbell domains Ω_ϵ, $\epsilon > 0$, satisfies the H-Condition. Then for every $n \in \mathbb{N}$ we have $\lambda_n(\Omega_\epsilon) \geq \lambda_n^\epsilon - o(1)$ as $\epsilon \to 0$.*

Proof. By Theorem 1 and its proof we know that both $\lambda_i(\Omega_\epsilon)$ and λ_i^ϵ are uniformly bounded in ϵ. Then, for each subsequence ϵ_k we can find a subsequence (which we still call ϵ_k), sequences of real numbers $(\lambda_i)_{i\in\mathbb{N}}$, $(\widehat{\lambda}_i)_{i\in\mathbb{N}}$, and sequences of $H^2(\Omega)$ functions $(\phi_i)_{i\in\mathbb{N}}$, $(\widehat{\varphi}_i)_{i\in\mathbb{N}}$, such that the following conditions are satisfied:

(i) $\lambda_i^{\epsilon_k} \longrightarrow \lambda_i$, for all $i \geq 1$;

(ii) $\lambda_i(\Omega_{\epsilon_k}) \longrightarrow \widehat{\lambda}_i$, for all $i \geq 1$;

(iii) $\xi_i^{\epsilon_k}|_\Omega \longrightarrow \phi_i$ strongly in $H^2(\Omega)$, for all $i \geq 1$;

(iv) $\varphi_i^{\epsilon_k}|_\Omega \longrightarrow \widehat{\varphi}_i$ weakly in $H^2(\Omega)$, for all $i \geq 1$;

Note that (iii) immediately follows by recalling that $\xi_i^{\epsilon_k}|_\Omega$ either it is zero or it coincides with φ_i^Ω. Then (iv) is deduced by the estimate $\|\varphi_i^{\epsilon_k}\|_{H^2(\Omega_{\epsilon_k})} \leq c \, \lambda_i(\Omega_{\epsilon_k})$ and by the boundedness of the sequence $\lambda_i(\Omega_{\epsilon_k})$, $k \in \mathbb{N}$.

We plan to prove that $\widehat{\lambda}_i = \lambda_i$ for all $i \geq 1$. We do it by induction. For $i = 1$ we clearly have $\lambda_1 = \lambda_1(\Omega) = 1 = \lambda(\Omega_{\epsilon_k})$ for all k; hence, passing to the limit as $k \to \infty$ in the right-hand side of the former equality we get $\lambda_1 = \widehat{\lambda}_1$. Then, we assume by induction hypothesis that $\widehat{\lambda}_i = \lambda_i$ for all $i = 1, \ldots, n$ and we prove that $\widehat{\lambda}_{n+1} = \lambda_{n+1}$. There are two possibilities: either $\lambda_n = \lambda_{n+1}$ or $\lambda_n < \lambda_{n+1}$. In the first case we deduce by (2.3) that

$$\lambda_n = \widehat{\lambda}_n \leq \widehat{\lambda}_{n+1} \leq \lambda_{n+1} = \lambda_n,$$

hence all the inequalities are equalities and in particular $\widehat{\lambda}_{n+1} = \lambda_{n+1}$. Consequently we can assume without loss of generality that $\lambda_n < \lambda_{n+1}$. In this case we must have $\widehat{\lambda}_{n+1} \in [\lambda_n, \lambda_{n+1}]$ because $\lambda_n = \widehat{\lambda}_n$ and $\lambda_n(\Omega_{\epsilon_k}) \leq \lambda_{n+1}(\Omega_{\epsilon_k}) \leq \lambda_{n+1}^{\epsilon_k} + o(1)$ as $k \to \infty$. Let $r = \max\{\lambda_i : i < n, \lambda_i < \lambda_n\}$. Then $\lambda_r < \lambda_{r+1} = \cdots = \lambda_n < \lambda_{n+1}$. In particular we can apply Proposition 1 with n replaced by r in order to get

$$\|\varphi_i^{\epsilon_k} - P_r \varphi_i^{\epsilon_k}\|_{H^2(\Omega) \oplus H^2(R_{\epsilon_k})} \to 0 \tag{2.27}$$

as $k \to \infty$, for all $i = 1, \ldots, r$. We now divide the proof in two steps.

Step 1: we prove that $\lambda_n < \widehat{\lambda}_{n+1}$.

Let us assume by contradiction that $\lambda_n = \widehat{\lambda}_{n+1}$; then $\widehat{\lambda}_{r+1} = \cdots = \widehat{\lambda}_n = \widehat{\lambda}_{n+1}$. Define the subspace S by $S = [\varphi_{r+1}^{\epsilon_k}, \ldots, \varphi_{n+1}^{\epsilon_k}]$. Hence, S is $(n - r + 1)$-dimensional. We then choose $\chi_{\epsilon_k} \in S$ with the following properties:

(I) $\|\chi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k})} = 1$.

(II) $\chi_{\epsilon_k} \perp \phi_{r+1}^{\epsilon_k}, \ldots, \phi_n^{\epsilon_k}$ in $L^2(\Omega_{\epsilon_k})$.

This choice is possible because $[\phi_{r+1}^{\epsilon_k}, \ldots, \phi_n^{\epsilon_k}]$ is $(n - r)$-dimensional. Moreover, we have that

$$(\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0 \tag{2.28}$$

as $k \to \infty$, for all $i = 1, \ldots, r$. To see this, recall that $\chi_{\epsilon_k} \in S$, hence

$$(\chi_{\epsilon_k}, \varphi_j^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0, \quad \forall j \leq r. \tag{2.29}$$

By (2.27) and (2.29), we have

$$(\chi_{\epsilon_k}, P_r \varphi_j^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \quad \forall j \leq r$$

as $k \to \infty$. Thus,

$$\sum_{l=1}^{r} (\varphi_j^{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} (\chi_{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \quad \forall j \leq r, \tag{2.30}$$

12

as $k \to \infty$. We can rewrite (2.30) as $A^t b \to 0$ as $k \to \infty$, where A is the matrix defined in Remark 2 and $b \in \mathbb{R}^r$ is the vector defined by $b_l = ((\chi_{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})})_l$ for all $l \in \{1, \dots, r\}$. Hence, also $AA^t b \to 0$ as $k \to \infty$ and by Remark 2 we deduce that $AA^t b = \big(\mathbb{I} + o(1)\big)b = b + o(1) \to 0$ as $k \to \infty$, since b is bounded in k. This implies that each component of b, which is $(\chi_{\epsilon_k}, \phi_l^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})}$ tends to zero as $k \to \infty$, which is (2.28).

It is now clear that (2.28) and property (II) of χ_{ϵ_k} yield

$$(\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \quad \text{for all } i = 1, \dots, n \tag{2.31}$$

as $k \to \infty$. Since $\|\chi_{\epsilon_k}\|_{H^2(\Omega)} \le C \max_{r+1 \le j \le n+1} \|\varphi_j^{\epsilon_k}\|_{H^2(\Omega)} < \infty$ there exists a function $\chi \in H^2(\Omega)$ such that possibly passing to a subsequence

$$\chi_{\epsilon_k}|_\Omega \rightharpoonup \chi \quad \text{in } H^2(\Omega), \tag{2.32}$$

as $k \to \infty$. By (2.31) and (2.32) we deduce that $(\chi, \phi_i)_{L^2(\Omega)} = 0$, for all $i = 1, \dots, n$. By Lemma 1 χ is a n-th eigenfunction of $(\Delta^2 - \tau\Delta + \mathbb{I})_{N(\sigma)}$ in Ω associated with $\widetilde{\lambda}_n$ which is orthogonal to $\phi_1, \dots, \phi_n$, among which there are all the possible n-th eigenfunctions. Since $\lambda_n < \lambda_{n+1}$, the only way to avoid a contradiction is that $\chi \equiv 0$ in Ω, that is

$$\|\chi_{\epsilon_k}\|_{L^2(\Omega)} \to 0, \qquad \|\chi_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} \to 1 \tag{2.33}$$

as $k \to \infty$. We use now the H-Condition; let us choose a sequence of functions $\overline{\chi}_{\epsilon_k} \in H^2_{L_{\epsilon_k}}(R_{\epsilon_k})$ such that $\|\chi_{\epsilon_k} - \overline{\chi}_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} \to 0$ as $k \to \infty$ and

$$[\overline{\chi}_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} \le [\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} + o(1) \tag{2.34}$$

as $k \to \infty$. Then we can extend by zero $\overline{\chi}_{\epsilon_k}$ to get a function (that we still call $\overline{\chi}_{\epsilon_k}$) in $H^2(\Omega_{\epsilon_k})$. Hence,

$$\begin{aligned} (\overline{\chi}_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} &= (\overline{\chi}_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(R_{\epsilon_k})} \\ &= (\overline{\chi}_{\epsilon_k} - \chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(R_{\epsilon_k})} + (\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} - (\chi_{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega)} \end{aligned} \tag{2.35}$$

for all $i = 1, \dots, n$. By (2.31), (2.33), and the definition of $\overline{\chi}_{\epsilon_k}$ the right hand side of (2.35) tends to 0 as $k \to \infty$, for all $i = 1, \dots, n$. Thus, $\overline{\chi}_{\epsilon_k}$ is asymptotically orthogonal to $\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}$. In particular, by the variational characterization of the eigenvalues $\lambda_i^{\epsilon_k}$ we get that

$$[\overline{\chi}_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} \ge \lambda_{n+1}^{\epsilon_k} \|\overline{\chi}_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} - o(1) \ge \lambda_{n+1} \|\overline{\chi}_{\epsilon_k}\|_{L^2(R_{\epsilon_k})} - o(1) \tag{2.36}$$

On the other hand, by (2.34) we deduce that

$$\begin{aligned} [\overline{\chi}_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(R_{\epsilon_k})} &\le [\chi_{\epsilon_k}]^2_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})} + o(1) \\ &= \lambda_n \|\chi_{\epsilon_k}\|^2_{L^2(\Omega_{\epsilon_k})} + o(1) = \lambda_n \|\overline{\chi}_{\epsilon_k}\|^2_{L^2(R_{\epsilon_k})} + o(1). \end{aligned}$$

This is a contradiction to (2.36) because $\lambda_n < \lambda_{n+1}$. Step 1 is complete.

Step 2: we prove that $\widehat{\lambda}_{n+1} = \lambda_{n+1}$.

Assume by contradiction that $\widehat{\lambda}_{n+1} < \lambda_{n+1}$. Let us note that as a consequence of Step 1 we can use Proposition 1 for the n-th eigenvalues in order to obtain

$$\|\varphi_i^{\epsilon_k} - P_n \varphi_i^{\epsilon_k}\|_{H^2(\Omega) \oplus H^2(R_{\epsilon_k})} \to 0, \tag{2.37}$$

for all $i = 1, \ldots, n$. Then we can use the same argument we used in Step 1 for χ_{ϵ_k} to show that

$$\|\varphi_{n+1}^{\epsilon_k}\|_{L^2(\Omega)} \longrightarrow 0, \tag{2.38}$$

as $k \to \infty$. To see this, just note that $\varphi_{n+1}^{\epsilon_k}$ is orthogonal to $\varphi_1^{\epsilon_k}, \ldots, \varphi_n^{\epsilon_k}$, and by (2.37) we deduce that $(\varphi_{n+1}^{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \to 0$, as $k \to \infty$, for all $i = 1, \ldots, n$. Moreover,

$$(\varphi_{n+1}^{\epsilon_k}, \phi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \tag{2.39}$$

as $k \to \infty$. Indeed, looking at the weak formulation of problem (1.2) and denoting by B_U denotes the quadratic problem associated with the operator $\Delta^2 - \tau\Delta + I$ on an open set U, we deduce both

$$B_{\Omega_{\epsilon_k}}(\varphi_{n+1}^{\epsilon_k}, \phi_{n+1}^{\epsilon_k}) = \lambda_{n+1}(\Omega_{\epsilon_k})(\varphi_{n+1}^{\epsilon_k}, \phi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} + o(1)$$

and

$$B_{\Omega_{\epsilon_k}}(\phi_{n+1}^{\epsilon_k}, \varphi_{n+1}^{\epsilon_k}) = \lambda_{n+1}^{\epsilon_k}(\phi_{n+1}^{\epsilon_k}, \varphi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} + o(1),$$

and subtracting the above equalities and passing to the limit as $k \to \infty$ we obtain $(\widehat{\lambda}_{n+1} - \lambda_{n+1}) \lim_{k\to\infty}(\phi_{n+1}^{\epsilon_k}, \varphi_{n+1}^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} = 0$, which implies (2.39). Then

$$(\varphi_{n+1}^{\epsilon_k}, \phi_i^{\epsilon_k})_{L^2(\Omega_{\epsilon_k})} \longrightarrow 0, \tag{2.40}$$

as $k \to \infty$, for all $i = 1, \ldots, n+1$. Passing to the limit in k we have $(\widehat{\varphi}_{n+1}, \phi_i)_{L^2(\Omega)} = 0$ for all $i = 1, \ldots, n+1$. However, as in Step 1 we would have $[\widehat{\varphi}_{n+1}]_{H^2_{\sigma,\tau}(\Omega)}^2 = \widehat{\lambda}_{n+1}\|\widehat{\varphi}_{n+1}\|_{L^2(\Omega)}$, which contradicts the assumption $\widehat{\lambda}_{n+1} < \lambda_{n+1}$ unless $\widehat{\varphi}_{n+1} \equiv 0$, which gives (2.38).

Now we use the H-Condition and (2.38) in order to find a function $\overline{\varphi}_{n+1}^{\epsilon_k} \in H^2_{L_{\epsilon_k}}(R_{\epsilon_k})$ such that $\|\overline{\varphi}_{n+1}^{\epsilon_k}\|_{L^2(R_{\epsilon_k})} = 1 + o(1)$ and

$$[\overline{\varphi}_{n+1}^{\epsilon_k}]_{H^2_{\sigma,\tau}(R_{\epsilon_k})}^2 \leq [\varphi_{n+1}^{\epsilon_k}]_{H^2_{\sigma,\tau}(\Omega_{\epsilon_k})}^2 + o(1) = \lambda_{n+1}(\Omega_{\epsilon_k}) + o(1) \leq \widehat{\lambda}_{n+1} + o(1),$$

as $k \to \infty$. On the other hand, by the variational characterization of $\lambda_{n+1}^{\epsilon_k}$ and by (2.38), (2.40) we deduce that $[\overline{\varphi}_{n+1}^{\epsilon_k}]_{H^2_{\sigma,\tau}(R_{\epsilon_k})}^2 \geq \lambda_{n+1}^{\epsilon_k}\|\overline{\varphi}_{n+1}^{\epsilon_k}\|_{L^2(R_{\epsilon_k})} - o(1) \geq \lambda_{n+1} - o(1)$ as $k \to \infty$, hence $\lambda_{n+1} \leq \widehat{\lambda}_{n+1}$, a contradiction. Thus it must be $\lambda_{n+1} = \widehat{\lambda}_{n+1}$. $\qquad\square$

We will say that $x_\epsilon \in (0, \infty)$ *divides the spectrum* of a family of nonnegative self-adjoint operators A_ϵ, $\epsilon > 0$, with compact resolvents in $L^2(\Omega_\epsilon)$ if there exist $\delta, M, N, \epsilon_0 > 0$ such that

$$[x_\epsilon - \delta, x_\epsilon + \delta] \cap \{\lambda_n^\epsilon\}_{n=1}^\infty = \emptyset, \quad \forall \epsilon < \epsilon_0 \tag{2.41}$$

$$x_\epsilon \leq M, \quad \forall \epsilon < \epsilon_0 \tag{2.42}$$

$$N(x_\epsilon) := \#\{\lambda_i^\epsilon : \lambda_i^\epsilon \leq x_\epsilon\} \leq N < \infty. \tag{2.43}$$

If x_ϵ divides the spectrum we define the projector P_{x_ϵ} from $L^2(\Omega_\epsilon)$ onto the linear span $[\phi_1^\epsilon, \ldots, \phi_{N(x_\epsilon)}^\epsilon]$ of the first $N(x_\epsilon)$ eigenfunctions by

$$P_{x_\epsilon}g = \sum_{i=1}^{N(x_\epsilon)} (g, \phi_i^\epsilon)_{L^2(\Omega_\epsilon)}\phi_i^\epsilon,$$

for all $g \in L^2(\Omega_\epsilon)$. Then, recalling Theorem 1 and Theorem 2 we deduce the following.

Theorem 3 ((Decomposition of the eigenvalues)). *Let Ω_ϵ, $\epsilon > 0$, be a family of dumbbell domains satisfying the H-Condition. Then the following statements hold:*

(i) $\lim_{\epsilon \to 0} |\lambda_n(\Omega_\epsilon) - \lambda_n^\epsilon| = 0$, *for all $n \in \mathbb{N}$.*

(ii) *For any x_ϵ dividing the spectrum, $\lim_{\epsilon \to 0} \|\varphi_{r_\epsilon}^\epsilon - P_{x_\epsilon} \varphi_{r_\epsilon}^\epsilon\|_{H^2(\Omega) \oplus H^2(R_\epsilon)} = 0$, for all $r_\epsilon = 1, \ldots, N(x_\epsilon)$.*

3   Proof of the H-Condition for regular dumbbells

The goal of this section is to prove that the H-Condition holds for regular dumbbell domains. More precisely, we will consider channels R_ϵ such that the profile function g has the following monotonicity property:

(MP): *there exists $\delta \in]0, 1/2[$ such that g is decreasing on $[0, \delta)$ and increasing on $(1 - \delta, 1]$.*

If (MP) is satisfied then the set $A_\epsilon = \{(x, y) \in \mathbb{R}^2 : x \in (0, \delta) \cup (1 - \delta, 1), 0 < y < \epsilon g(x)\}$ is contained in the union of the two rectangles $[0, \delta] \times [0, \epsilon g(0)]$ and $[1 - \delta, 1] \times [0, \epsilon g(1)]$. This fact will be used in the proof of the following theorem in order to control the H^2 norm of the candidate function $\overline{u}_\epsilon$ appearing in the H-Condition.

Theorem 4. *The validity of condition (MP) implies the validity of the H-Condition.*

Before writing the proof of this theorem we need to introduce some notation. First, for the sake of clarity we will consider a "one-sided" dumbbell $\Omega_\epsilon = \Omega \cup \overline{R_\epsilon}$ where Ω is a smooth bounded domain in $\mathbb{R}^2$ such that the segment $\{0\} \times [-1, 1]$ is contained in the boundary of Ω, $\Omega \cap \{x \geq 0\} = \emptyset$ and R_ϵ is defined as in (1.1). We will assume that R_ϵ satisfies the (MP) condition on $0 < x < \delta$ only. Let L_ϵ be the segment $\{0\} \times (0, \epsilon g(0))$.

For any $\gamma \in (0, 1)$, we define a function $\chi_\epsilon^\gamma \in C^{1,1}[-\epsilon^\gamma, 1]$, such that $\chi_\epsilon^\gamma(-\epsilon^\gamma) = (\chi_\epsilon^\gamma)'(-\epsilon^\gamma) = 0$, $\chi_\epsilon^\gamma(x) \equiv 1$ for all $0 \leq x \leq 1$ and such that the following bounds on the derivatives

$$\|(\chi_\epsilon^\gamma)'\|_{L^\infty(-\epsilon^\gamma, 0)} \leq \frac{c_1}{\epsilon^\gamma}, \quad \|(\chi_\epsilon^\gamma)''\|_{L^\infty(-\epsilon^\gamma, 0)} \leq \frac{c_2}{\epsilon^{2\gamma}},$$

are satisfied for some positive real numbers c_1, c_2. A possible choice for χ_ϵ^γ is

$$\chi_\epsilon^\gamma(x) = \begin{cases} -2\left(\dfrac{x + \epsilon^\gamma}{\epsilon^\gamma}\right)^3 + 3\left(\dfrac{x + \epsilon^\gamma}{\epsilon^\gamma}\right)^2, & x \in (-\epsilon^\gamma, 0), \\[4mm] 1, & x \in (0, 1), \end{cases}$$

which gives the (non-optimal) bounds $c_1 = 3/2$, $c_2 = 6$. For any $\gamma, \beta > 0$ we define the function $f_{\gamma,\beta} \in C^{1,1}(0, 1)$ by setting

$$f = f_{\gamma,\beta}(x) = \begin{cases} -\epsilon^\gamma\left(\dfrac{x}{\epsilon^\beta}\right)^2 + (\epsilon^\beta + 2\epsilon^\gamma)\left(\dfrac{x}{\epsilon^\beta}\right) - \epsilon^\gamma, & x \in (0, \epsilon^\beta), \\[4mm] x, & x \in (\epsilon^\beta, 1). \end{cases} \tag{3.1}$$

Note that f is a $C^{1,1}$-diffeomorphism from $(0, \epsilon^\beta)$ onto $(-\epsilon^\gamma, \epsilon^\beta)$. Then,

$$f'(x) = \begin{cases} 1 + 2\epsilon^{\gamma-\beta}\left(1 - \dfrac{x}{\epsilon^\beta}\right), & x \in (0, \epsilon^\beta), \\[4mm] 1, & x \in (\epsilon^\beta, 1), \end{cases}$$

and

$$f''(x) = \begin{cases} -2\epsilon^{\gamma-2\beta}, & x \in (0, \epsilon^\beta), \\ 0, & x \in (\epsilon^\beta, 1), \end{cases}$$

which implies that $|f'(x) - 1| \le 2\epsilon^{\gamma-\beta}$, for all $x \in (0,1)$, and $|f''(x)| \le 2\epsilon^{\gamma-2\beta}$, for all $x \in (0,1)$. Thus, if $\gamma > \beta$ then

$$f'(x) = 1 + o(1) \quad \text{as } \epsilon \to 0. \tag{3.2}$$

For any $\theta \in (0,1)$, we define the following sets:

$$K_\epsilon^\theta = \{(x,y) \in \Omega : -\epsilon^\theta < x < 0, \, 0 < y < \epsilon g(0)\},$$
$$\Gamma_\epsilon^\theta = \{(-\epsilon^\theta, y) : 0 < y < \epsilon g(0)\},$$
$$J_\epsilon^\theta = \{(x,y) \in R_\epsilon : 0 < x < \epsilon^\theta\},$$
$$Q_\epsilon^\theta = \{(x,y) \in \mathbb{R}^2 : 0 < x < \epsilon^\theta, \, 0 < y < \epsilon g(0)\}.$$

Finally, if $\gamma/3 < \beta < \gamma/2$, for every $u_\epsilon \in H^2(\Omega_\epsilon)$ we define the function $\bar{u}_\epsilon \in H^2(R_\epsilon)$ by setting

$$\bar{u}_\epsilon(x,y) = u_\epsilon(f(x), y)\,\chi_\epsilon^\gamma(f(x)), \tag{3.3}$$

for all $(x,y) \in R_\epsilon$. Function $\bar{u}_\epsilon$ will be used to prove the validity of the H-Condition. Before doing so, we need to prove the following proposition.

Proposition 2. *Let $\Omega_\epsilon = \Omega \cup R_\epsilon$ with R_ϵ satisfying the (MP) condition. Let $u_\epsilon \in H^2(\Omega_\epsilon)$ be such that $\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \le R$ for all $\epsilon > 0$. Then, with the notation above and for $0 < \theta < \frac{1}{3}$, we have*

$$\|u_\epsilon\|_{L^2(J_\epsilon^\theta)} = O(\epsilon^{2\theta}), \quad \|\nabla u_\epsilon\|_{L^2(J_\epsilon^\theta)} = O(\epsilon^\theta), \text{ as } \epsilon \to 0 \tag{3.4}$$

Proof. We define the function $u_\epsilon^s \in H^2(J_\epsilon^\theta)$ by setting

$$u_\epsilon^s(x,y) = -3u_\epsilon(-x,y) + 4u_\epsilon\left(-\frac{x}{2}, y\right)$$

for all $(x,y) \in J_\epsilon^\theta$. The function u_ϵ^s can be viewed as a higher order reflection of u_ϵ with respect to the y-axis. Let us note that we can estimate the L^2 norm of u_ϵ^s, of its gradient and of its derivatives of order 2, in the following way:

$$\|u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\|u_\epsilon\|_{L^2(K_\epsilon^\theta)}, \tag{3.5}$$

$$\|\nabla u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\|\nabla u_\epsilon\|_{L^2(K_\epsilon^\theta)}, \tag{3.6}$$

$$\|D^\alpha u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\|D^\alpha u_\epsilon\|_{L^2(K_\epsilon^\theta)}, \tag{3.7}$$

for any multiindex α of length 2 and for some constant C independent of ϵ. To obtain the three inequalities above, we are using that the image of K_ϵ^θ under the reflexion about the y-axis contains J_ϵ^θ. This is a consequence of (MP). Since the L^2 norms on the right-hand sides of the inequalities above are taken on a subset of Ω, we can improve the estimate of (3.5) and (3.6) using Hölder's inequality and Sobolev embeddings to obtain

$$\|u_\epsilon\|_{L^2(K_\epsilon^\theta)} \le |K_\epsilon^\theta|^{1/2}\|u_\epsilon\|_{L^\infty(\Omega)} \le c\left(\epsilon^{\theta+1}\right)^{1/2}\|u_\epsilon\|_{H^2(\Omega)} \tag{3.8}$$

and in a similar way

$$\|\nabla u_\epsilon\|_{L^2(K_\epsilon^\theta)} \le |K_\epsilon^\theta|^{\frac{1}{2}-\frac{1}{p}}\|\nabla u_\epsilon\|_{L^p(\Omega)} \le c\left(\epsilon^{\theta+1}\right)^{\frac{1}{2}-\frac{1}{p}}\|u_\epsilon\|_{H^2(\Omega)} \tag{3.9}$$

for any $2 < p < \infty$. Thus

$$\|u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\epsilon^{\frac{\theta+1}{2}}\|u_\epsilon\|_{H^2(\Omega)}, \quad \text{and} \quad \|\nabla u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\left(\epsilon^{\theta+1}\right)^{\frac{1}{2}-\frac{1}{p}}\|u_\epsilon\|_{H^2(\Omega)}. \tag{3.10}$$

We also get

$$\|D^\alpha u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le C\|u_\epsilon\|_{H^2(\Omega)}. \tag{3.11}$$

We define now the function

$$\psi_\epsilon = (u_\epsilon - u_\epsilon^s)|_{J_\epsilon^\theta} \in H^2(J_\epsilon^\theta).$$

Then $\psi_\epsilon = 0 = \nabla\psi_\epsilon$ on L_ϵ. Let us first estimate $\|\nabla u_\epsilon\|_{L^2(J_\epsilon^\theta)}$. Since we have

$$\|\nabla u_\epsilon\|_{L^2(J_\epsilon^\theta)}^2 = \sum_{i=1}^{2} \int_{J_\epsilon^\theta} \left|\frac{\partial u_\epsilon}{\partial x_i}\right|^2 dx,$$

we can directly estimate the L^2-norm of the partial derivatives. Since $\partial_{x_i}\psi_\epsilon = 0$ on L_ϵ for all $i = 1, 2$ we apply a one-dimensional Poincaré inequality in the x-direction. We proceed as follows. For each $x_2 \in (\epsilon g(\epsilon^\theta), \epsilon g(0))$ we denote by $h_\epsilon(x_2)$ the unique number such that $\epsilon g(h_\epsilon(x_2)) = x_2$ (that is, the inverse function of $\epsilon g(\cdot)$, which exists because of hypothesis (MP)). For $x_2 \in (0, \epsilon g(\epsilon^\theta))$ we define $h_\epsilon(x_2) = \epsilon^\theta$. Observe that $0 \le h_\epsilon(x_2) \le \epsilon^\theta$ and that J_ϵ^θ can be expressed as $J_\epsilon^\theta = \{(x_1, x_2) : 0 < x_2 < \epsilon g(0); 0 < x_1 < h_\epsilon(x_2)\}$. Hence, for $i = 1, 2$ we have

$$\left\|\frac{\partial\psi_\epsilon}{\partial x_i}(\cdot, x_2)\right\|_{L^2(0, h_\epsilon(x_2))}^2 \le \frac{1}{\lambda_1(h_\epsilon(x_2))}\left\|\frac{\partial}{\partial x_1}\left(\frac{\partial\psi_\epsilon}{\partial x_i}(\cdot, x_2)\right)\right\|_{L^2(0, h_\epsilon(x_2))}^2 \tag{3.12}$$

where $\lambda_1(\rho) = \left(\frac{\pi}{2}\right)^2\rho^{-2}$ is the first eigenvalue of the problem

$$\begin{cases} -v'' = \lambda v, & \text{in } (0, \rho), \\ v(0) = 0, \\ v'(\rho) = 0. \end{cases}$$

Since $0 \le h_\epsilon(x_2) \le \epsilon^\theta$, we get the bound $\lambda_1(h_\epsilon(x_2)) \ge \left(\frac{\pi}{2\epsilon^\theta}\right)^2$ and integrating in (3.12) with respect to $x_2 \in (0, \epsilon g(0))$, we get

$$\left\|\frac{\partial\psi_\epsilon}{\partial x_i}\right\|_{L^2(J_\epsilon^\theta)}^2 \le \left(\frac{2\epsilon^\theta}{\pi}\right)^2\left\|\frac{\partial^2\psi_\epsilon}{\partial x \partial x_i}\right\|_{L^2(J_\epsilon^\theta)}^2. \tag{3.13}$$

Now note that $\left\|\frac{\partial^2\psi_\epsilon}{\partial x \partial x_i}\right\|_{L^2(J_\epsilon^\theta)} \le C\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \le CR$ for all $\epsilon > 0$, where we have used (3.11). Hence we rewrite inequality (3.13) in the following way:

$$\left\|\frac{\partial\psi_\epsilon}{\partial x_i}\right\|_{L^2(J_\epsilon^\theta)} \le \frac{2}{\pi}\epsilon^\theta(CR + o(1)) = O(\epsilon^\theta) \tag{3.14}$$

as $\epsilon \to 0$, for $i = 1, 2$.

Finally, by the inequalities (3.10), (3.14) we deduce that

$$\|\nabla u_\epsilon\|_{L^2(J_\epsilon^\theta)} \le \|\nabla\psi_\epsilon\|_{L^2(J_\epsilon^\theta)} + \|\nabla u_\epsilon^s\|_{L^2(J_\epsilon^\theta)}$$

$$\le O(\epsilon^\theta) + C\left(\epsilon^{\theta+1}\right)^{\frac{1}{2}-\frac{1}{p}}\|u_\epsilon\|_{H^2(\Omega)} \le O(\epsilon^\theta), \tag{3.15}$$

where we have used that $(\theta + 1)(1/2 - 1/p) > \theta$ for large enough p.

It remains to prove that $\|u_\epsilon\|_{L^2(J_\epsilon^\theta)} = O(\epsilon^{2\theta})$ as $\epsilon \to 0$. We can repeat the argument for u_ϵ instead of $\partial_{x_i} u_\epsilon$, with the difference that now we can improve the decay of $\|\psi_\epsilon\|_{L^2(J_\epsilon^\theta)}$ by using the one-dimensional Poincaré inequality twice. More precisely we have that

$$\|\psi_\epsilon\|_{L^2(J_\epsilon^\theta)} \le \left(\frac{2}{\pi}\right)^2 \epsilon^{2\theta} \left\|\frac{\partial^2 \psi_\epsilon}{\partial x^2}\right\|_{L^2(J_\epsilon^\theta)}$$

from which we deduce $\|\psi_\epsilon\|_{L^2(J_\epsilon^\theta)} = O(\epsilon^{2\theta})$ as $\epsilon \to 0$. Hence,

$$\|u_\epsilon\|_{L^2(J_\epsilon^\theta)} \le \|\psi_\epsilon\|_{L^2(J_\epsilon^\theta)} + \|u_\epsilon^s\|_{L^2(J_\epsilon^\theta)} \le O(\epsilon^{2\theta}) + C\epsilon^{\frac{\theta+1}{2}} \|u_\epsilon\|_{H^2(\Omega)} = O(\epsilon^{2\theta}) \qquad (3.16)$$

as $\epsilon \to 0$, concluding the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We can now give a proof of Theorem 4.

Proof of Theorem 4. Let $u_\epsilon \in H^2(\Omega_\epsilon)$ be such that $\|u_\epsilon\|_{H^2(\Omega_\epsilon)} \le R$ for any $\epsilon > 0$. We prove that the H-Condition holds if we choose $\overline{u}_\epsilon$ as in (3.3) with $\gamma < 1/3$. Note that $u_\epsilon \equiv \overline{u}_\epsilon$ on $R_\epsilon \setminus J_\epsilon^\beta$. Let us first estimate $\|\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)}$. By a change of variable and by (3.2) we deduce that

$$
\begin{aligned}
\|\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)}^2 &= \int_0^{\epsilon^\beta} \int_0^{\epsilon g(x)} |(u_\epsilon \chi_\epsilon^\gamma)(f(x), y)|^2 \, dy dx \\
&= \int_{-\epsilon^\gamma}^{\epsilon^\beta} \int_0^{\epsilon g(f^{-1}(z))} |(u_\epsilon \chi_\epsilon^\gamma)(z, y)|^2 |f'(f^{-1}(z))|^{-1} \, dy dz \\
&\le (1 + o(1)) \int_{-\epsilon^\gamma}^{\epsilon^\beta} \int_0^{\epsilon g(f^{-1}(z))} |(u_\epsilon \chi_\epsilon^\gamma)(z, y)|^2 dy dz \\
&\le (1 + o(1)) \|u_\epsilon\|_{L^2(Z_\epsilon^\gamma)}^2,
\end{aligned}
\qquad (3.17)
$$

where $Z_\epsilon^\gamma = \{(x, y) \in \Omega_\epsilon : -\epsilon^\gamma < x < \epsilon^\beta, 0 < y < \epsilon g(f^{-1}(x))\}$. Note that since the function g is non increasing, then $Z_\epsilon^\gamma \subset K_\epsilon^\gamma \cup J_\epsilon^\beta$. Hence,

$$\|\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)}^2 \le (1 + o(1))(\|u_\epsilon\|_{L^2(K_\epsilon^\gamma)}^2 + \|u_\epsilon\|_{L^2(J_\epsilon^\beta)}^2). \qquad (3.18)$$

Note that the last summand in the right-hand side of (3.18) behaves as $O(\epsilon^{4\beta})$ as $\epsilon \to 0$ because of Proposition 2. Also by (3.8) with θ replaced by γ, we get

$$\|u_\epsilon\|_{L^2(K_\epsilon^\gamma)} \le c\epsilon^{\frac{\gamma+1}{2}} \|u_\epsilon\|_{H^2(\Omega)},$$

Thus,

$$\|\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)}^2 \le (1 + o(1))(O(\epsilon^{4\beta}) + O(\epsilon^{\gamma+1}) = O(\epsilon^{4\beta})$$

as $\epsilon \to 0$. We then have by Proposition 2 that

$$\|u_\epsilon - \overline{u}_\epsilon\|_{L^2(R_\epsilon)} = \|u_\epsilon - \overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} \le \|u_\epsilon\|_{L^2(J_\epsilon^\beta)} + \|\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} = O(\epsilon^{2\beta})$$

as $\epsilon \to 0$. This concludes the proof of (i) in the H-Condition.

In order to prove (ii) from Definition 2, we first need to compute $\|\nabla \bar{u}_\epsilon\|_{L^2(J_\epsilon^\beta)}$ and $\|D^2 \bar{u}_\epsilon\|_{L^2(J_\epsilon^\beta)}$. We have

$$\frac{\partial \bar{u}_\epsilon}{\partial x}(x,y) = \left[\left(\frac{\partial u_\epsilon}{\partial x}\chi_\epsilon^\gamma\right)(f(x),y) + (u_\epsilon(\chi_\epsilon^\gamma)')(f(x),y)\right]f'(x)$$

$$\frac{\partial \bar{u}_\epsilon}{\partial y}(x,y) = \left(\frac{\partial u_\epsilon}{\partial y}\chi_\epsilon^\gamma\right)(f(x),y).$$

Hence,

$$
\begin{aligned}
\|\nabla \bar{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} &\leq \|f'\|_{L^\infty}\left(\|\nabla u_\epsilon(f(\cdot),\cdot)\|_{L^2(J_\epsilon^\beta)} + \|(u_\epsilon(\chi_\epsilon^\gamma)')(f(\cdot),\cdot)\|_{L^2(J_\epsilon^\beta)}\right) \\
&\leq \|f'\|_{L^\infty}\|f'\|_{L^\infty}^{-1/2}\left(\|\nabla u_\epsilon\|_{L^2(K_\epsilon^\gamma \cup J_\epsilon^\beta)} + c_1\|\epsilon^{-\gamma}u_\epsilon\|_{L^2(K_\epsilon^\gamma)}\right) \qquad (3.19)\\
&\leq (1+o(1))\left(\|\nabla u_\epsilon\|_{L^2(K_\epsilon^\gamma)} + \|\nabla u_\epsilon\|_{L^2(J_\epsilon^\beta)} + c_1\epsilon^{-\gamma}\|u_\epsilon\|_{L^2(K_\epsilon^\gamma)}\right)
\end{aligned}
$$

where we have used the definition of χ_ϵ^γ and the change of variables $(f(x),y) \mapsto (x,y)$. By Proposition 2 we know that $\|\nabla u_\epsilon\|_{L^2(J_\epsilon^\beta)} = O(\epsilon^\beta)$ as $\epsilon \to 0$. Moreover, by (3.8), (3.9) with θ replaced by γ, we deduce that

$$\|u_\epsilon\|_{L^2(K_\epsilon^\gamma)} = O(\epsilon^{\frac{\gamma+1}{2}}), \qquad \|\nabla u_\epsilon\|_{L^2(K_\epsilon^\gamma)} = O(\epsilon^{\gamma_p}),$$

for any $p < \infty$, where we have set

$$\gamma_p = \left(\frac{1}{2} - \frac{1}{p}\right)(\gamma + 1).$$

Finally, we deduce by (3.19) that

$$\|\nabla \bar{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} \leq (1+o(1))(O(\epsilon^{\gamma_p}) + O(\epsilon^\beta) + \epsilon^{-\gamma}O(\epsilon^{\gamma_p})) = O(\epsilon^\beta) \qquad (3.20)$$

because $\gamma_p - \gamma > \beta$, for sufficiently large p (note that $\beta < (1-\gamma)/2$ for $\gamma < 1/3$).

We now estimate the L^2 norm of $D^2 \bar{u}_\epsilon$. In order to simplify our notation we write $F(x,y) = (f(x),y)$, $\chi_\epsilon^\gamma = \chi$, $\bar{u}_\epsilon = \bar{u}$, $u_\epsilon = u$ and we use the subindex notation for the partial derivatives, that is, $u_x = \frac{\partial u}{\partial x}$ and so on. First, note that

$$
\begin{aligned}
\bar{u}_{xx} &= \left[\left(u_{xx}\chi + 2u_x\chi' + u\chi''\right) \circ F\right] \cdot |f'|^2 + \left[\left(u_x\chi + u\chi'\right) \circ F\right] \cdot f'', \\
\bar{u}_{xy} &= \left[\left(u_{xy}\chi + u_y\chi'\right) \circ F\right] \cdot f', \qquad\qquad\qquad (3.21)\\
\bar{u}_{yy} &= \left(u_{yy}\chi\right) \circ F,
\end{aligned}
$$

and we may write

$$\bar{u}_{xx} = [u_{xx}\chi \circ F] \cdot |f'|^2 + R_1, \quad \bar{u}_{xy} = [u_{xy}\chi \circ F] \cdot f' + R_2, \quad \bar{u}_{yy} = u_{yy}\chi \circ F.$$

where

$$
\begin{aligned}
R_1 &= \left[\left(2u_x\chi' + u\chi''\right) \circ F\right] \cdot |f'|^2 + \left[\left(u_x\chi + u\chi'\right) \circ F\right] \cdot f'', \\
R_2 &= u_y\chi' \circ F \cdot f'.
\end{aligned}
$$

We now show that $\|R_1\|_{L^2(J_\epsilon^\beta)} = o(1)$, $\|R_2\|_{L^2(J_\epsilon^\beta)} = o(1)$ as $\epsilon \to 0$. For this, we will prove that each single term in R_1 and R_2 is $o(1)$ as $\epsilon \to 0$. Recall that $f'(x) = 1 + o(1)$ and

$f''(x) = o(1)$, $\chi' = O(\epsilon^{-\gamma})$ and $\chi'' = O(\epsilon^{-2\gamma})$ for $x \in (0, \epsilon^{\beta})$. By a change of variables, by the Sobolev Embedding Theorem and the definition of χ it is easy to deduce that

$$\|(u_x\chi') \circ F\|_{L^2(J_\epsilon^\beta)} \leq (1 + o(1))\|u_x\chi'\|_{L^2(K_\epsilon^\gamma)} \leq CR\epsilon^{\gamma_p - \gamma} = O(\epsilon^\beta)$$

$$\|(u\chi'') \circ F\|_{L^2(J_\epsilon^\beta)} \leq c_2(1 + o(1))\|u\epsilon^{-2\gamma}\|_{L^2(K_\epsilon^\gamma)} \leq CR\epsilon^{\frac{1-3\gamma}{2}}$$

$$\|(u_y\chi') \circ F\|_{L^2(J_\epsilon^\beta)} \leq c_1(1 + o(1))\|\epsilon^{-\gamma}u_y\|_{L^2(K_\epsilon^\gamma)} \leq CR\epsilon^{\gamma_p - \gamma} = O(\epsilon^\beta).$$

By (3.20) we also have

$$\|(u_x\chi + u\chi') \circ F\|_{L^2(J_\epsilon^\beta)} \leq (1 + o(1))\|\nabla\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} = O(\epsilon^\beta). \tag{3.22}$$

Hence the L^2 norms of R_1, R_2 vanish as $\epsilon \to 0$. In particular,

$$\|D^2\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} = (1 + o(1))\|D^2 u_\epsilon\|_{L^2(K_\epsilon^\gamma \cup J_\epsilon^\beta)} + O(\epsilon^{\frac{1-3\gamma}{2}}) + O(\epsilon^\beta),$$

as $\epsilon \to 0$. In a similar way we can also prove that

$$\|\Delta\overline{u}_\epsilon\|_{L^2(J_\epsilon^\beta)} = (1 + o(1))\|\Delta u_\epsilon\|_{L^2(K_\epsilon^\gamma \cup J_\epsilon^\beta)} + O(\epsilon^{\frac{1-3\gamma}{2}}) + O(\epsilon^\beta),$$

as $\epsilon \to 0$. Hence,

$$(1 - \sigma)\|D^2\overline{u}_\epsilon\|^2_{L^2(J_\epsilon^\beta)} + \sigma\|\Delta\overline{u}_\epsilon\|^2_{L^2(J_\epsilon^\beta)} + \tau\|\nabla\overline{u}_\epsilon\|^2_{L^2(J_\epsilon^\beta)}$$
$$= (1 - \sigma)\|D^2 u_\epsilon\|^2_{L^2(K_\epsilon^\gamma \cup J_\epsilon^\beta)} + \sigma\|\Delta u_\epsilon\|^2_{L^2(K_\epsilon^\gamma \cup J_\epsilon^\beta)} + o(1). \tag{3.23}$$

By adding to both handsides of (3.23) $(1 - \sigma)\|D^2\overline{u}_\epsilon\|^2_{L^2(R_\epsilon \setminus J_\epsilon^\beta)}$, $\sigma\|\Delta\overline{u}_\epsilon\|^2_{L^2(R_\epsilon \setminus J_\epsilon^\beta)}$ and the lower order term $\tau\|\nabla\overline{u}_\epsilon\|^2_{L^2(R_\epsilon \setminus J_\epsilon^\beta)}$, and keeping in account that $\overline{u}_\epsilon \equiv u_\epsilon$ on $R_\epsilon \setminus J_\epsilon^\beta$ we deduce that

$$(1 - \sigma)\|D^2\overline{u}_\epsilon\|^2_{L^2(R_\epsilon)} + \sigma\|\Delta\overline{u}_\epsilon\|^2_{L^2(R_\epsilon)} + \tau\|\nabla\overline{u}_\epsilon\|^2_{L^2(R_\epsilon)}$$
$$= (1 - \sigma)\|D^2 u_\epsilon\|^2_{L^2(K_\epsilon^\gamma \cup R_\epsilon)} + \sigma\|\Delta u_\epsilon\|^2_{L^2(K_\epsilon^\gamma \cup R_\epsilon)} + \tau\|\nabla u_\epsilon\|^2_{L^2(R_\epsilon \setminus J_\epsilon^\beta)} + o(1)$$
$$\leq (1 - \sigma)\|D^2 u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \sigma\|\Delta u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + \tau\|\nabla u_\epsilon\|^2_{L^2(\Omega_\epsilon)} + o(1), \tag{3.24}$$

as $\epsilon \to 0$, concluding the proof of (ii) in the H-Condition. Note that in (3.24), we have used the monotonicity of the quadratic form with respect to inclusion of sets. Such property is straightforward for $\sigma \in [0, 1)$. In the case $\sigma \in (-1, 0)$ it follows by observing that

$$(1 - \sigma)\left[u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2\right] + \sigma\left[u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2\right]$$
$$\geq u_{xx}^2 + u_{yy}^2 + \sigma(u_{xx}^2 + u_{yy}^2) = (1 + \sigma)(u_{xx}^2 + u_{yy}^2) > 0,$$

for all $u \in H^2(\Omega_\epsilon)$. $\qquad\square$

4   Asymptotic analysis on the thin domain

The purpose of this section is to study the convergence of the eigenvalue problem (1.9) as $\epsilon \to 0$. Since the thin domain R_ϵ is shrinking to the segment $(0, 1)$ as $\epsilon \to 0$, we plan to

identify the limiting problem in $(0, 1)$ and to prove that the resolvent operator of problem (1.9) converges as $\epsilon \to 0$ to the resolvent operator of the limiting problem in a suitable sense which guarantees the spectral convergence.

More precisely, we shall prove that the the limiting eigenvalue problem in $[0, 1]$ is

$$
\begin{cases}
\frac{1-\sigma^2}{g}(gh'')'' - \frac{\tau}{g}(gh')' + h = \theta h, & \text{in } (0, 1), \\
h(0) = h(1) = 0, \\
h'(0) = h'(1) = 0.
\end{cases} \tag{4.1}
$$

Note that the weak formulation of (4.1) is

$$
(1 - \sigma^2) \int_0^1 h''\psi'' g \, dx + \tau \int_0^1 h'\psi' g \, dx + \int_0^1 h\psi g \, dx = \theta \int_0^1 h\psi g \, dx,
$$

for all $\psi \in H_0^2(0, 1)$, where h is to be found in the Sobolev space $H_0^2(0, 1)$. In the sequel, we shall denote by $L_g^2(0, 1)$ the Hilbert space $L^2((0, 1); g(x)dx)$.

4.1 Finding the limiting problem

In order to use thin domain techniques in the spirit of [17], we need to fix a reference domain R_1 and pull-back the eigenvalue problem defined on R_ϵ onto R_1 by means of a suitable diffeomorphism.

Let R_1 be the rescaled domain obtained by setting $\epsilon = 1$ in the definition of R_ϵ (see (1.1)). For any fixed $\epsilon > 0$, let Φ_ϵ be the map from R_1 to R_ϵ defined by $\Phi_\epsilon(x', y') = (x', \epsilon y') = (x, y)$ for all $(x', y') \in R_1$. We consider the composition operator T_ϵ from $L^2(R_\epsilon; \epsilon^{-1}dxdy)$ to $L^2(R_1)$ defined by

$$
T_\epsilon u(x', y') = u \circ \Phi_\epsilon(x', y') = u(x', \epsilon y'),
$$

for all $u \in L^2(R_\epsilon)$, $(x', y') \in R_1$. We also endow the spaces $H^2(R_1)$ and $H^2(R_\epsilon)$ with the norms defined by

$$
\begin{aligned}
\|\varphi\|_{H_{\epsilon,\sigma,\tau}^2(R_1)}^2 = \int_{R_1} \Bigg((1 - \sigma)\Bigg[\left|\frac{\partial^2 \varphi}{\partial x^2}\right|^2 + \frac{2}{\epsilon^2}\left|\frac{\partial^2 \varphi}{\partial x \partial y}\right|^2 + \frac{1}{\epsilon^4}\left|\frac{\partial^2 \varphi}{\partial y^2}\right|^2 \Bigg] \\
+ \sigma\left|\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{\epsilon^2}\frac{\partial^2 \varphi}{\partial y^2}\right|^2 + \tau\Bigg[\left|\frac{\partial \varphi}{\partial x}\right|^2 + \frac{1}{\epsilon}\left|\frac{\partial \varphi}{\partial y}\right|^2 \Bigg] + |\varphi|^2 \Bigg) dxdy,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
\|\varphi\|_{H_{\sigma,\tau}^2(R_\epsilon)}^2 = \int_{R_\epsilon} \Bigg((1 - \sigma)\Bigg[\left|\frac{\partial^2 \varphi}{\partial x^2}\right|^2 + 2\left|\frac{\partial^2 \varphi}{\partial x \partial y}\right|^2 + \left|\frac{\partial^2 \varphi}{\partial y^2}\right|^2 \Bigg] \\
+ \sigma\left|\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right|^2 + \tau\Bigg[\left|\frac{\partial \varphi}{\partial x}\right|^2 + \left|\frac{\partial \varphi}{\partial y}\right|^2 \Bigg] + |\varphi|^2 \Bigg) dxdy.
\end{aligned} \tag{4.3}
$$

It is not difficult to see that if $\varphi \in H^2(R_\epsilon)$ then

$$
\|T_\epsilon \varphi\|_{H_{\epsilon,\sigma,\tau}^2(R_1)}^2 = \epsilon^{-1}\|\varphi\|_{H_{\sigma,\tau}^2(R_\epsilon)}^2.
$$

We consider the following Poisson problem with datum $f_\epsilon \in L^2(R_\epsilon)$:

$$
\begin{cases}
\Delta^2 v_\epsilon - \tau \Delta v_\epsilon + v_\epsilon = f_\epsilon, & \text{in } R_\epsilon, \\
(1 - \sigma)\frac{\partial^2 v_\epsilon}{\partial n_\epsilon^2} + \sigma \Delta v_\epsilon = 0, & \text{on } \Gamma_\epsilon, \\
\tau\frac{\partial v_\epsilon}{\partial n_\epsilon} - (1 - \sigma)\operatorname{div}_{\partial\Omega_\epsilon}(D^2 v_\epsilon \cdot n_\epsilon)_{\partial\Omega_\epsilon} - \frac{\partial(\Delta v_\epsilon)}{\partial n_\epsilon} = 0, & \text{on } \Gamma_\epsilon, \\
v = 0 = \frac{\partial v_\epsilon}{\partial n_\epsilon}, & \text{on } L_\epsilon.
\end{cases} \tag{4.4}
$$

Note that the energy space associated with Problem (4.4) is exactly $H^2_{L_\epsilon}(R_\epsilon)$. By setting $\tilde{v}_\epsilon = v_\epsilon(x', \epsilon y')$, $\tilde{f}_\epsilon = f(x', \epsilon y')$ and pulling-back problem (4.4) to R_1 by means of Φ_ϵ, we get the following equivalent problem in R_1 in the unknown $\tilde{v}_\epsilon$ (we use again the variables (x, y) instead of (x', y') to simplify the notation):

$$
\begin{cases}
\dfrac{\partial^4 \tilde{v}_\epsilon}{\partial x^4} + \dfrac{2}{\epsilon^2} \dfrac{\partial^4 \tilde{v}_\epsilon}{\partial x^2 \partial y^2} + \dfrac{1}{\epsilon^4} \dfrac{\partial^4 \tilde{v}_\epsilon}{\partial y^4} - \tau\left(\dfrac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} + \dfrac{1}{\epsilon^2} \dfrac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \right) + \tilde{v}_\epsilon = \tilde{f}_\epsilon, & \text{in } R_1, \\[2mm]
(1 - \sigma)\left(\dfrac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} \tilde{n}_x^2 + \dfrac{2}{\epsilon} \dfrac{\partial^2 \tilde{v}_\epsilon}{\partial x \partial y} \tilde{n}_x \tilde{n}_y + \dfrac{1}{\epsilon^2} \dfrac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \tilde{n}_y^2 \right) + \sigma\left(\dfrac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} + \dfrac{1}{\epsilon^2} \dfrac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \right) = 0, & \text{on } \Gamma_1, \\[2mm]
\tau\left(\dfrac{\partial \tilde{v}_\epsilon}{\partial x} \tilde{n}_x + \dfrac{1}{\epsilon} \dfrac{\partial \tilde{v}_\epsilon}{\partial y} \tilde{n}_y \right) - (1 - \sigma)\, \mathrm{div}_{\Gamma_{1,\epsilon}}(D^2_\epsilon \tilde{v}_\epsilon \cdot \tilde{n})_{\Gamma_{1,\epsilon}} - \nabla_\epsilon(\Delta_\epsilon \tilde{v}_\epsilon) \cdot \tilde{n} = 0, & \text{on } \Gamma_1, \\[2mm]
\tilde{v}_\epsilon = 0 = \dfrac{\partial \tilde{v}_\epsilon}{\partial x} n_x + \dfrac{1}{\epsilon} \dfrac{\partial \tilde{v}_\epsilon}{\partial y} \tilde{n}_y, & \text{on } L_1.
\end{cases}
\tag{4.5}
$$

Here $\tilde{n} = (\tilde{n}_x, \tilde{n}_y) = (n_x, \epsilon^{-1} n_y)$ and the operators $\Delta_\epsilon, \nabla_\epsilon$ are the standard differential operators associated with $(\partial_x, \epsilon^{-1}\partial_y)$. Moreover,

$$
\mathrm{div}_{\Gamma_{1,\epsilon}} F = \frac{\partial F_1}{\partial x} + \frac{1}{\epsilon} \frac{\partial F_2}{\partial y} - \tilde{n}_\epsilon \nabla_\epsilon F \, \tilde{n}_\epsilon
$$

and $(F)_{\Gamma_{1,\epsilon}} = F - (F, \tilde{n})\, \tilde{n}$ for any vector field $F = (F_1, F_2)$.

Assume now that the data f_ϵ, $\epsilon > 0$ are such that $(\tilde{f}_\epsilon)_{\epsilon > 0}$ is an equibounded family in $L^2(R_1)$, i.e.,

$$
\int_{R_1} |\tilde{f}_\epsilon|^2 \, dx dy' \le c, \quad \text{or equivalently} \quad \int_{R_\epsilon} |f_\epsilon|^2 dx dy \le c\epsilon,
\tag{4.6}
$$

for all $\epsilon > 0$, where c is a positive constant not depending on ϵ.

We plan to pass to the limit in (4.5) as $\epsilon \to 0$ by arguing as follows. If $\tilde{v}_\epsilon \in H^2_{L_1}(R_1)$ is the solution to problem (4.5), then we have the following integral equality

$$
(1 - \sigma) \int_{R_1} \frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{2}{\epsilon^2} \frac{\partial^2 \tilde{v}_\epsilon}{\partial x \partial y} \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{1}{\epsilon^4} \frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \frac{\partial^2 \varphi}{\partial y^2} dx
$$

$$
+ \sigma \int_{R_1} \left(\frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} + \frac{1}{\epsilon^2} \frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \right)\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{\epsilon^2} \frac{\partial^2 \varphi}{\partial y^2} \right) dx
$$

$$
+ \tau \int_{R_1} \frac{\partial \tilde{v}_\epsilon}{\partial x} \frac{\partial \varphi}{\partial x} + \frac{1}{\epsilon^2} \frac{\partial \tilde{v}_\epsilon}{\partial y} \frac{\partial \varphi}{\partial y} dx + \int_{R_1} \tilde{v}_\epsilon \varphi dx = \int_{R_1} \tilde{f}_\epsilon \varphi dx
\tag{4.7}
$$

for all $\varphi \in H^2_{L_1}(R_1)$. By choosing $\varphi = \tilde{v}_\epsilon$ we deduce the following apriori estimate:

$$
(1 - \sigma) \int_{R_1} \left| \frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} \right|^2 + \frac{2}{\epsilon^2} \left| \frac{\partial^2 \tilde{v}_\epsilon}{\partial x \partial y} \right|^2 + \frac{1}{\epsilon^4} \left| \frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \right|^2 dx + \sigma \int_{R_1} \left| \frac{\partial^2 \tilde{v}_\epsilon}{\partial x^2} + \frac{1}{\epsilon^2} \frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \right|^2 dx
$$

$$
+ \tau \int_{R_1} \left| \frac{\partial \tilde{v}_\epsilon}{\partial x} \right|^2 + \frac{1}{\epsilon^2} \left| \frac{\partial \tilde{v}_\epsilon}{\partial y} \right|^2 dx + \int_{R_1} |\tilde{v}_\epsilon|^2 dx \le \frac{1}{2} \int_{R_1} |\tilde{f}_\epsilon|^2 dx + \frac{1}{2} \int_{R_1} |\tilde{v}_\epsilon|^2 dx
\tag{4.8}
$$

for all $\epsilon > 0$. This implies that $\|\tilde{v}_\epsilon\|_{H^2_{\epsilon,\sigma,\tau}(R_1)} \le C$ for all $\epsilon > 0$, in particular $\|\tilde{v}_\epsilon\|_{H^2(R_1)} \le C(\sigma, \tau)$ for all $\epsilon > 0$; hence, there exists $v \in H^2(R_1)$ such that, up to a subsequence $\tilde{v}_\epsilon \to v$, weakly in $H^2(R_1)$, strongly in $H^1(R_1)$. Moreover from (4.8) we deduce that

$$
\left\| \frac{\partial^2 \tilde{v}_\epsilon}{\partial x \partial y} \right\|_{L^2(R_1)} \le C\epsilon, \qquad \left\| \frac{\partial \tilde{v}_\epsilon}{\partial y} \right\|_{L^2(R_1)} \le C\epsilon,
\tag{4.9}
$$

$$
\left\| \frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \right\|_{L^2(R_1)} \le C\epsilon^2,
\tag{4.10}
$$

for all $\epsilon > 0$, hence there exists $u \in L^2(R_1)$ such that, up to a subsequence

$$\frac{1}{\epsilon^2} \frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \rightharpoonup u, \text{ weakly in } L^2(R_1) \tag{4.11}$$

as $\epsilon \to 0$. By (4.9) we deduce that the limit function v is constant in y. Indeed, if we choose any function $\phi \in C_c^\infty(R_1)$, then

$$\int_{R_1} v \frac{\partial \phi}{\partial y} = \lim_{\epsilon \to 0} \int_{R_1} \tilde{v}_\epsilon \frac{\partial \phi}{\partial y} = -\lim_{\epsilon \to 0} \int_{R_1} \frac{\partial \tilde{v}_\epsilon}{\partial y} \phi = 0,$$

hence $\frac{\partial v}{\partial y} = 0$ and then $v(x, y) \equiv v(x)$ for almost all $(x, y) \in R_1$. This suggests to choose test functions ψ depending only on x in the weak formulation (4.7). Possibly passing to a subsequence, there exists $f \in L^2(R_1)$ such that

$$\tilde{f}_\epsilon \rightharpoonup f \qquad \text{in } L^2(R_1), \text{ as } \epsilon \to 0.$$

Let $\psi \in H_0^2(0, 1)$. Then $\psi \in H^2(R_1)$ (here it is understood that the function is extended to the whole of R_1 by setting $\psi(x, y) = \psi(x)$ for all $(x, y) \in R_1$) and clearly $\psi \equiv 0$ on L_1. Use ψ as a test function in (4.7), pass to the limit as $\epsilon \to 0$ and consider (4.11) to get

$$\int_0^1 \left(\frac{\partial^2 v}{\partial x^2} \frac{\partial^2 \psi}{\partial x^2} + \sigma \mathcal{M}(u) \frac{\partial^2 \psi}{\partial x^2} + \tau \frac{\partial v}{\partial x} \frac{\partial \psi}{\partial x} + v\psi \right) g(x) \, dx = \int_0^1 \mathcal{M}(f) \psi g(x) \, dx \tag{4.12}$$

for all $\psi \in H_0^2(0, 1)$. Here, the averaging operator $\mathcal{M}$ is defined from $L^2(R_1)$ to $L_g^2(0, 1)$ by

$$\mathcal{M}h(x) = \frac{1}{g(x)} \int_0^{g(x)} h(x, y) \, dy,$$

for all $h \in L^2(R_1)$ and for almost all $x \in (0, 1)$.

From (4.12) we deduce that

$$\frac{1}{g}(v''g)'' + \frac{\sigma}{g}(\mathcal{M}(u)g)'' - \frac{\tau}{g}(v'g)' + v = \mathcal{M}(f), \qquad \text{in } (0,1),$$

where the equality is understood in the sense of distributions.

Coming back to (4.7) we may also choose test functions $\varphi(x, y) = \epsilon^2 \zeta(x, y)$, where $\zeta \in H_{L_1}^2(R_1)$. Using (4.9), (4.10) and letting $\epsilon \to 0$ we deduce

$$(1 - \sigma) \int_{R_1} u \frac{\partial^2 \zeta}{\partial y^2} + \sigma \int_{R_1} \left(\frac{\partial^2 v}{\partial x^2} \frac{\partial^2 \zeta}{\partial y^2} + u \frac{\partial^2 \zeta}{\partial y^2} \right) = 0$$

which can be rewritten as

$$\int_{R_1} \left(u + \sigma \frac{\partial^2 v}{\partial x^2} \right) \frac{\partial^2 \zeta}{\partial y^2} = 0 \tag{4.13}$$

for all $\zeta \in H_{L_1}^2(R_1)$. In particular this holds for all $\zeta \in C_c^\infty(R_1)$, hence there exists the second order derivative

$$\frac{\partial^2}{\partial y^2} \left(u + \sigma \frac{\partial^2 v}{\partial x^2} \right) = 0. \tag{4.14}$$

Hence, $u(x, y) + \sigma \frac{\partial^2 v}{\partial x^2} = \psi_1(x) + \psi_2(x)y$ for almost all $(x, y) \in R_1$ and for some functions $\psi_1, \psi_2 \in L^2(R_1)$, and then (4.13) can be written as

$$\int_{R_1} (\psi_1(x) + y\psi_2(x)) \frac{\partial^2 \zeta}{\partial y^2} = 0 \tag{4.15}$$

Integrating twice by parts in y in equation (4.15) we deduce that

$$-\int_{\partial R_1} \psi_2(x)\zeta n_y dS + \int_{\partial R_1} (\psi_1(x) + y\psi_2(x))\frac{\partial \zeta}{\partial y} n_y dS = 0 \tag{4.16}$$

for all $\zeta \in H^2_{L_1}(R_1)$. We are going to choose now particular functions ζ in (4.16). Consider first $b = \frac{1}{2}\min_{x\in[0,1]} g(x) > 0$ so that the rectangle $(0,1) \times (0,b) \subset R_1$ and consider a function $\eta = \eta(y)$ with $\eta \in C^\infty(0,b)$ such that $\eta(y) = 1 + \alpha y$ for $0 < y < b/4$, where $\alpha \in \mathbb{R}$ is a parameter, and $\eta(y) \equiv 0$ for $y \in (\frac{3}{4}b, b)$. If we define $\zeta(x,y) = \theta(x)\eta(y)$ for $(x,y) \in (0,1) \times (0,b)$ where $\theta \in C_c^\infty(0,1)$ and we extend this function ζ by 0 to all of R_1, then we can use ζ in (4.16) in order to obtain

$$\alpha \int_0^1 \psi_1(x)\theta(x)dx - \int_0^1 \psi_2(x)\theta(x)dx = 0$$

for all $\alpha \in \mathbb{R}$ and all $\theta \in C_0^\infty(0,1)$. But this easily implies that $\psi_1 \equiv \psi_2 \equiv 0$. Thus, we obtain

$$u(x,y) = u(x) = -\sigma\frac{\partial^2 v(x)}{\partial x^2}$$

for almost all $(x,y) \in R_1$, i.e., $\frac{1}{\epsilon^2}\frac{\partial^2 \tilde{v}_\epsilon}{\partial y^2} \rightharpoonup -\sigma\frac{\partial^2 v(x)}{\partial x^2}$ in $L^2(R_1)$. Hence v solves the following limit problem

$$\begin{cases} \frac{1-\sigma^2}{g}(gv'')'' - \frac{\tau}{g}(gv')' + v = \mathcal{M}(f), & \text{in } (0,1) \\ v(0) = v(1) = 0, \\ v'(0) = v'(1) = 0, \end{cases} \tag{4.17}$$

and then by regularity theory we deduce that $v \in H^4(0,1)$.

4.2 Spectral convergence

We aim at proving the spectral convergence of the eigenvalues and eigenfunctions of problem (1.9) to the corresponding eigenvalues and eigenfunctions of the one dimensional problem (1.10). To do so we shall prove the compact convergence of the associated resolvent operators combined with the computations carried out in the previous section. Note that the domain R_ϵ varies with ϵ, hence the corresponding Hilbert spaces vary as well. To bypass this problem we will use the notion of $\mathcal{E}$-convergence of the resolvent operators in L^2. We recall the basic definitions and results.

Let $\mathcal{H}_\epsilon$, $\epsilon > 0$, be a family of Hilbert spaces. We assume the existence of a family of linear operators $\mathcal{E}_\epsilon \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_\epsilon)$, $\epsilon > 0$, such that

$$\|\mathcal{E}_\epsilon u_0\|_{\mathcal{H}_\epsilon} \to \|u_0\|_{\mathcal{H}_0}, \quad \text{as } \epsilon \to 0, \tag{4.18}$$

for all $u_0 \in \mathcal{H}_0$.

Definition 3. *Let $\mathcal{H}_\epsilon$ and $\mathcal{E}_\epsilon$ be as above.*

(i) *Let $u_\epsilon \in \mathcal{H}_\epsilon$, $\epsilon > 0$. We say that u_ϵ $\mathcal{E}$-converges to u as $\epsilon \to 0$ if $\|u_\epsilon - \mathcal{E}_\epsilon u\|_{\mathcal{H}_\epsilon} \to 0$ as $\epsilon \to 0$. We write $u_\epsilon \xrightarrow{E} u$.*

(ii) *Let $B_\epsilon \in \mathcal{L}(\mathcal{H}_\epsilon)$, $\epsilon > 0$. We say that B_ϵ $\mathcal{E}\mathcal{E}$-converges to a linear operator $B_0 \in \mathcal{L}(\mathcal{H}_0)$ if $B_\epsilon u_\epsilon \xrightarrow{E} B_0 u$ whenever $u_\epsilon \xrightarrow{E} u \in \mathcal{H}_0$. We write $B_\epsilon \xrightarrow{EE} B_0$.*

(iii) Let $B_\epsilon \in \mathcal{L}(\mathcal{H}_\epsilon)$, $\epsilon > 0$. We say that B_ϵ compactly converges to $B_0 \in \mathcal{L}(\mathcal{H}_0)$ (and we write $B_\epsilon \xrightarrow{C} B_0$) if the following two conditions are satisfied:

    (a) $B_\epsilon \xrightarrow{EE} B_0$ as $\epsilon \to 0$;

    (b) for any family $u_\epsilon \in \mathcal{H}_\epsilon$, $\epsilon > 0$, such that $\|u_\epsilon\|_{\mathcal{H}_\epsilon} = 1$ for all $\epsilon \in (0, 1)$, there exists a subsequence $B_{\epsilon_k} u_{\epsilon_k}$ of $B_\epsilon u_\epsilon$ and $\bar{u} \in \mathcal{H}_0$ such that $B_{\epsilon_k} u_{\epsilon_k} \xrightarrow{E} \bar{u}$ as $k \to \infty$.

For any $\epsilon \geq 0$, let A_ϵ be a (densely defined) closed, nonnegative differential operator on $\mathcal{H}_\epsilon$ with domain $\mathscr{D}(A_\epsilon) \subset \mathcal{H}_\epsilon$. We assume for simplicity that 0 does not belong to the spectrum of A_ϵ and that

$$\text{(H1): } A_\epsilon \text{ has compact resolvent } B_\epsilon := A_\epsilon^{-1} \text{ for any } \epsilon \in [0, 1),$$

and

$$\text{(H2): } B_\epsilon \xrightarrow{C} B_0, \text{ as } \epsilon \to 0.$$

Given an eigenvalue λ of A_0 we consider the generalized eigenspace $S(\lambda, A_0) := Q(\lambda, A_0)\mathcal{H}_0$, where

$$Q(\lambda, A_0) = \frac{1}{2\pi i} \int_{|\xi - \lambda| = \delta} (\xi \mathbb{I} - A_0)^{-1} d\xi$$

and $\delta > 0$ is such that the disk $\{\xi \in \mathbb{C} : |\xi - \lambda| \leq \delta\}$ does not contain any eigenvalue except for λ. In a similar way, if (H1),(H2) hold, then we can define $S(\lambda, A_\epsilon) := Q(\lambda, A_\epsilon)\mathcal{H}_\epsilon$, where

$$Q(\lambda, A_\epsilon) = \frac{1}{2\pi i} \int_{|\xi - \lambda| = \delta} (\xi \mathbb{I} - A_\epsilon)^{-1} d\xi.$$

This definition makes sense because for ϵ small enough $(\xi \mathbb{I} - A_\epsilon)$ is invertible for all ξ such that $|\xi - \lambda| = \delta$, see [4, Lemma 4.9]. Then the following theorem holds.

Theorem 5. *Let A_ϵ, A_0 be operators as above satisfying conditions (H1), (H2). Then the operators A_ϵ are spectrally convergent to A_0 as $\epsilon \to 0$, i.e., the following statements hold:*

  (i) *If λ_0 is an eigenvalue of A_0, then there exists a sequence of eigenvalues λ_ϵ of A_ϵ such that $\lambda_\epsilon \to \lambda_0$ as $\epsilon \to 0$. Conversely, if λ_ϵ is an eigenvalue of A_ϵ for all $\epsilon > 0$, and $\lambda_\epsilon \to \lambda_0$, then λ_0 is an eigenvalue of A_0.*

  (ii) *There exists $\epsilon_0 > 0$ such that the dimension of the generalized eigenspace $S(\lambda_0, A_\epsilon)$ equals the dimension of $S(\lambda_0, A_0)$, for any eigenvalue λ_0 of A_0, for any $\epsilon \in [0, \epsilon_0)$.*

  (iii) *If $\varphi_0 \in S(\lambda_0, A_0)$ then for any $\epsilon > 0$ there exists $\varphi_\epsilon \in S(\lambda_0, A_\epsilon)$ such that $\varphi_\epsilon \xrightarrow{E} \varphi_0$ as $\epsilon \to 0$.*

  (iv) *If $\varphi_\epsilon \in S(\lambda_0, A_\epsilon)$ satisfies $\|\varphi_\epsilon\|_{\mathcal{H}_\epsilon} = 1$ for all $\epsilon > 0$, then φ_ϵ, $\epsilon > 0$, has an $\mathcal{E}$-convergent subsequence whose limit is in $S(\lambda_0, A_0)$.*

Proof. See [5, Theorem 4.10]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now apply Theorem 5 to problem (1.9). To do so, we consider the following Hilbert spaces

$$\mathcal{H}_\epsilon = L^2(R_\epsilon; \epsilon^{-1} dx dy), \text{ and } \mathcal{H}_0 = L_g^2(0, 1),$$

and we denote by $\mathcal{E}_\epsilon$ the extension operator from $L^2_g(0,1)$ to $L^2(R_\epsilon; \epsilon^{-1}dxdy)$, defined by

$$(\mathcal{E}_\epsilon v)(x,y) = v(x), \tag{4.19}$$

for all $v \in L^2_g(0,1)$, for almost all $(x,y) \in R_\epsilon$. Clearly $\|E_\epsilon u_0\|_{(R_\epsilon; \epsilon^{-1}dxdy)} = \|u_0\|_{L^2_g(0,1)}$, hence $\mathcal{E}_\epsilon$ trivially satisfies property (4.18).

We consider the operators $A_\epsilon = (\Delta^2 - \tau\Delta + I)_{L_\epsilon}$, $A_0 = (\Delta^2 - \tau\Delta + I)_D$ on $\mathcal{H}_\epsilon$ and $\mathcal{H}_0$ respectively, associated with the eigenvalue problems (1.9) and (1.10), respectively. Namely, $(\Delta^2 - \tau\Delta + I)_{L_\epsilon}$ is the operator $\Delta^2 - \tau\Delta + I$ on R_ϵ subject to Dirichlet boundary conditions on L_ϵ and Neumann boundary conditions on $\partial R_\epsilon \setminus L_\epsilon$ as described in (1.9). Similarly, $(\Delta^2 - \tau\Delta + I)_D$ is the operator $\Delta^2 - \tau\Delta + I$ on $(0,1)$ subject to Dirichlet boundary conditions as described in (1.10).

Then we can prove the following

Theorem 6. *The operators* $(\Delta^2 - \tau\Delta + I)_{L_\epsilon}$ *spectrally converge to* $(\Delta^2 - \tau\Delta + I)_D$ *as* $\epsilon \to 0$, *in the sense of Theorem 5.*

Proof. In view of Theorem 5, it is sufficient to prove the following two facts:

(1) if $f_\epsilon \in L^2(R_\epsilon; \epsilon^{-1}dxdy)$ is such that $\epsilon^{-1/2}\|f_\epsilon\|_{L^2(R_\epsilon)} = 1$ for any $\epsilon > 0$, and v_ϵ is the corresponding solutions of Problem (4.4), then there exists a subsequence $\epsilon_k \to 0$ as $k \to \infty$ and $\bar{v} \in L^2_g(0,1)$ such that v_{ϵ_k} $\mathcal{E}$-converge to $\bar{v}$ as $k \to \infty$.

(2) if $f_\epsilon \in L^2(R_\epsilon; \epsilon^{-1}dxdy)$ and $f_\epsilon \xrightarrow{E} f$ as $\epsilon \to 0$, then the corresponding solutions v_ϵ of Problem (4.4) $\mathcal{E}$-converge to the solution of Problem (4.17) with datum f.

Note that (1) follows immediately from the computations in Section 4.1. Indeed, if $f_\epsilon \in L^2(R_\epsilon; \epsilon^{-1}dxdy)$ is as in (1), up to a subsequence, $\tilde{f}_\epsilon \rightharpoonup f$ in $L^2(R_1)$, which implies that $\tilde{v}_\epsilon \rightharpoonup v_0 \in H^2_0(0,1)$ in $H^2(R_1)$, where v_0 is the solution of Problem (4.17). This implies that $\|v_\epsilon - \mathcal{E}v_0\|_{L^2(R_\epsilon; \epsilon^{-1}dxdy)} \to 0$, hence (1) is proved.
In order to show (2) we take a sequence of functions $f_\epsilon \in L^2(R_\epsilon; \epsilon^{-1}dxdy)$ and $f \in L^2_g(0,1)$ such that $\epsilon^{-1/2}\|f_\epsilon - \mathcal{E}_\epsilon f\|_{L^2(R_\epsilon)} \to 0$ as $\epsilon \to 0$. After a change of variable, this is equivalent to $\|\tilde{f}_\epsilon - \mathcal{E}f\|_{L^2(R_1)} \to 0$ as $\epsilon \to 0$. Arguing as in Section 4.1, one show that the $\tilde{v}_\epsilon \rightharpoonup v \in L^2_g(0,1)$ in $H^2(R_1)$ and that v solves problem (4.17). Hence $\|\tilde{v}_\epsilon - \mathcal{E}v\|_{L^2(R_1)} \to 0$ as $\epsilon \to 0$, or equivalently, $\|v_\epsilon - \mathcal{E}_\epsilon v\|_{L^2(R_\epsilon; \epsilon^{-1}dxdy)} \to 0$ as $\epsilon \to 0$, proving (2). $\qquad\square$

5 Conclusion

Recall that the eigenpairs of problems (1.2), (1.8) are denoted by $(\lambda_n(\Omega_\epsilon), \varphi_n^\epsilon)$, $(\omega_n, \varphi_n^\Omega)_{n\geq 1}$ respectively, where the two families of eigenfunctions φ_n^ϵ, φ_n^Ω are complete orthonormal bases of the spaces $L^2(\Omega_\epsilon)$, $L^2(\Omega)$, respectively. Denote now by $(h_n, \theta_n)_{n\geq 1}$ the eigenpairs of problem (1.10) where the eigenfunctions h_n define an orthonormal basis of the space $L^2_g(0,1)$. In the spirit of the definition of λ_n^ϵ given in Section 2, we set now $(\lambda_n^0)_{n\geq 1} = (\omega_k)_{k\geq 1} \cup (\theta_l)_{l\geq 1}$, where it is understood that the eigenvalues are arranged in increasing order and repeated according to their multiplicity. For each λ_n^0 we define the function $\phi_n^0 \in H^2(\Omega) \oplus H^2(R_\epsilon)$ in the following way:

$$\phi_n^0 = \begin{cases} \varphi_k^\Omega, & \text{in } \Omega \\ 0, & \text{in } R_\epsilon, \end{cases}$$

if $\lambda_n^0 = \omega_k$, for some $k \in \mathbb{N}$; otherwise

$$\phi_n^0 = \begin{cases} 0, & \text{in } \Omega, \\ \epsilon^{-1/2} \mathcal{E}_\epsilon h_l, & \text{in } R_\epsilon \end{cases}$$

if $\lambda_n^\epsilon = \theta_l$, for some $l \in \mathbb{N}$ (here we agree to order the eigenvalues and the eigenfunctions following the same rule used in the definition of λ_n^ϵ and ϕ_n^ϵ in Section 2).

Finally, if $x > 0$ divides the spectrum $\lambda_n(\Omega_\epsilon)$ for all $\epsilon > 0$ sufficiently small (see the end of Section 2) and $N(x)$ is the number of eigenvalues with $\lambda_n(\Omega_\epsilon) \leq x$ (counting their multiplicity), we define the projector P_x^0 from $L^2(\Omega_\epsilon)$ onto the linear span $[\phi_1^0, \ldots, \phi_{N(x)}^0]$ by setting

$$P_x^0 u = \sum_{i=1}^{N(x)} (u, \phi_i^0)_{L^2(\Omega_\epsilon)} \phi_i^0$$

for all $u \in L^2(\Omega_\epsilon)$. (Note that choosing x independent of ϵ is possible by the limiting behaviour of the eigenvalues.) Then, using Theorems 3 and 6 we deduce the following.

Theorem 7. *Let Ω_ϵ, $\epsilon > 0$, be a family of dumbbell domains satisfying the H-Condition. Then the following statements hold:*

(i) $\lim_{\epsilon \to 0} |\lambda_n(\Omega_\epsilon) - \lambda_n^0| = 0$, *for all $n \in \mathbb{N}$.*

(ii) *For any x dividing the spectrum,* $\lim_{\epsilon \to 0} \|\varphi_n^\epsilon - P_x^0 \varphi_n^\epsilon\|_{H^2(\Omega) \oplus L^2(R_\epsilon)} = 0$, *for all $n = 1, \ldots, N(x)$.*

Proof. The convergence of the eigenvalues follows directly by Theorems 3 and 6. Indeed, by Theorem 3 we know that $|\lambda_n(\Omega_\epsilon) - \lambda_n^\epsilon| \to 0$ as $\epsilon \to 0$. If $\lambda_n^\epsilon = \omega_k$ for some $k \in \mathbb{N}$, for all sufficiently small ϵ, then we are done; otherwise, if $\lambda_n^\epsilon = \theta_l^\epsilon$ for some $l \in \mathbb{N}$, definitely in ϵ, by Theorem 6 we deduce that $\theta_l^\epsilon \to \theta_l$ as $\epsilon \to 0$, hence $|\lambda_n(\Omega_\epsilon) - \theta_l| \leq |\lambda_n(\Omega_\epsilon) - \theta_l^\epsilon| + |\theta_l^\epsilon - \theta_l| \to 0$ as $\epsilon \to 0$.

Consider now the convergence of the eigenfunctions. By Theorems 5, 6 it follows that for any $\epsilon > 0$ there exists an orthonormal sequence of generalized eigenfunctions δ_j^ϵ in $L^2(R_\epsilon, \epsilon^{-1} dx dy)$ associated with the eigenvalues θ_j^ϵ of problem (1.9) such that for every $j \in \mathbb{N}$

$$\|\delta_j^\epsilon - \mathcal{E}_\epsilon h_j\|_{L^2(R_\epsilon, \epsilon^{-1} dx dy)} \to 0, \tag{5.1}$$

as $\epsilon \to 0$. Recall that a generalized eigenfunction is an element of a generalized eigenspace, see Section 4.2. We set $\gamma_j^\epsilon = \epsilon^{-1/2} \delta_j^\epsilon$ and we note that γ_j^ϵ is a sequence of generalized eigenfunctions of Problem (1.9) which is orthonormal in $L^2(R_\epsilon)$, as required in Theorem 3.

Thus by Theorem 3 (ii), we deduce that

$$\left\|\varphi_n^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \epsilon^{-1/2}\mathcal{E}_\epsilon h_i\right\|_{L^2(R_\epsilon)}$$

$$\leq \left\|\varphi_n^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon)_{L^2(R_\epsilon)} \gamma_i^\epsilon\right\|_{L^2(R_\epsilon)}$$

$$+ \left\|\sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon)_{L^2(R_\epsilon)} \gamma_i^\epsilon - \sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \epsilon^{-1/2}\mathcal{E}_\epsilon h_i\right\|_{L^2(R_\epsilon)} \quad (5.2)$$

$$\leq o(1) + \left\|\sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} (\gamma_i^\epsilon - \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)\right\|_{L^2(R_\epsilon)}$$

$$+ \left\|\sum_{i=1}^{N(x)} (\varphi_n^\epsilon, \gamma_i^\epsilon - \epsilon^{-1/2}\mathcal{E}_\epsilon h_i)_{L^2(R_\epsilon)} \gamma_i^\epsilon\right\|_{L^2(R_\epsilon)}$$

$$\leq o(1) + C \sum_{i=1}^{N(x)} \|\gamma_i^\epsilon - \epsilon^{-1/2}\mathcal{E}_\epsilon h_i\|_{L^2(R_\epsilon)} = o(1) + C \sum_{i=1}^{N(x)} \|\delta_i^\epsilon - \mathcal{E}_\epsilon h_i\|_{L^2(R_\epsilon, \epsilon^{-1}dxdy)}.$$

Since the right-hand side of the last inequality in (5.2) goes to zero as $\epsilon \to 0$ by (5.1), we conclude that $\lim_{\epsilon\to 0} \|\varphi_n^\epsilon - P_x^0\varphi_n^\epsilon\|_{L^2(R_\epsilon)} = 0$. Finally, the fact that $\lim_{\epsilon\to 0} \|\varphi_n^\epsilon - P_x^0\varphi_n^\epsilon\|_{H^2(\Omega)} = 0$ follows directly from Theorem 3. $\qquad\square$

Acknowledgment

References

[1] J.M. Arrieta, *Spectral properties of Schrödinger operators under perturbations of the domain*, Doctoral Dissertation, Georgia Institute of Technology (1991).

[2] J.M. Arrieta, *Neumann eigenvalue problems on exterior perturbations of the domain*, J. Differential Equations **117** (1995).

[3] J.M. Arrieta, *Rates of eigenvalues on a dumbbell domain. Simple eigenvalue case*, Trans. Amer. Math. Soc. **347(9)** (1995), 3503–3531.

[4] J.M. Arrieta and A.N. Carvalho, *Spectral convergence and nonlinear dynamics of reaction-diffusion equations under perturbations of the domain*, J.Differential Equations **199** (2004), 143-178.

[5] J.M. Arrieta, A.N. Carvalho and G. Lozada-Cruz, *Dynamics in dumbbell domains I. Continuity of the set of equilibria*, J. Differential Equations **231(2)** (2006), 551-597.

[6] J.M. Arrieta, J.K. Hale and Q. Han, *Eigenvalue problems for nonsmoothly perturbed domains*, J. Differential Equations **91** (1991), 24-52.

[7] J.M. Arrieta and P.D. Lamberti, *Spectral stability results for higher-order operators under perturbations of the domain*, C.R. Math. Acad. Sci. Paris **351(19-20)** (2013), 725-730.

[8] J.M. Arrieta and P.D. Lamberti, *Higher order elliptic operators on variable domains. Stability results and boundary oscillations for intermediate problems*, preprint, online at arXiv:1502.04373v2 [math.AP]

[9] D. Buoso and P.D. Lamberti, *Eigenvalues of polyharmonic operators on variable domains*, ESAIM. Control, Optimisation and Calculus of Variations, **19(4)** (2013), 1225–1235.

[10] D. Buoso and L. Provenzano, *A few shape optimization results for a biharmonic Steklov problem*, J. Differential Equations **259(5)** (2015), 1778–1818.

[11] D. Buoso, L.M. Chasman and L. Provenzano, *On the stability of some isoperimetric inequalities for the fundamental tones of free plates*, to appear in the Journal of Spectral Theory.

[12] V. Burenkov and P.D. Lamberti, *Sharp spectral stability estimates via the Lebesgue measure of domains for higher order elliptic operators*, Rev. Mat. Complut. **25(2)** (2012), 435-457.

[13] L.M. Chasman, *An isoperimetric inequality for fundamental tones of free plates with nonzero Poisson's ratio*, Applicable Analysis (2015), DOI:10.1080/00036811.2015.1068299.

[14] L.M. Chasman, *An isoperimetric inequality for fundamental tones of free plates*, Comm. Math. Phys. **303(2)** (2011), 421–449.

[15] R. Courant and D. Hilbert, *Methods of mathematical physics Vol.I*, Wiley-Interscience, New York, 1953.

[16] E.B. Davies, *Spectral theory and differential operators.* Cambridge Studies in Advanced Mathematics **42**, Cambridge University Press, Cambridge, 1995.

[17] J. Hale and G. Raugel, *Partial differential equations on thin domains* Differential equations and mathematical physics, Math. Sci. Engrg. **186**(1992), 63–97.

[18] R.R. Gadyl'shin, *Scattering by bodies with narrow channels.* (Russian) Mat. Sb. **185** (1994), no. 7, 39–62; translation in Russian Acad. Sci. Sb. Math. **82** (1995), no. 2, 293-313

[19] R.R. Gadyl'shin, *On the eigenvalues of the Neumann problem in a domain with a narrow connecting channel.* (Russian) Dokl. Akad. Nauk **387** (2002), no. 1, 26-30.

[20] R.R. Gadyl'shin, *On domains with a narrow isthmus in the critical case.* Proc. Steklov Inst. Math. 2003, Asymptotic Expansions. Approximation Theory. Topology, suppl. 1, S68-S74.

[21] R.R Gadyl'shin, *On the eigenvalues of a "dumbbell with a thin handle"*. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **69** (2005), no. 2, 45–110; translation in Izv. Math. **69** (2005), no. 2, 265-329

[22] F. Gazzola, H.-C. Grunau and G. Sweers, *'Polyharmonic boundary value problems - Positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010

[23] S. Jimbo, *The singularly perturbed domain and the characterization for the eigenfunctions with Neumann boundary conditions*, J. Differential Equations **77** (1989), 322-350.

[24] S. Jimbo and S. Kosugi, *Spectra of domains with partial degeneration*, J. Math. Sci. Univ. Tokyo **16(3)** (2009), 269-414.

[25] K. Kishimoto and H.F. Weinberger, *The spatial homogeneity of stable equilibria of some reaction-diffusion systems on convex domains*, J. Differential Equations **58(1)** (1985), 15–21.

[26] P.D. Lamberti and L. Provenzano, *A maximum principle in spectral optimization problems for elliptic operators subject to mass density perturbations*, Eurasian Mathematical Journal, **4(3)** (2013), 70–83.

[27] P.D. Lamberti and M. Lanza de Cristoforis, *A real analyticity result for symmetric functions of the eigenvalues of a domain-dependent Neumann problem for the Laplace operator*, Mediterr. J. Math. **4(4)** (2007), 435-449.

[28] J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Masson et Cie, Éditeurs, Paris; Academia, Éditeurs, Prague 1967.

[29] L. Provenzano, *A note on the Neumann eigenvalues of the biharmonic operator*, to appear in Mathematical Methods in the Applied Sciences(2016).

[30] J. Taylor, *Convergence of Dirichlet eigenvalues for elliptic systems on perturbed domains*, J. Spectr. Theory **3(3)** (2013), 293-316.