Generalized 1-motivic sheaves

A. Bertapelle[1]

Dipartimento di Matematica, Università degli Studi di Padova, Via Trieste, 63, Padova I-35121, Italy

Abstract

Let k be a field of characteristic 0. We extend the definition of the category of 1-motivic sheaves, introduced by Barbieri-Viale and Kahn, allowing quotients of connected algebraic k-groups by formal k-groups. We show that its bounded derived category is equivalent to the bounded derived category of the category of generalized 1-motives with torsion.

Keywords: group schemes, 1-motives, 1-motivic sheaves, derived categories
2010 MSC: 14C15,18E30,14F42,14L15

1. Introduction

Let k be a field. There are several generalizations of the category of Deligne 1-motives $\mathcal{M}_1$, i.e., the category of complexes $[L \to G]$, in degrees $-1, 0$, where G is a semiabelian k-variety and L is an étale group scheme over k, isomorphic to a $\mathbb{Z}^r$ over the algebraic closure of k. One generalization is the abelian category of 1-motives with torsion ${}^t\mathcal{M}_1$. This category was first introduced by Barbieri-Viale, Rosenschon and Saito for k of characteristic 0. It is the localization at the class of quasi-isomorphisms of the category of complexes $[L \to G]$ where G is as above, but torsion étale group schemes may appear in degree -1. Over fields of positive characteristic p the category ${}^t\mathcal{M}_1$ was defined by Barbieri-Viale and Kahn in [3] (inverting p in the Hom groups) and by the author in [5] without inverting p-multiplication, but accepting connected finite k-group schemes in degree -1. In [3] the authors also introduced the category of 1-motivic sheaves Shv_1 and showed that $D^b(\mathrm{Shv}_1)$ is equivalent to $D^b({}^t\mathcal{M}_1)$ and, moreover, equivalent to the thick subcategory of Voevodsky's triangulated category of motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}(k)$ generated by motives of smooth curves. The first equivalence was extended in [5] to perfect fields of positive characteristic that are transcendental over their prime field, without inverting p.

Now let k be of characteristic 0. In [10] Laumon defined another generalization $\mathcal{M}_1^{\mathrm{a}}$ of $\mathcal{M}_1$. Its objects are complexes $[L \to G]$ where G is any connected commutative algebraic k-group and L is a formal k-group, in particular it may contain copies of the additive formal group $\hat{\mathbb{G}}_a$. In [2] the abelian category of generalized 1-motives with torsion ${}^t\mathcal{M}_1^{\mathrm{a}}$ was studied; it contains both ${}^t\mathcal{M}_1$ and $\mathcal{M}_1^{\mathrm{a}}$. In this paper we adapt several results in [5] to the "Laumon context". More precisely, we define the abelian category of generalized 1-motivic sheaves $\mathrm{Shv}_1^{\mathrm{a}}$, which generalizes the one in [3], and our main result is the equivalence between $D^b(\mathrm{Shv}_1^{\mathrm{a}})$ and $D^b({}^t\mathcal{M}_1^{\mathrm{a}})$. The key idea of the proof is to introduce a full subcategory $\mathcal{M}_1^{\mathrm{a}\star}$ of $\mathcal{M}_1^{\mathrm{a}}$ equivalent to a full subcategory $\mathrm{Shv}_1^{\mathrm{a}\star}$ of $\mathrm{Shv}_1^{\mathrm{a}}$, and then to reduce the equivalence result on bounded derived categories

Email address: alessandra.bertapelle@unipd.it (A. Bertapelle)

[1] Partially supported by Progetto di Eccellenza Cariparo 2008-2009 "Differential methods in Arithmetics, Geometry and Algebra".

to an equivalence result on suitable localizations of the categories $K^b(\mathcal{M}_1^{a\star})$ and $K^b(\mathrm{Shv}_1^{a\star})$. The interest in Laumon 1-motives arises from the expectation that they will play an important role in a contravariant theory of motivic cohomology for singular varieties. See [2], [11], for results in this direction.

Throughout the paper, k will always denote a field of characteristic 0.

2. Generalized 1-motives with torsion

Any formal (commutative) k-group L may be viewed as an fppf sheaf on the category of affine k-schemes and hence it provides an fppf sheaf on the category of k-schemes. It is $L = L^0 \times L^{\text{ét}}$ with L^0 the maximal connected subgroup of L and $L^{\text{ét}} := L/L^0$ étale. Furthermore, let $L^{\text{ét}}_{\text{tor}}$ be the maximal torsion subgroup of $L^{\text{ét}}$ and $L^{\text{ét}}_{\text{fr}} := L^{\text{ét}}/L^{\text{ét}}_{\text{tor}}$. We will extend constructions and results of [3] and [5] to Laumon 1-motives. In the following by connected algebraic k-group we mean a connected, smooth, commutative k-group scheme.

Definition 1. A *generalized effective* 1*-motive with torsion* is a complex of fppf sheaves on Spec(k), $M = [L \to G]$, where L is a formal k-group and G is a connected algebraic k-group. An *effective morphism* is a map of complexes.

These 1-motives admit unipotent groups, i.e., copies of $\mathbb{G}_a$, in the component of degree 0 and formal connected k-groups, i.e., powers of $\hat{\mathbb{G}}_a$, in the component of degree -1. See [12] for a generalization. Let ${}^t\mathcal{M}_1^{\text{a,eff}}$ denote the category of generalized effective 1-motives with torsion. The category of Laumon 1-motives $\mathcal{M}_1^{\text{a}}$ is the full subcategory of ${}^t\mathcal{M}_1^{\text{a,eff}}$ consisting of those $[L \to G]$ with $L^{\text{ét}}$ torsion free ([10], 5.1.1). Furthermore, if one requires L to be étale and torsion free, and G to be semiabelian, one gets the full subcategory of Deligne 1-motives $\mathcal{M}_1$.

We now define the category of 1-motives with torsion. Recall first that an effective morphism $(f, g)\colon M \to M'$ of (generalized) effective 1-motives with torsion is a *quasi-isomorphism (q.i.)* if f is a surjective map of formal k-groups, g is an isogeny, and $\ker(f) = \ker(g)$ is a finite group scheme.

Definition 2. The category of *generalized* 1*-motives with torsion* ${}^t\mathcal{M}_1^{\text{a}}$ is the localization of ${}^t\mathcal{M}_1^{\text{a,eff}}$ with respect to the multiplicative class of quasi-isomorphisms.

The abelian category ${}^t\mathcal{M}_1^{\text{a}}$ was introduced in [2] and shown to be equivalent to the category of Formal Hodge Structures of level ≤ 1 when $k = \mathbb{C}$; see also [1] for the "torsion free" case. This result generalizes equivalences in [4] and [8] for Deligne's 1-motives.

Let $\varphi = (f, g)\colon M \to M'$ be an effective map. Its kernel $[\ker^0(f) \to \ker^0(g)]$ is defined as the effective 1-motive with torsion that is the identity component of $\ker(g)$ in degree 0, and the pull-back of $\ker^0(g)$ along $\ker(f) \to \ker(g)$ in degree -1. The cokernel of φ in ${}^t\mathcal{M}_1^{\text{a,eff}}$ is the effective 1-motive with torsion $[\operatorname{coker}(f) \to \operatorname{coker}(g)]$. One sees quite easily that the localization functor ${}^t\mathcal{M}_1^{\text{a,eff}} \to {}^t\mathcal{M}_1^{\text{a}}$ preserves kernels. For cokernels this is true when φ is *strict*, i.e., if g has smooth connected kernel (cf. [3], C.5.2). In general, the fact that any effective map factors as a strict morphism followed by a quasi-isomorphism is often useful ([2], 1.4.2).

In order to give a nice description of the bounded derived category of ${}^t\mathcal{M}_1^{\text{a}}$ in Section 4 we introduce the following category.

Definition 3. Let $\mathcal{M}_1^{a\star}$ denote the full subcategory of $\mathcal{M}_1^{\text{a}}$ whose objects are those $[L \xrightarrow{u} G]$ with $\ker(u) = 0$.

Observe that there are no non-trivial quasi-isomorphisms in $\mathcal{M}_1^{a\star}$.

Lemma 4. *$\mathcal{M}_1^{a\star}$ and $\mathcal{M}_1^{a}$ are generating subcategories of ${}^t\mathcal{M}_1^{a}$ closed under kernels and closed under extensions.*

Proof. Closure under kernels is immediate and closure under extensions descends from [2], 1.4.5. To check that $\mathcal{M}_1^{a\star}$ is a generating subcategory of ${}^t\mathcal{M}_1^{a}$, we need to show that for any $M = [u\colon \hat{\mathbb{G}}_a^r \times L^{ét} \to G]$ in ${}^t\mathcal{M}_1^{a}$ there exists a 1-motive M' in $\mathcal{M}_1^{a\star}$ and an epimorphism $(f, g)\colon M' \to M$ in ${}^t\mathcal{M}_1^{a}$. By replacing, if necessary, $L^{ét}$ by a torsion free étale formal k-group that dominates it, we may assume that $L^{ét}$ is torsion free. Define $M' = [u'\colon \hat{\mathbb{G}}_a^r \times L^{ét} \to \mathbb{G}_a^r \times T \times G]$, $u'(x, y) = (\iota(x), v(y), u(x, y))$ where $\iota\colon \hat{\mathbb{G}}_a^r \to \mathbb{G}_a^r$ is the canonical embedding and $v\colon L^{ét} \to T$ is an embedding of $L^{ét}$ in a torus; clearly such an embedding exists over a finite extension k' of k, say $L^{ét}_{k'} = \mathbb{Z}^r \to \mathbb{G}^r_{m,k'}$ mapping $1 \mapsto 1$ in each component, and then consider the induced homomorphism $L^{ét} \to T := \mathfrak{R}_{k'/k}(\mathbb{G}^r_{m,k'})$ by restriction of scalars. The map $(\mathrm{id}, p_G)\colon M' \to M$, with p_G the projection on the third component, is a strict epimorphism of effective 1-motives and hence it remains an epimorphism in ${}^t\mathcal{M}_1^{a}$. Therefore $\mathcal{M}_1^{a\star}$ is a generating subcategory, and hence so too is $\mathcal{M}_1^{a}$.

3. Generalized 1-motivic sheaves

In this section we define the abelian category of generalized 1-motivic sheaves over k. It contains the category of 1-motivic sheaves introduced in [3], [5]. We will show in Section 4 that its bounded derived category is equivalent to $D^b({}^t\mathcal{M}_1^{a})$.

Definition 5. An fppf sheaf $\mathcal{F}$ on Spec(k) is a *generalized* 1*-motivic sheaf* if (i) there exists a morphism of sheaves $b\colon G \to \mathcal{F}$ with G a connected algebraic k-group and $E = \mathrm{coker}(b)$, $L = \ker(b)$ formal k-groups, (ii) the 2-fold exact sequence of sheaves

$$\eta\colon\ 0 \to L \xrightarrow{a} G \xrightarrow{b} \mathcal{F} \xrightarrow{c} E \to 0, \tag{1}$$

becomes trivial when restricted (by pull-back) to E^0. The morphism b is said to be *normalized* if $\ker(b)$ is torsion free.

Observe that one may always choose b normalized simply dividing G by the image of L_{tor}. Furthermore, condition (ii) above is superfluous if $\mathrm{Ext}^2(\hat{\mathbb{G}}_a, L) = 0$ for any formal k-group L. Unfortunately we have no proof for the vanishing of the above 2-fold extension group of fppf sheaves.

Let Shv_1^{a} denote the full subcategory of fppf sheaves on Spec(k) consisting of generalized 1-motivic sheaves and Shv_0^{a} the subcategory consisting of those $\mathcal{F}$ with $G = 0$. The latter category is equivalent to the category of formal k-groups.

Examples 6. • The category $\mathrm{Shv}_1^{\mathrm{fppf}}$ introduced in [5] is equivalent to the full subcategory of Shv_1^{a} consisting of those $\mathcal{F}$ with G semiabelian, L and E étale.

• Let $M = [L \to G]$ be a Deligne 1-motive over k. It is shown in [6], 5.12, that for $M^\natural = [u\colon L \to \mathbf{G}^\natural]$ a universal $\mathbb{G}_a$-extension of M the sheaf coker(u) is isomorphic to the sheaf of $\natural$-extensions of M^* by $\mathbb{G}_m$ (i.e., $\mathbb{G}_m$-extensions of the dual 1-motive M^* endowed with a connection). Since $\mathbf{G}^\natural$ is a connected algebraic k-group (but, in general, not a semiabelian variety), and L is a discrete group, coker(u) is an example of a generalized 1-motivic sheaf that is not 1-motivic in the sense of [5], 2.1.

• Let A, A', be dual abelian varieties over k and $\hat{A}'$ the completion of A' along the identity. The Laumon 1-motive $[\hat{A}' \xrightarrow{\iota} A']$, for ι the canonical embedding, is the Cartier dual of $[0 \to A^\natural]$ where $A^\natural$ is the universal $\mathbb{G}_a$-extension of A (*cf.* [10]). The sheaf coker(ι) is a generalized 1-motivic sheaf.

Propositions 3.2.3 in [3] and 2.3 in [5] generalize easily.

Proposition 7. *(i) A morphism in* $\mathrm{Shv}_1^{\mathrm{a}}$ *is uniquely determined by a morphism of the complexes* (1) *with normalized b's.*

(ii) Given a generalized 1*-motivic sheaf* $\mathcal{F}$*, a morphism* $b\colon G \to \mathcal{F}$ *as above with b normalized is uniquely determined by* $\mathcal{F}$.

(iii) Let $\varphi\colon \mathcal{F}_1 \to \mathcal{F}_2$ *be a morphism of fppf sheaves, where* $\mathcal{F}_i$ *fits into an exact sequence* $\eta_i\colon 0 \to L_i \to G_i \to \mathcal{F}_i \to E_i \to 0$*,* $i = 1, 2$*, of the type* (1)*. Suppose that* φ *induces an injective map* $\varphi_L\colon L_1 \to L_2$*; if* $\mathcal{F}_2$ *is generalized* 1*-motivic, then so too is* $\mathcal{F}_1$*. Suppose that* $\varphi_E\colon E_1 \to E_2$*, induced by* φ*, is surjective; if* $\mathcal{F}_1$ *is generalized* 1*-motivic, then the same holds for* $\mathcal{F}_2$.

(iv) $\mathrm{Shv}_1^{\mathrm{a}}$ *and* $\mathrm{Shv}_0^{\mathrm{a}}$ *are exact abelian subcategories of the category of fppf sheaves on* $\mathrm{Spec}(k)$.

Proof. Assertions (i),(ii) and (iv) are proved as in [5], 2.3, using [2], Lemma A.4.5, for the vanishing of $\mathrm{Ext}^1(G_1, L_2)$. For (iii): Let η_i^0 denote the pull-back of η_i along $E_i^0 \to E_i$. Then η_1^0 maps term by term to η_2^0. Suppose $\mathcal{F}_2$ generalized 1-motivic. Then η_2^0 is trivial and the same holds for the push-out of η_1^0 along φ_L. By the injectivity of φ_L we deduce that η_1^0 is obtained from an extension of E_1^0 by the formal k-group L_2/L_1. Hence it is trivial. The proof of the other case is analogous.

Definition 8. Let $\mathrm{Shv}_1^{\mathrm{a}\star}$ be the full subcategory of $\mathrm{Shv}_1^{\mathrm{a}}$ consisting of those generalized 1-motivic sheaves such that $\mathrm{coker}(b) = 0$, b as in (1).

It follows from (1) that any generalized 1-motivic sheaf $\mathcal{F}$ can be viewed as an extension of a formal k-group E by a 1-motivic sheaf $\mathcal{F}^\star$ in $\mathrm{Shv}_1^{\mathrm{a}\star}$. The importance of the category $\mathrm{Shv}_1^{\mathrm{a}\star}$ is due to the following equivalence result:

Proposition 9. *The functor* H_0 *mapping a generalized effective* 1*-motive with torsion* $[L \to G]$ *to the generalized* 1*-motivic sheaf* G/L *provides an equivalence between* $\mathcal{M}_1^{\mathrm{a}\star}$ *and* $\mathrm{Shv}_1^{\mathrm{a}\star}$.

Proof. It follows from Proposition 7 that $\mathrm{H}_0\colon \mathcal{M}_1^{\mathrm{a}\star} \to \mathrm{Shv}_1^{\mathrm{a}\star}$ is fully faithful. Now, given a 1-motivic sheaf $\mathcal{F}$ in $\mathrm{Shv}_1^{\mathrm{a}\star}$ and a normalized morphism $b\colon G \to \mathcal{F}$ as in (1), the 1-motive $[u\colon \ker(b) \to G]$ satisfies $\ker(u) = 0$ and $\mathrm{coker}(u) = \mathcal{F}$. Hence H_0 is essentially surjective.

Before proving that the category $\mathrm{Shv}_1^{\mathrm{a}\star}$ has the "dual" property of $\mathcal{M}_1^{\mathrm{a}\star}$, i.e., it is a cogenerating subcategory of $\mathrm{Shv}_1^{\mathrm{a}}$, we need some vanishing results and a different description of 1-motivic sheaves.

Lemma 10. *Let* $\mathcal{F}$ *be a generalized* 1*-motivic sheaf and* $\eta\colon 0 \to L \to G \to \mathcal{F} \to E \to 0$ *the exact sequence in* (1)*. Then, there exists an epimorphism* $\varphi\colon \tilde{E} \to E$ *of formal k-groups,* $\tilde{E}$ *torsion free, such that the pull-back of* η *along* φ *is isomorphic to the trivial extension.*

Proof. The case E connected follows from condition (ii) in Definition 5. We may then assume $E = E^{\text{ét}}$ and, up to an epimorphism, torsion free. Let $f\colon \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ be a finite Galois extension such that f^*E is constant free and there exists a section σ of the epimorphism $f^*\mathcal{F} \to f^*E$. Clearly k' exists if L is étale. For $L = \hat{\mathbb{G}}_a$ we have $\mathrm{H}^1(k, G) = \mathrm{H}^1(k, G/L)$ because $\mathrm{H}^i(k, \hat{\mathbb{G}}_a) = 0, i = 1, 2$, using the spectral sequence $\mathrm{H}^p(k, \underline{\mathrm{Ext}}^q(\mathbb{G}_a, \mathbb{G}_m)) \Rightarrow \mathrm{Ext}_k^{p+q}(\mathbb{G}_a, \mathbb{G}_m)$ and the vanishings $\underline{\mathrm{Ext}}^1(\mathbb{G}_a, \mathbb{G}_m) = 0$ and $\mathrm{Ext}_k^n(\mathbb{G}_a, \mathbb{G}_m) = 0$ for any n (cf. [7]). Let $\varphi\colon \tilde{E} := f_*f^*E \to E$ be the trace map. Then the composition of $f_*\sigma\colon \tilde{E} \to f_*f^*\mathcal{F}$ with the trace map $f_*f^*\mathcal{F} \to \mathcal{F}$ is a lifting of φ and hence we have the vanishing of the pull-back of η along φ.

Lemma 11. *Let G be a connected algebraic k-group. Then* $\mathrm{Ext}^1(\hat{\mathbb{G}}_a, G) = 0$.

Proof. The cases $G = \mathbb{G}_a, \mathbb{G}_m$, were treated in [2], A.4.6. The proof there extends to any connected algebraic k-group G showing that $\mathrm{Ext}^1(\hat{\mathbb{G}}_a, G) = \mathrm{Ext}^1(\hat{\mathbb{G}}_a, \hat{G}) = 0$ with $\hat{G}$ the formal completion along the identity of G.

We will need later the following description of generalized 1-motivic sheaves as quotients of algebraic k-groups (non necessarily connected) by formal k-groups (cf. [5], 4.7).

Lemma 12. *An fppf sheaf $\mathcal{F}$ on* $\mathrm{Spec}(k)$ *is generalized* 1*-motivic if and only if* $\mathcal{F} = \mathrm{coker}(u\colon F_1 \to F_0)$ *where F_1 is a formal k-group, F_0 is an extension of a formal k-group by a connected algebraic k-group G, and u is a monomorphism.*

Proof. To prove sufficiency, let b be the composition $G \to F_0 \to \mathcal{F}$. Then $\mathcal{F}$ fits into an exact sequence $\eta\colon\ 0 \to L \to G \to \mathcal{F} \to E \to 0$ as in (1) whose pull-back along $F_0/G \to E$ is trivial. Applying Proposition 7 (iii) one gets that $\mathcal{F}$ is 1-motivic.

We now prove necessity of the condition. Let $\mathcal{F}$ be a generalized 1-motivic sheaf and consider the associated exact sequence η as in (1). By Lemma 10 there exists an epimorphism $\psi\colon \tilde{E} \to E$ of formal k-groups such that

$$\psi^*(\eta)\colon 0 \to L \to G \to \tilde{\mathcal{F}} \to \tilde{E} \to 0$$

is isomorphic to the trivial 2-fold exact sequence and hence there exists an extension F_0 of $\tilde{E}$ by G such that $\tilde{\mathcal{F}} = F_0/L$. Then the composition $F_0 \to \tilde{\mathcal{F}} \to \mathcal{F}$ is an epimorphism whose kernel F_1 is a formal k-group. This completes the proof.

We are now ready to prove the key result on $\mathrm{Shv}_1^{\mathrm{a}\star}$.

Lemma 13. $\mathrm{Shv}_1^{\mathrm{a}\star}$ *is a cogenerating subcategory of* $\mathrm{Shv}_1^{\mathrm{a}}$*, closed under cokernels and closed under extensions.*

Proof. The only non trivial fact is that $\mathrm{Shv}_1^{\mathrm{a}\star}$ is cogenerating, i.e., for any generalized 1-motivic sheaf $\mathcal{F}$ there exists a $\mathcal{F}'$ in $\mathrm{Shv}_1^{\mathrm{a}\star}$ and a monomorphism $\varphi\colon \mathcal{F} \to \mathcal{F}'$. To prove this, consider a presentation $F_1 \to F_0$ of $\mathcal{F}$ as in Lemma 12. It is sufficient to prove that F_0 embeds in a 1-motivic sheaf $\mathcal{F}_0$ in $\mathrm{Shv}_1^{\mathrm{a}\star}$ and then set $\mathcal{F}' = \mathcal{F}_0/F_1$. So assume F_0 is extension of a formal k-group E by G. If $E = E^0 \cong \hat{\mathbb{G}}_a^s$, then F_0 is the product $G \times E^0$ by Lemma 11 and we can embed F_0 into $G \times \mathbb{G}_a^s$. If $E = E^{\text{ét}}$ the proof works as the analogous one in [5], 2.8. The general case is obtained by devissage from the connected and étale cases.

4. Equivalence on bounded derived categories

We can now generalize the results in [3], 3.9.2, and [5], 4.3, on the equivalence of bounded derived categories to Laumon 1-motives. In view of results in Sections 2 and 3 the main arguments used in [5] §4 still apply.

Lemma 14. *(i) Let $N^b(\mathrm{Shv}_1^{\mathrm{a}\star})$ be the full subcategory of $K^b(\mathrm{Shv}_1^{\mathrm{a}\star})$ consisting of complexes that are acyclic as complexes of generalized* 1*-motivic sheaves. The natural functor* $K^b(\mathrm{Shv}_1^{\mathrm{a}\star})/N^b(\mathrm{Shv}_1^{\mathrm{a}\star}) \to D^b(\mathrm{Shv}_1^{\mathrm{a}})$ *is an equivalence of categories.*

(ii) Let $N^b(\mathcal{M}_1^{a\star})$ (respectively $N^b(\mathcal{M}_1^{a})$) denote the full subcategory of $K^b(\mathcal{M}_1^{a\star})$ (respectively of $K^b(\mathcal{M}_1^{a})$) consisting of complexes that are acyclic as complexes of generalised 1-motives with torsion. The natural functors $K^b(\mathcal{M}_1^{a\star})/N^b(\mathcal{M}_1^{a\star}) \to D^b({}^t\mathcal{M}_1^{a})$, $K^b(\mathcal{M}_1^{a})/N^b(\mathcal{M}_1^{a}) \to D^b({}^t\mathcal{M}_1^{a})$, are equivalences of categories.

Proof. Assertion (i) follows immediately from Lemma 13 and [9], Lemma 13.2.2. Assertion (ii) for $\mathcal{M}_1^{a\star}$ is "dual" to the (i). The"dual" conditions required in [9], 13.2.2 ii), are satisfied thanks to Lemma 4. Furthermore, the "dual" statement of [9], 13.2.1, holds as well, and finally one applies [9], 10.2.7 ii). Since Lemma 4 applies to $\mathcal{M}_1^{a}$ as well, we can replace $\mathcal{M}_1^{a\star}$ by $\mathcal{M}_1^{a}$ in the proof.

To conclude the comparison between the bounded derived categories of ${}^t\mathcal{M}_1^{a}$ and Shv_1^{a} there remains the verification that the functor H_0 in Proposition 9 induces an equivalence on localizations.

Lemma 15. *Let $M^\bullet$ be a complex in $K^b(\mathcal{M}_1^{a\star})$. Then $M^\bullet \in N^b(\mathcal{M}_1^{a\star})$ if and only if $\mathrm{H}_0(M^\bullet) \in N^b(\mathrm{Shv}_1^{a\star})$. In particular H_0 induces an equivalence of categories*

$$K^b(\mathcal{M}_1^{a\star})/N^b(\mathcal{M}_1^{a\star}) \to K^b(\mathrm{Shv}_1^{a\star})/N^b(\mathrm{Shv}_1^{a\star}).$$

Proof. One may consider the proof in [5], 3.5, and replace $\mathcal{M}_1$ with $\mathcal{M}_1^{a}$ and Shv_1 with Shv_1^{a}. Since [5], Lemma 1.15, Proposition 2.3, Theorem 1.12, used there have their "Laumon" counterparts, precisely, Lemma 4, Proposition 7 in this paper, and [3], Theorem 1.4.5, the result follows.

We can now prove the main result of the paper, thus generalizing the first equivalence in [3], 3.9.2 to Laumon 1-motives; see also [5], Theorem 3.6.

Theorem 16. *Let $N^b(\mathcal{M}_1^{a})$ be the subcategory of $K^b(\mathcal{M}_1^{a})$ consisting of complexes that are acyclic as complexes in $K^b({}^t\mathcal{M}_1^{a})$. Let $D^b(\mathcal{M}_1^{a})$ denote the localization $K^b(\mathcal{M}_1^{a})/N^b(\mathcal{M}_1^{a})$. One then has the following equivalences of categories*

$$D^b(\mathcal{M}_1^{a}) \cong D^b({}^t\mathcal{M}_1^{a}) \cong D^b(\mathrm{Shv}_1^{a}).$$

Proof. Thanks to Lemmas 14 and 15 we have

$$D^b(\mathcal{M}_1^{a}) := K^b(\mathcal{M}_1^{a})/N^b(\mathcal{M}_1^{a}) \cong D^b({}^t\mathcal{M}_1^{a}) \cong K^b(\mathcal{M}_1^{a\star})/N^b(\mathcal{M}_1^{a\star}) \cong K^b(\mathrm{Shv}_1^{a\star})/N^b(\mathrm{Shv}_1^{a\star}) \cong D^b(\mathrm{Shv}_1^{a}).$$

Acknowledgements

The author is grateful to L. Barbieri Viale for directing her attention to this subject and for helpful discussions. The author would like to thank the referee for several comments and suggestions that have improved the exposition of this paper. She also thanks P. Jossen for pointing out a mistake in an earlier version of the paper.

References

[1] L. Barbieri-Viale, Formal Hodge theory, Math. Res. Lett. 14 (2007) 385–394.

[2] L. Barbieri-Viale, A. Bertapelle, Sharp de Rham realization, Adv. Math. 222 (2009) 1308–1338.

[3] L. Barbieri-Viale, B. Kahn, On the derived category of 1-motives, arXiv:1009.1900v1 [math.AG].

[4] L. Barbieri-Viale, A. Rosenschon, M. Saito, Deligne's Conjecture on 1-motives, Ann. of Math. 158 (2003) 593–633.

[5] A. Bertapelle, Remarks on 1-motivic sheaves, Journal of K-Theory 12 (2013) 363–380.

[6] A. Bertapelle, Delignes duality for de Rham realizations of 1-motives, Math. Nachr. 282 (2009) 1637–1655.

[7] L. Breen, Extensions of abelian sheaves and Eilenberg-MacLane algebras, Invent. Math. 9 (1969) 15–44.

[8] P. Deligne, Théorie de Hodge. III, Publ. Math., IHES 44 (1974) 5–77.

[9] M. Kashiwara, P. Schapira, Categories and sheaves, Grundlehren der mathematischen Wissenschaftern 332, Springer-Verlag, Berlin, 2006.

[10] G. Laumon, Transformation de Fourier généralisée, arXiv:alg-geom/9603004v1 [math.AG].

[11] S. Lekaus, On Albanese and Picard 1-motives with $\mathbb{G}_a$-factors, Manuscripta math. 130 (2009) 495–522.

[12] H. Russell, Albanese varieties with modulus over a perfect field. Algebra Number Theory 7 (2013) 853–892.