A NEW CHARACTERIZATION OF COMPLETE HEYTING AND CO-HEYTING ALGEBRAS

FRANCESCO RANZATO

Dipartimento di Matematica, University of Padova, Italy
e-mail address: francesco.ranzato@unipd.it

ABSTRACT. We give a new order-theoretic characterization of a complete Heyting and co-Heyting algebra C. This result provides an unexpected relationship with the field of Nash equilibria, being based on the so-called Veinott ordering relation on subcomplete sublattices of C, which is crucially used in Topkis' theorem for studying the order-theoretic stucture of Nash equilibria of supermodular games.

INTRODUCTION

Complete Heyting algebras — also called frames, while locales is used for complete co-Heyting algebras — play a fundamental role as algebraic model of intuitionistic logic and in pointless topology [Johnstone 1982, Johnstone 1983]. To the best of our knowledge, no characterization of complete Heyting and co-Heyting algebras has been known. As reported in [Balbes and Dwinger 1974], a sufficient condition has been given in [Funayama 1959] while a necessary condition has been given by [Chang and Horn 1962].

We give here an order-theoretic characterization of complete Heyting and co-Heyting algebras that puts forward an unexected relationship with Nash equilibria. Topkis' theorem [Topkis 1998] is well known in the theory of supermodular games in mathematical economics. This result shows that the set of solutions of a supermodular game, *i.e.*, its set of pure-strategy Nash equilibria, is nonempty and contains a greatest element and a least one [Topkis 1978]. Topkis' theorem has been strengthned by [Zhou 1994], where it is proved that this set of Nash equilibria is indeed a complete lattice. These results rely on so-called Veinott's ordering relation (also called strong set relation). Let $\langle C, \leq, \wedge, \vee \rangle$ be a complete lattice. Then, the relation $\leq^v \subseteq \wp(C) \times \wp(C)$ on subsets of C, according to Topkis [Topkis 1978], has been introduced by Veinott [Topkis 1998, Veinott 1989]: for any $S, T \in \wp(C)$,

$$S \leq^v T \;\overset{\triangle}{\Longleftrightarrow}\; \forall s \in S. \forall t \in T.\, s \wedge t \in S \;\;\&\;\; s \vee t \in T.$$

This relation $\leq^v$ is always transitive and antisymmetric, while reflexivity $S \leq^v S$ holds if and only if S is a sublattice of C. If $\mathrm{SL}(C)$ denotes the set of nonempty subcomplete sublattices of C then $\langle \mathrm{SL}(C), \leq^v \rangle$ is therefore a poset. The proof of Topkis' theorem is then based on the fixed points of a certain mapping defined on the poset $\langle \mathrm{SL}(C), \leq^v \rangle$.

1998 ACM Subject Classification: F.3.0 [Theory of computation]: Logics and meanings of programs—General.
Key words and phrases: Complete Heyting algebra, Veinott ordering.

LOGICAL METHODS
IN COMPUTER SCIENCE

To the best of our knowledge, no result is available on the order-theoretic properties of the Veinott poset $\langle \mathrm{SL}(C), \leq^v \rangle$. When is this poset a lattice? And a complete lattice? Our efforts in investigating these questions led to the following main result: the Veinott poset $\mathrm{SL}(C)$ is a complete lattice if and only if C is a complete Heyting and co-Heyting algebra. This finding therefore reveals an unexpected link between complete Heyting algebras and Nash equilibria of supermodular games. This characterization of the Veinott relation $\leq^v$ could be exploited for generalizing a recent approach based on abstract interpretation for approximating the Nash equilibria of supermodular games introduced by [Ranzato 2016].

1. Notation

If $\langle P, \leq \rangle$ is a poset and $S \subseteq P$ then $\mathrm{lb}(S)$ denotes the set of lower bounds of S, *i.e.*, $\mathrm{lb}(S) \triangleq \{x \in P \mid \forall s \in S.\, x \leq s\}$, while if $x \in P$ then $\downarrow x \triangleq \{y \in P \mid y \leq x\}$.
Let $\langle C, \leq, \wedge, \vee \rangle$ be a complete lattice. A nonempty subset $S \subseteq C$ is a subcomplete sublattice of C if for all its nonempty subsets $X \subseteq S$, $\wedge X \in S$ and $\vee X \in S$, while S is merely a sublattice of C if this holds for all its nonempty and finite subsets $X \subseteq S$ only. If $S \subseteq C$ then the nonempty Moore closure of S is defined as $\mathcal{M}^*(S) \triangleq \{\wedge X \in C \mid X \subseteq S, X \neq \varnothing\}$. Let us observe that $\mathcal{M}^*$ is an upper closure operator on the poset $\langle \wp(C), \subseteq \rangle$, meaning that: (1) $S \subseteq T \;\Rightarrow\; \mathcal{M}^*(S) \subseteq \mathcal{M}^*(T)$; (2) $S \subseteq \mathcal{M}^*(S)$; (3) $\mathcal{M}^*(\mathcal{M}^*(S)) = \mathcal{M}^*(S)$.
We define

$$\mathrm{SL}(C) \triangleq \{S \subseteq C \mid S \neq \varnothing,\ S \text{ subcomplete sublattice of } C\}.$$

Thus, if $\leq^v$ denotes the Veinott ordering defined in Section then $\langle \mathrm{SL}(C), \leq^v \rangle$ is a poset.
C is a complete Heyting algebra (also called frame) if for any $x \in C$ and $Y \subseteq C$, $x \wedge (\bigvee Y) = \bigvee_{y \in Y} x \wedge y$, while it is a complete co-Heyting algebra (also called locale) if the dual equation $x \vee (\bigwedge Y) = \bigwedge_{y \in Y} x \vee y$ holds. Let us recall that these two notions are orthogonal, for example the complete lattice of open subsets of $\mathbb{R}$ ordered by $\subseteq$ is a complete Heyting algebra, but not a complete co-Heyting algebra. C is (finitely) distributive if for any $x, y, z \in C$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
Let us also recall that C is completely distributive if for any family $\{x_{j,k} \mid j \in J, k \in K(j)\} \subseteq C$, we have that

$$\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in J \rightsquigarrow K} \bigwedge_{j \in J} x_{j,f(j)}$$

where J and, for any $j \in J$, $K(j)$ are sets of indices and $J \rightsquigarrow K \triangleq \{f : J \to \cup_{j \in J} K(j) \mid \forall j \in J.\ f(j) \in K(j)\}$ denotes the set of choice functions. It turns out that the class of completely distributive complete lattices is strictly contained in the class of complete Heyting and co-Heyting algebras. Clearly, any completely distribuitive lattice is a complete Heyting and co-Heyting algebra. On the other hand, this containment turns out to be strict, as shown by the following counterexample.

Example 1.1. Let us recall that a subset $S \subseteq [0,1]$ of real numbers is a regular open set if S is open and S coincides with the interior of the closure of S. Let us consider $\mathcal{C} = \langle \{S \subseteq [0,1] \mid S \text{ is a regular open set}\}, \subseteq \rangle$. It is known that $\mathcal{C}$ is a complete Boolean algebra (see e.g. [Vladimirov 2002, Theorem 12, Section 2.5]), where $\neg S$ denotes the complement of $S \in \mathcal{C}$, $\varnothing$ is the least element and $(0,1)$ is the greatest element. As a consequence, $\mathcal{C}$ is a complete Heyting and co-Heyting algebra (see e.g. [Vladimirov 2002, Theorem 3, Section 0.2.3]). It also known that a complete Boolean algebra is completely distributive if and only if it is atomic (see [Koppelberg 1989, Theorem 14.5, Chapter 5]). Recall that an element $a \in C$ in a complete lattice is an atom if a is different from the least element $\bot_C$ of C and for any $x \in C$, if $\bot_C < x \leq a$ then $x = a$, while C is atomic if for any $x \in C^+ \triangleq C \smallsetminus \{\bot_C\}$ there exists an atom $a \in C$ such that $a \leq x$. Let us show that $\mathcal{C}$ is not completely distributive. We clearly have that $\bigwedge_{S\in\mathcal{C}^+} \vee\{S, \neg S\} = (0,1)$. Let us assume, by contradiction, that $\mathcal{C}$ is completely distributive. Then, we have that

$$(0,1) = \bigvee_{f\in\mathcal{C}^+\to\{\mathrm{tt},\mathrm{ff}\}} \bigwedge_{S\in\mathcal{C}^+} T_{S,f(S)}$$

where for any $S \in \mathcal{C}^+$,

$$T_{S,f(S)} = \begin{cases} S & \text{if } f(S) = \mathrm{tt} \\ \neg S & \text{if } f(S) = \mathrm{ff} \end{cases}$$

First, let us observe that for any $V \in \mathcal{C}^+$,

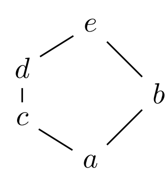
$$V = V \wedge (0,1) = V \wedge \Big(\bigvee_{f\in\mathcal{C}^+\to\{\mathrm{tt},\mathrm{ff}\}} \bigwedge_{S\in\mathcal{C}^+} T_{S,f(S)} \Big)$$

so that it must exist some $f_V \in \mathcal{C}^+ \to \{\mathrm{tt},\mathrm{ff}\}$ such that $\bigwedge_{S\in\mathcal{C}^+} T_{S,f_V(S)} \neq \varnothing$. It turns out that for any $V \in \mathcal{C}^+$, $\bigwedge_{S\in\mathcal{C}^+} T_{S,f_V(S)}$ is an atom of $\mathcal{C}$. In fact, if $U \in \mathcal{C}^+$ is such that $\varnothing \subsetneq U \subseteq \bigwedge_{S\in\mathcal{C}^+} T_{S,f_V(S)}$ then $U \subseteq T_{U,f_V(U)}$, so that $T_{U,f_V(U)} = U$, thus implying that $U = \bigwedge_{S\in\mathcal{C}^+} T_{S,f_V(S)}$. This implies that $\mathcal{C}$ is atomic, which is a contradiction. $\square$

2. The Sufficient Condition

To the best of our knowledge, no result is available on the order-theoretic properties of the Veinott poset $\langle \mathrm{SL}(C), \leq^v \rangle$. The following example shows that, in general, $\langle \mathrm{SL}(C), \leq^v \rangle$ is not a lattice.

Example 2.1. Consider the nondistributive pentagon lattice N_5, where, to use a compact notation, subsets of N_5 are denoted by strings of letters.



Consider $ed, abce \in \mathrm{SL}(N_5)$. It turns out that ${\downarrow} ed = \{a, c, d, ab, ac, ad, cd, ed, acd, ade, cde, abde, acde, abcde\}$ and ${\downarrow} abce = \{a, ab, ac, abce\}$. Thus, $\{a, ab, ac\}$ is the set of common lower bounds of ed and $abce$. However, the set $\{a, ab, ac\}$ does not include a greatest element, since $a \leq^v ab$ and $a \leq^v ac$ while ab and ac are incomparable. Hence, ab and c are maximal lower bounds of ed and $abce$, so that $\langle \mathrm{SL}(N_5), \leq^v \rangle$ is not a lattice. $\square$

Indeed, the following result shows that if $\mathrm{SL}(C)$ turns out to be a lattice then C must necessarily be distributive.

Lemma 2.2. If $\langle \mathrm{SL}(C), \leq^v \rangle$ is a lattice then C is distributive.

Proof. By the basic characterization of distributive lattices, we know that C is not distributive iff either the pentagon N_5 is a sublattice of C or the diamond M_3 is a sublattice of C. We consider separately these two possibilities.

(N_5) Assume that N_5, as depicted by the diagram in Example 2.1, is a sublattice of C. Following Example 2.1, we consider the sublattices $ed, abce \in \langle \mathrm{SL}(C), \leq^v \rangle$ and we prove that their meet does not exist. By Example 2.1, $ab, ac \in \mathrm{lb}(\{ed, abce\})$. Consider any $X \in \mathrm{SL}(C)$ such that $X \in \mathrm{lb}(\{ed, abce\})$. Assume that $ab \leq^v X$. If $x \in X$ then, by $ab \leq^v X$, we have that $b \vee x \in X$.

Moreover, by $X \leq^v abce$, $b \vee x \in \{a,b,c,e\}$. If $b \vee x = e$ then we would have that $e \in X$, and in turn, by $X \leq^v ed$, $d = e \wedge d \in X$, so that, by $X \leq^v abce$, we would get the contradiction $d = d \vee c \in \{a,b,c,e\}$. Also, if $b \vee x = c$ then we would have that $c \in X$, and in turn, by $ab \leq^v X$, $e = b \wedge c \in X$, so that, as in the previous case, we would get the contradiction $d = d \vee c \in \{a,b,c,e\}$. Thus, we necessarily have that $b \vee x \in \{a,b\}$. On the one hand, if $b \vee x = b$ then $x \leq b$ so that, by $ab \leq^v X$, $x = b \wedge x \in \{a,b\}$. On the other hand, if $b \vee x = a$ then $x \leq a$ so that, by $ab \leq^v X$, $x = a \wedge x \in \{a,b\}$. Hence, $X \subseteq \{a,b\}$. Since $X \neq \varnothing$, suppose that $a \in X$. Then, by $ab \leq^v X$, $b = b \vee a \in X$. If, instead, $b \in X$ then, by $X \leq^v abce$, $a = b \wedge a \in X$. We have therefore shown that $X = ab$. An analogous argument shows that if $ac \leq^v X$ then $X = ac$. If the meet of ed and $abce$ would exist, call it $Z \in \mathrm{SL}(C)$, from $Z \in \mathrm{lb}(\{ed, abce\})$ and $ab, ac \leq^v Z$ we would get the contradiction $ab = Z = ac$.

(M_3) Assume that the diamond M_3, as depicted by the following diagram, is a sublattice of C.

$$\begin{array}{ccccc} & & e & & \\ & \diagup & \mid & \diagdown & \\ b & & c & & d \\ & \diagdown & \mid & \diagup & \\ & & a & & \end{array}$$

In this case, we consider the sublattices $eb, ec \in \langle \mathrm{SL}(C), \leq^v \rangle$ and we prove that their meet does not exist. It turns out that $abce, abcde \in \mathrm{lb}(\{eb, ec\})$ while $abce$ and $abcde$ are incomparable. Consider any $X \in \mathrm{SL}(C)$ such that $X \in \mathrm{lb}(\{eb, ec\})$. Assume that $abcde \leq^v X$. If $x \in X$ then, by $X \leq^v eb, ec$, we have that $x \wedge b, x \wedge c \in X$, so that $x \wedge b \wedge c = x \wedge a \in X$. From $abcde \leq^v X$, we obtain that for any $y \in \{a,b,c,d,e\}$, $y = y \vee (x \wedge a) \in X$. Hence, $\{a,b,c,d,e\} \subseteq X$. From $X \leq^v eb$, we derive that $x \vee b \in \{e,b\}$, and, from $abcde \leq^v X$, we also have that $x \vee b \in X$. If $x \vee b = e$ then $x \leq e$, so that, from $abcde \leq^v X$, we obtain $x = e \wedge x \in \{a,b,c,d,e\}$. If, instead, $x \vee b = b$ then $x \leq b$, so that, from $abcde \leq^v X$, we derive $x = b \wedge x \in \{a,b,c,d,e\}$. In both cases, we have that $X \subseteq \{a,b,c,d,e\}$. We thus conclude that $X = abcde$. An analogous argument shows that if $abce \leq^v X$ then $X = abce$. Hence, similarly to the previous case (N_5), the meet of eb and ec does not exist. □

Moreover, we show that if we require $\mathrm{SL}(C)$ to be a complete lattice then the complete lattice C must be a complete Heyting and co-Heyting algebra. Let us remark that this proof makes use of the axiom of choice.

Theorem 2.3. If $\langle \mathrm{SL}(C), \leq^v \rangle$ is a complete lattice then C is a complete Heyting and co-Heyting algebra.

Proof. Assume that the complete lattice C is not a complete co-Heyting algebra. If C is not distributive, then, by Lemma 2.2, $\langle \mathrm{SL}(C), \leq^v \rangle$ is not a complete lattice. Thus, let us assume that C is distributive. The (dual) characterization in [Gierz *et al.* 1980, Remark 4.3, p. 40] states that a complete lattice C is a complete co-Heyting algebra iff C is distributive and join-continuous (*i.e.*, the join distributes over arbitrary meets of directed subsets). Consequently, it turns out that C is not join-continuous. Thus, by the result in [Bruns 1967] on directed sets and chains (see also [Gierz *et al.* 1980, Exercise 4.9, p. 42]), there exists an infinite descending chain $\{a_\beta\}_{\beta<\alpha} \subseteq C$, for some ordinal $\alpha \in \mathrm{Ord}$, such that if $\beta < \gamma < \alpha$ then $a_\beta > a_\gamma$, and an element $b \in C$ such that $\bigwedge_{\beta<\alpha} a_\beta \leq b < \bigwedge_{\beta<\alpha}(b \vee a_\beta)$. We observe the following facts:

(A) α must necessarily be a limit ordinal (so that $|\alpha| \geq |\mathbb{N}|$), otherwise if α is a successor ordinal then we would have that, for any $\beta < \alpha$, $a_{\alpha-1} \leq a_\beta$, so that $\bigwedge_{\beta<\alpha} a_\beta = a_{\alpha-1} \leq b$, and in turn we would obtain $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = b \vee a_{\alpha-1} = b$, *i.e.*, a contradiction.

(B) We have that $\bigwedge_{\beta<\alpha} a_\beta < b$, otherwise $\bigwedge_{\beta<\alpha} a_\beta = b$ would imply that $b \leq a_\beta$ for any $\beta < \alpha$, so that $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = \bigwedge_{\beta<\alpha} a_\beta = b$, which is a contradiction.

(C) Firstly, observe that $\{b \vee a_\beta\}_{\beta<\alpha}$ is an infinite descending chain in C. Let us consider a limit ordinal $\gamma < \alpha$. Without loss of generality, we assume that the glb's of the subchains $\{a_\rho\}_{\rho<\gamma}$ and $\{b \vee a_\rho\}_{\rho<\gamma}$ belong, respectively, to the chains $\{a_\beta\}_{\beta<\alpha}$ and $\{b \vee a_\beta\}_{\beta<\alpha}$. For our purposes, this is not a restriction because the elements $\bigwedge_{\rho<\gamma} a_\rho$ and $\bigwedge_{\rho<\gamma}(b \vee a_\rho)$ can be added to the respective chains $\{a_\beta\}_{\beta<\alpha}$ and $\{b \vee a_\beta\}_{\beta<\alpha}$ and these extensions would preserve both the glb's of the chains $\{a_\beta\}_{\beta<\alpha}$ and $\{b \vee a_\beta\}_{\beta<\alpha}$ and the inequalities $\bigwedge_{\beta<\alpha} a_\beta < b < \bigwedge_{\beta<\alpha}(b \vee a_\beta)$. Hence, by this nonrestrictive assumption, we have that for any limit ordinal $\gamma < \alpha$, $\bigwedge_{\rho<\gamma} a_\rho = a_\gamma$ and $\bigwedge_{\rho<\gamma}(b \vee a_\rho) = b \vee a_\gamma$ hold.

(D) Let us consider the set $S = \{a_\beta \mid \beta < \alpha,\ \forall \gamma \geq \beta.\ b \not\leq a_\gamma\}$. Then, S must be nonempty, otherwise we would have that for any $\beta < \alpha$ there exists some $\gamma_\beta \geq \beta$ such that $b \leq a_{\gamma_\beta} \leq a_\beta$, and this would imply that for any $\beta < \alpha$, $b \vee a_\beta = a_\beta$, so that we would obtain $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = \bigwedge_{\beta<\alpha} a_\beta$, which is a contradiction. Since any chain in (*i.e.*, subset of) S has an upper bound in S, by Zorn's Lemma, S contains the maximal element $a_{\bar\beta}$, for some $\bar\beta < \alpha$, such that for any $\gamma < \alpha$ and $\gamma \geq \bar\beta$, $b \not\leq a_\gamma$. We also observe that $\bigwedge_{\beta<\alpha} a_\beta = \bigwedge_{\bar\beta \leq \gamma<\alpha} a_\gamma$ and $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = \bigwedge_{\bar\beta\leq\gamma<\alpha}(b \vee a_\gamma)$. Hence, without loss of generality, we assume that the chain $\{a_\beta\}_{\beta<\alpha}$ is such that, for any $\beta < \alpha$, $b \not\leq a_\beta$ holds.

For any ordinal $\beta < \alpha$ — therefore, we remark that the limit ordinal α is not included — we define, by transfinite induction, the following subsets $X_\beta \subseteq C$:

– $\beta = 0 \ \Rightarrow\ X_\beta \triangleq \{a_0,\ b \vee a_0\}$;
– $\beta > 0 \ \Rightarrow\ X_\beta \triangleq \bigcup_{\gamma<\beta} X_\gamma \cup \{b \vee a_\beta\} \cup \{(b \vee a_\beta) \wedge a_\delta \mid \delta \leq \beta\}$.

Observe that, for any $\beta > 0$, $(b \vee a_\beta) \wedge a_\beta = a_\beta$ and that the set $\{b \vee a_\beta\} \cup \{(b \vee a_\beta) \wedge a_\delta \mid \delta \leq \beta\}$ is indeed a chain. Moreover, if $\delta \leq \beta$ then, by distributivity, we have that $(b \vee a_\beta) \wedge a_\delta = (b \wedge a_\delta) \vee (a_\beta \wedge a_\delta) = (b \wedge a_\delta) \vee a_\beta$. Moreover, if $\gamma < \beta < \alpha$ then $X_\gamma \subseteq X_\beta$.

We show, by transfinite induction on β, that for any $\beta < \alpha$, $X_\beta \in \mathrm{SL}(C)$. Let $\delta \leq \beta$ and $\mu \leq \gamma < \beta$. We notice the following facts:

(1) $(b \vee a_\beta) \wedge (b \vee a_\gamma) = b \vee a_\beta \in X_\beta$
(2) $(b \vee a_\beta) \vee (b \vee a_\gamma) = b \vee a_\gamma \in X_\gamma \subseteq X_\beta$
(3) $(b \vee a_\beta) \wedge \big((b \vee a_\gamma) \wedge a_\mu\big) = (b \vee a_\beta) \wedge a_\mu \in X_\beta$
(4) $(b \vee a_\beta) \vee \big((b \vee a_\gamma) \wedge a_\mu\big) = (b \vee a_\beta) \vee (b \wedge a_\mu) \vee a_\gamma = b \vee a_\gamma \in X_\gamma \subseteq X_\beta$
(5) $\big((b \vee a_\beta) \wedge a_\delta\big) \wedge \big((b \vee a_\gamma) \wedge a_\mu\big) = (b \vee a_\beta) \wedge a_{\max(\delta,\mu)} \in X_\beta$
(6) $\big((b\vee a_\beta)\wedge a_\delta\big) \vee \big((b\vee a_\gamma)\wedge a_\mu\big) = \big((b\wedge a_\delta)\vee a_\beta\big) \vee \big((b\wedge a_\mu)\vee a_\gamma\big) = (b \wedge a_{\min(\delta,\mu)}) \vee a_\gamma = (b \vee a_\gamma) \wedge a_{\min(\delta,\mu)} \in X_\gamma \subseteq X_\beta$
(7) if β is a limit ordinal then, by point (C) above, $\bigwedge_{\rho<\beta}(b \vee a_\rho) = b \vee a_\beta$ holds; therefore, $\bigwedge_{\rho<\beta}\big((b\vee a_\rho)\wedge a_\delta\big) = \big(\bigwedge_{\rho<\beta}(b\vee a_\rho)\big)\wedge a_\delta = (b\vee a_\beta)\wedge a_\delta \in X_\beta$; in turn, by taking the glb of these latter elements in X_β, we have that $\bigwedge_{\delta\leq\beta}\big((b\vee a_\beta)\wedge a_\delta\big) = (b\vee a_\beta)\wedge\big(\bigwedge_{\delta\leq\beta} a_\delta\big) = (b \vee a_\beta) \wedge a_\beta = a_\beta \in X_\beta$

Since $X_0 \in \mathrm{SL}(C)$ obviously holds, the points (1)-(7) above show, by transfinite induction, that for any $\beta < \alpha$, X_β is closed under arbitrary lub's and glb's of nonempty subsets, *i.e.*, $X_\beta \in \mathrm{SL}(C)$. In the following, we prove that the glb of $\{X_\beta\}_{\beta<\alpha} \subseteq \mathrm{SL}(C)$ in $\langle \mathrm{SL}(C), \leq^v\rangle$ does not exist.

Recalling, by point (A) above, that α is a limit ordinal, we define $A \triangleq \mathcal{M}^*(\bigcup_{\beta<\alpha} X_\beta)$. By point (C) above, we observe that for any limit ordinal $\gamma < \alpha$, the $\bigcup_{\beta<\alpha} X_\beta$ already contains the glb's

$$\bigwedge_{\rho<\gamma}(b \vee a_\rho) = b \vee a_\gamma \in X_\gamma, \qquad \bigwedge_{\rho<\gamma} a_\rho = a_\gamma \in X_\gamma,$$

$$\{\big(\bigwedge_{\rho<\gamma}(b \vee a_\rho)\big) \wedge a_\delta \mid \delta < \gamma\} = \{(b \vee a_\gamma) \wedge a_\delta \mid \delta < \gamma\} \subseteq X_\gamma.$$

Hence, by taking the glb's of all the chains in $\bigcup_{\beta<\alpha} X_\beta$, A turns out to be as follows:

$$A = \bigcup_{\beta<\alpha} X_\beta \cup \{\bigwedge_{\beta<\alpha}(b \vee a_\beta), \ \bigwedge_{\beta<\alpha} a_\beta\} \cup \{\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta \mid \delta < \alpha\}.$$

Let us show that $A \in \mathrm{SL}(C)$. First, we observe that $\bigcup_{\beta<\alpha} X_\beta$ is closed under arbitrary nonempty lub's. In fact, if $S \subseteq \bigcup_{\beta<\alpha} X_\beta$ then $S = \bigcup_{\beta<\alpha}(S \cap X_\beta)$, so that

$$\bigvee S = \bigvee \bigcup_{\beta<\alpha}(S \cap X_\beta) = \bigvee_{\beta<\alpha} \bigvee S \cap X_\beta.$$

Also, if $\gamma < \beta < \alpha$ then $S \cap X_\gamma \subseteq S \cap X_\beta$ and, in turn, $\bigvee S \cap X_\gamma \leq \bigvee S \cap X_\beta$, so that $\{\bigvee S \cap X_\beta\}_{\beta<\alpha}$ is an increasing chain. Hence, since $\bigcup_{\beta<\alpha} X_\beta$ does not contain infinite increasing chains, there exists some $\gamma < \alpha$ such that $\bigvee_{\beta<\alpha} \bigvee S \cap X_\beta = \bigvee S \cap X_\gamma \in X_\gamma$, and consequently $\bigvee S \in \bigcup_{\beta<\alpha} X_\beta$. Moreover, $\{\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta\}_{\delta<\alpha} \subseteq A$ is a chain whose lub is $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_0$ which belongs to the chain itself, while its glb is

$$\bigwedge_{\delta<\alpha}\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge \bigwedge_{\delta<\alpha} a_\delta = \bigwedge_{\delta<\alpha} a_\delta \in A.$$

Finally, if $\delta \leq \gamma < \alpha$ then we have that:

(8) $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge (b \vee a_\gamma) = \bigwedge_{\beta<\alpha}(b \vee a_\beta) \in A$

(9) $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee (b \vee a_\gamma) = b \vee a_\gamma \in X_\gamma \subseteq A$

(10) $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge \big((b \vee a_\gamma) \wedge a_\delta\big) = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta \in A$

(11) We have that $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee \big((b \vee a_\gamma) \wedge a_\delta\big) = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee (b \wedge a_\delta) \vee a_\gamma = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee a_\gamma$. Moreover, $b \vee a_\gamma \leq \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee a_\gamma \leq (b \vee a_\gamma) \vee a_\gamma = b \vee a_\gamma$; hence, $\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \vee \big((b \vee a_\gamma) \wedge a_\delta\big) = b \vee a_\gamma \in X_\gamma \subseteq A$.

Summing up, we have therefore shown that $A \in \mathrm{SL}(C)$.

We now prove that A is a lower bound of $\{X_\beta\}_{\beta<\alpha}$, *i.e.*, we prove, by transfinite induction on β, that for any $\beta < \alpha$, $A \leq^v X_\beta$.

- $\big(A \leq^v X_0\big)$: this is a consequence of the following easy equalities, for any $\delta \leq \beta < \alpha$: $(b \vee a_\beta) \wedge a_0 \in X_\beta \subseteq A$; $(b \vee a_\beta) \vee a_0 = b \vee a_0 \in X_0$; $(b \vee a_\beta) \wedge (b \vee a_0) = b \vee a_\beta \in X_\beta \subseteq A$; $(b \vee a_\beta) \vee (b \vee a_0) = b \vee a_0 \in X_0$; $\big((b \vee a_\beta) \wedge a_\delta\big) \wedge a_0 = (b \vee a_\beta) \wedge a_\delta \in X_\beta \subseteq A$; $\big((b \vee a_\beta) \wedge a_\delta\big) \vee a_0 = a_0 \in X_0$; $\big((b \vee a_\beta) \wedge a_\delta\big) \wedge (b \vee a_0) = (b \vee a_\beta) \wedge a_\delta \in X_\beta \subseteq A$; $\big((b \vee a_\beta) \wedge a_\delta\big) \vee (b \vee a_0) = b \vee a_0 \in X_0$.
- $\big(A \leq^v X_\beta,\ \beta > 0\big)$: Let $a \in A$ and $x \in X_\beta$. If $x \in \bigcup_{\gamma<\beta} X_\gamma$ then $x \in X_\gamma$ for some $\gamma < \beta$, so that, since by inductive hypothesis $A \leq^v X_\gamma$, we have that $a \wedge x \in A$ and $a \vee x \in X_\gamma \subseteq X_\beta$. Thus, assume that $x \in X_\beta \smallsetminus \big(\bigcup_{\gamma<\beta} X_\gamma\big)$. If $a \in X_\beta$ then $a \wedge x \in X_\beta \subseteq A$ and $a \vee x \in X_\beta$. If $a \in X_\mu$, for some $\mu > \beta$, then $a \wedge x \in X_\mu \subseteq A$, while points (2), (4) and (6) above show that $a \vee x \in X_\beta$. If $a = \bigwedge_{\beta<\alpha}(b \vee a_\beta)$ then points (8)-(11) above show that $a \wedge x \in A$ and $a \vee x \in X_\beta$. If $a = \big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu$, for some $\mu < \alpha$, and $\delta \leq \beta$ then we have that:

(12) $\big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu\big) \wedge (b \vee a_\beta) = \big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu \in A$

(13) $\big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu\big) \vee (b \vee a_\beta) = \big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \vee (b \vee a_\beta)\big) \wedge (a_\mu \vee (b \vee a_\beta)) = (b \vee a_\beta) \wedge (b \vee a_{\min(\mu,\beta)}) = b \vee a_\beta \in X_\beta$

(14) $\big(\big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_\mu\big) \wedge \big((b \vee a_\beta) \wedge a_\delta\big) = \big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\big) \wedge a_{\max(\mu,\delta)} \in A$

(15)
$$\Big(\Big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\Big) \wedge a_\mu\Big) \vee \big((b \vee a_\beta) \wedge a_\delta\big) =$$
$$\Big(\Big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\Big) \vee (b \vee a_\beta)\Big) \wedge \Big(\Big(\bigwedge_{\gamma<\alpha}(b \vee a_\gamma)\Big) \vee a_\delta\Big) \wedge \big(a_\mu \vee (b \vee a_\beta)\big) \wedge (a_\mu \vee a_\delta) =$$
$$(b \vee a_\beta) \wedge (b \vee a_\delta) \wedge \big(b \vee a_{\min(\mu,\beta)}\big) \wedge a_{\min(\mu,\delta)} =$$
$$(b \vee a_\beta) \wedge a_{\min(\mu,\delta)} \in X_\beta$$

Finally, if $a = \bigwedge_{\gamma<\alpha} a_\gamma$ and $x \in X_\beta$ then $a \leq x$ so that $a \wedge x = a \in A$ and $a \vee x = x \in X_\beta$. Summing up, we have shown that $A \leq^v X_\beta$.

Let us now prove that $b \notin A$. Let us first observe that for any $\beta < \alpha$, we have that $a_\beta \not\leq b$: in fact, if $a_\gamma \leq b$, for some $\gamma < \alpha$ then, for any $\delta \leq \gamma$, $b \vee a_\delta = b$, so that we would obtain $\bigwedge_{\beta<\alpha}(b \vee a_\beta) = b$, which is a contradiction. Hence, for any $\beta < \alpha$ and $\delta \leq \beta$, it turns out that $b \neq b \vee a_\beta$ and $b \neq (b \wedge a_\delta) \vee a_\beta = (b \vee a_\beta) \wedge a_\delta$. Moreover, by point (B) above, $b \neq \bigwedge_{\beta<\alpha}(b \vee a_\beta)$, while, by hypothesis, $b \neq \bigwedge_{\beta<\alpha} a_\beta$. Finally, for any $\delta < \alpha$, if $b = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta$ then we would derive that $b \leq a_\delta$, which, by point (D) above, is a contradiction.

Now, we define $B \triangleq \mathcal{M}^*(A \cup \{b\})$, so that
$$B = A \cup \{b\} \cup \{b \wedge a_\delta \mid \delta < \alpha\}.$$
Observe that for any $a \in A$, with $a \neq \bigwedge_{\beta<\alpha} a_\beta$, and for any $\delta < \alpha$, we have that $b \wedge a_\delta \leq a$, while $b \vee \Big(\big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge a_\delta\Big) = \Big(b \vee \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big)\Big) \wedge (b \vee a_\delta) = \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) \wedge (b \vee a_\delta) = \bigwedge_{\beta<\alpha}(b \vee a_\beta) \in B$. Also, for any $\delta \leq \beta < \alpha$, we have that $b \vee \big((b \vee a_\beta) \wedge a_\delta\big) = \big(b \vee (b \vee a_\beta)\big) \wedge (b \vee a_\delta) = b \vee a_\delta \in B$. Also, $b \vee \big(\bigwedge_{\beta<\alpha}(b \vee a_\beta)\big) = \bigwedge_{\beta<\alpha}(b \vee a_\beta) \in B$ and $b \vee \bigwedge_{\beta<\alpha} a_\beta = b \in B$. We have thus checked that B is closed under lub's (of arbitrary nonempty subsets), *i.e.*, $B \in \mathrm{SL}(C)$. Let us check that B is a lower bound of $\{X_\beta\}_{\beta<\alpha}$. Since we have already shown that A is a lower bound, and since $b \wedge a_\delta \leq b$, for any $\delta < \alpha$, it is enough to observe that for any $\beta < \alpha$ and $x \in X_\beta$, $b \wedge x \in B$ and $b \vee x \in X_\beta$. The only nontrivial case is for $x = (b \vee a_\beta) \wedge a_\delta$, for some $\delta \leq \beta < \alpha$. On the one hand, $b \wedge \big((b \vee a_\beta) \wedge a_\delta\big) = b \wedge a_\delta \in B$, on the other hand, $b \vee \big((b \vee a_\beta) \wedge a_\delta\big) = b \vee \big((b \wedge a_\delta) \vee a_\beta\big) = b \vee a_\beta \in X_\beta$.

Let us now assume that there exists $Y \in \mathrm{SL}(C)$ such that Y is the glb of $\{X_\beta\}_{\beta<\alpha}$ in $\langle \mathrm{SL}(C), \leq^v \rangle$. Therefore, since we proved that A is a lower bound, we have that $A \leq^v Y$. Let us consider $y \in Y$. Since $b \vee a_0 \in A$, we have that $b \vee a_0 \vee y \in Y$. Since $Y \leq^v X_0 = \{a_0, b \vee a_0\}$, we have that $b \vee a_0 \vee y \vee a_0 = b \vee a_0 \vee y \in \{a_0, b \vee a_0\}$. If $b \vee a_0 \vee y = a_0$ then $b \leq a_0$, which, by point (D), is a contradiction. Thus, we have that $b \vee a_0 \vee y = b \vee a_0$, so that $y \leq b \vee a_0$ and $b \vee a_0 \in Y$. We know that if $x \in X_\beta$, for some $\beta < \alpha$, then $x \leq b \vee a_0$, so that, from $Y \leq^v X_\beta$, we obtain that $(b \vee a_0) \wedge x = x \in Y$, that is, $X_\beta \subseteq Y$. Thus, we have that $\bigcup_{\beta<\alpha} X_\beta \subseteq Y$, and, in turn, by subset monotonicity of $\mathcal{M}^*$, we get $A = \mathcal{M}^*(\bigcup_{\beta<\alpha} X_\beta) \subseteq \mathcal{M}^*(Y) = Y$. Moreover, from $y \leq b \vee a_0$, since $A \leq^v Y$ and $b \vee a_0 \in A$, we obtain $(b \vee a_0) \wedge y = y \in A$, that is $Y \subseteq A$. We have therefore shown that $Y = A$. Since we proved that B is a lower bound, $B \leq^v Y = A$ must hold. However, it turns out that $B \leq^v A$ is a contradiction: by considering $b \in B$ and $\bigwedge_{\beta<\alpha} a_\beta \in A$, we would have that $b \vee \big(\bigwedge_{\beta<\alpha} a_\beta\big) = b \in A$, while we have shown above that $b \notin A$. We have therefore shown that the glb of $\{X_\beta\}_{\beta<\alpha}$ in $\langle \mathrm{SL}(C), \leq^v \rangle$ does not exist.

To close the proof, it is enough to observe that if $\langle C, \leq \rangle$ is not a complete Heyting algebra then, by duality, $\langle \mathrm{SL}(C), \leq^v \rangle$ does not have lub's. $\square$

3. The Necessary Condition

It turns out that the property of being a complete lattice for the poset $\langle \mathrm{SL}(C), \leq^v \rangle$ is a necessary condition for a complete Heyting and co-Heyting algebra C.

Theorem 3.1. If C is a complete Heyting and co-Heyting algebra then $\langle \mathrm{SL}(C), \leq^v \rangle$ is a complete lattice.

Proof. Let $\{A_i\}_{i\in I} \subseteq \mathrm{SL}(C)$, for some family of indices $I \neq \varnothing$. Let us define

$$G \triangleq \{x \in \mathcal{M}^*(\cup_{i\in I} A_i) \mid \forall k \in I.\ \mathcal{M}^*(\cup_{i\in I} A_i) \cap {\downarrow} x \leq^v A_k\}.$$

The following three points show that G is the glb of $\{A_i\}_{i\in I}$ in $\langle \mathrm{SL}(C), \leq^v \rangle$.

(1) We show that $G \in \mathrm{SL}(C)$. Let $\bot \triangleq \bigwedge_{i\in I} \bigwedge A_i$. First, G is nonempty because it turns out that $\bot \in G$. Since, for any $i \in I$, $\bigwedge A_i \in A_i$ and $I \neq \varnothing$, we have that $\bot \in \mathcal{M}^*(\cup_i A_i)$. Let $y \in \mathcal{M}^*(\cup_i A_i) \cap {\downarrow}\bot$ and, for some $k \in I$, $a \in A_k$. On the one hand, we have that $y \wedge a \in \mathcal{M}^*(\cup_i A_i) \cap {\downarrow}\bot$ trivially holds. On the other hand, since $y \leq \bot \leq a$, we have that $y \vee a = a \in A_k$.

Let us now consider a set $\{x_j\}_{j\in J} \subseteq G$, for some family of indices $J \neq \varnothing$, so that, for any $j \in J$ and $k \in I$, $\mathcal{M}^*(\cup_i A_i) \cap {\downarrow} x_j \leq^v A_k$.

First, notice that $\bigwedge_{j\in J} x_j \in \mathcal{M}^*(\cup_i A_i)$ holds. Then, since ${\downarrow}(\bigwedge_{j\in J} x_j) = \bigcap_{j\in J} {\downarrow} x_j$ holds, we have that $\mathcal{M}^*(\cup_i A_i) \cap {\downarrow}(\bigwedge_{j\in J} x_j) = \mathcal{M}^*(\cup_i A_i) \cap (\bigcap_{j\in J} {\downarrow} x_j)$, so that, for any $k \in I$, $\mathcal{M}^*(\cup_i A_i) \cap {\downarrow}(\bigwedge_{j\in J} x_j) \leq^v A_k$, that is, $\bigwedge_{j\in J} x_j \in G$.

Let us now prove that $\bigvee_{j\in J} x_j \in \mathcal{M}^*(\cup_i A_i)$ holds. First, since any $x_j \in \mathcal{M}^*(\cup_{i\in I} A_i)$, we have that $x_j = \bigwedge_{i\in K(j)} a_{j,i}$, where, for any $j \in J$, $K(j) \subseteq I$ is a nonempty family of indices in I such that for any $i \in K(j)$, $a_{j,i} \in A_i$. For any $i \in I$, we then define the family of indices $L(i) \subseteq J$ as follows: $L(i) \triangleq \{j \in J \mid i \in K(j)\}$. Observe that it may happen that $L(i) = \varnothing$. Since for any $i \in I$ such that $L(i) \neq \varnothing$, $\{a_{j,i}\}_{j\in L(i)} \subseteq A_i$ and A_i is meet-closed, we have that if $L(i) \neq \varnothing$ then $\hat{a}_i \triangleq \bigwedge_{l\in L(i)} a_{l,i} \in A_i$. Since, given $k \in I$ such that $L(k) \neq \varnothing$, for any $j \in J$, $\mathcal{M}^*(\cup_{i\in I} A_i) \cap {\downarrow} x_j \leq^v A_k$, we have that for any $j \in J$, $x_j \vee \hat{a}_k \in A_k$. Since A_k is join-closed, we obtain that $\bigvee_{j\in J}(x_j \vee \hat{a}_k) = (\bigvee_{j\in J} x_j) \vee \hat{a}_k \in A_k$. Consequently,

$$\bigwedge_{\substack{k\in I,\\ L(k)\neq\varnothing}} \big((\bigvee_{j\in J} x_j) \vee \hat{a}_k\big) \in \mathcal{M}^*(\cup_{i\in I} A_i).$$

Since C is a complete co-Heyting algebra,

$$\bigwedge_{\substack{k\in I,\\ L(k)\neq\varnothing}} \big((\bigvee_{j\in J} x_j) \vee \hat{a}_k\big) = (\bigvee_{j\in J} x_j) \vee (\bigwedge_{\substack{k\in I,\\ L(k)\neq\varnothing}} \hat{a}_k).$$

Thus, since, for any $j \in J$,

$$\bigwedge_{\substack{k\in I,\\ L(k)\neq\varnothing}} \hat{a}_k = \bigwedge_{j\in J} \textstyle\bigwedge_{i\in K(j)} a_{j,i} \leq x_j,$$

we obtain that $(\bigvee_{j\in J} x_j) \vee (\bigwedge_{\substack{k\in I,\\ L(k)\neq\varnothing}} \hat{a}_k) = \bigvee_{j\in J} x_j$, so that $\bigvee_{j\in J} x_j \in \mathcal{M}^*(\cup_{i\in I} A_i)$.

Finally, in order to prove that $\bigvee_{j\in J} x_j \in G$, let us show that for any $k \in I$, $\mathcal{M}^*(\cup_i A_i) \cap \downarrow (\bigvee_{j\in J} x_j) \leq^v A_k$. Let $y \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow (\bigvee_{j\in J} x_j)$ and $a \in A_k$. For any $j \in J$, $y \wedge x_j \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow (\bigvee_{j\in J} x_j)$, so that $(y \wedge x_j) \vee a \in A_k$. Since A_k is join-closed, we obtain that $\bigvee_{j\in J} \big((y \wedge x_j) \vee a\big) = a \vee \big(\bigvee_{j\in J}(y \wedge x_j)\big) \in A_k$. Since C is a complete Heyting algebra, $a \vee \big(\bigvee_{j\in J}(y \wedge x_j)\big) = a \vee \big(y \wedge (\bigvee_{j\in J} x_j)\big)$. Since $y \wedge (\bigvee_{j\in J} x_j) = y$, we derive that $y \vee a \in A_k$. On the other hand, $y \wedge a \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow (\bigvee_{j\in J} x_j)$ trivially holds.

(2) We show that for any $k \in I$, $G \leq^v A_k$. Let $x \in G$ and $a \in A_k$. Hence, $x \in \mathcal{M}^*(\cup_i A_i)$ and for any $j \in I$, $\mathcal{M}^*(\cup_i A_i) \cap \downarrow x \leq^v A_j$. We first prove that $\mathcal{M}^*(\cup_i A_i) \cap \downarrow x \subseteq G$. Let $y \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow x$, and let us check that for any $j \in I$, $\mathcal{M}^*(\cup_i A_i) \cap \downarrow y \leq^v A_j$: if $z \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow y$ and $u \in A_j$ then $z \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow x$ so that $z \vee u \in A_j$ follows, while $z \wedge u \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow y$ trivially holds. Now, since $x \wedge a \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow x$, we have that $x \wedge a \in G$. On the other hand, since $x \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow x \leq^v A_k$, we also have that $x \vee a \in A_k$.

(3) We show that if $Z \in \mathrm{SL}(C)$ and, for any $i \in I$, $Z \leq^v A_i$ then $Z \leq^v G$. By point (1), $\bot = \bigwedge_{i\in I} \bigwedge A_i \in G$. We then define $Z^\bot \subseteq C$ as follows: $Z^\bot \triangleq \{x \vee \bot \mid x \in Z\}$. It turns out that $Z^\bot \subseteq \mathcal{M}^*(\cup_i A_i)$: in fact, since C is a complete co-Heyting algebra, for any $x \in Z$, we have that $x \vee (\bigwedge_{i\in I} \bigwedge A_i) = \bigwedge_{i\in I}(x \vee \bigwedge A_i)$, and since $x \in Z$, for any $i \in I$, $\bigwedge A_i \in A_i$, and $Z \leq^v A_i$, we have that $x \vee \bigwedge A_i \in A_i$, so that $\bigwedge_{i\in I}(x \vee \bigwedge A_i) \in \mathcal{M}^*(\cup_i A_i)$. Also, it turns out that $Z^\bot \in \mathrm{SL}(C)$. If $Y \subseteq Z^\bot$ and $Y \neq \varnothing$ then $Y = \{x \vee \bot\}_{x\in X}$ for some $X \subseteq Z$ with $X \neq \varnothing$. Hence, $\bigvee Y = \bigvee_{x\in X}(x \vee \bot) = (\bigvee X) \vee \bot$, and since $\bigvee X \in Z$, we therefore have that $\bigvee Y \in Z^\bot$. On the other hand, $\bigwedge Y = \bigwedge_{x\in X}(x \vee \bot)$, and, as C is a complete co-Heyting algebra, $\bigwedge_{x\in X}(x \vee \bot) = (\bigwedge X) \vee \bot$, and since $\bigwedge X \in Z$, we therefore obtain that $\bigwedge Y \in Z^\bot$. We also observe that $Z \leq^v Z^\bot$. In fact, if $x \in Z$ and $y \vee \bot \in Z^\bot$, for some $y \in Z$, then, clearly, $x \vee y \vee \bot \in Z^\bot$, while, by distributivity of C, $x \wedge (y \vee \bot) = (x \wedge y) \vee \bot \in Z^\bot$. Next, we show that for any $i \in I$, $Z^\bot \leq^v A_i$. Let $x \vee \bot \in Z^\bot$, for some $z \in Z^\bot$, and $a \in A_i$. Then, by distributivity of C, $(x \vee \bot) \wedge a = (x \wedge a) \vee (\bot \wedge a) = (x \wedge a) \vee \bot$, and since, by $Z \leq^v A_i$, we know that $x \wedge a \in Z$, we also have that $(x \wedge a) \vee \bot \in Z^\bot$. On the other hand, $(x \vee \bot) \vee a = (x \vee a) \vee \bot$, and since, by $Z \leq^v A_i$, we know that $\bot \leq x \vee a \in A_i$, we obtain that $(x \vee a) \vee \bot = x \vee a \in A_i$.

Summing up, we have therefore shown that for any $Z \in \mathrm{SL}(C)$ such that, for any $i \in I$, $Z \leq^v A_i$, there exists $Z^\bot \in \mathrm{SL}(C)$ such that $Z^\bot \subseteq \mathcal{M}^*(\cup_i A_i)$ and, for any $i \in I$, $Z^\bot \leq^v A_i$. We now prove that $Z^\bot \subseteq G$. Consider $w \in Z^\bot$, and let us check that for any $i \in I$, $\mathcal{M}^*(\cup_i A_i) \cap \downarrow w \leq^v A_i$. Hence, consider $y \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow w$ and $a \in A_i$. Then, $y \wedge a \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow w$ follows trivially. Moreover, since $y \in \mathcal{M}^*(\cup_i A_i)$, there exists a subset $K \subseteq I$, with $K \neq \varnothing$, such that for any $k \in K$ there exists $a_k \in A_k$ such that $y = \bigwedge_{k\in K} a_k$. Thus, since, for any $k \in K$, $z \wedge a_k \in \mathcal{M}^*(\cup_i A_i) \cap \downarrow z \leq^v A_i$, we obtain that $\{(z \wedge a_k) \vee a\}_{k\in K} \subseteq A_i$. Since A_i is meet-closed, $\bigwedge_{k\in K} \big((w \wedge a_k) \vee a\big) \in A_i$. Since C is a complete co-Heyting algebra, $\bigwedge_{k\in K} \big((w \wedge a_k) \vee a\big) = a \vee \big(\bigwedge_{k\in K}(w \wedge a_k)\big) = a \vee \big(w \wedge (\bigwedge_{k\in K} a_k)\big) = a \vee (w \wedge y) = a \vee y$, so that $a \vee y \in A_i$ follows.

To close the proof of point (3), we show that $Z^\bot \leq^v G$. Let $z \in Z^\bot$ and $x \in G$. On the one hand, since $Z^\bot \subseteq G$, we have that $z \in G$, and, in turn, as G is join-closed, we obtain that $z \vee x \in G$. On the other hand, since $x \in \mathcal{M}^*(\cup_i A_i)$, there exists a subset $K \subseteq I$, with $K \neq \varnothing$, such that for any $k \in K$ there exists $a_k \in A_k$ such that $x = \bigwedge_{k\in K} a_k$. Thus, since $Z^\bot \leq^v A_k$, for any $k \in K$, we obtain that $z \wedge a_k \in Z^\bot$. Hence, since $Z^\bot$ is meet-closed, we have that $\bigwedge_{k\in K}(z \wedge a_k) = z \wedge \big(\bigwedge_{k\in K} a_k\big) = z \wedge x \in Z^\bot$.

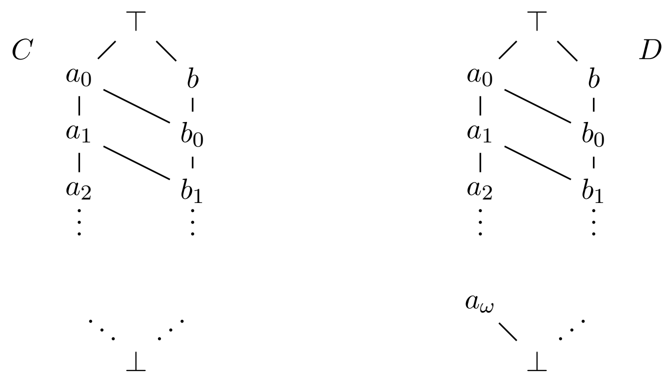
To conclude the proof, we notice that $\{\top_C\} \in \mathrm{SL}(C)$ is the greatest element in $\langle \mathrm{SL}(C), \leq^v \rangle$. Thus, since $\langle \mathrm{SL}(C), \leq^v \rangle$ has nonempty glb's and the greatest element, it turns out that it is a complete lattice. $\square$

We have thus shown the following characterization of complete Heyting and co-Heyting algebras.

Corollary 3.2. Let C be a complete lattice. Then, $\langle \mathrm{SL}(C), \leq^v \rangle$ is a complete lattice if and only if C is a complete Heyting and co-Heyting algebra.

To conclude, we provide an example showing that the property of being a complete lattice for the poset $\langle \mathrm{SL}(C), \leq^v \rangle$ cannot be a characterization for a complete Heyting (or co-Heyting) algebra C.

Example 3.3. Consider the complete lattice C depicted on the left.



C is distributive but not a complete co-Heyting algebra: $b \vee \big(\bigwedge_{i\geq 0} a_i\big) = b < \bigwedge_{i\geq 0}(b \vee a_i) = \top$. Let $X_0 \triangleq \{\top, a_0\}$ and, for any $i \geq 0$, $X_{i+1} \triangleq X_i \cup \{a_{i+1}\}$, so that $\{X_i\}_{i\geq 0} \subseteq \mathrm{SL}(C)$. Then, it turns out that the glb of $\{X_i\}_{i\geq 0}$ in $\langle \mathrm{SL}(C), \leq^v \rangle$ does not exist. This can be shown by mimicking the proof of Theorem 2.3. Let $A \triangleq \{\bot\} \cup \bigcup_{i\geq 0} X_i \in \mathrm{SL}(C)$. Let us observe that A is a lower bound of $\{X_i\}_{i\geq 0}$. Hence, if we suppose that $Y \in \mathrm{SL}(C)$ is the glb of $\{X_i\}_{i\geq 0}$ then $A \leq^v Y$ must hold. Hence, if $y \in Y$ then $\top \wedge y = y \in A$, so that $Y \subseteq A$, and $\top \vee y \in Y$. Since, $Y \leq^v X_0$, we have that $\top \vee y \vee \top = \top \vee y \in X_0 = \{\top, a_0\}$, so that necessarily $\top \vee y = \top \in Y$. Hence, from $Y \leq^v X_i$, for any $i \geq 0$, we obtain that $\top \wedge a_i = a_i \in Y$. Hence, $Y = A$. The whole complete lattice C is also a lower bound of $\{X_i\}_{i\geq 0}$, therefore $C \leq^v Y = A$ must hold: however, this is a contradiction because from $b \in C$ and $\bot \in A$ we obtain that $b \vee \bot = b \in A$.
It is worth noting that if we instead consider the complete lattice D depicted on the right of the above figure, which includes a new glb a_ω of the chain $\{a_i\}_{i\geq 0}$, then D becomes a complete Heyting and co-Heyting algebra, and in this case the glb of $\{X_i\}_{i\geq 0}$ in $\langle \mathrm{SL}(D), \leq^v \rangle$ turns out to be $\{\top\} \cup \{a_i\}_{i\geq 0} \cup \{a_\omega\}$. $\square$

ACKNOWLEDGEMENTS

The author has been partially supported by the University of Padova under the 2014 PRAT project "ANCORE".

REFERENCES

[Balbes and Dwinger 1974] R. Balbes and P. Dwinger. *Distributive Lattices*. University of Missouri Press, Columbia, Missouri, 1974.

[Bruns 1967] G. Bruns. A lemma on directed sets and chains. *Archiv der Mathematik*, 18(6):561-563, 1967.

[Chang and Horn 1962] C.C. Chang and A. Horn. On the representation of α-complete lattices. *Fund. Math.*, 51:254-258, 1962.

[Funayama 1959] N. Funayama. Imbedding infinitely distributive lattices completely isomorphically into Boolean algebras. *Nagoya Math. J.*, 15:71-81, 1959.

[Gierz *et al.* 1980] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *A Compendium of Continuous Lattices*. Springer, Berlin, 1980.

[Johnstone 1982] P.T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[Johnstone 1983] P.T. Johnstone. The point of pointless topology. *Bull. Amer. Math. Soc.*, 8(1):41-53, 1983.

[Koppelberg 1989] S. Koppelberg. *Handbook of Boolean algebras, vol. 1*. Edited by J.D. Monk and R. Bonnet, North-Holland, 1989.

[Ranzato 2016] F. Ranzato. Abstract interpretation of supermodular games. In X. Rival editor, *Proceedings of the 23rd International Static Analysis Symposium (SAS'16)*, Edinburgh, UK, LNCS vol. 9837, pages 403-423, Springer, 2016.

[Topkis 1978] D.M. Topkis. Minimizing a submodular function on a lattice. *Operations Research*, 26(2):305–321, 1978.

[Topkis 1998] D.M. Topkis. *Supermodularity and Complementarity*. Princeton University Press, 1998.

[Veinott 1989] A.F. Veinott. *Lattice programming*. Unpublished notes from lectures at Johns Hopkins University, 1989.

[Vladimirov 2002] D.A. Vladimirov. *Boolean Algebras in Analysis*. Springer Netherlands, 2002.

[Zhou 1994] L. Zhou. The set of Nash equilibria of a supermodular game is a complete lattice. *Games and Economic Behavior*, 7(2):295-300, 1994.