SOME REMARKS ON HARMONIC PROJECTION OPERATORS ON SPHERES
ALCUNE OSSERVAZIONI SUI PROIETTORI ARMONICI SU SFERE

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ABSTRACT. We give a survey of recent works concerning the mapping properties of joint harmonic projection operators, mapping the space of square integrable functions on complex and quaternionic spheres onto the eigenspaces of the Laplace–Beltrami operator and of a suitably defined subLaplacian. In particular, we discuss similarities and differences between the real, the complex and the quaternionic framework.

SUNTO. Presentiamo un sunto di alcuni risultati recenti relativi alle proprietà degli operatori di proiezione armonica, che mappano lo spazio delle funzioni a quadrato sommabile sulla sfera unitaria complessa e quaternionica sopra gli autospazi congiunti per l'operatore di Laplace–Beltrami e per un sublaplaciano. Discutiamo, in particolare, analogie e differenze fra il caso reale, quello complesso e quello quaternionico.

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1. INTRODUCTION

According to a classical result (see [16, 14]), the space of square-integrable functions on S^{dn-1}, where S^{dn-1} denotes the unit sphere in $\mathbb{R}^{dn}$, $d = 1, 2, 4$, may be decomposed as direct sum of certain orthogonal subspaces, that is,

$$L^2(S^{dn-1}) = \bigoplus_{\rho \in \mathcal{F}} V^\rho \,, \tag{1}$$

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where $\mathcal{F} = \mathbb{N}$ if $d = 1$, $\mathcal{F} = \mathbb{N} \times \mathbb{N}$ if $d = 2$, and $\mathcal{F} = \{(\ell, \ell') \in \mathbb{N} \times \mathbb{N} : \ell \geq \ell' \geq 0)\}$ if $d = 4$.

The central concern of this paper consists in comparing the mapping properties of the projection operators

$$\pi_\rho : L^2(S^{dn-1}) \to V^\rho \tag{2}$$

as operators from $L^p(S^{dn-1})$ to $L^2(S^{dn-1})$, $1 \leq p \leq 2$, in the real, complex and quaternionic framework.

At the present stage, such properties are well known in the real and in the complex case. More precisely, if $d = 1$, then π_ρ is the projection operator mapping $L^2(S^{n-1})$ onto the subspace of all spherical harmonics of fixed degree ρ, which is also, as is well known, an eigenspace for the Laplace-Beltrami operator on the unit sphere S^{n-1}. In this framework, C. Sogge [17, 20] proved that the $L^p - L^2$ norm of π_ρ may be controlled by a suitable power of the corresponding eigenvalue.

If $d = 2$, then π_ρ is the projection operator mapping $L^2(S^{2n-1})$ onto the subspace consisting of all complex spherical harmonics of fixed bidegree. It turns out that that V^ρ is also a joint eigenspace for the the Laplace-Beltrami operator $\Delta_{S^{2n-1}}$ and for a suitably defined subLaplacian $\mathcal{L}_{S^{2n-1}}$. Sharp estimates for $\|\pi_\rho\|_{(p,2)}$ in terms on the corresponding eigenvalues were proved by the author [5], [6].

If $d = 4$, a "polynomial" description (that is, in terms of spherical harmonics) of V^ρ is more involved; in the quaternionic framework V^ρ is instead usually characterized as the joint eigenspace of the Laplace Beltrami operator $\Delta_{S^{4n-1}}$ and of a subLaplacian $L_{S^{4n-1}}$.

Recently, in collaboration with Paolo Ciatti we started the study of the quaternionic joint spectral projection π_ρ. In the quaternionic context the picture is much more involved than in the complex framework, mainly due to the loss of symmetry between the indices ℓ and ℓ'. In this note, we briefly review the results in [8] and discuss some interesting differences and analogies with respect to the real and the complex case. In a forthcoming paper [9] we shall provide a detailed proof of sharp estimates for $\|\pi_\rho\|_{(p,2)}$ for all $1 \leq p \leq 2$ and for all possible values of ρ.

The bounds for $\|\pi_\rho\|_{(p,2)}$, interesting in themselves, also provide an essential tool to approach some problems in harmonic analysis. In the real and complex framework, in particular, they have been successfully applied in different contexts, such as the L^p summability of Bochner–Riesz means [18, 10], the unique continuation problem [13, 19, 20], and Strichartz estimates for solutions of dispersive equations [4, 11].

The plan of the paper is as follows. In Section 2 we briefly describe some classical results concerning the analysis on the spheres S^{dn-1}. In Section 3 we recall what is known about the norm of π_ρ, defined by (2), as an operator from $L^p(S^{dn-1})$ to $L^2(S^{dn-1})$, when $1 \leq p \leq 2$ and $d = 1, 2, 4$. In Section 4 we illustrate the numerology linking exponents and critical points of the estimates, introduced in the previous section. Finally, in Section 5 we discuss the sharpness, highlighting the peculiarity of the quaternionic bounds with respect to the real and the complex ones.

In the following, the symbol e_1 will denote the north pole of S^{4n-1}, that is $e_1 := (1, 0, \dots, 0)$. The letter C and variants such as $C(n)$ denote constants, always assumed to be positive, which may vary from one occurrence to the next. The symbol $\simeq$ between two positive expressions means that their ratio is bounded above and below.

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2. The spheres S^{dn-1}

2.1. **The real sphere.** When $d = 1$, (1) reduces to the classical decomposition

$$L^2(S^{n-1}) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}^\ell ,$$

$\mathcal{H}^\ell$ denoting the subspace of spherical harmonics of degree ℓ (that is, the space of the restriction to S^{n-1} of polynomials $P = P(x_1, \dots, x_n)$ homogeneous of degree ℓ and harmonic). Each subspace $\mathcal{H}^\ell$ is an eigenspace for the Laplace–Beltrami operator $\Delta_{S^{n-1}}$ for the eigenvalue $\nu_\ell = \ell(\ell + n - 2)$ and is invariant under the action of the orthogonal group $\mathrm{O}(n)$. Moreover, the representation of $\mathrm{O}(n)$ on $\mathcal{H}^\ell$ is irreducible (see [23, Ch. 4]).

Define the harmonic projection operator

$$\pi_\ell :\ L^2(S^{n-1}) \to \mathcal{H}^\ell\,.$$

It is well known [23, Ch. 4] that

$$\pi_\ell f(x) = \int_{S^{n-1}} f(y)\mathbb{Z}_\ell^{(x)}(y)\, d\sigma(y)\,, \tag{3}$$

where the integral kernel of π_ℓ is a zonal spherical harmonic of degree ℓ with pole x and $d\sigma$ represents the Lebesgue measure on S^{n-1}, given, up to a constant, by

$$d\sigma_{n-1} = \sin^{n-2}\theta\, d\sigma_{n-2}\,,$$

with respect to the standard system of spherical coordinates

$$\begin{cases} x_1 & = \cos\theta_1 \\ \vdots & \\ x_{n-1} & = \sin\theta_1 \ldots \sin\theta_{n-2}\cos\theta_{n-1} \\ x_n & = \sin\theta_1 \ldots \sin\theta_{n-2}\, \sin\theta_{n-1}\,, \end{cases} \tag{4}$$

with $\theta_j \in [0,\pi]$, $j = 1,\ldots,n-2$, $\theta_{n-1} \in [0,2\pi]$.

The zonal function $\mathbb{Z}_\ell := \mathbb{Z}_\ell^{e_1}$ only depends on x_1 and may be explicitly expressed as

$$\begin{aligned} \mathbb{Z}_\ell(\theta_1) &= \frac{2\ell+n-2}{(n-2)\omega_{n-1}} C_\ell^{\frac{n-2}{2}}(\cos\theta_1) \\ &= \frac{d_\ell}{\omega_{n-1}} \times \left(\binom{(n-3)/2+\ell}{\ell} \right)^{-1} P_\ell^{(\frac{n-3}{2},\frac{n-3}{2})}(\cos\theta_1)\,, \end{aligned} \tag{5}$$

where

$$d_\ell := \dim \mathcal{H}^\ell = \binom{n+\ell-3}{\ell-1}\Big(\frac{n+2\ell-2}{\ell}\Big)\,,$$

ω_{n-1} denotes the surface area of S^{n-1}, and $C_\ell^{\frac{n-2}{2}}$ and $P_\ell^{(\frac{n-3}{2},\frac{n-3}{2})}$ denote, respectively, the Gegenbauer and the Jacobi polynomial of degree ℓ (we refer to [24] for the precise definition).

2.2. **The complex sphere.** For $n \geq 2$ we denote by $\mathcal{S}^{2n-1}$ the unit sphere S^{2n-1} in $\mathbb{C}^n$, that is,

$$\mathcal{S}^{2n-1} := \left\{ z = (z_1, \dots, z_n) \in \mathbb{C}^n : \langle z, z \rangle = 1 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product defined by $\langle z, w \rangle = z_1 \overline{w_1} + \ldots + z_n \overline{w_n}$, $z, w \in \mathbb{C}^n$. The decomposition (1) reads in this case as

$$L^2\left(\mathcal{S}^{2n-1}\right) = \bigoplus_{\ell,\ell'=0}^{+\infty} \mathcal{V}^{\ell\ell'},$$

where $\mathcal{V}^{\ell\ell'}$ is the space of *complex spherical harmonics of bidegree* (ℓ, ℓ'), $\ell, \ell' \geq 0$, that is, the space of the restrictions to $\mathcal{S}^{2n-1}$ of polynomials $P = P(z, \bar{z})$ harmonic in $\mathbb{C}^n$, homogeneous of degree ℓ in z and ℓ' in $\bar{z}$. It is easy to check that $\mathcal{V}^{\ell\ell'}$ is invariant under the action of the unitary group $\mathrm{U}(n)$ and that the representation of $\mathrm{U}(n)$ on $\mathcal{H}^{\ell\ell'}$ is irreducible.

Moreover, each subspace $\mathcal{V}^{\ell\ell'}$ is a joint eigenspace for the Laplace–Beltrami operator $\Delta_{\mathcal{S}^{2n-1}}$ and for a subLaplacian. Following Geller [12], we define the subLaplacian $\mathcal{L}_{\mathcal{S}^{2n-1}}$ on $\mathcal{S}^{2n-1}$ as

$$\mathcal{L}_{\mathcal{S}^{2n-1}} = -\sum_{j<k} M_{jk}\overline{M}_{jk} + \overline{M}_{jk} M_{jk},$$

where

$$M_{jk} := \overline{z}_j \partial_{z_k} - \overline{z}_k \partial_{z_j}.$$

The eigenvalues of $\Delta_{\mathcal{S}^{2n-1}}$ and $\mathcal{L}_{\mathcal{S}^{2n-1}}$ are given, respectively, by $\eta_{\ell\ell'} = (\ell+\ell')(\ell + \ell' + 2n - 2)$ and $\theta_{\ell\ell'} = 2\ell\ell' + (n-1)(\ell + \ell')$.

Then we introduce the joint spectral projector $\mathcal{P}_{\ell\ell'}$, mapping $L^2(\mathcal{S}^{2n-1})$ onto $\mathcal{V}^{\ell\ell'}$. It turns out that

$$\mathcal{P}_{\ell\ell'} f(w) = \int_{\mathcal{S}^{2n-1}} f(z) \overline{\mathcal{Z}^w_{\ell\ell'}(z)}\, d\sigma(z), \tag{6}$$

for all $f \in L^2(\mathcal{S}^{2n-1})$, where the integral kernel $\mathcal{Z}^w_{\ell\ell'}$ is the so called zonal spherical harmonic of bidegree (ℓ, ℓ'), with pole w [15, Ch. 11]. $\mathcal{Z}^w_{\ell\ell'}$ may be explicitly written in terms of certain orthogonal polynomials, called the Zernike polynomials [25]. To be more precise, we call Zernike polynomials or disc polynomials the polynomials

$$P^{\alpha}_{m.n}(z,\bar z) := \Big(1/\binom{m+\alpha}{m}\Big)\bar z^{n-m}P^{(\alpha,n-m)}_{m}(2z\bar z-1)\,,$$

where $z\in\mathbb{C}$, $|z|\le 1$, and $P^{(\alpha,m-n)}_n$ denotes a Jacobi polynomial of degree n. The Zernike polynomials are orthogonal in the unit disc with respect to the weight $(1-z\bar z)^{\alpha}$.

Let us introduce the following coordinates system on $\mathcal{S}^{2n-1}$

$$\begin{cases} z_1 = e^{i\varphi_1}\ \cos\theta_1 \\ z_2 = e^{i\varphi_2}\sin\theta_1\cos\theta_2 \\ \vdots \\ z_n = e^{i\varphi_n}\sin\theta_1\ldots\sin\theta_{n-2}\ \sin\theta_{n-1}\,, \end{cases} \tag{7}$$

with $\varphi_k\in[0,2\pi]$, $k=1,\ldots,n$, and $\theta_j\in[0,\pi/2]$, $j=1,\ldots,n-1$. Then the normalized invariant measure $d\sigma_{\mathcal{S}^{2n-1}}$ on $\mathcal{S}^{2n-1}$ in these coordinates is, up to a constant $C=C(n)$,

$$\sin^{2n-3}\theta_1\cos\theta_1\,d\theta_1\,d\varphi_1\,d\sigma_{\mathcal{S}^{2n-3}}\,. \tag{8}$$

By taking $w=e_1$, the zonal function $\mathcal{Z}_{\ell\ell'}=\mathcal{Z}^{e_1}_{\ell\ell'}$ in $\mathcal{V}^{\ell\ell'}$, which only depends on $\langle z,e_1\rangle$, may be expressed as

$$\begin{aligned}\mathcal{Z}_{\ell\ell'}(\theta_1,\varphi_1) &:= \frac{d_{\ell,\ell'}}{\omega_{2n-1}}P^{n-2}_{\ell',\ell}(z_1,\bar z_1)\\ &= \frac{d_{\ell,\ell'}}{\omega_{2n-1}}\frac{\ell'(n-2)!}{(\ell'+n-2)!}e^{i(\ell'-\ell)\varphi_1}(\cos\theta_1)^{\ell-\ell'}P^{(n-2,\ell-\ell')}_{\ell'}(\cos 2\theta_1)\end{aligned} \tag{9}$$

where $\ell\ge\ell'\ge 1$, ω_{2n-1} denotes the surface area of $\mathcal{S}^{2n-1}$, and

$$d_{\ell,\ell'} := \dim\mathcal{V}^{\ell\ell'} = (n-1)\cdot\frac{\ell+\ell'+n-1}{\ell\ell'}\binom{\ell+n-2}{\ell-1}\binom{\ell'+n-2}{\ell'-1}\ \text{ for all } \ell,\ell'\ge 1. \tag{10}$$

If $\ell<\ell'$, we use the fact the $\mathcal{Z}_{\ell,\ell'}=\overline{\mathcal{Z}_{\ell',\ell}}$.

Recall finally that $\mathcal{V}^{\ell,0}$ consists of holomorphic polynomials and $\mathcal{V}^{0,\ell}$ consists of polynomials whose complex conjugates are holomorphic. The dimension of the spaces is given by

$$\dim\mathcal{V}^{\ell,0} = \binom{\ell+n-2}{\ell}$$

and the zonal function is

$$\mathcal{Z}_{\ell,0}(\theta_1,\varphi_1) := \frac{1}{\omega_{2n-1}}\binom{\ell+n-2}{\ell} e^{-i\ell\varphi_1}(\cos\theta_1)^\ell, \quad \varphi_1\in[0,2\pi],\ \theta_1\in[0,\frac{\pi}{2}].$$

2.3. **The quaternionic sphere.** Let $\mathbb{H}$ denote the skew field of all quaternions $q = x_0 + x_1 i + x_2 j + x_3 k$ over $\mathbb{R}$, where x_0, x_1, x_2, x_3 are real numbers and i, j, k fulfill

$$i^2 = j^2 = k^2 = -1\,,\ ij = -ji = k\,,\ ik = -ki = -j\,,\ jk = -kj = i\,.$$

For $n \geq 1$ the symbol $\mathbb{H}^n$ shall denote the n-dimensional vector space over $\mathbb{H}$, consisting of all vectors $q = (q_1, \ldots, q_n)$, $q_j \in \mathbb{H}$, $j = 1, \ldots, n$. The conjugate $\overline{q}$ and the modulus $|q|$ are, respectively, defined by

$$\overline{q} = x_0 - x_1 i - x_2 j - x_3 k \quad \text{and} \quad |q|^2 = q\overline{q} = \sum_{j=0}^{3} x_j^2\,.$$

We endow $\mathbb{H}^n$ with the inner product $\langle\langle q, q'\rangle\rangle := q_1\overline{q'_1} + \ldots + q_n\overline{q'_n}$, $q,\ q' \in \mathbb{H}^n$.

Let $\mathbb{S}^{4n-1}$ denote the unit sphere S^{4n-1} in $\mathbb{H}^n$, that is,

$$\mathbb{S}^{4n-1} := \{q = (q_1, \ldots, q_n) \in \mathbb{H}^n \,:\, \langle\langle q, q\rangle\rangle = 1\}\,.$$

The decomposition (1) reads in the quaternionic framework as

$$L^2\left(\mathbb{S}^{4n-1}\right) = \bigoplus_{\ell,\ell'=0}^{+\infty} \mathcal{H}^{\ell\ell'}$$

where $\mathcal{H}^{\ell\ell'}$ is the joint eigenspace of the Laplace Beltrami operator $\Delta_{\mathbb{S}^{4n-1}}$ and of a sub-Laplacian $\mathcal{L}_{\mathbb{S}^{4n-1}}$, with eigenvalues given, respectively, by $\mu_{\ell,\ell'} = (\ell+\ell')(\ell+\ell'+4n-2)$ and $\lambda_{\ell,\ell'} = 4(\ell\ell' + (n-1)\ell + n\ell')$. For a precise definition of $\mathcal{L}_{\mathbb{S}^{4n-1}}$ we refer to [2, p. 250] and [3, Section 2.2]; we only recall that $\mathcal{L}$ is a positive definite, selfadjoint, subelliptic operator. We also recall here that $\mathcal{H}^{\ell\ell'}$ is irreducible under the action of $Sp(n)\times Sp(1)$, $Sp(n)$ denoting the quaternionic unitary group.

It is worth noticing that $\mathcal{H}^{\ell\ell'}$ is also an eigenspace (with respect to the eigenvalue $\gamma_{\ell,\ell'} = (\ell-\ell')(\ell-\ell'+2))$ for the operator Γ, which may be defined in the following way. We write $q \in \mathbb{H}^n$ as

$$q = (z_1 + jz_{n+1}, z_2 + jz_{n+2}, \ldots, z_n + jz_{2n})\,, \quad z_1, \ldots, z_{2n} \in \mathbb{C}\,, \tag{11}$$

and set

$$\Gamma = (\bar{D} - D)^2 - 2D_1\bar{D}_1 - 2\bar{D}_1 D_1\,, \tag{12}$$

where $D = \sum_{\ell=1}^{2n} z_\ell \partial_\ell$, $D_1 = \sum_{\ell=1}^{n} \overline{z}_\ell \partial_{n+\ell} - \overline{z}_{n+\ell}\partial_\ell$, $\overline{D} = \sum_{\ell=1}^{2n} \overline{z}_\ell \overline{\partial}_\ell$, and $\overline{D}_1 = \sum_{\ell=1}^{n} z_\ell \overline{\partial}_{n+\ell} - z_{n+\ell}\overline{\partial}_\ell$.

By the symbol $\pi_{\ell\ell'}$ we denote the joint spectral projector mapping $L^2(\mathbb{S}^{4n-1})$ onto $\mathcal{H}^{\ell\ell'}$. It turns out that

$$\pi_{\ell\ell'} f(w) = \int_{\mathbb{S}^{4n-1}} f(z)\overline{\mathbb{Z}^w_{\ell\ell'}(z)}\, d\sigma(z)\,, \tag{13}$$

for all $f \in L^2(\mathbb{S}^{4n-1})$, where the integral kernel $\mathbb{Z}^w_{\ell\ell'}$ is called quaternionic zonal function of bidegree (ℓ, ℓ') with pole w.

A system of spherical coordinates on the sphere $\mathbb{S}^{4n-1}$ is given by

$$\begin{cases} q_1 = \cos\theta\,(\cos t + \tilde{q}\sin t) \\ q_s = \sigma_s \sin\theta\,, \end{cases} \tag{14}$$

where $\theta \in [0, \pi/2]$, $t \in [0, \pi]$, $\tilde{q} \in \mathbb{H}$ with $|\tilde{q}|^2 = 1$ and $\Re\tilde{q} = 0$, $\sigma_s \in \mathbb{H}$ with $\sum_{s=2}^{n} |\sigma_s|^2 = 1$. Then the normalized invariant measure $d\sigma_{\mathbb{S}^{4n-1}}$ on $\mathbb{S}^{4n-1}$ in these coordinates is, up to a constant $C = C(n)$,

$$\sin^{4n-5}\theta \cos^3\theta\, d\theta\, \sin^2 t\, dt\, d\sigma_{S^{4n-5}}\,. \tag{15}$$

By choosing e_1 as pole, an explicit formula for the zonal function $\mathbb{Z}^{e_1}_{\ell\ell'}$ in $\mathcal{H}^{\ell\ell'}$ was obtained in [14]. We have

$$\mathbb{Z}_{\ell\ell'}(\theta, t) := \frac{D_{\ell\ell'}}{\omega_{4n-1}} \frac{\sin\left((\ell - \ell' + 1)t\right)}{(\ell - \ell' + 1)\,\sin t} (\cos\theta)^{\ell-\ell'} \times \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\mathbf{1})}\,, \tag{16}$$

where $\ell \geq \ell' \geq 0$, $t \in [0, \pi/2]$, $\theta \in [0, \frac{\pi}{2}]$, ω_{4n-1} denotes the surface area of $\mathbb{S}^{4n-1}$, $P_{\ell'}^{(2n-3,\ell-\ell'+1)}$ is the Jacobi polynomial and

$$D_{\ell\ell'} := (\ell + \ell' + 2n - 1)(\ell - \ell' + 1)^2 \frac{(\ell + 2n - 2)!}{(\ell + 1)!(2n - 3)!} \frac{(\ell' + 2n - 3)!}{\ell'!(2n - 1)!}\,, \quad \ell \geq \ell' \geq 0,$$

is the dimension of $\mathcal{H}^{\ell\ell'}$. Observe that

$$D_{\ell\ell'} \simeq \begin{cases} (\ell-\ell'+1)^2\ell^{4n-5} & \text{if } (1-\varepsilon_0)\ell \le \ell' \le \ell \\ \ell^{4n-3} & \text{if } \varepsilon_0\ell' \le (1-\varepsilon_0)\ell \\ \ell^{2n}\ell'^{2n-2} & \text{if } 1 \le \ell' \le \varepsilon_0\ell\,, \end{cases} \tag{17}$$

which yields a first clue to the symmetry breaking occurring in the harmonic analysis in the quaternionic context.

When $\ell' = 0$, the dimension of the space $\mathcal{H}^{\ell,0}$ is given by

$$\dim \mathcal{H}^{\ell,0} = \binom{\ell+2n}{\ell},$$

and the zonal function is

$$\mathbb{Z}_{\ell,0}(\theta,t) := \frac{1}{\omega_{4n-1}}\binom{\ell+2n}{\ell}\frac{\sin((\ell+1)t)}{(\ell+1)\sin t}(\cos\theta)^{\ell}, \quad t\in[0,\frac{\pi}{2}],\ \theta\in[0,\frac{\pi}{2}].$$

3. The estimates

In this section, we briefly recall the estimates which are known for the norm of the projector π_ρ, defined by (2), as an operator from $L^p(S^{dn-1})$ to $L^2(S^{dn-1})$, when $1 \le p \le 2$ and $d = 1, 2, 4$.

In the real case C. Sogge proved the following sharp result.

Theorem 3.1. [17, 20] *For $1 \le p \le 2$ and for all $\ell \in \mathbb{N}$*

$$\|\pi_\ell\|_{(p,2)} \le C(n,p)\,\ell^{\sigma(1/p,n)}\,, \tag{18}$$

where

$$\sigma(\frac{1}{p},n) = \begin{cases} (n-1)(\frac{1}{p}-\frac{1}{2})-\frac{1}{2} & \text{if} \quad 1 \le p < p_{\mathbb{R}}(n) \\ (\frac{n}{2}-1)(\frac{1}{p}-\frac{1}{2}) & \text{if} \quad p_{\mathbb{R}}(n) \le p \le 2\,, \end{cases}$$

with $p_{\mathbb{R}}(n) := \frac{2n}{n+2}$.

In the case of the complex spheres sharp estimates for $\|\mathcal{P}_{\ell,\ell'}\|_{(p,2)}$, where $\mathcal{P}_{\ell\ell'}$ is defined by (6), were obtained by the author in [5, 6].

Theorem 3.2. *For $1 \leq p \leq 2$ and for all $0 \leq \ell' \leq \ell$, we have*

$$\|\mathcal{P}_{\ell,\ell'}\|_{(p,2)} \leq C(n,p)\ (1+\ell')^{\alpha(p)}\,(1+\ell)^{(n-1)\left(\frac{1}{p}-\frac{1}{2}\right)}\ ,$$

where

$$\alpha(p) = \begin{cases} (n-1)(\frac{1}{p}-\frac{1}{2}) - \frac{1}{2} & \text{if} \quad 1 \leq p < p_{\mathbb{C}}(n) \\ -\frac{1}{2}\left(\frac{1}{p}-\frac{1}{2}\right) & \text{if} \quad p_{\mathbb{C}}(n) \leq p \leq 2\,, \end{cases}$$

with $p_{\mathbb{C}}(n) := 2\frac{2n-1}{2n+1}$. These bounds are sharp.

It is worth mentioning that there is an intimate connection between the reduced Heisenberg group $\mathfrak{h}_n$ and the unit complex sphere. By exploiting the fact that $\mathfrak{h}_n$ is a contraction of $\mathcal{S}^{2n-1}$, in [7] we deduced from Theorem 3.2 sharp bounds for analogous joint spectral projectors in the Heisenberg framework.

Let us consider now the quaternionic case. In [8] we proved sharp bounds for the $L^p - L^2$ norm of the projector $\pi_{\ell\ell'}$, defined by (13), in two particular cases. First of all, we considered the case when $\ell - \ell' \leq c_0$, for some non negative constant c_0.

Theorem 3.3. *Let $n \geq 2$, $1 \leq p \leq 2$, and let ℓ,ℓ' be integer numbers such that $\ell \geq \ell' \geq 0$, $\ell - \ell' \leq c_0$ for some non negative constant c_0. Then there exists some constant C, only depending on n, p and c_0, such that the following estimate holds*

$$||\pi_{\ell\ell'}f||_2 \leq C(n,p,c_0)\,(1+\ell)^{A(\frac{1}{p},n)}(1+\ell')^{B(\frac{1}{p},n)}(c_0+1)^{C(\frac{1}{p},n)}\,||f||_p\,, \tag{19}$$

with

$$A\Big(\frac{1}{p},n\Big) := 2(n-1)\Big(\frac{1}{p}-\frac{1}{2}\Big)\,, \quad C\Big(\frac{1}{p}\Big) = 2\Big(\frac{1}{p}-\frac{1}{2}\Big)\,,\ \textit{for all } 1 \leq p \leq 2\,,\ \textit{and}$$

$$B\Big(\frac{1}{p},n\Big) := \begin{cases} 2(n-1)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{1}{2} & \text{if } 1 \leq p \leq p_{\mathbb{H}}(n) \\ \frac{1}{2}\big(\frac{1}{2}-\frac{1}{p}\big) & \text{if } p_{\mathbb{H}}(n) \leq p \leq 2\,. \end{cases}$$

where $p_{\mathbb{H}}(n) := 2\frac{4n-3}{4n-1}$.

Then we considered the case when ℓ', the minimum between ℓ and ℓ', is bounded.

Theorem 3.4. *Let $n \geq 2$, $1 \leq p \leq 2$, and let ℓ,ℓ' be integer numbers such that $\ell \geq \ell' \geq 0$, $\ell' \leq c_1$, for some non negative constant c_1. Then there exists some constant C, only depending on n, p and c_1, such that the following estimate holds*

$$||\pi_{\ell\ell'} f||_2 \leq C(n,p,c_1)\,(1+\ell)^{A'}(1+\ell')^{(2n-3)(\frac{1}{p}-\frac{1}{2})}\,||f||_p\,, \tag{20}$$

where

$$A'(\frac{1}{p},n) := \begin{cases} (2n+1)\big(\frac{1}{p}-\frac{1}{2}\big)-\frac{1}{2} & \text{if } 1 \leq p \leq p_c \\ (2n-1)\big(\frac{1}{p}-\frac{1}{2}\big) & \text{if } p_c \leq p \leq 2, \end{cases}$$

with $p_c = \frac{4}{3}$ and constants $C(n,p,c_1)$ only depending on n, p and c_1.

The proofs of Theorems 3.2, 3.3, 3.4, are inspired by Sogge's proof of Theorem 3.1. They are both quite long and technical and can not be summarized here. Anyway, they are essentially based on two ingredients: interpolation arguments and fine estimates for the Jacobi and Zernike polynomials. We briefly sketch the main steps.

(1) The estimates for $p = 1$ and $p = 2$ are trivial, as a consequence of Young's inequality and of the fact that $\mathcal{P}_{\ell\ell'}$ and $\pi_{\ell\ell'}$ are projectors.

(2) Thus, by the Riesz-Thorin Theorem, it suffices to prove the bounds for p equal to the critical point, that is, $p = p_{\mathbb{C}}(n)$ in Theorem 3.2, $p = p_{\mathbb{H}}(n)$ in Theorem 3.3 and $p = 4/3$ in Theorem 3.4.

(3) We use the Stein–Tomas trick to reduce the $L^p - L^2$ bound to a $L^p - L^{p'}$ bound (here p is one of the critical points). Then, we introduce a suitable family of analytic operators $\{T^z\}$, which are related to the convolution with the zonal functions $\mathcal{Z}_{\ell\ell'}$ and $\mathbb{Z}_{\ell\ell'}$, defined for $z \in \mathbb{C}$, $\Re z \in [0,1]$, and invoke the Stein theorem on analytic interpolation [23].

   For $\Re ez = 0$ we prove a $L^1 - L^\infty$ bound for T^{iy}, while for $\Re ez = 1$ we have to prove a $L^2 - L^2$ bound for T^{1+iy}. Thus we need both pointwise and integral estimates for the zonal functions, and therefore for the Jacobi polynomials.

(4) In the complex case, we fix $\beta = \ell - \ell'$ and use the classical bounds for $P_k^{(\alpha,\beta)}$ holding for α and β larger that -1 and fixed, which may be found, for instance, in [24]. Since the estimates we obtain are independent of $\ell - \ell'$, we may easily deduce uniform bounds.

(5) In the quaternionic case,by fixing β and applying the classical bounds for $P_k^{(\alpha,\beta)}$ we get estimates for the L^2-L^2 norm of T^{1+iy}, that strongly depend on ℓ, ℓ', and $\ell-\ell'$. Thus we need some more refined and recent estimates for Jacobi polynomials with variable indices. We refer to [8] for a discussion of the problem.

4. The critical points

A recurrent thread among the estimates for different joint harmonic projection operators is the occurrence of critical points in the interval $(1,2)$ (denoted, respectively, by $p_{\mathbb{R}}(n)$, $p_{\mathbb{C}}(n)$, $p_{\mathbb{H}}(n)$ and p_c), where the behavior of $\|\pi_\rho\|_{(p,2)}$ changes. In this section we shall try to illustrate the numeralogy linking these different exponents.

Since the existence of such critical points is strictly related to the Stein–Tomas restriction theorem, we start by recalling it.

On the Euclidean space $\mathbb{R}^d$ the spectral resolution of the Laplacian $\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_d^2}$ may be expressed in terms of convolution with the Fourier transform of the measures $d\sigma_r$, induced on the spheres centered at the origin by the Lebesgue measure, since $\Delta f * \widehat{\sigma_r} = r^2 f * \widehat{\sigma_r}$. The Stein-Tomas theorem [22, Ch. 9], which primarily concerns the restriction properties of the Fourier transform, also yields therefore the mapping properties of the spectral resolution of the Laplacian.

Theorem 4.1. *Suppose that* $1 \le p \le p_*(m)$, *where*

$$p_*(m) := 2\frac{m+1}{m+3}\,,\ d \in \mathbb{N}\,, \tag{21}$$

and let $\frac{1}{p} + \frac{1}{p'} = 1$. *Then the estimate*

$$\|f * \widehat{\sigma_r}\|_{p'} \le C_r \|f\|_p \tag{22}$$

holds for all Schwartz functions on $\mathbb{R}^m$ *and all* $r > 0$.

According to the Knapp example [22], estimate (22) fails if $p > p_*$.

Let us consider now the real case. According to Sogge's bounds, the exponent in (18) is a piecewise affine function changing its slope at a point $p_{\mathbb{R}}(n) = \frac{2n}{n+2}$. Calling m the dimension of the real sphere (that is, setting $n-1=m$), we get $p_{\mathbb{R}}(m+1) = \frac{2(m+1)}{m+3}$, corrisponding to the restriction exponent $p_*(m)$. This fact is not surprising, since Theorem

3.1 may be considered for all intents and purposes a discrete version of the restriction estimates of Stein and Tomas (we refer to [21] for a thorough discussion of this analog).

Then we observe that the critical point in the complex case $p_{\mathbb{C}}(n) = 2\frac{2n-1}{2n+1}$ coincides with the critical point $p_{\mathbb{R}}(2n-1)$, as could be heuristically expected from the fact that

$$\mathcal{H}^k_{(2n)} = \bigoplus_{\ell+\ell'=k} \mathcal{V}^{\ell\ell'} ,$$

where $\mathcal{H}^k_{(2n)}$ denotes the space of real spherical harmonics of degree k in $\mathbb{R}^{2n}$.

The quaternionic case is more involved, due to the existence of two critical points, one, $p_{\mathbb{H}}(n)$, depending on n, the other one, p_c, independent of the dimension of the sphere. A possible interpretation is the following.

We observe that, up to some detail, the quaternionic zonal function $\mathbb{Z}_{\ell\ell'}(\theta,t)$, defined by (16), is essentially given by the product between the real spherical harmonic in $\mathbb{R}^4$, from now on denoted by $\mathbb{Z}^{(4)}_{\ell-\ell'}(t)$, and the complex spherical harmonic $\mathcal{Z}_{\ell,\ell'}$ in $\mathbb{C}^{2n-1}$, from now on denoted by $\mathcal{Z}^{(2n-1)}_{\ell,\ell'}$.

To justify our claim, we recall from (3) and (5) that on the real sphere $S^{n-1} \subseteq \mathbb{R}^n$ the spectral projector associated to the k-th eigenvalue is given by

$$\pi_k f = \mathbb{Z}_k * f \simeq k^{(n-1)/2} P_k^{(\frac{n-3}{2},\frac{n-3}{2})} * f ,$$

where we refer to [17] for an appropriate definition of a convolution product on the sphere. In particular, as a consequence of [24, (4.1.7)] for $n = 4$ and $k = \ell - \ell'$ we have

$$\mathbb{Z}^{(4)}_{\ell-\ell'}(t) \simeq (\ell-\ell'+1)^{3/2} P^{(\frac{1}{2},\frac{1}{2})}_{\ell-\ell'}(\cos t) \simeq \; (\ell-\ell'+1)^2 \, \frac{\sin\left((\ell-\ell'+1)t\right)}{(\ell-\ell'+1)\,\sin t} ,$$

for all $t \in [0,\pi]$, so that we may write

$$\mathbb{Z}_{\ell\ell'}(\theta,t) = C_n \, \mathbb{Z}^{(4)}_{\ell-\ell'}(t) \, \times (\ell+\ell'+1)\Big((\ell+1)(\ell'+1)\Big)^{2n-3} (\cos\theta)^{\ell-\ell'} \qquad (23)$$

$$\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} , \;\; \ell \geq \ell' \geq 0, \; t \in [0,\pi] , \;\; \theta \in [0, \frac{\pi}{2}] .$$

Now we observe from (10) that the dimension of the subspace $\mathcal{V}^{\ell\ell'}$ of $L^2(\mathcal{S}^{4n-3})$ has the same order of growth as $(\ell+\ell'+1)\Big((\ell+1)(\ell'+1)\Big)^{2n-3}$. Recall moreover that the critical point in $\mathbb{C}^{2n-1}$ is $p_{\mathbb{C}}(2n-1) = 2\frac{4n-3}{4n-1}$ and that the Lebesgue measure on the unit sphere in $\mathbb{C}^{2n-1}$ is essentially given by $(\sin\theta)^{4n-5}\cos\theta \, d\theta$.

Indeed, a standard recursive formula for the Jacobi polynomials (see, e.g., (22.7.16) in [1]) yields

$$(2n+\ell+\ell')\cos^2\theta\, P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta) = (\ell+2)P_{\ell'}^{(2n-3,\ell-\ell')}(\cos 2\theta) \\ + (\ell'+1)P_{\ell'+1}^{(2n-3,\ell-\ell')}(\cos 2\theta)\,.$$

Since the factors $\frac{\ell+2}{2n+\ell+\ell'}$ and $\frac{\ell'+1}{2n+\ell+\ell'}$ are bounded, multiplying by $\cos^2\theta$, we recover the measure (15) on $\mathbb{S}^{4n-1}\subseteq\mathbb{H}^n$.

In [9] we shall prove, in particular, that the bounds for $\|\pi_{\ell\ell'}\|_{(p,2)}$ in the general case may be deduced from a combination between Theorem 3.1 in $\mathbb{R}^4$ and Theorem 3.2 in $\mathbb{C}^{2n-1}$.

5. Optimality

In order to prove the optimality of the estimates for the joint harmonic projection operators π_ρ, defined by (2), we are led by the inequality

$$\|\pi_\rho\|_{(p,2)} \geq \frac{\|Y_\rho\|_{p'}}{\|Y_\rho\|_2}\,, \qquad (p'\geq 2\,,\, Y_\rho\in\mathcal{H}^\rho) \tag{24}$$

to study the L^q norms of the eigenfunctions $Y_\rho\in V^\rho$, for $q\geq 2$.

In the complex case, a careful analysis of the Lebesgue norm (see [5, 6]) of the Zernike polynomials shows that the zonal functions $\mathcal{Z}_{\ell\ell'}$ yield the sharpness on the interval $(1/p_{\mathbb{C}}(n),1)$. while the "highest weight spherical harmonics" $\mathcal{Q}_{\ell\ell'}=z_1^\ell\bar z_2^{\ell'}$ yield the sharpness on the interval $(1/2,1/p_{\mathbb{C}}(n))$.

Moreover, it is not difficult to check that the zonal functions are pointwise concentrated at the North Pole, while the highest weight spherical harmonics are concentrated in a small neighborhood around the Equator. Thus, as in the real case, we observe that for small p the estimates for $\|\mathcal{P}_{\ell\ell'}\|_{(p,2)}$ are sensitive to a high pointwise concentration, while for large p bounds in Theorem 3.2 are more sensitive to a "scattered" concentration along the Equator (or, more generally, to a high concentration along closed geodesics, as proved also in more general contexts by Sogge).

This scheme is true in the quaternionic framework as well, with some extra disadvantages due to the occurrence of two critical points.

To be more precise, when p is small, we prove that the estimates in Theorem 3.3 and in Theorem 3.4 are optimal by considering the quaternionic zonal functions $\mathbb{Z}_{\ell\ell'}$. When p is close to 2, we prove the sharpness by computing the Lebesgue norm of the quaternionic highest weight vectors in $\mathcal{H}^{\ell\ell'}$, which may be expressed, by using the coordinates (11), as

$$\mathbb{Q}_{\ell\ell'} = \bar{z}_{n+1}^{\ell-\ell'}(z_1\bar{z}_{n+2} - z_2\bar{z}_{n+1})^{\ell'} . \tag{25}$$

Just to give an idea of the results, we present one of the results proved in [8].

Proposition 5.1. *Let ℓ,ℓ' be integer numbers such that $\ell \geq \ell' \geq 0$, $\ell - \ell' \leq c_0$ for some non negative constant c_0, both sufficiently large. Let $\mathbb{Q}_{\ell\ell'}$ be the highest weight vector defined by* (25)*. For all $q > 2$ we have*

$$\|\mathbb{Q}_{\ell,\ell'}\|_q \geq C(n,q,c_0)\left(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell'+1)^{2n-2}\,(\ell-\ell'+1)}\right)^{\frac{1}{q}} . \tag{26}$$

Moreover, for $q = 2$ in (26) *the symbol $\geq$ may be replaced by $\simeq$.*

Then in the light of (24) Corollary 5.1 entails optimality for large p, at least when $\ell - \ell'$ is bounded. In the general case, anyway, when $\ell - \ell'$ is variable, the picture is much more tangled and we have to prove optimality on the intermediate interval $p \in (\frac{4}{3}, p_{\mathbb{H}}(n))$ as well. This case in treated in full generality in [9], where we prove the sharpness, by considering other joint eigenfunctions for the Laplace Beltrami operator $\Delta_{\mathbb{S}^{4n-1}}$ and for the subLaplacian $\mathcal{L}_{\mathbb{S}^{4n-1}}$, different from $\mathbb{Z}_{\ell\ell'}$ and $\mathbb{Q}_{\ell\ell'}$.

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