Transition fronts for the Fisher-KPP equation

François Hamel [a] and Luca Rossi [b] [*]

[a] Aix Marseille Université, CNRS, Centrale Marseille

Institut de Mathématiques de Marseille, UMR 7373, 13453 Marseille, France

[b] Dipartimento di Matematica P. e A., Università di Padova, via Trieste 63, 35121 Padova, Italy

Abstract

This paper is concerned with transition fronts for reaction-diffusion equations of the Fisher-KPP type. Basic examples of transition fronts connecting the unstable steady state to the stable one are the standard traveling fronts, but the class of transition fronts is much larger and the dynamics of the solutions of such equations is very rich. In the paper, we describe the class of transition fronts and we study their qualitative dynamical properties. In particular, we characterize the set of their admissible asymptotic past and future speeds and their asymptotic profiles and we show that the transition fronts can only accelerate. We also classify the transition fronts in the class of measurable superpositions of standard traveling fronts.

1 Introduction and main results

The purpose of this paper is to describe front-like solutions connecting 0 and 1 for the equation

$$u_t = u_{xx} + f(u), \quad t \in \mathbb{R}, \ x \in \mathbb{R}, \tag{1.1}$$

when f is of Fisher-KPP (for Kolmogorov-Petrovski-Piskunov) type [13, 24]. Namely, we will assume that

$$\begin{cases} f : [0,1] \to \mathbb{R} \text{ is of class } C^2 \text{ and concave,} \\ f(0) = f(1) = 0 \text{ and } f(s) > 0 \text{ for all } s \in (0,1). \end{cases} \tag{1.2}$$

A typical example of such a function f is the quadratic nonlinearity $f(s) = s(1-s)$ coming from the logistic equation. Equations of the type (1.1) are among the most studied reaction-diffusion equations and they are known to be good models to describe the propagation of fronts connecting the unstable steady state 0 and the stable steady state 1, see e.g. [11, 30, 41]. The solution u typically stands for the density of a species invading an open space and the fronts play a fundamental role in the description of the dynamical properties of the solutions. This fact was already understood in the pioneering paper [24], where the authors investigated the existence of solutions of (1.1) of the form $u(t,x) = \varphi(x - ct)$ with c constant and $0 = \varphi(+\infty) < \varphi(\xi) < \varphi(-\infty) = 1$ for all $\xi \in \mathbb{R}$. We will refer to such solutions as *standard traveling fronts* (connecting 0 and 1).

For Fisher-KPP functions f satisfying (1.2), a standard traveling front $u(t,x) = \varphi(x - ct)$ exists if and only if

$$c \geq 2\sqrt{f'(0)} =: c^*$$

and, for each $c \geq c^*$, the function $\varphi = \varphi_c$ is decreasing and unique up to shifts [2, 11, 24, 43]. Furthermore, these fronts $\varphi_c(x - ct)$ are known to be stable with respect to perturbations in some suitable weighted spaces and to attract the solutions of the associated Cauchy problem for a large class of exponentially decaying initial conditions, see e.g. [3, 7, 10, 18, 23, 24, 25, 35, 43]. For a given $c \in [c^*, +\infty)$, the standard traveling front $\varphi_c(x - ct)$ moves with the constant speed c, in the sense that its profile $\varphi_c(x)$ is translated in time towards right with speed c. In particular, its level sets move with speed c. The existence of a continuum of speeds is in sharp contrast with the uniqueness of the speed for bistable type nonlinearities $f(u)$, see e.g. [2, 12]. The standard traveling fronts do not exist in general if the coefficients of the equation depend on space or time. When the dependence is periodic, the notion which immediately generalizes that of standard traveling fronts is that of pulsating traveling fronts, see e.g. [4, 20, 26, 27, 31, 34, 42, 44] for the existence and stability properties of such pulsating traveling fronts for time and/or space periodic KPP type equations. The notion of pulsating traveling front can also be extended in order to deal with time almost-periodic, almost-automorphic, recurrent or uniquely ergodic media, see [22, 36, 37, 38, 39, 40].

However, even for the homogeneous equation (1.1), what was unexpected until quite recently was the existence of front-like solutions connecting 0 and 1 which are not standard traveling fronts and whose speed changes in time. More precisely, it was proved in [17] that, for any real numbers c_- and c_+ such that $2\sqrt{f'(0)} = c^* \leq c_- < c_+$, there exist solutions $0 < u < 1$ satisfying $u(t, -\infty) = 1$ and $u(t, +\infty) = 0$ locally uniformly in $t \in \mathbb{R}$, together with

$$\begin{cases} u(t,x) - \varphi_{c_-}(x - c_- t) \to 0 & \text{as } t \to -\infty, \\ u(t,x) - \varphi_{c_+}(x - c_+ t) \to 0 & \text{as } t \to +\infty, \end{cases} \quad \text{uniformly in } x \in \mathbb{R}, \qquad (1.3)$$

where $\varphi_{c_-}(x - c_- t)$ and $\varphi_{c_+}(x - c_+ t)$ are any two given standard traveling fronts with speeds c_- and c_+ respectively. Roughly speaking, (1.3) implies that the level sets of u move approximately with speed c_- for large negative times and with speed c_+ for large positive times, and this is why one can intuitively say that the speed of u changes in time.

Transition fronts

The non-uniformly translating front-like solutions satisfying (1.3) fall into a notion of fronts connecting 0 and 1 introduced some years after [17]. This new notion, called transition fronts after [5, 6], has also been introduced in order to cover the case of general space-time dependent equations. When applied to a one-dimensional equation such as (1.1), the definition of transition fronts reads as follows:

Definition 1.1. *For problem* (1.1)*, a transition front connecting* 0 *and* 1 *is a time-global classical solution* $u : \mathbb{R} \times \mathbb{R} \to [0, 1]$ *for which there exists a function* $X : \mathbb{R} \to \mathbb{R}$ *such that*

$$\begin{cases} u(t, X(t) + x) \to 1 & as\ x \to -\infty, \\ u(t, X(t) + x) \to 0 & as\ x \to +\infty, \end{cases} \quad uniformly\ in\ t \in \mathbb{R}. \qquad (1.4)$$

The real numbers $X(t)$ therefore reflect the positions of a transition front as time runs. The simplest examples of transition fronts connecting 0 and 1 are the standard traveling fronts $u(t, x) = \varphi_c(x - ct)$, for which one can set $X(t) = ct$ for all $t \in \mathbb{R}$ (in time or space-periodic media, pulsating traveling fronts connecting 0 and 1 are also transition fronts and the positions $X(t)$ can be written in the same form). Furthermore, the solutions u constructed in [17] and satisfying (1.3) are transition fronts connecting 0 and 1, with $X(t) = c_- t$ for $t < 0$ and $X(t) = c_+ t$ for $t \geq 0$. This example shows that the above definition of transition fronts is meaningful and does not reduce to standard traveling fronts even in the case of the homogeneous equation (1.1).

Definition 1.1 is actually a particular case of a more general definition given in [5, 6] in a broader framework. For the one-dimensional equation (1.1), the transition fronts connecting 0 and 1 correspond to the "wave-like" solutions defined in [36, 37] (see also [29] for a different notion involving the continuity with respect to the environment around the front position). Roughly speaking, property (1.4) means that the diameter of the transition zone between the sets where $u \simeq 1$ and $u \simeq 0$ is uniformly bounded in time; the position $X(t)$ is at bounded Hausdorff distance from this transition zone. The fundamental property in (1.4) is the uniformity of the limits with respect to time $t \in \mathbb{R}$ (we point out that there are solutions satisfying these limits pointwise in $t \in \mathbb{R}$, but not uniformly, as shown by Corollary 1.12 below). Another way of writing (1.4) is to say that, given any real numbers a and b such that $0 < a \leq b < 1$, there is a constant $C = C(u, a, b) \geq 0$ such that, for every $t \in \mathbb{R}$,

$$\{ x \in \mathbb{R};\ a \leq u(t, x) \leq b \} \subset \left[X(t) - C, X(t) + C \right].$$

It is easily seen that, for every transition front u connecting 0 and 1 and for every $C \geq 0$,

$$0 < \inf_{t \in \mathbb{R},\, x \in [X(t) - C, X(t) + C]} u(t, x) \leq \sup_{t \in \mathbb{R},\, x \in [X(t) - C, X(t) + C]} u(t, x) < 1. \qquad (1.5)$$

Indeed, if there were a sequence $(t_n, x_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^2$ such that $(x_n - X(t_n))_{n \in \mathbb{N}}$ is bounded and $u(t_n, x_n) \to 0$ as $n \to +\infty$, then standard parabolic estimates would imply that the functions $u_n(t, x) = u_n(t + t_n, x + x_n)$ would converge locally uniformly, up to extraction of a subsequence, to a solution $0 \leq u_\infty \leq 1$ of (1.1) such that $u_\infty(0, 0) = 0$, whence $u_\infty = 0$

in $\mathbb{R}^2$ from the strong maximum principle and the uniqueness of the bounded solutions of the Cauchy problem associated to (1.1). But (1.4) and the boundedness of the sequence $(x_n - X(t_n))_{n \in \mathbb{N}}$ would yield the existence of $M > 0$ such that $u(t_n, x_n - M) \geq 1/2$ for all $n \in \mathbb{N}$, whence $u_\infty(0, -M) \geq 1/2$, which is a contradiction. Therefore, the first inequality of (1.5) follows, and the last one can be proved similarly.

Notice also that, for a given transition front u, the family $(X(t))_{t \in \mathbb{R}}$ is not uniquely defined since, for any bounded function $\xi : \mathbb{R} \to \mathbb{R}$, the family $(X(t) + \xi(t))_{t \in \mathbb{R}}$ satisfies (1.4) if $(X(t))_{t \in \mathbb{R}}$ does. Conversely, if $(X(t))_{t \in \mathbb{R}}$ and $(\widetilde{X}(t))_{t \in \mathbb{R}}$ are associated with a given transition front u connecting 0 and 1 in the sense of (1.4), then it follows from (1.4) applied to the family $(X(t))_{t \in \mathbb{R}}$ and from (1.5) applied to the family $(\widetilde{X}(t))_{t \in \mathbb{R}}$ that

$$\sup_{t \in \mathbb{R}} \left| X(t) - \widetilde{X}(t) \right| < +\infty. \tag{1.6}$$

In other words, we can say that the family of positions $(X(t))_{t \in \mathbb{R}}$ of a given transition front connecting 0 and 1 is defined up to an additive bounded function.

The main goals of the paper are: describing the set of transition fronts for equation (1.1), providing conditions for a solution $0 < u < 1$ to be a transition front, studying the qualitative properties of the transition fronts, characterizing their asymptotic propagation rates for large negative and positive times, in a sense to be made more precise below, and finally describing their asymptotic profiles.

Boundedness of the local oscillations of the positions of a transition front

First of all, we claim that a given transition front cannot move infinitely fast. More precisely, even if the positions $(X(t))_{t \in \mathbb{R}}$ of a given transition front are only defined up to bounded functions, their local oscillations are uniformly bounded, in the sense of the following proposition.

Proposition 1.2. *Under the assumptions* (1.2)*, for any transition front of* (1.1) *connecting* 0 *and* 1 *in the sense of Definition* 1.1*, there holds*

$$\forall \tau \geq 0, \quad \sup_{(t,s) \in \mathbb{R}^2, |t-s| \leq \tau} |X(t) - X(s)| < +\infty. \tag{1.7}$$

Because this result is of independent interest, we will prove it under weaker hypotheses on f, in the case of time-space dependent equations, see Proposition 4.2 in Section 4.4 below. Applying recursively (1.7) with, say, $\tau = 1$, one deduces that

$$\limsup_{t \to \pm\infty} \left| \frac{X(t)}{t} \right| < +\infty, \tag{1.8}$$

that is, the propagation speeds of any transition front connecting 0 and 1 are asymptotically bounded as $t \to \pm\infty$. This property is in sharp contrast with some known acceleration phenomena and infiniteness of the asymptotic speed as $t \to +\infty$ for the solutions of the associated Cauchy problem when the initial condition $u_0(x)$ decays to 0 as $x \to +\infty$ more slowly than any exponential function: we refer to [19] for the existence of such accelerating solutions, which are not transition fronts connecting 0 and 1 on the time interval $[0, +\infty)$. These observations somehow mean that the transition fronts u connecting 0 and 1, as defined by (1.4), cannot decay too slowly to 0 as $x \to +\infty$, for any given time $t \in \mathbb{R}$.

4

Global mean speed

As it has become clear from the previous considerations, important notions associated with the transition fronts are those of their propagation speeds. We first define the notion of possible global mean speed.

Definition 1.3. *We say that a transition front connecting* 0 *and* 1 *for* (1.1) *has a* global mean speed $\gamma \in \mathbb{R}$ *if*

$$\frac{X(t) - X(s)}{t - s} \to \gamma \quad as \ t - s \to +\infty, \tag{1.9}$$

that is $(X(t + \tau) - X(t))/\tau \to \gamma$ *as* $\tau \to +\infty$ *uniformly in* $t \in \mathbb{R}$.

We point out that this definition slightly differs from the one used in [5, 6], where absolute values were taken in (1.9), that is $|X(t) - X(s)|/|t - s| \to |\gamma|$ as $t - s \to +\infty$ (a more general notion of global mean speed involving the Hausdorff distance of family of moving hypersurfaces is defined in [5, 6] for more general equations in higher dimensions). But, in this one-dimensional situation considered here, it is quite easy to see that the two definitions coincide. Notice that, due to Proposition 1.2, a given transition front connecting 0 and 1 for (1.1) cannot have an infinite global mean speed, that is γ in (1.9) cannot be equal to $\pm\infty$. Furthermore, if a transition front has a global mean speed γ, then this speed does not depend on the family $(X(t))_{t \in \mathbb{R}}$, due to the property (1.6). But the global mean speed, if any, does depend on the transition front. Actually, any standard traveling front $\varphi_c(x - ct)$ solving (1.1) has a global mean speed equal to c, and the set of all admissible global mean speeds among all standard traveling fronts is therefore equal to the semi-infinite interval $[c^*, +\infty)$ with $c^* = 2\sqrt{f'(0)}$.

Moreover, it turns out that the set of global mean speeds is not increased when passing from the set of standard traveling fronts to the larger class of transition fronts, and the standard traveling fronts are the only transition fronts with non-critical (that is, larger than c^*) global mean speed γ. Indeed, we derive the following.

Theorem 1.4. *Under the assumptions* (1.2), *the set of admissible global mean speeds of transition fronts connecting* 0 *and* 1 *for* (1.1) *is equal to the interval* $[c^*, +\infty)$. *Furthermore, if a transition front* u *connecting* 0 *and* 1 *has a global mean speed* $\gamma > c^*$, *then it is a standard traveling front of the type* $u(t, x) = \varphi_\gamma(x - \gamma t)$.

In Theorem 1.4, the fact that transition fronts cannot have global mean speeds less than c^* easily follows from standard spreading results of Aronson and Weinberger [2], see (1.13) below. The second statement of Theorem 1.4, classifying transition fronts with non-minimal global mean speed, is more intricate. It follows from additional results of [17] on the classification of solutions of (1.1) which move faster than c^* as $t \to -\infty$, in a sense to be made more precise later, and from a more detailed analysis of the convergence properties of such solutions, see Theorem 1.14 below. It is conjectured that the transition fronts with global mean speed $\gamma = c^*$ are also standard traveling fronts $\varphi_{c^*}(x - c^*t)$, but the proof of this conjecture would require further arguments.

5

Asymptotic past and future speeds and limiting profiles

The aforementioned example (1.3) shows that, even in a homogeneous medium, transition fronts may not admit a global mean speed, since the transition fronts u satisfying (1.3) are such that $X(t)/t \to c_-$ as $t \to -\infty$ and $X(t)/t \to c_+$ as $t \to +\infty$, with $c_+ > c_-$. This fact leads us naturally to define the notion of possible asymptotic speeds as $t \to -\infty$ and as $t \to +\infty$.

Definition 1.5. *We say that a transition front connecting* 0 *and* 1 *for the equation* (1.1) *has an* asymptotic past speed $c_- \in \mathbb{R}$, *resp. an* asymptotic future speed $c_+ \in \mathbb{R}$, *if*

$$\frac{X(t)}{t} \to c_- \ as \ t \to -\infty, \quad resp. \ \frac{X(t)}{t} \to c_+ \ as \ t \to +\infty. \tag{1.10}$$

As in the case of the global mean speed, the asymptotic past and future speeds, if any, of a given transition front do not depend on the family $(X(t))_{t \in \mathbb{R}}$. Clearly, if a transition front admits a global mean speed γ in the sense of (1.9), then it has asymptotic past and future speeds equal to γ. Furthermore, it follows from the proof of Theorem 1.4 (see Remark 4.1 below) that if a transition front u connecting 0 and 1 has asymptotic past and future speeds $c_\pm$ equal to the same value γ and if $\gamma > c^*$, then u is a standard traveling front of the type $\varphi_\gamma(x - \gamma t)$ and, in particular, u has a global mean speed equal to γ. Notice also that Proposition 1.2 implies that a given transition cannot have past or future speed equal to $\pm\infty$.

The asymptotic past and future speeds, if any, characterize the rate of expansion of a given transition front at large negative or positive times. These asymptotic speeds might not exist a priori and one could wonder whether these notions of speeds as $t \to \pm\infty$ would be sufficient to describe the large time dynamics of all transition fronts connecting 0 and 1 for (1.1). As a matter of fact, one of the main purposes of the present paper will be to characterize completely the set of all admissible asymptotic speeds and to show that the asymptotic speeds exist and characterize the asymptotic profiles of the fronts, at least for all fronts which are non-critical as $t \to -\infty$.

We first begin with the complete characterization of the set of admissible asymptotic past and future mean speeds.

Theorem 1.6. *Under the assumptions* (1.2)*, transition fronts connecting* 0 *and* 1 *for* (1.1) *and having asymptotic past and future speeds* $c_\pm$ *as* $t \to \pm\infty$ *exist if and only if*

$$c^* \leq c_- \leq c_+ < +\infty.$$

In this theorem, the sufficiency condition is clear. Indeed, firstly, for any $c \in [c^*, +\infty)$, the standard traveling fronts $\varphi_c(x - ct)$ are transition fronts with asymptotic past and future speeds equal to c. Secondly, for any $c^* \leq c_- < c_+ < +\infty$, the transition fronts solving (1.3) have asymptotic past and future speeds equal to $c_\pm$. However, unlike the Liouville-type result stated in Theorem 1.4 for the fronts with non-critical global mean speeds, transition fronts having prescribed asymptotic past and future speeds $c_- < c_+$ in $[c^*, +\infty)$ are not unique up to shifts in time or space: as a matter of fact, there exists an infinite-dimensional

manifold of solutions satisfying (1.3) for any prescribed speeds $c_- < c_+$ in $[c^*, +\infty)$, as will be seen later.

The new part in Theorem 1.6 is the necessity condition, that is transition fronts connecting 0 and 1 and having asymptotic past and future speeds can only accelerate and never decelerate, in the sense that the asymptotic future speed cannot be smaller than the past one. This property can have the following heuristic explanation: if one believes that a transition front u is a superposition of a family of standard fronts $(\varphi_c(x - ct))_c$ (and this is generally the case, in a sense that will be explained later on in this section), then this family should not contain fronts with c less than the asymptotic past speed c_- of u (otherwise, since they are slower, they would be dominant as $t \to -\infty$). Therefore, for very negative times t, the tail of $u(t, \cdot)$ at $+\infty$ should not involve the faster decay rates λ_c, with $\lambda_c > \lambda_{c_-}$, of the traveling fronts with smaller speeds $c < c_-$. In other words, for very negative times t, $u(t, \cdot)$ is expected to decay at $+\infty$ with a rate not larger than λ_{c_-}. But since such tails give rise in future times to solutions which move not slower than c_-, by [43], u is then expected to move faster than c_- as $t \to +\infty$.

Actually, the necessity condition of Theorem 1.6 is a consequence of the following more general and more precise result.

Theorem 1.7. *Under the assumptions* (1.2), *for any transition front u connecting 0 and 1 for* (1.1), *there holds*

$$c^* \le \liminf_{t \to -\infty} \frac{X(t)}{t} \le \liminf_{t \to +\infty} \frac{X(t)}{t} \le \limsup_{t \to +\infty} \frac{X(t)}{t} < +\infty. \qquad (1.11)$$

Furthermore, if $c^* < \liminf_{t \to -\infty} X(t)/t$, then u has asymptotic past and future speeds, that is the limits $c_\pm = \lim_{t \to \pm\infty} X(t)/t$ exist in $\mathbb{R}$, with $c_- \le c_+$, and there exists a bounded function $\xi : \mathbb{R} \to \mathbb{R}$ such that

$$u(t, X(t) + \xi(t) + \cdot) \to \varphi_{c_\pm} \quad in \ C^2(\mathbb{R}) \quad as \ t \to \pm\infty. \qquad (1.12)$$

Some comments are in order. The first inequality of (1.11) will follow immediately from some spreading estimates of [2], see (1.13) below. The last inequality is a direct consequence of Proposition 1.2. The difficult inequality is the second one. Actually, if $c^* = \liminf_{t \to -\infty} X(t)/t$, then the inequality $\liminf_{t \to -\infty} X(t)/t = c^* \le \liminf_{t \to +\infty} X(t)/t$ also follows from [2]. Therefore, the real interest of Theorem 1.7 lies in the second assertion. Among other things, its proof uses the decomposition of supercritical transition fronts as superposition of standard traveling fronts. This decomposition, given in [17], will be explained at the end of this section and it is conjectured to hold for *every* solution $0 < u < 1$ of (1.1). Therefore, we conjecture that the following property holds, even in the critical case where $\liminf_{t \to -\infty} X(t)/t = c^*$.

Conjecture 1.8. *Under the assumptions* (1.2), *any transition front u connecting 0 and 1 for* (1.1) *has asymptotic past and future speeds $c_\pm$ such that $c^* \le c_- \le c_+ < +\infty$ and the convergence* (1.12) *to the limiting profiles $\varphi_{c_\pm}$ holds.*

Theorem 1.7 shows the existence of the asymptotic past and future speeds for transition fronts u such that $\liminf_{t \to -\infty} X(t)/t$ is not critical (that is, larger than c^*). Thus, for

such fronts, some complex dynamics where the set of limiting values of $X(t)/t$ as $t \to -\infty$ and $t \to +\infty$ would not be reduced to a singleton are impossible. As a matter of fact, such infinitely many oscillations are possible in general for the solutions of the Cauchy problem associated with (1.1), as shown in [16, 45]. This occurs when the initial condition $u_0(x)$ has itself a complex behavior as $x \to +\infty$ with infinitely many oscillations on some larger and larger intervals between two exponentially decaying functions (see also [14] for the existence of asymptotic oscillations between two different speeds for the solutions of x-dependent Cauchy problems when the initial condition $u_0(x)$ is a Heaviside-type function and $f(x,u)$ is slowly and non-periodically oscillating as $x \to +\infty$). Here, we consider time-global solutions defined for all $t \in \mathbb{R}$ and, roughly speaking, Theorem 1.7 means that, at least when the transition front is not critical as $t \to -\infty$, the possible oscillations of the tails as $x \to +\infty$ are somehow annihilated by the fact that the solution is ancient.

Characterization of transition fronts and their asymptotic past and future speeds

Transition fronts connecting 0 and 1 for (1.1) are particular time-global solutions $0 < u < 1$ satisfying (1.4) for some family $(X(t))_{t \in \mathbb{R}}$. One can now wonder what condition would guarantee a given time-global solution u of (1.1) to be a transition front. First of all, the spreading results of [2] imply that any solution $0 < u(t,x) < 1$ of (1.1) is such that

$$\forall\, c \in [0, c^*), \quad \max_{[-c|t|, c|t|]} u(t, \cdot) \to 0 \text{ as } t \to -\infty. \tag{1.13}$$

The next result provides a characterization of the transition fronts connecting 0 and 1 in the class of solutions $0 < u < 1$ of (1.1) which satisfy a condition slightly stronger than (1.13).

Theorem 1.9. *Assume that f satisfies (1.2) and let $0 < u < 1$ be a solution of (1.1) such that*

$$\exists\, c > c^*, \quad \max_{[-c|t|, c|t|]} u(t, \cdot) \to 0 \quad \text{as } t \to -\infty. \tag{1.14}$$

Then the limit

$$\lambda := -\lim_{x \to +\infty} \frac{\ln u(0,x)}{x} \tag{1.15}$$

exists, satisfies $\lambda \in \left[0, \sqrt{f'(0)}\right)$ and u is a transition front connecting 0 and 1 if and only if $\lambda > 0$. Furthermore, if $\lambda > 0$, the transition front u admits asymptotic past and future speeds c_- and c_+ given by

$$c^* < c_- = \sup\left\{c \geq 0, \ \lim_{t \to -\infty} \max_{[-c|t|, c|t|]} u(t, \cdot) = 0\right\} \leq c_+ = \lambda + \frac{f'(0)}{\lambda}, \tag{1.16}$$

and (1.12) holds for some bounded function $\xi : \mathbb{R} \to \mathbb{R}$.

The time $t = 0$ in condition (1.15) can obviously be replaced by an arbitrary time $t_0 \in \mathbb{R}$. Moreover, $-\lim_{x \to +\infty}(\ln u(t_0, x))/x = \lambda$ does not depend on $t_0 \in \mathbb{R}$. Indeed, u is a transition front connecting 0 and 1 if and only if $u^{t_0} : (t, x) \mapsto u(t + t_0, x)$ is too for every $t_0 \in \mathbb{R}$ and, in that case, u^{t_0} has the same asymptotic future speed c_+ as u. Therefore, if $\lambda = -\lim_{x \to +\infty}(\ln u(0, x))/x > 0$, then u and u^{t_0} are transition fronts connecting 0 and 1,

8

for every $t_0 \in \mathbb{R}$. Hence, in that case, $\lambda_{t_0} := -\lim_{x \to +\infty} (\ln u(t_0, x))/x \in (0, \sqrt{f'(0)})$ and u^{t_0} has an asymptotic future speed equal to $c_+ = \lambda_{t_0} + f'(0)/\lambda_{t_0} = \lambda + f'(0)/\lambda$, whence $\lambda_{t_0} = \lambda$. Similarly, if $\lambda = 0$, then $\lambda_{t_0} = 0$ for every $t_0 \in \mathbb{R}$. To sum up, $-(\ln u(t_0, x))/x \to \lambda \in [0, \sqrt{f'(0)})$ as $x \to +\infty$ for all $t_0 \in \mathbb{R}$.

The fact that it is sufficient to know at least some informations on the profile of u at a given time in order to infer what its asymptotic future speed c_+ will be is not surprising, due to the parabolic maximum principle. A formula for the past speed c_- which would be as simple as the one for c_+ is far to be trivial: our characterization of c_- requires a knowledge of the profile of u as $t \to -\infty$, but it seems hopeless to be able to derive the past speed from the profile at a given time.

A consequence of Theorem 1.9 is that, under the assumption (1.14), if $0 < u < 1$ solves (1.1) but is not a transition front connecting 0 and 1, then, for every $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that $u(0, x) \geq C_\varepsilon \, e^{-\varepsilon x}$ for all $x \geq 0$. In other words, for any solution u which is not a transition front and for any time, say $t = 0$, then $u(0, x)$ either decays to 0 as $x \to +\infty$ more slowly than any exponentially decaying function, or is away from 0 at $+\infty$ (in the latter case, it follows from the proof of Theorem 1.9 that $\lim_{x \to +\infty} u(0, x)$ exists in $(0, 1]$).

Hypothesis (1.14) is required in the proof of Theorem 1.9 in order to apply the result of classification of such solutions $0 < u < 1$ of (1.1), see [17] and the end of this section. If, as conjectured, the characterization holds without this additional condition (1.14), then we can reasonably state the following conjecture.

Conjecture 1.10. *Under the assumptions* (1.2), *for any solution* $0 < u < 1$ *of* (1.1), *the limit* $\lambda = -\lim_{x \to +\infty} \ln u(0, x)/x$ *exists in* $[0, \sqrt{f'(0)}]$ *and* u *is a transition front connecting 0 and 1 if and only if* $\lambda > 0$. *Furthermore, if* $\lambda > 0$, *then the transition front* u *has asymptotic past and future speeds* c_- *and* c_+, *which satisfy* (1.16) *with* $c^* \leq c_-$, *and* (1.12) *holds.*

Finally, we point out that Theorem 1.9 will easily imply the second part of the conclusion of Theorem 1.7 since, for a given transition front connecting 0 and 1 such that $\liminf_{t \to -\infty} X(t)/t > c^*$, the condition (1.14) is automatically fulfilled.

Transition fronts as superposition of standard traveling fronts

An important tool in the proof of the above theorems is the construction of time-global solutions $0 < u < 1$ of (1.1) as superposition of standard traveling fronts. We recall that $\varphi_c(x - ct)$ denotes a standard traveling front for (1.1) such that $\varphi_c(-\infty) = 1 > \varphi_c > \varphi_c(+\infty) = 0$. Such front exists if and only if $c \geq c^* = 2\sqrt{f'(0)}$. Furthermore, for each $c \geq c^*$, the function φ_c is decreasing, unique up to shift, and one can assume without loss of generality that it satisfies

$$\begin{cases} \varphi_c(\xi) \sim e^{-\lambda_c \xi} & \text{if } c > c^*, \\ \varphi_{c^*}(\xi) \sim \xi \, e^{-\lambda_{c^*} \xi}, \end{cases} \qquad \text{as } \xi \to +\infty, \tag{1.17}$$

where

$$\lambda_c = \frac{c - \sqrt{c^2 - 4f'(0)}}{2} \quad \text{for } c \geq c^*. \tag{1.18}$$

Notice in particular that $\lambda_{c^*} = c^*/2 = \sqrt{f'(0)}$. With the normalization (1.17), it is known that $\varphi_c(\xi) \le e^{-\lambda_c \xi}$ for all $\xi \in \mathbb{R}$ if $c > c^*$. Lastly, let $\theta : \mathbb{R} \to (0,1)$ be the unique solution of

$$\theta'(t) = f(\theta(t)), \ t \in \mathbb{R},$$

such that $\theta(t) \sim e^{f'(0)t}$ as $t \to -\infty$.

Let us now introduce some notations which are used for the construction of other solutions of (1.1) given as measurable interactions of some shifts of the space- independent solution $\theta(t)$ and the right- or left-moving standard traveling fronts $\varphi_c(\pm x - ct)$. More precisely, let first $\Psi : [-\lambda_{c^*}, \lambda_{c^*}] \to X := (-\infty, -c^*] \cup [c^*, +\infty) \cup \{\infty\}$ be the bijection defined by

$$\begin{cases} \Psi(\lambda) & = \ \lambda + \dfrac{f'(0)}{\lambda} & \text{if } \lambda \in [-\lambda_{c^*}, \lambda_{c^*}] \backslash \{0\}, \\ \Psi(0) & = \ \infty, \end{cases}$$

and let us endow X with the topology induced by the image by Ψ of the Borel topology of $[-\lambda_{c^*}, \lambda_{c^*}]$. In other words, a subset O of X is open if $\Psi^{-1}(O)$ is open relatively in $[-\lambda_{c^*}, \lambda_{c^*}]$, that is the intersection of $[-\lambda_{c^*}, \lambda_{c^*}]$ with an open set of $\mathbb{R}$. The bijection $\Psi^{-1} : X \to [-\lambda_{c^*}, \lambda_{c^*}]$ denotes the reciprocal of the function Ψ and it is given by $\Psi^{-1}(c) = \lambda_c := (|c| - \sqrt{c^2 - 4f'(0)}) \times (\operatorname{sgn} c)/2$ if $c \in X \backslash \{\infty\}$ and $\Psi^{-1}(\infty) = 0$, where $\operatorname{sgn} c = c/|c|$ denotes the sign of c. Notice that the notation for λ_c is coherent with (1.18) when $c \in [c^*, +\infty)$. Lastly, let $\mathcal{M}$ be the set of all nonnegative Borel measures μ on X such that $0 < \mu(X) < +\infty$.

It follows from Theorem 1.2 of [17] and formula (30) of [17] that there is a one-to-one map

$$\mu \mapsto u_\mu$$

from $\mathcal{M}$ to the set of solutions $0 < u < 1$ of (1.1). Furthermore, for each $\mu \in \mathcal{M}$, denoting $M = \mu(X \backslash \{-c^*, c^*\})$, the solution u_μ constructed in [17] satisfies

$$
\begin{aligned}
\max \Big(&\varphi_{c^*}\big(x - c^*t - c^* \ln \mu(c^*)\big), \ \varphi_{c^*}\big(-x - c^*t - c^* \ln \mu(-c^*)\big), \\
&M^{-1} \int_{\mathbb{R} \backslash [-c^*, c^*]} \varphi_{|c|}\big((\operatorname{sgn} c)x - |c|t - |c| \ln M\big) \, d\mu(c) \ + \ M^{-1}\theta(t + \ln M)\,\mu(\infty) \Big) \\
\le u_\mu(t,x) \le \ &\varphi_{c^*}\big(x - c^*t - c^* \ln \mu(c^*)\big) + \varphi_{c^*}\big(-x - c^*t - c^* \ln \mu(-c^*)\big) \\
&+ M^{-1} \int_{\mathbb{R} \backslash [-c^*, c^*]} e^{-\lambda_{|c|}((\operatorname{sgn} c)x - |c|t - |c| \ln M)} d\mu(c) \ + \ M^{-1} e^{f'(0)(t + \ln M)} \mu(\infty)
\end{aligned}
\tag{1.19}
$$

for all $(t,x) \in \mathbb{R}^2$, under the convention that $\mu(\pm c^*) = \mu(\{\pm c^*\})$, that $\mu(\infty) = \mu(\{\infty\})$ and that the terms involving $\ln 0$ are not present. The estimate (1.19) reflects different types of contributions weighted by μ: critical standard traveling fronts, supercritical standard traveling fronts and spatially homogeneous solutions. More precisely, from Section 3 of [17], each u_μ is constructed as the monotone $C_{loc}^{1,2}(\mathbb{R} \times \mathbb{R})$ limit as $n \to +\infty$ of the functions u_μ^n

defined in $[-n, +\infty) \times \mathbb{R}$ as the solutions of the Cauchy problems

$$
\begin{cases}
(u_\mu^n)_t = (u_\mu^n)_{xx} + f(u_\mu^n), \quad t > -n, \ x \in \mathbb{R}, \\
u_\mu^n(-n, x) = \max\Big(\varphi_{c^*}\big(x + c^*n - c^* \ln \mu(c^*)\big), \ \varphi_{c^*}\big(-x + c^*n - c^* \ln \mu(-c^*)\big), \\
\qquad\qquad M^{-1}\displaystyle\int_{\mathbb{R}\setminus[-c^*,c^*]} \varphi_{|c|}\big((\operatorname{sgn} c)x + |c|n - |c|\ln M\big)\, d\mu(c) \\
\qquad\qquad + M^{-1}\theta(-n + \ln M)\, \mu(\infty)\Big), \quad x \in \mathbb{R}.
\end{cases}
\tag{1.20}
$$

In particular, for each $\mu \in \mathcal{M}$, one has $u_\mu^n(t, x) \to u_\mu(t, x)$ as $n \to +\infty$ locally uniformly in $\mathbb{R}^2$. Throughout the paper, the notation u_μ will refer to such solutions, which all fulfill $0 < u_\mu < 1$ in $\mathbb{R}^2$. A straightforward consequence of the maximum principle is that the solutions u_μ are nonincreasing (resp. nondecreasing) with respect to x if $\mu\big((-\infty, -c^*]\big) = 0$ (resp. if $\mu\big([c^*, +\infty)\big) = 0$), that is if u_μ is a measurable interaction of right-moving (resp. left-moving) spatially decreasing (resp. increasing) traveling fronts $\varphi_c(x - ct)$ (resp. $\varphi_c(-x - ct)$) and the spatially uniform function θ. Before going further on, let us just mention the simplest example of such solutions u_μ. Namely, if $\mu = m\, \delta_c$ with $m > 0$ and $c \in (-\infty, -c^*] \cup [c^*, +\infty)$, where δ_c denotes the Dirac measure at $\{c\}$, then $u_\mu^n(-n, x) = \varphi_{|c|}\big((\operatorname{sgn} c)x + |c|n - |c|\ln m\big)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, whence $u_\mu(t, x) = \varphi_{|c|}\big((\operatorname{sgn} c)x - |c|t - |c|\ln m\big)$ for all $(t, x) \in \mathbb{R}^2$. For more general measures μ, we refer to [17] for additional monotonicity properties, continuity with respect to μ and asymptotic properties of u_μ as $t \to \pm\infty$.

Lastly, we point out that these solutions u_μ almost describe the set of all solutions of (1.1). Indeed, reminding the general property (1.13), the following almost-uniqueness result was proved in [17]: if a solution $0 < u(t, x) < 1$ of (1.1) satisfies (1.14), that is

$$
\exists\, c > c^*, \quad \max_{[-c|t|, c|t|]} u(t, \cdot) \to 0 \ \text{ as } t \to -\infty,
$$

then there is a measure $\mu \in \mathcal{M}$ such that $u = u_\mu$ and the support of μ does not intersect the interval $(-c, c)$. For the one-dimensional equation (1.1), it is conjectured that this classification result holds in full generality, that is, without assuming (1.14).

This almost-uniqueness result motived us to investigate more deeply the properties of the solutions of (1.1) of the type u_μ, starting from the following very natural question: under which condition on μ is u_μ a transition front? The answer is given by the following statement.

Theorem 1.11. *Assume that f satisfies (1.2) and let u_μ be the solution of (1.1) associated with a measure $\mu \in \mathcal{M}$. Then u_μ is a transition front connecting 0 and 1 if and only if the support of μ is bounded and is included in $[c^*, +\infty)$. In such a case, u_μ is decreasing with respect to the x variable.*

Theorem 1.11 has the following intuitive explanations: since right-moving traveling fronts $\varphi_c(x - ct)$ connect 1 at $-\infty$ and 0 at $+\infty$, while the left-moving ones $\varphi_c(-x - ct)$ connect 0 at $-\infty$ and 1 at $+\infty$, it is natural that transition fronts connecting 1 (at $-\infty$) and 0 (at $+\infty$) only involve right-moving traveling fronts $\varphi_c(x - ct)$. Furthermore, the faster the standard traveling fronts $\varphi_c(x - ct)$, the flatter, since the quantities $|c\varphi_c'(x - ct)| = \partial_t \varphi_c(x - ct) = \partial_{xx}\varphi_c(x - ct) + f(\varphi_c(x - ct))$ remain bounded as $c \to +\infty$,

by standard parabolic estimates. Therefore, for a solution u_μ of (1.1) to have a uniformly bounded transition region between values close to 1 and values close to 0, it is natural that the set of speeds of the standard traveling fronts involved in the measure μ be bounded.

Based on Theorem 1.11, another important example of transition fronts connecting 0 and 1 of the type u_μ is the following one: namely, if $c_- < c_+ \in [c^*, +\infty)$ are given, if $\widetilde{\mu} \in \mathcal{M}$ is given with $\widetilde{\mu}\big(X \backslash (c_-, c_+)\big) = 0$, then for any positive real numbers $m_\pm$, the solution u_μ with $\mu = m_- \delta_{c_-} + m_+ \delta_{c_+} + \widetilde{\mu}$ satisfies

$$\sup_{x \in \mathbb{R}} \left| u_\mu(t, x) - \varphi_{c_\pm}\big(x - c_\pm t - (c_\pm - \lambda_{c_\pm}^{-1}) \ln(m_- + m_+ + \widetilde{\mu}(X)) - \lambda_{c_\pm}^{-1} \ln m_\pm \big) \right| \to 0$$

as $t \to \pm\infty$, as follows easily from (1.19) and Theorem 1.14 below; in particular, there are some positive real numbers $m_\pm$ such that u_μ satisfies (1.3). Thus, since two different such measures $\widetilde{\mu}$ give rise to two different solutions u_μ, which are transition fronts connecting 0 and 1 by Theorem 1.11, one infers that, for any two given speeds $c_- < c_+$ in $[c^*, +\infty)$, there is an infinite-dimensional manifold of transition fronts of (1.1) satisfying (1.3), thus having asymptotic past and future speeds $c_\pm$.

An immediate byproduct of Theorem 1.11, which is interesting by itself but was a priori not obvious, is that the spatial monotonicity together with the existence of the limits 0 and 1 as $x \to \pm\infty$ pointwise in $t \in \mathbb{R}$ do not guarantee the transition front property (1.4). Indeed, the following corollary holds.

Corollary 1.12. *Under the assumption* (1.2), *equation* (1.1) *admits some solutions* $0 < u < 1$ *such that, for every* $t \in \mathbb{R}$, $u(t, \cdot)$ *is decreasing with* $u(t, -\infty) = 1$ *and* $u(t, +\infty) = 0$, *but which are not transition fronts connecting* 0 *and* 1.

Proof. It suffices to take $u = u_\mu$, where μ is any measure in $\mathcal{M}$ whose support is included in $[c^*, +\infty)$ but is not compact. Indeed, on the one hand, for such a μ, we know from (1.20) and the maximum principle that u is nonincreasing with respect to x. Furthermore, the inequalities (1.19), together with $\mu\big((-\infty, -c^*] \cup \{\infty\}\big) = 0$ and Lebesgue's dominated convergence theorem, yield $u(t, -\infty) = 1$ and $u(t, +\infty) = 0$ for all $t \in \mathbb{R}$ (for the limit $u(t, +\infty) = 0$, one uses the upper bound in (1.19) and the fact that $0 < \lambda_c c = \lambda_c^2 + f'(0) \le 2f'(0)$ for all $c \in [c^*, +\infty)$). In particular, for any fixed time, the space derivative $v = (u_\mu)_x$ - which is a nonpositive solution of a linear parabolic equation - cannot be identically equal to 0 and is thus negative in $\mathbb{R}^2$ by the strong maximum principle. On the other hand, since μ is not compactly supported, u_μ is not a transition front, by Theorem 1.11. $\square$

Remark 1.13. It actually follows from part (ii) of Theorem 1.1 of Zlatoš [46] (after some changes of notations) that, if the measure $\mu \in \mathcal{M}$ is compactly supported in $(c^*, +\infty)$, then u_μ is a transition front connecting 0 and 1 for (1.1). The paper [46] is actually concerned with more general heterogeneous KPP equations. In the case of the homogeneous equation (1.1), the proof of the sufficiency condition of Theorem 1.11 above is actually alternate to that of [46] and it covers the case where the leftmost point of the support of μ is equal to c^*. The proof of Theorem 1.11, given in Section 2 below, is based on some explicit comparisons with some standard traveling fronts (see in particular Proposition 2.1 below), which have their own interest. More precisely, for the proof of the sufficiency condition of Theorem 1.11, we will

show, thanks to an intersection number argument, that, if a measure $\mu \in \mathcal{M}$ has a bounded support included in $[c^*, +\infty)$, then the solution u_μ is bounded from above (resp. from below) by a suitably shifted standard traveling front on the right (resp. on the left) of any point, at any time. On the other hand, the proof of the necessity condition of Theorem 1.11 is much shorter (see Section 2.1) and it is based directly on the estimates (1.19) and on Proposition 1.2 (for measures μ supported on $(-\infty, -c^* - \varepsilon] \cup [c^* + \varepsilon, +\infty)$ for some $\varepsilon > 0$, instead of using (1.19), an alternate proof of the necessity condition of Theorem 1.11 could be done by using Theorem 1.3 of [46]).

We next derive several properties of the solution u_μ of (1.1) in the case it is a transition front connecting 0 and 1, that is when the measure μ is compactly supported in $[c^*, +\infty)$.

Theorem 1.14. *Assume that f satisfies (1.2) and let u_μ be the solution of (1.1) associated with a measure $\mu \in \mathcal{M}$ whose support is a compact subset of $[c^*, +\infty)$. Calling c_- and c_+ the leftmost and rightmost points of the support of μ, the following properties hold:*

(i) *The front u_μ has an asymptotic past speed equal to c_- and an asymptotic future speed equal to c_+.*

(ii) *The positions $(X(t))_{t \in \mathbb{R}}$ satisfy*

$$\limsup_{t \to \pm\infty} |X(t) - c_\pm t| < +\infty \quad \text{if } \mu(c_\pm) > 0 \quad \text{and} \quad \lim_{t \to \pm\infty} (X(t) - c_\pm t) = -\infty \quad \text{if } \mu(c_\pm) = 0.$$

(iii) *If $c_- > c^*$, then there is a bounded function $\xi : \mathbb{R} \to \mathbb{R}$ such that*

$$u_\mu(t, X(t) + \xi(t) + \cdot) \to \varphi_{c_\pm} \quad \text{in } C^2(\mathbb{R}) \quad \text{as } t \to \pm\infty. \tag{1.21}$$

(iv) *If $c_- = c^*$ and $\mu(c^*) > 0$, then*

$$u_\mu(t, c^*t + c^* \ln \mu(c^*) + \cdot) \to \varphi_{c^*} \quad \text{in } C^2(\mathbb{R}) \quad \text{as } t \to -\infty. \tag{1.22}$$

Theorems 1.11 and 1.14 are the keystones of our paper and they are actually the first results we will prove in the sequel. The other theorems can be viewed as their corollaries. We chose to present Theorems 1.11 and 1.14 only at the end of the introduction because their contribution to the understanding of general propagation properties for the equation (1.1) is less evident than that of their consequences stated before. Indeed, even though the aforementioned quasi-uniqueness result of [17] almost completely characterize the class of solutions $0 < u < 1$ of (1.1) in terms of the functions u_μ, recovering μ from u - that is describing the inverse of the mapping $\mu \mapsto u_\mu$ - is a difficult task. Nevertheless, if u satisfies (1.14), Theorem 1.9 above provides a simple criterion to know whether u is a transition front or not and to identify the least upper bound of the support of the associated measure μ.

An important point in Theorem 1.14 is the existence and characterization of the asymptotic past and future speeds when the support of μ is compactly included in $[c^*, +\infty)$, as well as the characterization of the boundedness of the asymptotic shifts $X(t) - c_\pm t$ as $t \to \pm\infty$. In particular, an immediate consequence of statement (ii) of Theorem 1.14 is that, even when

13

the limits $\lim_{t\to\pm\infty} X(t)/t = c_\pm$ exist, the distance between $X(t)$ and $c_- t$ (resp. $c_+ t$) may or may not bounded as $t \to -\infty$ (resp. as $t \to +\infty$). Actually, the four possible cases can occur, as it is seen by applying Theorem 1.14 with the following measures:

$$\mu_{(\alpha,\beta)} = \alpha\,\delta_{c_-} + \mathbb{1}_{(c_-,c_+)} + \beta\,\delta_{c_+}, \quad \alpha,\beta \in \{0,1\},$$

where $\mathbb{1}_{(c_-,c_+)}$ denotes the measure whose density is given by the characteristic function of (c_-, c_+).

In parts (iii) an (iv) regarding the convergence of u_μ to the profiles $\varphi_{c_\pm}$ as $t \to \pm\infty$, the case $c_- = c^*$ and $\mu(c^*) = 0$ is not treated (and neither is the case $\mu(c^*) > 0$ for the convergence as $t \to +\infty$), the difficulty coming from the fact that the classification of the solutions of (1.1) is still an open question when one does not assume (1.14). If, as conjectured, all solutions of (1.1) were known to be of the type u_μ, then (1.21) would hold without any restriction on c_-. However, even the case $c_- > c^*$, the proof of the convergence formula (1.21) to the asymptotic profiles $\varphi_{c_\pm}$ as $t \to \pm\infty$ is far from obvious, due to the fact that in general the quantities $X(t) - c_\pm t$ are not bounded as $t \to \pm\infty$ and the positions $X(t)$ are not those of the traveling fronts $\varphi_{c_\pm}(x - c_\pm t)$ up to bounded functions. The proof of (1.21) is obtained after some passages to the limit and the decomposition of the measure μ into some restricted measures. The proof also uses some intersection number arguments.

Remark 1.15. All the above results have a specular counterpart for left-moving transition fronts connecting 0 at $-\infty$ and 1 at $+\infty$, as it is immediately seen by performing the reflection $x \mapsto -x$.

Remark 1.16. In time-periodic or space-periodic media, as already emphasized, standard traveling fronts are replaced by pulsating fronts. For Fisher-KPP equations of the type (1.1) and (1.2) with periodicity in t or x, the existence of pulsating fronts connecting the unstable steady state 0 and the stable steady state 1 is well-known, see [4, 26, 27, 31, 44]. The existence of time-global front-like solutions converging to two pulsating fronts as in (1.3) can easily be obtained. However, although reasonably expected, the extension of our results and those of [17] on the classification of transition fronts having a global mean speed, on the classification of all admissible past and future speeds of transition fronts, on the determination of their asymptotic profiles, as well as on the decomposition of the solutions into measurable superpositions of pulsating fronts, would require proofs which are beyond the scope of the current paper and are left for further studies. On the other hand, in our next paper [21], we consider the case of time-dependent Fisher-KPP type equations which are asymptotically homogeneous as $t \to \pm\infty$, with two possibly different limits: standard traveling or pulsating traveling fronts do not exist anymore, but transition fronts still exist and their properties and asymptotic dynamics are analyzed in [21]. In particular, whereas it can be conjectured that in periodic media pulsating fronts are the only transition fronts having a global mean speed, this property is immediately not satisfied anymore in non-periodic media, as in [21, 33].

Outline of the paper. In Section 2, we will prove Theorem 1.11 on the characterization of transition fronts given as measurable superpositions of standard traveling fronts of the type $\varphi_c(\pm x - ct)$. Section 3 is devoted to the proof of Theorem 1.14 on the existence and characterization of the asymptotic past and future speeds and the asymptotic profiles of

the transition fronts of the type u_μ. Lastly, we will carry out the proofs of the remaining results, that is Theorems 1.4, 1.7, 1.9 and Proposition 1.2 (in a more general framework), in Section 4. We recall that Theorem 1.6 essentially follows from Theorem 1.7, as explained after the statement of Theorem 1.6.

2 Proof of Theorem 1.11

We first prove the necessity condition in Section 2.1. Section 2.2 is devoted to the proof of the sufficiency condition. It is based on the auxiliary Proposition 2.1 below, which is proved in Section 2.3.

2.1 Proof of the necessity condition

Here and in the sequel, we use the notation taken from [17] and recalled in Section 1. Let $\mu \in \mathcal{M}$ be given and assume that u_μ is a transition front connecting 0 and 1 for the equation (1.1), that is $0 < u_\mu < 1$ in $\mathbb{R}^2$ and there is a family of real numbers $(X(t))_{t\in\mathbb{R}}$ such that (1.4) holds. We will show that the support of μ is bounded and that it is included in $[c^*, +\infty)$, the latter meaning that $\mu\big((-\infty, -c^*] \cup \{\infty\}\big) = 0$. First of all, if $\mu(-c^*) > 0$, then (1.19) yields

$$u_\mu(0, x) \geq \varphi_{c^*}(-x - c^* \ln \mu(-c^*)) \to 1 \quad \text{as } x \to +\infty,$$

which immediately contradicts (1.4). On the other hand, if $\mu\big((-\infty, -c^*) \cup \{\infty\}\big) > 0$, then $M = \mu\big(X\backslash\{-c^*, c^*\}\big) > 0$ and it follows from (1.19) that

$$u_\mu(0, x) \geq M^{-1} \int_{(-\infty, -c^*)} \varphi_{|c|}\big(-x - |c| \ln M\big)\, d\mu(c) \,+\, M^{-1}\theta(\ln M)\, \mu(\infty)$$

for all $x \in \mathbb{R}$. If $\mu(\infty) > 0$, then $\inf_{\mathbb{R}} u_\mu(0, \cdot) \geq M^{-1}\theta(\ln M)\, \mu(\infty) > 0$, which contradicts (1.4). If $\mu\big((-\infty, -c^*)\big) > 0$, then applying Lebesgue's dominated convergence theorem, one deduces that $\liminf_{x\to+\infty} u_\mu(0, x) \geq M^{-1}\mu\big((-\infty, -c^*)\big) > 0$, which is again impossible. Finally, one has shown that

$$\mu\big((-\infty, -c^*] \cup \{\infty\}\big) = 0,$$

that is μ is supported in $[c^*, +\infty)$.

In order to prove that the support of μ is bounded, it remains to show that its least upper bound, denoted by m, is less than $+\infty$. The first inequality in (1.19) implies that, for all $\gamma \in [c^*, +\infty)$ and $t \in \mathbb{R}$,

$$u_\mu(t, \gamma t) \geq M^{-1} \int_{(\gamma, +\infty)} \varphi_c\big(\gamma t - ct - c \ln M\big)\, d\mu(c),$$

whence $\liminf_{t\to+\infty} u_\mu(t, \gamma t) \geq M^{-1}\mu\big((\gamma, +\infty)\big)$ by Lebesgue's dominated convergence theorem. If $\gamma < m$ then the term $M^{-1}\mu\big((\gamma, +\infty)\big)$ is positive and thus (1.4) implies that $\liminf_{t\to+\infty}\big(X(t) - \gamma t\big) > -\infty$, which in turn yields $\liminf_{t\to+\infty} X(t)/t \geq \gamma$. Since this is true for any $\gamma < m$, but on the other hand we know that X satisfies (1.8) (the property (1.8) follows from Proposition 1.2 which is proved independently in Section 4.4), we infer that $m < +\infty$. The proof of the necessity condition of Theorem 1.11 is complete. $\qquad\square$

15

2.2 Proof of the sufficiency condition

The proof is based on the following proposition, which provides uniform lower and upper bounds of any solution u_μ of (1.1) on left and right of its level sets, when the measure $\mu \in \mathcal{M}$ is compactly supported in $[c^*, +\infty) = [2\sqrt{f'(0)}, +\infty)$. These bounds say that the u_μ is steeper than the standard front associated with any speed larger than its support, in the sense of (2.23) below, and they imply especially that the x-derivative of u_μ is bounded away from 0 along any of its level sets. The following proposition has its own interest and it will actually be used again in the proof of Theorem 1.14 in Section 3.

Proposition 2.1. *Under the assumptions* (1.2) *and the notations of Section* 1*, let μ be any measure in $\mathcal{M}$ that is supported in $[c^*, \gamma]$ for some $\gamma \in [c^*, +\infty)$, and let $0 < u_\mu < 1$ be the solution of* (1.1) *that is associated to the measure μ. Then, for every $(t, X) \in \mathbb{R}^2$,*

$$
\begin{cases}
u_\mu(t, X + x) \geq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, X)) + x\big) & \text{for all } x \leq 0, \\
u_\mu(t, X + x) \leq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, X)) + x\big) & \text{for all } x \geq 0,
\end{cases}
\tag{2.23}
$$

where $\varphi_\gamma^{-1} : (0, 1) \to \mathbb{R}$ denotes the reciprocal of the function φ_γ.

Remark 2.2. Under the assumptions of Proposition 2.1, it follows from (2.23) that the spatial derivative of u_μ at any given point $(t, X) \in \mathbb{R}^2$ is bounded from above by a negative real number that only depends on the value of u_μ at this point, in the sense that $(u_\mu)_x(t, X) \leq \varphi_\gamma'\big(\varphi_\gamma^{-1}(u_\mu(t, X))\big) < 0$ for all $(t, X) \in \mathbb{R}^2$.

Postponing the proof of Proposition 2.1, let us first complete the proof of the sufficiency condition of Theorem 1.11. Namely, let μ be any measure in $\mathcal{M}$ which is supported in $[c^*, \gamma]$ for some $\gamma \in [c^*, +\infty)$ and let us show that the function u_μ is a transition front connecting 0 and 1 for (1.1).

As already recalled in the general properties before the statement of Theorem 1.11, one knows that the function u_μ ranges in $(0, 1)$ and is nonincreasing with respect to x. Furthermore, the inequalities (1.19) read in this case

$$
\max \left(\varphi_{c^*}\big(x - c^*t - c^* \ln \mu(c^*)\big), M^{-1} \int_{(c^*, \gamma]} \varphi_c\big(x - ct - c \ln M\big) \, d\mu(c) \right)
$$

$$
\leq u_\mu(t, x) \leq \varphi_{c^*}\big(x - c^*t - c^* \ln \mu(c^*)\big) + M^{-1} \int_{(c^*, \gamma]} e^{-\lambda_c(x - ct - c \ln M)} d\mu(c)
$$

under the usual convention of neglecting the terms involving $\ln 0$. Therefore, $u_\mu(t, -\infty) = 1$ and $u_\mu(t, +\infty) = 0$ for every $t \in \mathbb{R}$, from Lebesgue's dominated convergence theorem. Thus, the function $v = (u_\mu)_x$, which solves $v_t = v_{xx} + f'(u_\mu)v$ in $\mathbb{R}^2$, cannot be identically equal to 0 and is therefore negative in $\mathbb{R}^2$ from the strong maximum principle. In particular, for every $m \in (0, 1)$ and $t \in \mathbb{R}$, there is a unique $X_m(t) \in \mathbb{R}$ such that

$$
u_\mu(t, X_m(t)) = m.
\tag{2.24}
$$

We claim that u_μ is a transition front connecting 0 and 1 in the sense of (1.4), with, say, $X(t) = X_{\varphi_\gamma(0)}(t)$. Indeed, applying (2.23) in Proposition 2.1 with $X = X(t)$, we infer that, for every $t \in \mathbb{R}$,

$$\begin{cases} u_\mu(t, X(t) + x) \geq \varphi_\gamma(x) & \text{for all } x \leq 0, \\ u_\mu(t, X(t) + x) \leq \varphi_\gamma(x) & \text{for all } x \geq 0. \end{cases}$$

From this (1.4) follows because $\varphi_\gamma(-\infty) = 1$ and $\varphi_\gamma(+\infty) = 0$. The proof of the sufficiency condition of Theorem 1.11 is thereby complete. □

2.3   Proof of Proposition 2.1

Proposition 2.1 states that u_μ is steeper than any standard front with a speed lying on the right of the support of μ. It is natural to expect that u_μ satisfies this property, since it is constructed as a superposition of standard fronts with speeds contained in the support of μ, and it is known, for instance from [2], that, if $c < \gamma$, then φ_c is steeper than φ_γ in the sense that (2.23) holds with u_μ replaced by φ_c. However, the fact that this property is preserved after the averaging process in the definition of u_μ^n is higly non-trivial. In order to prove it, we first show some bounds similar to (2.23) with any speed $\gamma + \varepsilon$ larger than γ and with an approximating sequence $(u_{\mu_k})_{k \in \mathbb{N}}$. The approximated bounds will actually be uniform with respect to $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}$ and we will finally pass to the limit as $\varepsilon \to 0^+$ and $k \to +\infty$. The proof is based on an intersection number argument and on some uniform decay estimates of the fronts φ_c.

Step 1: uniform decay estimates of the traveling fronts with speeds $c \in [c^*, \gamma]$

Let us first fix any $\varepsilon > 0$ (ε will be fixed till almost the end of the proof of Proposition 2.1) and call, in accord with (1.18),

$$\lambda_{\gamma+\varepsilon} = \frac{\gamma + \varepsilon - \sqrt{(\gamma + \varepsilon)^2 - 4f'(0)}}{2}, \tag{2.25}$$

Notice that $0 < \lambda_{\gamma+\varepsilon} < \lambda_\gamma \leq \lambda_c \leq \lambda_{c^*}$ for all $c \in [c^*, \gamma]$. The introduction of the parameter $\varepsilon > 0$ is used for the following auxiliary result.

Lemma 2.3. *There is $A \in \mathbb{R}$ such that*

$$\varphi_c'(x) < -\lambda_{\gamma+\varepsilon}\, \varphi_c(x) \quad \text{for all } x \geq A \text{ and for all } c \in [c^*, \gamma]. \tag{2.26}$$

Proof. Assume that there is no real number A such that (2.26) holds. Then there are two sequences $(x_p)_{p \in \mathbb{N}}$ in $\mathbb{R}$ and $(c_p)_{p \in \mathbb{N}}$ in $[c^*, \gamma]$ such that $x_p \to +\infty$ as $p \to +\infty$ and

$$\varphi_{c_p}'(x_p) \geq -\lambda_{\gamma+\varepsilon}\, \varphi_{c_p}(x_p) \quad \text{for all } p \in \mathbb{N}. \tag{2.27}$$

As already recalled in the introduction, one knows that $\varphi_c(x) \leq e^{-\lambda_c x}$ for all $c > c^*$ and for all $x \in \mathbb{R}$ (and $\varphi_{c^*}(+\infty) = 0$). Furthermore, $0 < \lambda_\gamma \leq \lambda_c \leq \lambda_{c^*}$ for all $c \in [c^*, \gamma]$. Therefore, $\varphi_{c_p}(x_p) \to 0^+$ as $p \to +\infty$. Define, for every $p \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\psi_p(x) = \frac{\varphi_{c_p}(x + x_p)}{\varphi_{c_p}(x_p)} \quad (> 0).$$

Since each function φ_c is positive and satisfies $\varphi_c'' + c\varphi_c' + f(\varphi_c) = 0$ in $\mathbb{R}$ and since the sequence $(c_p)_{p \in \mathbb{N}}$ is bounded, it follows from Harnack inequality that, for every $R > 0$, the sequence $(\psi_p)_{p \in \mathbb{N}}$ is bounded in $L^\infty([-R, R])$. On the other hand, each function ψ_p satisfies

$$\psi_p'' + c_p\psi_p' + \frac{f\big(\varphi_{c_p}(x_p)\,\psi_p\big)}{\varphi_{c_p}(x_p)} = 0 \ \text{ in } \mathbb{R}.$$

Standard elliptic estimates yield the existence of a function $\psi \in C^2(\mathbb{R})$ and a real number $c \in [c^*, \gamma]$ such that, up to extraction of a subsequence, $\psi_p \to \psi$ in $C^2_{loc}(\mathbb{R})$ and $c_p \to c$ as $p \to +\infty$. The function ψ is nonnegative and obeys

$$\psi'' + c\psi' + f'(0)\psi = 0 \ \text{ in } \mathbb{R}.$$

Moreover, $\psi(0) = 1$ (whence ψ is positive in $\mathbb{R}$ from the strong maximum principle) and

$$\psi'(0) \geq -\lambda_{\gamma+\varepsilon}, \tag{2.28}$$

by (2.27). On the one hand, if $c > c^* = 2\sqrt{f'(0)}$, then $\psi(x) = C_1 e^{-\lambda_c x} + C_2 e^{-\widetilde{\lambda}_c x}$ for all $x \in \mathbb{R}$, for some constants C_1 and $C_2 \in \mathbb{R}$, where

$$\widetilde{\lambda}_c = \frac{c + \sqrt{c^2 - 4f'(0)}}{2} > \frac{c - \sqrt{c^2 - 4f'(0)}}{2} = \lambda_c.$$

In this case, $1 = \psi(0) = C_1 + C_2$ and C_2 is nonnegative since $\psi(x)$ is positive in $\mathbb{R}$ and in particular $\psi(x)$ remains positive as $x \to -\infty$. Therefore,

$$\psi'(0) = -\lambda_c C_1 - \widetilde{\lambda}_c C_2 \leq -\lambda_c C_1 - \lambda_c C_2 = -\lambda_c \leq -\lambda_\gamma < -\lambda_{\gamma+\varepsilon}$$

from the choice of $\lambda_{\gamma+\varepsilon}$ in (2.25). This contradicts (2.28). On the other hand, if $c = c^* = 2\sqrt{f'(0)}$, then $\psi(x) = (Cx + 1)\,e^{-\lambda_{c^*} x}$ for all $x \in \mathbb{R}$, for some constant $C \in \mathbb{R}$, with $\lambda_{c^*} = \sqrt{f'(0)}$. The positivity of ψ in $\mathbb{R}$ and in particular as $x \to -\infty$ implies that C is nonpositive. Therefore, $\psi'(0) = C - \lambda_{c^*} \leq -\lambda_{c^*} < -\lambda_{\gamma+\varepsilon}$, and one has again reached a contradiction. The proof of the claim (2.26) is thereby complete. $\square$

Step 2: approximation of u_μ by u_{μ_k}

Now, we approximate u_μ by some functions u_{μ_k}. Namely, when $\mu(c^*) = 0$, we consider any fixed sequence $(\sigma_k)_{k \in \mathbb{N}}$ of positive real numbers converging to 0^+, and we define

$$\mu_k = \mu + \sigma_k \delta_{c^*}$$

for every $k \in \mathbb{N}$. When $\mu(c^*) > 0$, we simply set $\sigma_k = 0$ and $\mu_k = \mu$. The advantage of adding a positive weight to the point c^* if $\mu(c^*) = 0$ is that it forces the associated function u_{μ_k} to have an interface close to $c^* t$ for large negative t. This property, that can be viewed as a consequence of Theorem 1.14 part (ii), will be crucial in the sequel. Notice that

18

the measures μ_k belong to $\mathcal{M}$. Furthermore, with the definitions (1.20), for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$, there holds

$$u_\mu^n(-n,x) = \max\left(\varphi_{c^*}\big(x+c^*n-c^*\ln\mu(c^*)\big), M^{-1}\int_{(c^*,\gamma]}\varphi_c\big(x+cn-c\ln M\big)d\mu(c)\right)$$

$$\leq \max\left(\varphi_{c^*}\big(x+c^*n-c^*\ln(\mu(c^*)+\sigma_k)\big), M^{-1}\int_{(c^*,\gamma]}\varphi_c\big(x+cn-c\ln M\big)d\mu(c)\right) \quad (2.29)$$

$$= u_{\mu_k}^n(-n,x)$$

for all $x \in \mathbb{R}$, since φ_{c^*} is positive and decreasing, under the usual convention that the terms involving $\ln 0$ are absent. For every $k \in \mathbb{N}$, there always holds $u_\mu^n(t,x) \leq u_{\mu_k}^n(t,x)$ for all $(t,x) \in [-n,+\infty) \times \mathbb{R}$ from the maximum principle, and

$$u_\mu(t,x) \leq u_{\mu_k}(t,x) \quad \text{for all } (t,x) \in \mathbb{R}^2 \qquad (2.30)$$

by passing to the limit $n \to +\infty$.

Furthermore, the solutions u_{μ_k} converge to u_μ locally uniformly in $\mathbb{R}^2$ as $k \to +\infty$.

Lemma 2.4. *One has*

$$u_{\mu_k}(t,x) \to u_\mu(t,x) \quad \text{as } k \to +\infty, \text{ locally uniformly in } (t,x) \in \mathbb{R}^2. \qquad (2.31)$$

Proof. If $\mu(c^*) > 0$, then $\mu_k = \mu$ and $u_{\mu_k} = u_\mu$, so there is nothing to prove. Consider the case $\mu(c^*) = 0$ (whence $M = \mu\big((c^*,+\infty)\big) > 0$ in this case). By (2.29), there holds

$$u_{\mu_k}^n(-n,x) \leq \min\big(\varphi_{c^*}(x+c^*n-c^*\ln\sigma_k)+u_\mu^n(-n,x), 1\big)$$

for every $k, n \in \mathbb{N}$ and $x \in \mathbb{R}$. For every k and $n \in \mathbb{N}$, the function $\overline{u}$ defined in $[-n,+\infty) \times \mathbb{R}$ by

$$\overline{u}(t,x) = \min\big(\varphi_{c^*}(x-c^*t-c^*\ln\sigma_k)+u_\mu^n(t,x), 1\big)$$

is a supersolution for the Cauchy problem satisfied by $u_{\mu_k}^n$ in $[-n,+\infty) \times \mathbb{R}$ since $f(1) = 0$ and, when $\overline{u}(t,x) < 1$,

$$\overline{u}_t(t,x) - \overline{u}_{xx}(t,x) - f(\overline{u}(t,x)) = f\big(\varphi_{c^*}(x-c^*t-c^*\ln\sigma_k)\big) + f\big(u_\mu^n(t,x)\big)$$
$$-f\big(\varphi_{c^*}(x-c^*t-c^*\ln\sigma_k)+u_\mu^n(t,x)\big),$$

which is nonnegative because, for $a, b \geq 0$ with $a + b \leq 1$, the concavity of f and $f(0) = 0$ yield

$$f(a) + f(b) = \int_0^a f'(s)ds + \int_0^b f'(s)ds \geq \int_0^{a+b} f'(s)ds = f(a+b).$$

It follows from the maximum principle that, for every $k \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$u_{\mu_k}^n(t,x) \leq \overline{u}(t,x) = \min\big(\varphi_{c^*}(x-c^*t-c^*\ln\sigma_k)+u_\mu^n(t,x), 1\big) \text{ for all } (t,x) \in [-n,+\infty) \times \mathbb{R},$$

whence

$$u_{\mu_k}(t,x) \leq \min\big(\varphi_{c^*}(x-c^*t-c^*\ln\sigma_k)+u_\mu(t,x), 1\big) \quad \text{for all } (t,x) \in \mathbb{R}^2$$

by passing to the limit as $n \to +\infty$. Together with (2.30), one gets that

$$0 \leq u_{\mu_k}(t,x) - u_\mu(t,x) \leq \varphi_{c^*}(x-c^*t-c^*\ln\sigma_k)$$

for every $k \in \mathbb{N}$ and $(t,x) \in \mathbb{R}^2$, and the conclusion (2.31) follows since $\varphi_{c^*}(+\infty) = 0$ and $\sigma_k \to 0^+$. The proof of Lemma 2.4 is thereby complete. $\qquad \square$

Step 3: the solutions u_{μ_k} are steeper than $\varphi_{\gamma+\varepsilon}$

The goal of this step is to derive an analogue of Proposition 2.1 for the approximated speed $\gamma + \varepsilon$ and the approximated measures μ_k. Namely:

Lemma 2.5. *For every $k \in \mathbb{N}$ and $(t, X) \in \mathbb{R}^2$, there holds*

$$
\begin{cases}
u_{\mu_k}(t, X + x) \geq \varphi_{\gamma+\varepsilon}\big(\varphi_{\gamma+\varepsilon}^{-1}(u_{\mu_k}(t, X)) + x\big) & \textit{for all } x \leq 0, \\
u_{\mu_k}(t, X + x) \leq \varphi_{\gamma+\varepsilon}\big(\varphi_{\gamma+\varepsilon}^{-1}(u_{\mu_k}(t, X)) + x\big) & \textit{for all } x \geq 0,
\end{cases}
\tag{2.32}
$$

whence $(u_{\mu_k})_x(t, X) \leq \varphi'_{\gamma+\varepsilon}\big(\varphi_{\gamma+\varepsilon}^{-1}(u_{\mu_k}(t, X))\big) < 0.$

Proof. Throughout the proof, k denotes any fixed integer. Define

$$
B_k = \max\big(A, -c^* \ln(\mu(c^*) + \sigma_k) + \gamma|\ln M| + A\big),
$$

where $A \in \mathbb{R}$ is as in (2.26), and let $u_{k,0} : \mathbb{R} \to (0, 1)$ be the continuous function defined by

$$
u_{k,0}(x) =
\begin{cases}
\varphi_{c^*}(B_k) & \text{for } x < 0, \\
\varphi_{c^*}(B_k)\, e^{-\lambda_{\gamma+\varepsilon} x} & \text{for } x \geq 0.
\end{cases}
\tag{2.33}
$$

Let us show in this first paragraph that for all $n \in \mathbb{N}$ the functions $u_{\mu_k}^n(-n, \cdot)$ given in (2.29) are steeper than any translate $u_{k,0}(\cdot + x_0)$ of $u_{k,0}$, in the sense that if the graphs of $u_{\mu_k}^n(-n, \cdot)$ and $u_{k,0}(\cdot + x_0)$ intersect, then they have exactly one intersection point and $u_{\mu_k}^n(-n, \cdot)$ is larger (resp. smaller) than $u_{k,0}(\cdot + x_0)$ on the left (resp. right) of this intersection point. So, let $n \in \mathbb{N}$ and $x_0 \in \mathbb{R}$ be arbitrary and assume that there is $x_1 \in \mathbb{R}$ such that

$$
u_{\mu_k}^n(-n, x_1) = u_{k,0}(x_1 + x_0).
\tag{2.34}
$$

It follows from the definition of the function $u_{\mu_k}^n(-n, \cdot)$ in (2.29) that $u_{\mu_k}^n(-n, \cdot)$ is continuous and decreasing in $\mathbb{R}$, and that $u_{\mu_k}^n(-n, -\infty) = 1$ and $u_{\mu_k}^n(-n, +\infty) = 0$. Therefore, there is a unique $\xi_{k,n} \in \mathbb{R}$ such that

$$
\begin{cases}
u_{\mu_k}^n(-n, \xi_{k,n}) & = & \varphi_{c^*}(B_k), \\
u_{\mu_k}^n(-n, x) & > & \varphi_{c^*}(B_k)\ (\geq u_{k,0}(x + x_0)) & \text{for all } x < \xi_{k,n}, \\
u_{\mu_k}^n(-n, x) & < & \varphi_{c^*}(B_k) & \text{for all } x > \xi_{k,n}.
\end{cases}
\tag{2.35}
$$

Since $u_{k,0} \leq \varphi_{c^*}(B_k)$ in $\mathbb{R}$ by definition, one infers from (2.34) and (2.35) that $x_1 \geq \xi_{k,n}$. Furthermore, for all $x \geq \xi_{k,n}$, one has

$$
\varphi_{c^*}\big(x + c^* n - c^* \ln(\mu(c^*) + \sigma_k)\big) \leq u_{\mu_k}^n(-n, x) \leq \varphi_{c^*}(B_k)
$$

by (2.29) and (2.35), whence

$$
\begin{aligned}
x &\geq -c^* n + c^* \ln(\mu(c^*) + \sigma_k) + B_k \\
&\geq \max\big(-c^* n + c^* \ln(\mu(c^*) + \sigma_k) + A, -c^* n + \gamma|\ln M| + A\big)
\end{aligned}
$$

20

since φ_{c^*} is decreasing. Thus, (2.26) implies that, for all $x \geq \xi_{k,n}$,

$$
\begin{cases}
\varphi'_{c^*}\big(x + c^*n - c^* \ln(\mu(c^*) + \sigma_k)\big) & < & -\lambda_{\gamma+\varepsilon}\,\varphi_{c^*}\big(x + c^*n - c^* \ln(\mu(c^*) + \sigma_k)\big), \\
\varphi'_c(x + cn - c\ln M) & < & -\lambda_{\gamma+\varepsilon}\,\varphi_c(x + cn - c\ln M) \quad \text{for all } c^* < c \leq \gamma
\end{cases}
$$

by (2.26). As a consequence of the definition (2.29) of $u^n_{\mu_k}(-n, \cdot)$, it follows that

$$
u^n_{\mu_k}(-n, x') < u^n_{\mu_k}(-n, x)\, e^{-\lambda_{\gamma+\varepsilon}(x'-x)} \quad \text{for all } \xi_{k,n} \leq x < x'.
$$

On the other hand, by definition of $u_{k,0}$ in (2.33), there holds

$$
u_{k,0}(x' + x_0) \geq u_{k,0}(x + x_0)\, e^{-\lambda_{\gamma+\varepsilon}(x'-x)} \quad \text{for all } x < x' \in \mathbb{R}.
$$

From the definition of x_1 in (2.34) and the fact that $x_1 \geq \xi_{k,n}$, one concludes that x_1 is necessarily unique, and that

$$
\begin{cases}
u^n_{\mu_k}(-n, x) > u_{k,0}(x + x_0) & \text{for all } x < \xi_{k,n} \text{ (from (2.35))}, \\
u^n_{\mu_k}(-n, x) > u_{k,0}(x + x_0) & \text{for all } \xi_{k,n} \leq x < x_1, \\
u^n_{\mu_k}(-n, x) < u_{k,0}(x + x_0) & \text{for all } x > x_1.
\end{cases}
$$

Finally, for every $n \in \mathbb{N}$ and $x_0 \in \mathbb{R}$, the function $u^n_{\mu_k}(-n, \cdot)$ is steeper than $u_{k,0}(\cdot + x_0)$ in the above sense.

Let now u_k be the solution of the Cauchy problem (1.1) with initial condition $u_{k,0}$ defined in (2.33). It follows from [1, 8, 28], as in [9, 15], that, for every $x_0 \in \mathbb{R}$, $n \in \mathbb{N}$ and $t > -n$, the function $u^n_{\mu_k}(t, \cdot)$ is steeper than $u_k(t+n, \cdot + x_0)$: in other words, if $u^n_{\mu_k}(t, X) = u_k(t+n, X+x_0)$ for some $X \in \mathbb{R}$, then

$$
\begin{cases}
u^n_{\mu_k}(t, X + x) > u_k(t + n, X + x_0 + x) & \text{for all } x < 0, \\
u^n_{\mu_k}(t, X + x) < u_k(t + n, X + x_0 + x) & \text{for all } x > 0.
\end{cases} \tag{2.36}
$$

On the other hand, by (2.33) and since $0 < \lambda_{\gamma+\varepsilon} < \lambda_\gamma \leq \lambda_{c^*}$ and $\gamma + \varepsilon = \lambda_{\gamma+\varepsilon} + f'(0)/\lambda_{\gamma+\varepsilon}$, it is also known [23, 43] that there exists a real number ζ_k such that

$$
u_k(s, x) - \varphi_{\gamma+\varepsilon}\big(x - (\gamma + \varepsilon)s + \zeta_k\big) \to 0 \quad \text{as } s \to +\infty, \text{ uniformly in } x \in \mathbb{R}. \tag{2.37}
$$

Finally, let (t, X) be any fixed pair in $\mathbb{R}^2$. Since $u^n_{\mu_k}(t, X) \to u_{\mu_k}(t, X) \in (0, 1)$ as $n \to +\infty$, there holds

$$
0 < u^n_{\mu_k}(t, X) < \frac{1 + u_{\mu_k}(t, X)}{2} \ (< 1) \quad \text{for all } n \text{ large enough}.
$$

Furthermore, for each $n > \max(0, -t)$, the function $u_k(t+n, \cdot)$ is decreasing and continuous in $\mathbb{R}$, while $\sup_{\mathbb{R}} u_k(t+n, \cdot) \to 1$ as $n \to +\infty$ by (2.37), and $\inf_{\mathbb{R}} u_k(t+n, \cdot) = u_k(t+n, +\infty) = 0$ for every $n > \max(0, -t)$. Therefore, for all n large enough, there is a unique $x_n \in \mathbb{R}$ such that

$$
u_k(t + n, X + x_n) = u^n_{\mu_k}(t, X).
$$

It follows then from (2.36) that, for all n large enough,

$$\begin{cases} u_{\mu_k}^n(t, X + x) > u_k(t + n, X + x_n + x) & \text{for all } x < 0, \\ u_{\mu_k}^n(t, X + x) < u_k(t + n, X + x_n + x) & \text{for all } x > 0. \end{cases}$$

Thus, by (2.37), there exists a sequence $(h_n)_{n \in \mathbb{N}}$ converging to 0^+ such that

$$\begin{cases} u_{\mu_k}^n(t, X + x) \geq \varphi_{\gamma + \varepsilon}\big(X + x_n + x - (\gamma + \varepsilon)(t + n) + \zeta_k\big) - h_n & \text{for all } x \leq 0, \\ u_{\mu_k}^n(t, X + x) \leq \varphi_{\gamma + \varepsilon}\big(X + x_n + x - (\gamma + \varepsilon)(t + n) + \zeta_k\big) + h_n & \text{for all } x \geq 0. \end{cases} \tag{2.38}$$

From this at $x = 0$ and the convergence $u_{\mu_k}^n \to u_{\mu_k}$ as $n \to +\infty$ locally uniformly in $\mathbb{R}^2$, we deduce that

$$\varphi_{\gamma + \varepsilon}\big(X + x_n - (\gamma + \varepsilon)(t + n) + \zeta_k\big) \to u_{\mu_k}(t, X) \in (0, 1) \quad \text{as } n \to +\infty,$$

whence

$$X + x_n - (\gamma + \varepsilon)(t + n) + \zeta_k \to \varphi_{\gamma + \varepsilon}^{-1}(u_{\mu_k}(t, X)) \quad \text{as } n \to +\infty.$$

As a consequence, by passing to the limit as $n \to +\infty$ in (2.38), the conclusion (2.32) follows. The proof of Lemma 2.5 is thereby complete. $\square$

Step 4: passages to the limit as $k \to +\infty$ and $\varepsilon \to 0^+$ in the inequalities (2.32)

Let (t, X) be arbitrary in $\mathbb{R}^2$. Since $u_{\mu_k} \to u_\mu$ locally uniformly in $\mathbb{R}^2$ as $k \to +\infty$ by Lemma 2.4, passing to the limit as $k \to +\infty$ in (2.32) yields

$$\begin{cases} u_\mu(t, X + x) \geq \varphi_{\gamma + \varepsilon}\big(\varphi_{\gamma + \varepsilon}^{-1}(u_\mu(t, X)) + x\big) & \text{for all } x \leq 0, \\ u_\mu(t, X + x) \leq \varphi_{\gamma + \varepsilon}\big(\varphi_{\gamma + \varepsilon}^{-1}(u_\mu(t, X)) + x\big) & \text{for all } x \geq 0. \end{cases} \tag{2.39}$$

On the other hand, by standard elliptic estimates and the fact that all solutions $0 < v < 1$ of $v'' + cv' + f(v) = 0$ in $\mathbb{R}$ with $v(-\infty) = 1$ and $v(+\infty) = 0$ are translations of the function φ_c, it follows easily that, for every $m \in (0, 1)$ and for every sequence $(c_p)_{p \in \mathbb{N}}$ in $[c^*, +\infty)$ such that $c_p \to c \in [c^*, +\infty)$ as $p \to +\infty$, there holds

$$\varphi_{c_p}\big(\varphi_{c_p}^{-1}(m) + x\big) \to \varphi_c\big(\varphi_c^{-1}(m) + x\big) \quad \text{as } p \to +\infty, \text{ uniformly with respect to } x \in \mathbb{R}.$$

Therefore, the desired inequalities (2.23) follow from (2.39) by passing to the limit as $\varepsilon \to 0^+$. The proof of Proposition 2.1 is thereby complete. $\square$

3 Proof of Theorem 1.14

In this section, we carry out the proof of Theorem 1.14. Namely, when the measure $\mu \in \mathcal{M}$ is compactly supported in $[c^*, +\infty) = [2\sqrt{f'(0)}, +\infty)$, we prove that u_μ has some asymptotic past and future speeds and we identify these speeds as the leftmost and rightmost points of the support of μ. Furthermore, we prove the convergence of u_μ to some well identified profiles along its level sets as $t \to \pm\infty$.

As in the previous section, an important step in the proof will be the following proposition, which provides uniform lower and upper bounds of any solution u_μ of (1.1) on left and right of its level sets, when the measure $\mu \in \mathcal{M}$ is supported in $[\gamma, +\infty)$ for some $\gamma \geq c^*$. Roughly speaking, the following proposition is the counterpart of Proposition 2.1, in the sense that it states that u_μ is less steep than the standard front associated with any speed on the left of its support.

Proposition 3.1. *Under the assumptions* (1.2) *and the notations of Section* 1, *let* μ *be any measure in* $\mathcal{M}$ *that is supported in* $[\gamma, +\infty)$ *for some* $\gamma \geq c^*$ *and let* $0 < u_\mu < 1$ *be the solution of* (1.1) *that is associated to the measure* μ. *Then, for every* $(t, X) \in \mathbb{R}^2$,

$$\begin{cases} u_\mu(t, X + x) \leq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, X)) + x\big) & \text{for all } x \leq 0, \\ u_\mu(t, X + x) \geq \varphi_\gamma\big(\varphi_\gamma^{-1}(u_\mu(t, X)) + x\big) & \text{for all } x \geq 0. \end{cases} \tag{3.40}$$

3.1 The asymptotic past and future speeds and the limiting profiles

Postponing the proof of Proposition 3.1 in Section 3.2, let us first complete the proof of Theorem 1.14. Let $\mu \in \mathcal{M}$ be compactly supported in $[c^*, +\infty)$ and let $c_- \leq c_+ \in [c^*, +\infty)$ be the leftmost and rightmost points of the support of μ, that is μ is supported in $[c_-, c_+]$, and

$$\mu\big([c_-, c_- + \varepsilon]\big) > 0 \quad \text{and} \quad \mu\big([\max(c_+ - \varepsilon, c^*), c_+]\big) > 0$$

for every $\varepsilon > 0$. Denote $M = \mu\big(X \setminus \{-c^*, c^*\}\big) = \mu\big((c^*, +\infty)\big)$. As already proved in Section 2.2, we know that $(u_\mu)_x < 0$ in $\mathbb{R}^2$ and that $0 < u_\mu < 1$ is a transition front connecting 0 and 1 for (1.1), namely there is a family $(X(t))_{t \in \mathbb{R}}$ of real numbers such that (1.4) holds for u_μ. The fact that $c_\pm$ are the asymptotic past and future speeds of u_μ will follow from the fact that the fronts with speeds larger than c_- (resp. smaller than c_+) are hidden as $t \to -\infty$ (resp. as $t \to +\infty$) by the one with speed c_- (resp. c_+). The proof of the convergence to the asymptotic profiles is subtler and uses in particular the lower and upper bounds stated in Propositions 2.1 and 3.1.

Step 1: c_- **is the asymptotic past speed of** u_μ **and** $\limsup_{t \to -\infty} |X(t) - c_- t| < +\infty$ **if and only if** $\mu(c_-) = \mu\big(\{c_-\}\big) > 0$

We will consider three cases, according to the value of c_- and $\mu(c_-)$.

Case 1: $\mu(c_-) = 0$. In this case, $\mu(c^*) = 0$, $M > 0$ and the inequalities (1.19) amount to

$$0 < M^{-1} \int_{(c_-, c_+]} \varphi_c(x - ct - c \ln M) \, d\mu(c) \leq u_\mu(t, x) \leq M^{-1} \int_{(c_-, c_+]} e^{-\lambda_c(x - ct - c \ln M)} d\mu(c)$$

for all $(t, x) \in \mathbb{R}^2$. Lebesgue's dominated convergence theorem yields $u_\mu(t, c_- t) \to 0$ as $t \to -\infty$, whence $X(t) - c_- t \to -\infty$ as $t \to -\infty$ by (1.4) and (1.5). In particular, $\liminf_{t \to -\infty} X(t)/t \geq c_-$. Furthermore, for every $\varepsilon > 0$,

$$\begin{aligned} \liminf_{t \to -\infty} u_\mu(t, (c_- + \varepsilon)t) &\geq \liminf_{t \to -\infty} M^{-1} \int_{(c_-, c_- + \varepsilon)} \varphi_c\big((c_- + \varepsilon)t - ct - c \ln M\big) \, d\mu(c) \\ &= M^{-1} \mu\big((c_-, c_- + \varepsilon)\big) > 0, \end{aligned}$$

23

the positivity following from definition of c_- and the assumption $\mu(c_-) = 0$. Therefore, $\liminf_{t \to -\infty} \big(X(t) - (c_- + \varepsilon)t\big) > -\infty$, by (1.4), whence $\limsup_{t \to -\infty} X(t)/t \le c_- + \varepsilon$, for every $\varepsilon > 0$. Finally, $X(t)/t \to c_-$ as $t \to -\infty$.

Case 2: $\mu(c_-) > 0$ and $c_- > c^*$. In this case, $M > 0$ and the inequalities (1.19) read

$$M^{-1}\varphi_{c_-}(x - c_-t - c_- \ln M)\,\mu(c_-) + M^{-1}\int_{(c_-,c_+]} \varphi_c(x - ct - c \ln M)\,d\mu(c)$$

$$\le u_\mu(t,x) \le M^{-1}e^{-\lambda_{c_-}(x - c_-t - c_- \ln M)}\mu(c_-) + M^{-1}\int_{(c_-,c_+]} e^{-\lambda_c(x - ct - c \ln M)}d\mu(c).$$

Therefore, Lebesgue's dominated convergence theorem yields

$$
\begin{aligned}
0 \; < \; M^{-1}\varphi_{c_-}(1)\,\mu(c_-) \; &\le \; \liminf_{t \to -\infty} u_\mu(t, c_-t + c_- \ln M + 1)\\
&\le \; \limsup_{t \to -\infty} u_\mu(t, c_-t + c_- \ln M + 1) \; \le \; M^{-1}e^{-\lambda_{c_-}}\mu(c_-) \; < \; 1,
\end{aligned}
$$

whence $\limsup_{t \to -\infty}|X(t) - c_-t| < +\infty$ by (1.4) (in particular, $X(t)/t \to c_-$ as $t \to -\infty$).

Case 3: $\mu(c_-) > 0$ and $c_- = c^*$. In this case, the inequalities (1.19) amount to

$$
\begin{aligned}
0 < \max\left(\varphi_{c^*}(x - c^*t - c^* \ln \mu(c^*)), M^{-1}\int_{(c_-,c_+]} \varphi_c(x - ct - c \ln M)\,d\mu(c)\right) \\
\le u_\mu(t,x) \le \varphi_{c^*}(x - c^*t - c^* \ln \mu(c^*)) + M^{-1}\int_{(c_-,c_+]} e^{-\lambda_c(x - ct - c \ln M)}d\mu(c),
\end{aligned}
\tag{3.41}
$$

under the convention that the terms involving M are not present if $M = 0$ (which is the case if $c_+ = c_- = c^*$). It follows from Lebesgue's dominated convergence theorem that $u_\mu(t, c^*t) \to \varphi_{c^*}(-c^* \ln \mu(c^*)) \in (0,1)$ as $t \to -\infty$. Therefore, $\limsup_{t \to -\infty}|X(t) - c^*t| < +\infty$ by (1.4). In particular, $X(t)/t \to c^* = c_-$ as $t \to -\infty$.

Step 2: c_+ is the asymptotic future speed of u_μ and $\limsup_{t \to -\infty}|X(t) - c_+t| < +\infty$ if and only if $\mu(c_+) = \mu(\{c_+\}) > 0$

Namely, one can prove, as in Step 1, that $X(t)/t \to c_+$ as $t \to +\infty$. Furthermore, $\limsup_{t \to +\infty}|X(t) - c_+t| < +\infty$ if $\mu(c_+) > 0$, whereas $X(t) - c_+t \to -\infty$ as $t \to +\infty$ if $\mu(c_+) = 0$. This properties easily follow from the inequalities (1.19) and the analysis of the behavior of $u_\mu(t, c_+t)$ or $u_\mu(t, (c_+ - \varepsilon)t)$ as $t \to +\infty$.

Therefore, at this stage, parts (i) and (ii) of Theorem 1.14 have been proved.

Step 3: convergence to the profile φ_{c_-} as $t \to -\infty$

The strategy consists in passing to the limit as $t \to -\infty$ along the positions $X(t)$, that is in some functions $u_\mu(t + t_p, X(t_p) + x)$ with $t_p \to -\infty$ as $p \to +\infty$. We shall identify the limit as a solution u_{μ_∞} and, by comparisons with approximated solutions, we shall prove that the measure μ_∞ is a multiple of the Dirac mass at the point c_-, meaning that u_{μ_∞} is a standard traveling front with speed c_-.

We begin with two lemmas on some properties of solutions and limits of solutions of the type $v = u_{\widetilde\mu}$ when $\widetilde\mu \in \mathcal{M}$ is compactly supported in $[c^*, +\infty)$ (in this case, we recall that Theorem 1.11 implies that $u_{\widetilde\mu}$ is a transition front connecting 0 and 1).

Lemma 3.2. *Let $\widetilde{\mu}$ be a measure in $\mathcal{M}$ whose support is included in $[a, b] \subset [c^*, +\infty)$. Let $(\widetilde{X}(t))_{t\in\mathbb{R}}$ be a family such that (1.4) holds for the transition front $u_{\widetilde{\mu}}$ connecting 0 and 1 for (1.1). Then, for every $c < a$, there is $\tau_c > 0$ such that*

$$\frac{\widetilde{X}(t+\tau) - \widetilde{X}(t)}{\tau} > c \quad \text{for all } t \in \mathbb{R} \text{ and } \tau \geq \tau_c.$$

Proof. By (1.4) and (1.5), there exists a constant $m \in (0, 1)$ such that

$$u_{\widetilde{\mu}}(t, \widetilde{X}(t) + x) \geq m \quad \text{for all } t \in \mathbb{R} \text{ and } x \leq 0.$$

Thus, for any given $t \in \mathbb{R}$, Proposition 3.1 yields

$$u_{\widetilde{\mu}}(t, \widetilde{X}(t)+x) \geq \begin{cases} m \geq m\, \varphi_a\big(\varphi_a^{-1}(u_{\widetilde{\mu}}(t, \widetilde{X}(t)))+x\big) & \text{for all } x \leq 0, \\ \varphi_a\big(\varphi_a^{-1}(u_{\widetilde{\mu}}(t, \widetilde{X}(t)))+x\big) \geq m\, \varphi_a\big(\varphi_a^{-1}(u_{\widetilde{\mu}}(t, \widetilde{X}(t)))+x\big) & \text{for all } x \geq 0. \end{cases}$$

The function $m\varphi_a(x - at)$ is a subsolution of (1.1), because $f(0) = 0$ together with the concavity of f imply that $f(ms) \geq mf(s)$ for any $s \geq 0$. As a consequence of the comparison principle, we then derive

$$u_{\widetilde{\mu}}\big(t + \tau, \widetilde{X}(t) + x\big) \geq m\, \varphi_a\big(\varphi_a^{-1}(u_{\widetilde{\mu}}(t, \widetilde{X}(t))) + x - a\tau\big) \quad \text{for all } \tau > 0, \ x \in \mathbb{R}.$$

In particular, taking $x = a\tau$ yields

$$u_{\widetilde{\mu}}\big(t + \tau, \widetilde{X}(t) + a\tau\big) \geq m\, u_{\widetilde{\mu}}(t, \widetilde{X}(t)) \geq m^2 \quad \text{for all } \tau > 0.$$

It follows from (1.4) that there exists A, independent of t and τ, such that

$$\widetilde{X}(t) + a\tau - \widetilde{X}(t + \tau) \leq A.$$

The statement of the lemma is an immediate consequence of this inequality. $\qquad\square$

Lemma 3.3. *Let $\widetilde{\mu}$ be a measure in $\mathcal{M}$ whose support is included in $[a, b] \subset (c^*, +\infty)$. Let $(t_p, x_p)_{p\in\mathbb{N}}$ be any sequence in $\mathbb{R}^2$ and assume that the functions $v_p(t, x) = u_{\widetilde{\mu}}(t_p+t, x_p+x)$ converge in $C_{loc}^{1,2}(\mathbb{R}^2)$ to a solution $0 < v_\infty < 1$ of (1.1) in $\mathbb{R}^2$. Then*

$$v_\infty = u_{\widetilde{\mu}_\infty},$$

with a measure $\widetilde{\mu}_\infty \in \mathcal{M}$ which is supported in $[a, b]$.

Proof. Let first c and c' be any two real numbers such that $c^* < c' < c < a$ and let ε be any positive real number. Since $u_{\widetilde{\mu}}$ is a transition front connecting 0 and 1 for (1.1), there is a family $(\widetilde{X}(t))_{t\in\mathbb{R}}$ of real numbers such that (1.4) holds with $u_{\widetilde{\mu}}$. Since $u_{\widetilde{\mu}}(t_p, x_p) = v_p(0, 0) \to v_\infty(0, 0) \in (0, 1)$ as $p \to +\infty$, it follows from (1.4) that there is a real number $\widetilde{A} \geq 0$ such that

$$\big|x_p - \widetilde{X}(t_p)\big| \leq \widetilde{A} \quad \text{for all } p \in \mathbb{N}.$$

25

Furthermore, again from (1.4), there is $A \geq 0$ such that

$$u_{\widetilde{\mu}}(t, y) \leq \varepsilon \quad \text{for all } t \in \mathbb{R} \text{ and } y \geq \widetilde{X}(t) + A.$$

On the other hand, Lemma 3.2 yields the existence of $\tau_c > 0$ such that

$$\frac{\widetilde{X}(t + \tau) - \widetilde{X}(t)}{\tau} \geq c \quad \text{for all } t \in \mathbb{R} \text{ and } \tau \geq \tau_c.$$

Without loss of generality, one can assume that $\tau_c \geq (A + \widetilde{A})/(c - c')$. Let now (t, x) be any pair in $\mathbb{R}^2$ such that $t \geq \tau_c$ and $x \geq -c't$. For every $p \in \mathbb{N}$, there holds $\widetilde{X}(t_p) - \widetilde{X}(t_p - t) \geq ct$, whence

$$x_p + x \geq \widetilde{X}(t_p) - \widetilde{A} - c't \geq \widetilde{X}(t_p - t) + ct - c't - \widetilde{A} \geq \widetilde{X}(t_p - t) + (c - c')\tau_c - \widetilde{A}$$
$$\geq \widetilde{X}(t_p - t) + A$$

and thus $u_{\widetilde{\mu}}(t_p - t, x_p + x) \leq \varepsilon$. This means that $v_p(-t, x) \leq \varepsilon$, whence $v_\infty(-t, x) \leq \varepsilon$ by passing to the limit as $p \to +\infty$. In other words, $\sup_{x \geq -c't} v_\infty(-t, x) \leq \varepsilon$ for all $t \geq \tau_c$. Thus, since $v_\infty > 0$ in $\mathbb{R}^2$ and $\varepsilon > 0$ was arbitrary, one gets that

$$\sup_{x \geq -c'|t|} v_\infty(t, x) \to 0 \quad \text{as } t \to -\infty. \tag{3.42}$$

Now, since c' can be arbitrary in (c^*, a), Theorem 1.4 of [17] implies that v_∞ is of the type

$$v_\infty = u_{\widetilde{\mu}_\infty}$$

for some measure $\widetilde{\mu}_\infty \in \mathcal{M}$ that is supported in $(-\infty, -a] \cup [a, +\infty) \cup \{\infty\}$. Furthermore, Proposition 2.1 implies in particular that, for every $x \geq 0$ and $p \in \mathbb{N}$,

$$v_p(0, x) = u_{\widetilde{\mu}}(t_p, x_p + x) \leq \varphi_b\big(\varphi_b^{-1}(u_{\widetilde{\mu}}(t_p, x_p)) + x\big) = \varphi_b\big(\varphi_b^{-1}(v_p(0, 0)) + x\big),$$

whence

$$u_{\widetilde{\mu}_\infty}(0, x) = v_\infty(0, x) \leq \varphi_b\big(\varphi_b^{-1}(v_\infty(0, 0)) + x\big) \quad \text{for all } x \geq 0 \tag{3.43}$$

and $v_\infty(0, +\infty) = 0$. Therefore, as in the proof of the necessity condition of Theorem 1.11 in Section 2.1, it follows that the measure $\widetilde{\mu}_\infty$ is supported in $[a, +\infty)$ (otherwise, $\liminf_{x \to +\infty} v_\infty(t, x) > 0$ for every $t \in \mathbb{R}$).

Let us finally show that $\widetilde{\mu}_\infty$ is supported in the interval $[a, b]$. Assume by contradiction that there is $b' \in (b, +\infty)$ such that $\widetilde{\mu}_\infty((b', +\infty)) > 0$. It follows from the inequalities (1.19) applied to $v_\infty = u_{\widetilde{\mu}_\infty}$ that, for all $x \in \mathbb{R}$,

$$e^{\lambda_{b'}x} v_\infty(0, x) = e^{\lambda_{b'}x} u_{\widetilde{\mu}_\infty}(0, x) \geq \widetilde{M}_\infty^{-1} \int_{(b', +\infty)} \min\big(e^{\lambda_{b'}x} \varphi_c(x - c \ln \widetilde{M}_\infty), 1\big) d\widetilde{\mu}_\infty(c),$$

where $\widetilde{M}_\infty = \widetilde{\mu}_\infty((c^*, +\infty)) > 0$. But, for every $c > b' (> b \geq a > c^*)$, one has $0 < \lambda_c < \lambda_{b'}$, whence $e^{\lambda_{b'}x} \varphi_c(x - c \ln \widetilde{M}_\infty) \to +\infty$ as $x \to +\infty$ by (1.17). Therefore, Lebesgue's dominated

26

convergence theorem (in order to apply 1, we took the minimum with 1 in the above displayed formula) yields

$$\liminf_{x\to+\infty}\left(e^{\lambda_{b'}x}v_\infty(0,x)\right)\geq\widetilde{M}_\infty^{-1}\widetilde{\mu}_\infty\left((b',+\infty)\right)>0,$$

whence $e^{\lambda_b x}v_\infty(0,x)\to+\infty$ as $x\to+\infty$ since $0<\lambda_{b'}<\lambda_b$. But the inequality (3.43) implies actually that $v_\infty(0,x)\leq Ce^{-\lambda_b x}$ for all $x\geq0$, for some constant $C>0$. One has then reached a contradiction. Finally, $\widetilde{\mu}_\infty\left((b',+\infty)\right)=0$ for all $b'>b$ and $\widetilde{\mu}_\infty$ is supported in $[a,b]$ (in particular, by Theorem 1.11, the function $v_\infty=u_{\widetilde{\mu}_\infty}$ is a transition front connecting 0 and 1). The proof of Lemma 3.3 is thereby complete. $\qquad\square$

Let us now come back to our transition front u_μ with $\mu\in\mathcal{M}$ being supported in $[c_-,c_+]\subset[c^*,+\infty)$ and $c_\pm$ being the leftmost and rightmost points of the support of μ. We assume that either $c_->c^*$, or $c_-=c^*$ and $\mu(c^*)>0$. In order to show that

$$u_\mu(t,X(t)+\xi(t)+\cdot)\to\varphi_{c_-}\quad\text{in }C^2(\mathbb{R})\quad\text{as }t\to-\infty,\tag{3.44}$$

for some bounded function $\xi:(-\infty,0)\to\mathbb{R}$, as well as (1.22) if $c_-=c^*$ and $\mu(c^*)>0$, we are going to pass to the limit as $t\to-\infty$ along a given level set of u_μ.

First of all, if $c_-=c^*$ and $\mu(c^*)>0$, then the inequalities (3.41) and Lebesgue's dominated convergence theorem imply that, for every $A\in\mathbb{R}$,

$$\sup_{x\geq A}\left|u_\mu(t,x+c^*t)-\varphi_{c^*}(x-c^*\ln\mu(c^*))\right|\to0\quad\text{as }t\to-\infty.$$

But u_μ is decreasing in x and less than 1, while $\varphi_{c^*}(-\infty)=1$. Therefore,

$$\sup_{x\in\mathbb{R}}\left|u_\mu(t,x+c^*t)-\varphi_{c^*}(x-c^*\ln\mu(c^*))\right|\to0\quad\text{as }t\to-\infty.\tag{3.45}$$

Furthermore, the functions $u_\mu(t,c^*t+c^*\ln\mu(c^*)+\cdot)$, which converge uniformly to φ_{c^*} as $t\to-\infty$, actually converge in $C^2(\mathbb{R})$ by standard parabolic estimates. In other words, (1.22) holds and the proof of part (iv) of Theorem 1.14 is done.

Remark 3.4. Condition (1.22) implies (1.21) as $t\to-\infty$, with $c_-=c^*$. The function $t\mapsto\xi(t)$ is defined as follows, under the notations (2.24) in Section 2.2: set $m=\varphi_{c^*}(0)\in(0,1)$ and $\xi(t)=X_m(t)-X(t)$, for all $t\in\mathbb{R}$. Namely,

$$u_\mu(t,X(t)+\xi(t))=u_\mu(t,X_m(t))=m=\varphi_{c^*}(0).$$

Notice that the function ξ is bounded by (1.4). It follows from (1.22) and the strict monotonicity of φ_{c^*} that $X(t)+\xi(t)-c^*t-c^*\ln\mu(c^*)\to0$ as $t\to-\infty$. Thus, by (1.22) again and by uniform continuity of φ_{c^*}, one infers (1.21) as $t\to-\infty$, with $c_-=c^*$.

Let us now assume in the sequel, till the end of the proof of Theorem 1.14, that

$$c_->c^*,$$

and let us complete the proof of part (iii) of Theorem 1.14, that is (1.21).

27

For every $t \in \mathbb{R}$, let $Y(t) = X_{1/2}(t)$ be the unique real number such that

$$u_\mu(t, Y(t)) = \frac{1}{2}. \tag{3.46}$$

Pick any sequence $(t_p)_{p \in \mathbb{N}}$ of real numbers such that $t_p \to -\infty$ as $p \to -\infty$ and define

$$v_p(t, x) = u_\mu\big(t_p + t, Y(t_p) + x\big) \quad \text{for all } (t, x) \in \mathbb{R}^2. \tag{3.47}$$

From standard parabolic estimates, the functions v_p converge in $C^{1,2}_{loc}(\mathbb{R}^2)$, up to extraction of a subsequence, to a solution v_∞ of (1.1) such that $0 \leq v_\infty \leq 1$ in $\mathbb{R}^2$ and $v_\infty(0,0) = 1/2$, whence $0 < v_\infty < 1$ in $\mathbb{R}^2$ from the strong maximum principle.

The key-point of the proof of the convergence (3.44) is the following lemma, from which (3.44) will follow easily by uniqueness of the limit and the fact that the quantities $X(t) - Y(t)$ are bounded.

Lemma 3.5. *There holds*

$$v_\infty(t, x) = \varphi_{c_-}\Big(x - c_- t + \varphi_{c_-}^{-1}\Big(\frac{1}{2}\Big)\Big) \quad \text{for all } (t, x) \in \mathbb{R}^2. \tag{3.48}$$

Proof. As a matter of fact, the proof is easy if $\mu(c_-) > 0$, similarly as in the case $c_- = c^*$ and $\mu(c^*) > 0$ treated above. The general case $\mu(c_-) = 0$ is more difficult due to the fact that the position of $X(t)$ is not just $c_- t + O(1)$ as $t \to -\infty$, since $X(t) - c_- t \to -\infty$. The proof of this lemma is actually done simultaneously in both cases $\mu(c_-) > 0$ and $\mu(c_-) = 0$.

Since μ is supported in $[c_-, c_+] \subset (c^*, +\infty)$, it follows from Lemma 3.3 that v_∞ is a transition front connecting 0 and 1 of the type

$$v_\infty = u_{\mu_\infty},$$

for a measure $\mu_\infty \in \mathcal{M}$ which is supported in $[c_-, c_+]$. In order to conclude the proof of Lemma 3.5, we shall prove that the rightmost point of the support of μ_∞ is nothing but c_- itself, that is μ_∞ is a multiple of the Dirac mass at the point c_-. To do so, we shall show that $v_\infty = u_{\mu_\infty}$ can be approximated by some solutions of (1.1) which do not move too much faster than c_-.

If $c_+ = c_-$, then the initial measure μ is itself a multiple of the Dirac mass at the point c_-; in other words, in this case, there is $\xi \in \mathbb{R}$ such that $u_\mu(t, x) = \varphi_{c_-}(x - c_- t + \xi)$ for all $(t, x) \in \mathbb{R}^2$, whence $Y(t) = c_- t - \xi + \varphi_{c_-}^{-1}(1/2)$ for all $t \in \mathbb{R}$ and

$$v_p(t, x) = u_\mu\big(t_p + t, Y(t_p) + x\big) = \varphi_{c_-}\Big(x - c_- t + \varphi_{c_-}^{-1}\Big(\frac{1}{2}\Big)\Big) \quad \text{for all } p \in \mathbb{N} \text{ and } (t, x) \in \mathbb{R}^2;$$

the desired conclusion (3.48) is immediate in this case.

Therefore, one can focus in the sequel on the case

$$c_- < c_+.$$

Let c' be any real number such that $c_- < c' < c_+$ and denote

$$\begin{cases} \mu_1 = \mu \, \mathbb{1}_{[c_-, c']}, & M_1 = \mu\big([c_-, c']\big) = \mu_1(X), \\ \mu_2 = \mu \, \mathbb{1}_{(c', c_+]}, & M_2 = \mu\big((c', c_+]\big) = \mu_2(X). \end{cases} \tag{3.49}$$

Since $c_\pm$ are the leftmost and rightmost points of the support of μ, one has $M_1, M_2 > 0$ and $M_1 + M_2 = M = \mu([c_-, c_+])$. Remember now that $u^n_\mu(t, x) \to u_\mu(t, x)$ as $n \to +\infty$ for all $(t, x) \in \mathbb{R}^2$, where u^n_μ solves the Cauchy problem (1.20) in $[-n, +\infty) \times \mathbb{R}$ with initial condition

$$u^n_\mu(-n, x) = M^{-1} \int_{[c_-, c_+]} \varphi_c(x + cn - c \ln M) \, d\mu(c).$$

On the one hand, since all functions φ_c are decreasing and positive, one has

$$u^n_\mu(-n, x) \geq M^{-1} \int_{[c_-, c']} \varphi_c(x + cn - c \ln M_1) \, d\mu(c) = \frac{M_1}{M} u^n_{\mu_1}(-n, x) > 0$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. But, for every $n \in \mathbb{N}$, the positive function $(M_1/M) u^n_{\mu_1}$ is a subsolution of the equation satisfied by u^n_μ in $(-n, +\infty) \times \mathbb{R}$, since $f((M_1/M)s) \geq (M_1/M) f(s)$ for all $0 \leq s \leq 1$. The maximum principle yields $(M_1/M) u^n_{\mu_1}(t, x) \leq u^n_\mu(t, x)$ for all $n \in \mathbb{N}$ and $(t, x) \in [-n, +\infty) \times \mathbb{R}$, whence

$$0 < \frac{M_1}{M} u_{\mu_1}(t, x) \leq u_\mu(t, x) \quad \text{for all } (t, x) \in \mathbb{R}^2, \tag{3.50}$$

by passing to the limit as $n \to +\infty$.

On the other hand, calling

$$\xi_1 := c' \ln \frac{M_1}{M} \quad \text{and} \quad \xi_2 := c_+ \ln \frac{M_2}{M}, \tag{3.51}$$

one has, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$
\begin{aligned}
u^n_\mu(-n, x) \ &\leq \ M^{-1} \int_{[c_-, c']} \varphi_c(x + cn - c \ln M_1 + \xi_1) \, d\mu_1(c) \\
&\quad + M^{-1} \int_{(c', c_+]} \varphi_c(x + cn - c \ln M_2 + \xi_2) \, d\mu_2(c) \\
&\leq \ M_1^{-1} \int_{[c_-, c']} \varphi_c(x + cn - c \ln M_1 + \xi_1) \, d\mu_1(c) \\
&\quad + M_2^{-1} \int_{(c', c_+]} \varphi_c(x + cn - c \ln M_2 + \xi_2) \, d\mu_2(c) \\
&= \ u^n_{\mu_1}(-n, x + \xi_1) + u^n_{\mu_2}(-n, x + \xi_2).
\end{aligned}
$$

Since $u^n_\mu < 1$ in $[-n, +\infty) \times \mathbb{R}$, it follows then as in the proof of Lemma 2.4 that, for every $n \in \mathbb{N}$, the function $\overline{u}^n$ defined in $[-n, +\infty) \times \mathbb{R}$ by

$$\overline{u}^n(t, x) = \min \left(u^n_{\mu_1}(t, x + \xi_1) + u^n_{\mu_2}(t, x + \xi_2), 1 \right)$$

is a supersolution of the equation satisfied by u^n_μ. As a consequence,

$$u^n_\mu(t, x) \leq \min \left(u^n_{\mu_1}(t, x + \xi_1) + u^n_{\mu_2}(t, x + \xi_2), 1 \right)$$

for every $n \in \mathbb{N}$ and $(t, x) \in [-n, +\infty) \times \mathbb{R}$, whence

$$u_\mu(t, x) \leq \min \left(u_{\mu_1}(t, x + \xi_1) + u_{\mu_2}(t, x + \xi_2), 1 \right) \quad \text{for all } (t, x) \in \mathbb{R}^2 \qquad (3.52)$$

by passing to the limit as $n \to +\infty$.

Remembering the definition (3.47) of the functions v_p, define now, for every $p \in \mathbb{N}$ and $(t, x) \in \mathbb{R}^2$,

$$v_{p,1}(t, x) = u_{\mu_1}\left(t_p + t, Y(t_p) + x \right) \quad \text{and} \quad v_{p,2}(t, x) = u_{\mu_2}\left(t_p + t, Y(t_p) + x \right). \qquad (3.53)$$

One infers from (3.50) and (3.52) that, for every $p \in \mathbb{N}$ and $(t, x) \in \mathbb{R}^2$,

$$\frac{M_1}{M} \underbrace{u_{\mu_1}\left(t_p + t, Y(t_p) + x \right)}_{= v_{p,1}(t,x)} \leq \underbrace{u_\mu\left(t_p + t, Y(t_p) + x \right)}_{= v_p(t,x)}$$

$$\leq \underbrace{u_{\mu_1}\left(t_p + t, Y(t_p) + x + \xi_1 \right)}_{= v_{p,1}(t, x + \xi_1)} + \underbrace{u_{\mu_2}\left(t_p + t, Y(t_p) + x + \xi_2 \right)}_{= v_{p,2}(t, x + \xi_2)}.$$

Up to extraction of another subsequence, the functions $v_{p,1}$ and $v_{p,2}$ converge in $C_{loc}^{1,2}(\mathbb{R}^2)$ as $p \to +\infty$ to two solutions $0 \leq v_{\infty,1} \leq 1$ and $0 \leq v_{\infty,2} \leq 1$ of (1.1) such that

$$\frac{M_1}{M} v_{\infty,1}(t, x) \leq v_\infty(t, x) \leq v_{\infty,1}(t, x + \xi_1) + v_{\infty,2}(t, x + \xi_2) \quad \text{for all } (t, x) \in \mathbb{R}^2.$$

But the inequalities (1.19) applied to u_{μ_2} imply that, for every $p \in \mathbb{N}$ and $(t, x) \in \mathbb{R}^2$,

$$0 < v_{p,2}(t, x) = u_{\mu_2}(t_p + t, Y(t_p) + x) \leq M_2^{-1} \int_{(c', c_+]} e^{-\lambda_c (Y(t_p) + x - c(t_p + t) - c \ln M_2)} d\mu_2(c).$$

Since $c_- < c'$ and $Y(t_p)/t_p \to c_-$ as $p \to +\infty$ (because $X(t)/t \to c_-$ as $t \to -\infty$ from Step 1, and the quantities $Y(t) - X(t) = X_{1/2}(t) - X(t)$ are bounded in $\mathbb{R}$), it follows from Lebesgue's dominated convergence theorem that $v_{p,2}(t, x) \to 0$ as $p \to +\infty$ locally uniformly in $(t, x) \in \mathbb{R}^2$, whence $v_{\infty,2} = 0$ in $\mathbb{R}^2$ and

$$\frac{M_1}{M} v_{\infty,1}(t, x) \leq v_\infty(t, x) \leq v_{\infty,1}(t, x + \xi_1) \quad \text{for all } (t, x) \in \mathbb{R}^2. \qquad (3.54)$$

In particular, $v_{\infty,1} > 0$ in $\mathbb{R}^2$, while $v_{\infty,1}$ cannot be identically equal to 1 (otherwise, $\inf_{\mathbb{R}^2} v_\infty \geq M_1/M > 0$, which would contradict (3.42) for instance). Thus, there holds $0 < v_{\infty,1} < 1$ in $\mathbb{R}^2$ from the strong maximum principle.

Furthermore, since μ_1 is supported in $[c_-, c'] \subset (c^*, +\infty)$, it follows from Lemma 3.3 that $v_{\infty,1}$ is a transition front connecting 0 and 1 of the type

$$v_{\infty,1} = u_{\mu_{\infty,1}},$$

for a measure $\mu_{\infty,1} \in \mathcal{M}$ which is supported in $[c_-, c']$. From Steps 1 and 2 of the present proof of Theorem 1.14, the transition front $v_\infty = u_{\mu_\infty}$, resp. $v_{\infty,1} = u_{\mu_{\infty,1}}$, has asymptotic past and future speeds which are equal to the leftmost and rightmost points of the support

30

of μ_∞, resp. $\mu_{\infty,1}$. By (3.54), the asymptotic past and future speeds of v_∞ are equal to those of $v_{\infty,1}$. Therefore, the measures μ_∞ and $\mu_{\infty,1}$ have the same leftmost and rightmost points and, since $\mu_{\infty,1}$ is supported in $[c_-, c']$, so is μ_∞. Finally, since c' was arbitrarily chosen in (c_-, c_+), one concludes that the measure μ_∞ is nothing but a multiple of the Dirac mass at $\{c_-\}$. In other words, $v_\infty(t,x) = u_{\mu_\infty}(t,x) = \varphi_{c_-}(x - c_- t + \xi)$ in $\mathbb{R}^2$, for some $\xi \in \mathbb{R}$. Since $v_\infty(0,0) = 1/2$, one gets $\xi = \varphi_{c_-}^{-1}(1/2)$ and the desired conclusion (3.48) has been proved. The proof of Lemma 3.5 is thereby complete. $\qquad\square$

Let us finally complete the proof of the convergence of u_μ along the positions $X(t)$ to the shifted profile φ_{c_-} as $t \to -\infty$, that is (3.44), or (1.21) as $t \to -\infty$, in the case $c_- > c^*$. From the uniqueness of the limit of $u_\mu(t_p + t, Y(t_p) + x)$ as $p \to +\infty$ (with $t_p \to -\infty$), one gets that

$$u_\mu(t + t', Y(t) + x) \xrightarrow[t \to -\infty]{} \varphi_{c_-}\left(x - c_- t' + \varphi_{c_-}^{-1}\left(\frac{1}{2}\right)\right), \text{ locally uniformly in } (t', x) \in \mathbb{R}^2.$$

Denote $\xi^-(t) = Y(t) - X(t) - \varphi_{c_-}^{-1}(1/2)$ for all $t \in (-\infty, 0]$. By (1.4) and $u_\mu(t, Y(t)) = 1/2$, the function ξ^- is then bounded in $(-\infty, 0]$. Furthemore,

$$u_\mu(t, X(t) + \xi^-(t) + x) \to \varphi_{c_-}(x) \text{ as } t \to -\infty, \text{ locally uniformly in } x \in \mathbb{R}. \qquad (3.55)$$

Lastly, since $0 < u_\mu(t, \cdot) < 1$ is decreasing in $\mathbb{R}$ for every $t \in \mathbb{R}$, while $\varphi_{c_-}(-\infty) = 1$ and $\varphi_{c_-}(+\infty) = 0$, one concludes that the convergence (3.55) is actually uniform in $x \in \mathbb{R}$, and even holds in $C^2(\mathbb{R})$ from standard parabolic estimates.

Step 4: convergence to the profile φ_{c_+} along the level sets of u_μ as $t \to +\infty$.

In this last step, we prove the convergence of u_μ along the positions $X(t)$ as $t \to +\infty$. We recall that $c_- > c^*$ here. The proof follows the same lines as for $t \to -\infty$ in Step 3 above. Namely, let $(t_p)_{p \in \mathbb{N}}$ be any sequence of real numbers such that $t_p \to +\infty$ as $p \to +\infty$. Let $(Y(t))_{t \in \mathbb{R}}$ and $(v_p)_{p \in \mathbb{N}}$ be defined as in (3.46) and (3.47). Up to extraction of a subsequence, the functions v_p converge in $C_{loc}^{1,2}(\mathbb{R}^2)$ to a solution $0 < v_\infty < 1$ of (1.1) such that $v_\infty(0,0) = 1/2$. The analogue of Lemma 3.5 is the following result.

Lemma 3.6. *There holds*

$$v_\infty(t,x) = \varphi_{c_+}\left(x - c_+ t + \varphi_{c_+}^{-1}\left(\frac{1}{2}\right)\right) \text{ for all } (t,x) \in \mathbb{R}^2. \qquad (3.56)$$

Proof. As in the proof of Lemma 3.5, we do not distinguish the cases $\mu(c_+) = 0$ and $\mu(c_+) > 0$, the latter being actually much easier to handle. From Lemma 3.3, the function v_∞ is a transition front connecting 0 and 1 of the type

$$v_\infty = u_{\mu_\infty},$$

for a measure $\mu_\infty \in \mathcal{M}$ which is supported in $[c_-, c_+]$. This time, our goal is to show that μ_∞ is a multiple of the Dirac mass at the point $\{c_+\}$. If $c_- = c_+$, then the conclusion (3.56) is immediate, as in Lemma 3.5. Therefore, we focus in the sequel on the case $c_- < c_+$.

31

So, let c' be arbitrary in the interval (c_-, c_+) and let μ_1, μ_2, M_1, M_2, ξ_1 and ξ_2 be as in (3.49) and (3.51). The same arguments as in the proof of Lemma 3.5 imply that

$$0 < \frac{M_2}{M}\, u_{\mu_2}(t,x) \le u_\mu(t,x) \le \min\left(u_{\mu_1}(t,x+\xi_1) + u_{\mu_2}(t,x+\xi_2), 1\right)$$

for all $(t,x) \in \mathbb{R}^2$. Let the sequences $(v_{p,1})_{p\in\mathbb{N}}$ and $(v_{p,2})_{p\in\mathbb{N}}$ be defined as in (3.53). Up to extraction of a subsequence, they converge locally uniformly in $\mathbb{R}^2$ to some solutions $0 \le v_{\infty,1} \le 1$ and $0 \le v_{\infty,2} \le 1$ of (1.1) such that

$$\frac{M_2}{M}\, v_{\infty,2}(t,x) \le v_\infty(t,x) \le v_{\infty,1}(t,x+\xi_1) + v_{\infty,2}(t,x+\xi_2)$$

for all $(t,x) \in \mathbb{R}^2$. This time, one has

$$0 < v_{p,1}(t,x) = u_{\mu_1}(t_p + t, Y(t_p) + x) \le M_1^{-1} \int_{[c_-,c']} e^{-\lambda_c(Y(t_p)+x-c(t_p+t)-c\ln M_1)} d\mu_1(c),$$

but $Y(t_p)/t_p \to c_+$ as $p \to +\infty$ (since $t_p \to +\infty$ as $p \to +\infty$ and $X(t)/t \to c_+$ as $t \to +\infty$). Therefore, Lebesgue's dominated convergence theorem implies that $v_{p,1}(t,x) \to 0$ as $p \to +\infty$ locally unifiormly in $(t,x) \in \mathbb{R}^2$, whence $v_{\infty,1} = 0$ in $\mathbb{R}^2$ and

$$\frac{M_2}{M}\, v_{\infty,2}(t,x) \le v_\infty(t,x) \le v_{\infty,2}(t,x+\xi_2) \quad \text{for all } (t,x) \in \mathbb{R}^2. \tag{3.57}$$

As a consequence, $0 < v_{\infty,2} < 1$ in $\mathbb{R}^2$. This time, since μ_2 is supported in $[c', c_+] \subset (c^*, +\infty)$, it follows from Lemma 3.3 that $v_{\infty,2}$ is a transition front connecting 0 and 1 of the type

$$v_{\infty,2} = u_{\mu_{\infty,2}},$$

for a measure $\mu_{\infty,2} \in \mathcal{M}$ which is supported in $[c', c_+]$. From Steps 1 and 2, the transition front $v_\infty = u_{\mu_\infty}$, resp. $v_{\infty,2} = u_{\mu_{\infty,2}}$, has asymptotic past and future speeds which are equal to the leftmost and rightmost points of the support of μ_∞, resp. $\mu_{\infty,2}$. By (3.57), the asymptotic past and future speeds of v_∞ are equal to those of $v_{\infty,2}$. Therefore, the measures μ_∞ and $\mu_{\infty,2}$ have the same leftmost and rightmost points and, since $\mu_{\infty,2}$ is supported in $[c', c_+]$, so is μ_∞. Finally, since c' was arbitrarily chosen in (c_-, c_+), one concludes that the measure μ_∞ is nothing but a multiple of the Dirac mass at $\{c_+\}$. The conclusion of Lemma 3.6 follows immediately. □

By denoting $\xi^+(t) = Y(t) - X(t) - \varphi_{c_+}^{-1}(1/2)$ for all $t \in (0, +\infty)$, one concludes as at the end of Step 3 that the function ξ^+ is bounded in $(0, +\infty)$ and that

$$u_\mu(t, X(t) + \xi^+(t) + \cdot) \to \varphi_{c_+} \quad \text{in } C^2(\mathbb{R}) \quad \text{as } t \to +\infty.$$

Finally, (1.21) holds with $\xi(t) = \xi^-(t)$ for $t \le 0$ and $\xi(t) = \xi^+(t)$ for $t > 0$. The proof of Theorem 1.14 is thereby complete. □

3.2 Proof of Proposition 3.1

Let $\mu \in \mathcal{M}$ be supported in $[\gamma, +\infty)$ for some $\gamma \geq c^* = 2\sqrt{f'(0)}$ and let $0 < u_\mu < 1$ be the solution of (1.1) associated to the measure μ. We will prove that u_μ is less steep than the traveling front φ_γ associated to the speed γ. As for Proposition 2.1, the proof uses an intersection number argument. But the proof is a bit simpler here since one does not need to approximate u_μ by some u_{μ_k} nor to approximate the speed γ by $\gamma + \varepsilon$.

To do so, let us first show that every front φ_c, which is already known to be such that $\varphi_c' < 0$ in $\mathbb{R}$, does not decay logarithmically faster than its decay rate at $+\infty$. Namely, let c be any real number in $[c^*, +\infty)$. It is known, see e.g. [2], that $\phi_c(\xi) := \varphi_c'(\xi)/\varphi_c(\xi) \to -\lambda_c$ as $\xi \to +\infty$. We claim that

$$-\lambda_c < \frac{\varphi_c'(\xi)}{\varphi_c(\xi)} = \phi_c(\xi) \ (< 0) \quad \text{for all } \xi \in \mathbb{R}. \tag{3.58}$$

Indeed, if this were not true, then there would exist a real number ξ_m such that $\phi_c(\xi_m) \leq \phi_c(\xi) < 0$ for all $\xi \in \mathbb{R}$. But the function ϕ_c obeys

$$\phi_c'' + (2\phi_c + c)\phi_c' + \left(f'(\varphi_c) - \frac{f(\varphi_c)}{\varphi_c}\right)\phi_c = 0 \quad \text{in } \mathbb{R},$$

while $f'(\varphi_c) - f(\varphi_c)/\varphi_c \leq 0$ in $\mathbb{R}$ since $f(s)/s$ is nonincreasing on $(0,1]$. The strong maximum principle would imply that ϕ_c is constant, which is clearly impossible since $\phi_c(-\infty) = 0$ and $\phi_c < 0$ in $\mathbb{R}$. Therefore, the claim (3.58) holds.

Let now v be the solution of the Cauchy problem

$$\begin{cases} v_t &= v_{xx} + f(v), \quad t > 0, \ x \in \mathbb{R}, \\ v_0(x) &= \begin{cases} 1 & \text{for all } x \leq 0, \\ e^{-\lambda_\gamma x} & \text{for all } x > 0. \end{cases} \end{cases}$$

It follows from [43] that

$$\sup_{x \in \mathbb{R}} \left|v(t,x) - \varphi_\gamma(x - \gamma t + m(t))\right| \to 0 \quad \text{as } t \to +\infty, \tag{3.59}$$

where $m(t)$ can be taken to be 0 if $\gamma > c^*$ (remember that $\varphi_c(\xi) \sim e^{-\lambda_c \xi}$ as $\xi \to +\infty$ if $c > c^*$) and $m(t) = o(t)$ as $t \to +\infty$ if $\gamma = c^*$.

On the other hand, by definition, the solution u_μ of (1.1) is given as the limit as $n \to +\infty$ of the solutions u_μ^n of the Cauchy problems (1.20) in $[-n, +\infty) \times \mathbb{R}$ associated with the Cauchy data

$$u_\mu^n(-n, x) = \max \left(\varphi_{c^*}(x + c^* n - c^* \ln \mu(c^*)), M^{-1}\int_{(c^*, +\infty)} \varphi_c(x + cn - c \ln M)\, d\mu(c)\right)$$

for all $x \in \mathbb{R}$, under the convention that the terms involving $\ln 0$ are absent. For every $c \geq \gamma \ (\geq c^*)$, there holds $0 < \lambda_c \leq \lambda_\gamma$. Since the support of μ is included in $[\gamma, +\infty)$, the property (3.58) implies that, for all $n \in \mathbb{N}$,

$$u_\mu^n(-n, x') > u_\mu^n(-n, x)\, e^{-\lambda_\gamma (x' - x)} \quad \text{for all } x < x' \in \mathbb{R}.$$

Since the continuous function $u_\mu^n(-n, \cdot)$ ranges in $(0, 1)$, one infers that $v_0(x_0 + \cdot)$ is steeper than $u_\mu^n(-n, \cdot)$ for every $n \in \mathbb{N}$ and $x_0 \in \mathbb{R}$, in the sense that there is a unique $x_1 \in \mathbb{R}$ such that

$$
\begin{cases}
v_0(x_0 + x_1) &= u_\mu^n(-n, x_1), \\
v_0(x_0 + x_1 + x) &> u_\mu^n(-n, x_1 + x) \quad \text{for all } x < 0, \\
v_0(x_0 + x_1 + x) &< u_\mu^n(-n, x_1 + x) \quad \text{for all } x > 0.
\end{cases}
$$

As a consequence, as in the proof of Lemma 2.5 for instance, it follows from an intersection number argument that, for every $n \in \mathbb{N}$, $t > -n$ and $x_0 \in \mathbb{R}$, $v(t + n, x_0 + \cdot)$ is steeper than $u_\mu^n(t, \cdot)$.

Let finally (t, X) be any fixed pair in $\mathbb{R}^2$. For every $n \in \mathbb{N}$ such that $t > -n$, since $v(t + n, \cdot + X)$ is continuously decreasing and converges to 0 and 1 at $\pm\infty$, there is a unique real number x_n such that $v(t + n, x_n + X) = u_\mu^n(t, X)$. Since $v(t + n, x_n + \cdot)$ is steeper than $u_\mu^n(t, \cdot)$, one infers that

$$
\begin{cases}
v(t + n, x_n + X + x) > u_\mu^n(t, X + x) \quad \text{for all } x < 0, \\
v(t + n, x_n + X + x) < u_\mu^n(t, X + x) \quad \text{for all } x > 0.
\end{cases}
$$

Thus, by (3.59), there exists a sequence $(h_n)_{n \in \mathbb{N}}$ converging to 0^+ such that

$$
\begin{cases}
u_\mu^n(t, X + x) \leq \varphi_\gamma(x_n + X + x - \gamma(t + n) + m(t + n)) + h_n \quad \text{for all } x \leq 0, \\
u_\mu^n(t, X + x) \geq \varphi_\gamma(x_n + X + x - \gamma(t + n) + m(t + n)) - h_n \quad \text{for all } x \geq 0.
\end{cases} \tag{3.60}
$$

From this at $x = 0$ and the convergence $u_\mu^n \to u_\mu$ as $n \to +\infty$ locally uniformly in $\mathbb{R}^2$, we deduce that

$$
\varphi_\gamma(x_n + X - \gamma(t + n) + m(t + n)) \to u_\mu(t, X) \in (0, 1) \quad \text{as } n \to +\infty,
$$

whence

$$
x_n + X - \gamma(t + n) + m(t + n) \to \varphi_\gamma^{-1}(u_\mu(t, X)) \quad \text{as } n \to +\infty.
$$

Passing to the limit as $n \to +\infty$ in (3.60) leads to the desired conclusion (3.40) and the proof of Proposition 3.1 is thereby complete. $\qquad\square$

3.3 An additional comparison result for any time-global solution u of (1.1)

To complete this section, we include an additional comparison result which has its own interest. This result is of the same spirit as Proposition 3.1 but it holds for any solution $0 < u < 1$ of (1.1), even not a transition front or even not a solution which is a priori of the type u_μ. It says that the traveling front $\varphi_{c^*}(x - c^* t)$ is the steepest time-global solution (that is a critical front, in the sense of [32]). This result, stated in Proposition 3.7 below, actually holds under more general assumptions on f, for which solutions of the type u_μ are not known. Namely, we will consider $C^1([0, 1])$ functions f satisfying

$$
f(0) = f(1) = 0 \quad \text{and} \quad 0 < f(u) \leq f'(0)u \text{ for all } u \in (0, 1). \tag{3.61}
$$

34

For such functions f, standard traveling fronts $\varphi_c(x - ct)$ solving (1.1) and such that $1 = \varphi_c(-\infty) > \varphi_c > \varphi_c(+\infty) = 0$ still exist if and only if $c \geq c^* = 2\sqrt{f'(0)}$.

Proposition 3.7. *Let* $f : [0, 1] \to \mathbb{R}$ *be any* C^1 *function satisfying* (3.61). *Then, for every solution* $0 < u < 1$ *of* (1.1) *and for every* $(t, X) \in \mathbb{R}^2$, *there holds*

$$\begin{cases} u(t, X + x) \leq \varphi_{c^*}\left(\varphi_{c^*}^{-1}(u(t, X)) + x\right) & \text{for all } x \leq 0, \\ u(t, X + x) \geq \varphi_{c^*}\left(\varphi_{c^*}^{-1}(u(t, X)) + x\right) & \text{for all } x \geq 0. \end{cases} \tag{3.62}$$

Proof. In the proof, $0 < u < 1$ denotes any solution of (1.1) and (t, X) denotes any fixed pair in $\mathbb{R}^2$. The proof will again use an intersection number argument, by comparing this time u with the solution of some Cauchy problems with the steepest possible initial condition. Namely, let v be the solution of the Cauchy problem

$$\begin{cases} v_\tau = v_{xx} + f(v), & \tau > 0, \ x \in \mathbb{R}, \\ v(0, x) = \begin{cases} 1 & \text{for all } x < 0, \\ 0 & \text{for all } x \geq 0. \end{cases} \end{cases}$$

From the maximum principle and the parabolic regularity, the function $v(\tau, \cdot)$ is decreasing and continuous for every $\tau > 0$, while $v(\tau, -\infty) = 1$ and $v(\tau, +\infty) = 0$. Therefore, for every $\tau > 0$, there is a unique $m(\tau) \in \mathbb{R}$ such that

$$v(\tau, m(\tau)) = u(t, X) \ (\in (0, 1)).$$

It is also known, see e.g. [7, 9, 18, 24, 25, 43], that $m(\tau) = c^*\tau - (3/c^*) \ln \tau + O(1)$ as $\tau \to +\infty$ and that

$$v(\tau, m(\tau) + x) \to \varphi_{c^*}\left(\varphi_{c^*}^{-1}(u(t, X)) + x\right) \quad \text{as } t \to +\infty, \text{ uniformly in } x \in \mathbb{R}, \tag{3.63}$$

where $\varphi_{c^*}(x - c^*t)$ denotes any traveling front with critical speed $c^* = 2\sqrt{f'(0)}$ for the equation (1.1).

Now, owing to the definition of v_0, for every $s \in (-\infty, t)$, the shifted step-function $v(0, m(t - s) + \cdot)$ is steeper than $u(s, X + \cdot)$, in the sense that

$$\begin{cases} u(s, X + x) < 1 = v(0, m(t - s) + x) & \text{for all } x < -m(t - s), \\ u(s, X + x) > 0 = v(0, m(t - s) + x) & \text{for all } x > -m(t - s). \end{cases}$$

Therefore, it follows from an intersection number argument that $v(t - s, m(t - s) + \cdot))$ is steeper than $u(t, X + \cdot)$ and, since

$$v(t - s, m(t - s)) = u(t, X),$$

one infers that

$$\begin{cases} u(t, X + x) < v(t - s, m(t - s) + x) & \text{for all } x < 0, \\ u(t, X + x) > v(t - s, m(t - s) + x) & \text{for all } x > 0. \end{cases}$$

By passing to the limit as $s \to -\infty$ and using (3.63), the conclusion (3.62) follows and the proof of Proposition 3.7 is thereby complete. $\qquad\square$

4 Proofs of the other main results

In this section, we do the proofs of Theorems 1.4, 1.7, 1.9 and Proposition 1.2. Theorems 1.4, 1.7, 1.9 will chiefly follow from Theorems 1.11 and 1.14, while the proof of Proposition 1.2 will be done in a more general heterogeneous framework, in Section 4.4.

4.1 Proof of Theorem 1.9

The assumption (1.14) with $c > c^*$ implies, by Theorem 1.4 of [17], that u is of the type $u = u_\mu$, with a measure $\mu \in \mathcal{M}$ that is supported in $(-\infty, -c] \cup [c, +\infty) \cup \{\infty\}$. In particular, $M = \mu(X \backslash \{-c^*, c^*\}) = \mu(X) > 0$. Two cases may occur:

Case 1: $\mu\big((-\infty, -c] \cup \{\infty\}\big) > 0$. In this case, it follows from (1.19) and Lebesgue's dominated convergence theorem that

$$\liminf_{x \to +\infty} \underbrace{u(0,x)}_{=u_\mu(0,x)} \geq \frac{\mu\big((-\infty, -c]\big) + \theta(\ln M)\,\mu(\infty)}{M} > 0.$$

Since $u(0, x) < 1$ for all $x \in \mathbb{R}$, it follows then that

$$\frac{\ln u(0,x)}{x} \to 0 \ \text{ as } x \to +\infty.$$

Case 2: $\mu\big((-\infty, -c] \cup \{\infty\}\big) = 0$. In this case, μ is supported in $[c, +\infty) \subset (c^*, +\infty)$ and the inequalities (1.19) imply that

$$M^{-1} \int_{[c,+\infty)} \varphi_{c'}(x - c' \ln M)\, d\mu(c') \leq \underbrace{u(0,x)}_{=u_\mu(0,x)} \leq M^{-1} \int_{[c,+\infty)} e^{-\lambda_{c'}(x - c' \ln M)}\, d\mu(c')$$

for all $x \in \mathbb{R}$.

We now claim that, for every $\gamma \in (c^*, +\infty)$ such that $\mu\big([\gamma, +\infty)\big) > 0$, one has

$$\limsup_{x \to +\infty} -\frac{\ln u(0,x)}{x} \leq \lambda_\gamma = \frac{\gamma - \sqrt{\gamma^2 - 4f'(0)}}{2}. \tag{4.64}$$

Indeed, for such a γ and for any $\lambda' > \lambda_\gamma$, we have

$$e^{\lambda' x} u(0,x) \geq M^{-1} \int_{[\gamma,+\infty)} \min\big(e^{\lambda' x}\varphi_{c'}(x - c' \ln M), 1\big)\, d\mu(c') \ \text{ for all } x \in \mathbb{R}.$$

For every $c' \geq \gamma\,(> c^*)$, there holds $\varphi_{c'}(x - c' \ln M) \sim e^{-\lambda_{c'}(x - c' \ln M)}$ as $x \to +\infty$ with $0 < \lambda_{c'} \leq \lambda_\gamma < \lambda'$, whence

$$\min\big(e^{\lambda' x}\varphi_{c'}(x - c' \ln M), 1\big) \to 1 \ \text{ as } x \to +\infty.$$

Finally, Lebesgue's dominated convergence theorem shows that

$$\liminf_{x \to +\infty} \big(e^{\lambda' x} u(0,x)\big) \geq M^{-1}\mu\big([\gamma, +\infty)\big) > 0,$$

whence $\limsup_{x\to+\infty} -(\ln u(0,x))/x \leq \lambda'$. Since λ' can be arbitrary in $(\lambda_\gamma, +\infty)$, the claim (4.64) follows.

Subcase 2.1: the measure μ is not compactly supported. Therefore, it follows from (4.64) that, for every $\gamma \in (c^*, +\infty)$, $\limsup_{x\to+\infty} -(\ln u(0,x))/x \leq \lambda_\gamma = (\gamma - \sqrt{\gamma^2 - 4f'(0)})/2$, whence $\limsup_{x\to+\infty} -(\ln u(0,x))/x \leq 0$ by passing to the limit as $\gamma \to +\infty$. But since $u(0,x) \in (0,1)$ for all $x \in \mathbb{R}$, one concludes in this case that

$$\frac{\ln u(0,x)}{x} \to 0 \;\; \text{as } x \to +\infty.$$

Subcase 2.2: the measure μ is compactly supported. In this case, call c_+ the rightmost point of the measure μ and notice that $c^* < c \leq c_+ < +\infty$. It follows from Proposition 2.1 that

$$\underbrace{u(0,x)}_{=u_\mu(0,x)} \leq \varphi_{c_+}\big(\varphi_{c_+}^{-1}(u(0,0)) + x\big) \text{ for all } x \geq 0.$$

Since $\varphi_{c_+}(\xi) \sim e^{-\lambda_{c_+}\xi}$ as $\xi \to +\infty$, one infers that $\limsup_{x\to+\infty}(\ln u(0,x))/x \leq -\lambda_{c_+}$, that is $\liminf_{x\to+\infty}(-\ln u(0,x))/x \geq \lambda_{c_+}$. But $\mu\big([\gamma, +\infty)\big) > 0$ for every $\gamma \in (c^*, c_+)$, by definition of c_+. Thus, (4.64) implies that $\limsup_{x\to+\infty}(-\ln u(0,x))/x \leq \lambda_\gamma$ for every $\gamma \in (c^*, c_+)$, whence $\limsup_{x\to+\infty}(-\ln u(0,x))/x \leq \lambda_{c_+}$. Finally,

$$-\frac{\ln u(0,x)}{x} \to \lambda_{c_+} \;\; \text{as } x \to +\infty, \tag{4.65}$$

with $0 < \lambda_{c_+} \leq \lambda_c < \lambda_{c^*} = \sqrt{f'(0)}$.

Conclusion. To sum up, in all above cases, there holds $u = u_\mu$,

$$-\frac{\ln u(0,x)}{x} \to \lambda \in \big[0, \sqrt{f'(0)}\big) \;\; \text{as } x \to +\infty$$

and $\lambda > 0$ if and only if the measure μ is compactly supported in $[c^*, +\infty)$. In this case (that is subcase 2.2 with the previous notations), call c_- the leftmost point of the support of μ (one has $c^* < c \leq c_- \leq c_+ < +\infty$). It follows from Theorems 1.11 and 1.14 that u is a transition front connecting 0 and 1, that it has asymptotic past and future speeds equal to $c_\pm$, and that (1.21) holds for some bounded function ξ. Furthermore,

$$c_+ = \lambda_{c_+} + \frac{f'(0)}{\lambda_{c_+}} = \lambda + \frac{f'(0)}{\lambda}$$

by (4.65). On the other hand, since $X(t)/t \to c_-$ as $t \to -\infty$, where $(X(t))_{t\in\mathbb{R}}$ is a family of real numbers such that (1.4) holds, one has $\max_{[-c'|t|,c'|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for every $c' \in [0, c_-)$, while (1.5) implies that $\liminf_{t\to-\infty} \max_{[-c'|t|,c'|t|]} u(t,\cdot) > 0$ for every $c' > c_-$. As a conclusion,

$$c_- = \sup\Big\{ c \geq 0, \; \lim_{t\to-\infty} \max_{[-c|t|,c|t|]} u(t,\cdot) = 0 \Big\}.$$

The proof of Theorem 1.9 is thereby complete. $\qquad\square$

4.2   Proof of Theorem 1.7

Let u be a transition front connecting 0 and 1 for (1.1), and let $(X(t))_{t\in\mathbb{R}}$ be a family of real numbers such that (1.4) holds. It follows from (1.4) and (1.13) that $\liminf_{t\to-\infty} X(t)/t \geq c^*$, that is the first inequality in (1.11).

On the other hand, the last inequality in (1.11) holds by the consequence (1.8) of Proposition 1.2.

Furthermore, the spreading results of Aronson and Weinberger [2] also imply that $\min_{[-ct,ct]} u(t,\cdot) \to 1$ as $t \to +\infty$ for every $c \in [0,c^*)$, whence $\liminf_{t\to+\infty} X(t)/t \geq c^*$ by (1.4). In particular, if $c^* = \liminf_{t\to-\infty} X(t)/t$, then formula (1.11) holds.

Lastly, if $c^* < \liminf_{t\to-\infty} X(t)/t$, then (1.4) implies that $\max_{[-c|t|,c|t|]} u(t,\cdot) \to 0$ as $t \to -\infty$ for some $c > c^*$. It follows from Theorem 1.4 of [17] that $u = u_\mu$ for some measure $\mu \in \mathcal{M}$ that is supported in $(-\infty,-c] \cup [c,+\infty) \cup \{\infty\}$. From Theorem 1.11, one infers that μ is actually compactly supported in $[c,+\infty) \subset (c^*,+\infty)$ and the conclusion of Theorem 1.7 follows from Theorem 1.14. $\qquad\square$

4.3   Proof of Theorem 1.4

First of all, all speeds γ in $[c^*,+\infty)$ are admissible global mean speeds since every standard traveling front $\varphi_\gamma(x-\gamma t)$ has a global mean speed equal to γ. Furthermore, if a transition front connecting 0 and 1 has a global mean speed equal to γ then necessarily $\gamma \geq c^*$ by the spreading result [2] (in the form of (1.13), or even by Theorem 1.6). Now, let u be a transition front connecting 0 and 1 and having a global mean speed γ such that $\gamma > c^*$, whence $X(t)/t \to \gamma > c^*$ as $t \to -\infty$ and $\max_{[-c'|t|,+\infty)} u(t,\cdot) \to 0$ as $t \to -\infty$, for every $c' \in (c^*,\gamma)$. Theorem 1.4 of [17] then implies that u is of the type

$$u = u_\mu,$$

for a measure $\mu \in \mathcal{M}$ whose support is included in $(-\infty,-c'] \cup [c',+\infty) \cup \{\infty\}$. On the other hand, since $u = u_\mu$ is a transition front connecting 0 and 1 for (1.1), it follows from Theorem 1.11 that μ is compactly supported in $[c^*,+\infty)$. Since u_μ has asymptotic past and future speeds which are both equal to c, Theorem 1.14 implies that the leftmost and rightmost points of the support of μ are both equal to c. In other words, $\mu = m\,\delta_c$ for some $m > 0$ and the conclusion follows immediately. $\qquad\square$

Remark 4.1. The above proof actually implies the following slightly stronger version of Theorem 1.4. Namely, if a transition front u connecting 0 and 1 has asymptotic past and future speeds $c_\pm$ equal to the same quantity γ and if $\gamma > c^*$, then u is a standard traveling front of the type $u(t,x) = \varphi_\gamma(x-\gamma t)$ and, in particular, u has then a global mean speed equal to γ.

4.4   Proof of Proposition 1.2

Proposition 1.2 is actually a particular case of Proposition 4.2 below, which holds in a more general framework. More precisely, we consider in this section some heterogeneous reaction-

diffusion equations of the type

$$u_t(t,x) = a(t,x)\,u_{xx}(t,x) + b(t,x)\,u_x(t,x) + f(t,x,u(t,x)), \quad t \in \mathbb{R}, \ x \in \mathbb{R}, \qquad (4.66)$$

where $a : \mathbb{R}^2 \to \mathbb{R}$ is a bounded continuous function such that $\inf_{\mathbb{R}^2} a > 0$, $b : \mathbb{R}^2 \to \mathbb{R}$ is a bounded continuous function, and $f : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ is a bounded continuous function. For the equation (4.66), we say that a classical solution $0 \le u(t,x) \le 1$ is a transition front connecting 0 and 1 if there is a family of real numbers $(X(t))_{t\in\mathbb{R}}$ satisfying (1.4).

Proposition 4.2. *Under the above assumptions, for any transition front u of (4.66) connecting 0 and 1, there holds*

$$\forall\, \tau \ge 0, \quad \sup_{(t,s)\in\mathbb{R}^2,\, |t-s|\le\tau} |X(t) - X(s)| < +\infty.$$

Proof. In the proof, $u : \mathbb{R}^2 \to [0,1]$ denotes a transition front connecting 0 and 1 for (4.66). It follows first from (1.4) and the continuity of u that, for every $t \in \mathbb{R}$, the quantity

$$x(t) = \min\left\{ x \in \mathbb{R};\ u(t,x) = \frac{1}{2} \right\}$$

is a well defined real number such that $u(t,x(t)) = 1/2$. From the uniformity of the limits in (1.4), there is a constant $C_1 \ge 0$ such that

$$\left| X(t) - x(t) \right| \le C_1 \quad \text{for all } t \in \mathbb{R}. \qquad (4.67)$$

On the other hand, since $u : \mathbb{R}^2 \to [0,1]$ is bounded in $\mathbb{R}^2$ and $f \in L^\infty(\mathbb{R}^2 \times [0,1])$, standard parabolic estimates imply that u is uniformly continuous in $\mathbb{R}^2$. Hence, there is $\eta > 0$ such that

$$\frac{1}{4} \le u(t+t', x(t)) \le \frac{3}{4} \quad \text{for all } t \in \mathbb{R} \text{ and } t' \in [0,\eta].$$

By using again (1.4), one gets the existence of a constant $C_2 \ge 0$ such that

$$\left| X(t+t') - x(t) \right| \le C_2 \quad \text{for all } t \in \mathbb{R} \text{ and } t' \in [0,\eta].$$

Hence,
$$\left| X(t+t') - X(t) \right| \le C_1 + C_2 \quad \text{for all } t \in \mathbb{R} \text{ and } t' \in [0,\eta]. \qquad (4.68)$$

Therefore, if τ is a nonnegative real number and $k \in \mathbb{N}$ is such that $\tau \le k\eta$, one infers that

$$|X(t) - X(s)| \le k(C_1 + C_2)$$

for all $(t,s) \in \mathbb{R}^2$ such that $|t - s| \le \tau$. The proof of Proposition 4.2 is thereby complete. $\square$

References

[1] S.B. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine. Angew. Math. **390** (1988), 79-96.

[2] D.G. Aronson, H.F. Weinberger, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math. **30** (1978), 33-76.

[3] M. Bages, P. Martinez, J.-M. Roquejoffre, *How travelling waves attract the solutions of KPP-type equations*, Trans. Amer. Math. Soc. **364** (2012), 5415-5468.

[4] H. Berestycki, F. Hamel, *Front propagation in periodic excitable media*, Comm. Pure Appl. Math. **55** (2002), 949-1032.

[5] H. Berestycki, F. Hamel, *Generalized travelling waves for reaction-diffusion equations*, In: *Perspectives in Nonlinear Partial Differential Equations. In honor of H. Brezis*, Amer. Math. Soc., Contemp. Math. **446**, 2007, 101-123.

[6] H. Berestycki, F. Hamel, *Generalized transition waves and their properties*, Comm. Pure Appl. Math. **65** (2012), 592-648.

[7] M. Bramson, *Convergence of Solutions of the Kolmogorov Equation to Travelling Waves*, Memoirs Amer. Math. Soc. **44**, 1983.

[8] Y. Du, H. Matano, *Convergence and sharp thresholds for propagation in nonlinear diffusion problems*, J. Europ. Math. Soc. **12** (2010), 279-312.

[9] A. Ducrot, T. Giletti, H. Matano, *Existence and convergence to a propagating terrace in one-dimensional reaction-diffusion equations*, Trans. Amer. Math. Soc. **366** (2014), 5541-5566.

[10] U. Ebert, W. van Saarloos, *Front propagation into unstable states: universal algebraic convergence towards uniformly translating pulled fronts*, Physica D **146** (2000), 1-99.

[11] P.C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomathematics **28**, Springer Verlag, 1979.

[12] P.C. Fife, J.B. McLeod, *The approach of solutions of non-linear diffusion equations to traveling front solutions*, Arch. Ration. Mech. Anal. **65** (1977), 335-361.

[13] R.A. Fisher, *The advance of advantageous genes*, Ann. Eugenics **7** (1937), 335-369.

[14] J. Garnier, T. Giletti, G. Nadin, *Maximal and minimal spreading speeds for reaction-diffusion equations in nonperiodic slowly varying media*, J. Dyn. Diff. Eq. **24** (2012), 521-538.

[15] T. Giletti, *Convergence to pulsating traveling waves with minimal speed in some KPP heterogeneous problems*, Calc. Var. Part. Diff. Equations, forthcoming.

[16] F. Hamel, G. Nadin, *Spreading properties and complex dynamics for monostable reaction-diffusion equations*, Comm. Part. Diff. Eq. **37** (2012), 511-537.

[17] F. Hamel, N. Nadirashvili, *Travelling waves and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$*, Arch. Ration. Mech. Anal. **157** (2001), 91-163.

[18] F. Hamel, J. Nolen, J.-M. Roquejoffre, L. Ryzhik, *A short proof of the logarithmic Bramson correction in Fisher-KPP equations*, Netw. Heterog. Media **8** (2013), 275-289.

[19] F. Hamel, L. Roques, *Fast propagation for KPP equations with slowly decaying initial conditions*, J. Diff. Equations **249** (2010), 1726-1745.

[20] F. Hamel, L. Roques, *Uniqueness and stability properties of monostable pulsating fronts*, J. Europ. Math. Soc. **13** (2011), 345-390.

[21] F. Hamel, L. Rossi, *Admissible speeds of transition fronts for non-autonomous monostable equations*, preprint.

[22] J. Huang, W. Shen, *Speeds of spread and propagation for KPP models in time almost and space periodic media*, SIAM J. Appl. Dyn. Syst. **8** (2009), 790-821.

[23] Y. Kametaka, *On the nonlinear diffusion equation of Kolmogorov-Petrovski-Piskunov type*, Osaka J. Math. **13** (1976), 11-66.

[24] A.N. Kolmogorov, I.G. Petrovsky, and N.S. Piskunov, *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bull. Univ. État Moscou, Sér. Internat. A **1** (1937), 1-26.

[25] K.-S. Lau, *On the nonlinear diffusion equation of Kolmogorov, Petrovsky, and Piscounov*, J. Diff. Equations **59** (1985), 44-70.

[26] X. Liang, X.Q. Zhao, *Asymptotic speeds of spread and traveling waves for monotone semiflows with applications*, Comm. Pure Appl. Math. **60** (2007), 1-40.

[27] X. Liang, X.Q. Zhao, *Spreading speeds and traveling waves for abstract monostable evolution systems*, J. Func. Anal. **259** (2010), 857-903.

[28] H. Matano, *Convergence of solutions of one-dimensional semilinear parabolic equations*, J. Math. Kyoto Univ. **18** (1978), 221-227.

[29] H. Matano, Talks at various conferences.

[30] J.D. Murray, *Mathematical Biology*, Springer-Verlag, 2003.

[31] G. Nadin, *Travelling fronts in space-time periodic media*, J. Math. Pures Appl. **92** (2009), 232-262.

[32] G. Nadin, *Critical travelling waves for general heterogeneous one-dimensional reaction-diffusion equations*, Ann. Inst. H. Poincaré, Analyse Non Lin., forthcoming.

[33] G. Nadin, L. Rossi, *Propagation phenomena for time heterogeneous KPP reaction-diffusion equations*, J. Math. Pures Appl. **98** (2012), 633-653.

[34] J. Nolen, M. Rudd, J. Xin, *Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds*, Dyn. Part. Diff. Equations **2** (2005), 1-24.

[35] D.H. Sattinger, *Stability of waves of nonlinear parabolic systems*, Adv. Math. **22** (1976), 312-355.

[36] W. Shen, *Dynamical systems and traveling waves in almost periodic structures*, J. Diff. Equations **169** (2001), 493-548.

[37] W. Shen, *Traveling waves in diffusive random media*, J. Dyn. Diff. Equations **16** (2004), 1011-1060.

[38] W. Shen, *Variational principle for spreading speeds and generalized propagating speeds in time almost periodic and space periodic KPP models*, Trans. Amer. Math. Soc. **362** (2010), 5125-5168.

[39] W. Shen, *Existence, uniqueness, and stability of generalized traveling waves in time dependent monostable equations*, J. Dyn. Diff. Equations **23** (2011), 1-44.

[40] W. Shen, *Existence of generalized traveling waves in time recurrent and space periodic monostable equations*, J. Appl. Anal. Comput. **1** (2011), 69-93.

[41] N. Shigesada, K. Kawasaki, *Biological Invasions: Theory and Practice*, Oxford Series in Ecology and Evolution, Oxford: Oxford UP, 1997.

[42] N. Shigesada, K. Kawasaki, E. Teramoto, *Traveling periodic waves in heterogeneous environments*, Theor. Pop. Bio. **30** (1986), 143-160.

[43] K. Uchiyama, *The behavior of solutions of some semilinear diffusion equation for large time*, J. Math. Kyoto Univ. **18** (1978), 453-508.

[44] H.F. Weinberger, *On spreading speeds and traveling waves for growth and migration in periodic habitat*, J. Math. Biol. **45** (2002), 511-548.

[45] E. Yanagida, *Irregular behavior of solutions for Fisher's equation*, J. Dyn. Diff. Equations **19** (2007), 895-914.

[46] A. Zlatoš, *Transition fronts in inhomogeneous Fisher-KPP reaction-diffusion equations*, J. Math. Pures Appl. **98** (2012), 89-102.