

A unified formalism for the core mass-luminosity relations of shell-burning stars

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Abstract. The luminosity evolution of stars with highly condensed cores surrounded by nuclear-burning shell(s) is analytically investigated with the aid of homology relations. With respect to earlier works using a similar approach (e.g. Refsdal & Weigert 1970; Kippenhahn 1981), the major improvement is that we derive all the basic dependences (i.e. on core mass, core radius, and chemical composition) in a completely generalised fashion, then accounting for a large range of possible physical properties characterising the burning shell(s). Parameterised formulas for the luminosity are given as a function of the (i) relative contribution of the gas to the total pressure (gas plus radiation), (ii) opacity source, and (iii) dominant nuclear reaction rates.

In this way, the same formalism can be applied to shell-burning stars of various metallicities and in different evolutionary phases. In particular, we present some applications concerning the luminosity evolution of RGB and AGB stars with different chemical compositions, including the case of initial zero metallicity. It turns out that homology predictions provide a good approximation to the results of stellar model calculations.

Therefore, the proposed formalism is useful to understand the possible differences in the luminosity evolution of shell-burning stars within a unified interpretative framework, and can be as well adopted to improve the analytical description of stellar properties in synthetic models.

Key words: stars: evolution – stars: fundamental parameters – stars: interiors – stars: late-type – stars: AGB and post-AGB – stars: neutron

1. Motivation of the work

It is well known that the quiescent luminosity of a shell-burning star with degenerate core is essentially determined by its core mass, without any dependence on the envelope mass, as extensively described in several works carried out in the past (e.g. Eggleton 1967; Paczyński 1970; Tuchman et al. 1983). This property is usually referred to as the core mass-luminosity ($M_c - L$) relation, although other structural parameters may affect the luminosity evolution, as indicated in the following.

Stellar evolutionary calculations actually confirm that $M_c - L$ relations are generally followed, for instance, by (i) low-mass stars during their ascent on the Red Giant Branch (RGB) up to the He-flash (e.g. Boothroyd & Sackmann 1988); (ii) low- and intermediate-mass stars during the quiescent inter-pulse periods of their Thermally Pulsing Asymptotic Giant Branch (TP-AGB) evolution (e.g. Iben & Truran 1978, Wood & Zarro 1981, Forestini & Charbonnel 1997), provided that hot-bottom burning is not operating (see Blöcker & Schönberner 1991); (iii) Planetary Nebula nuclei as long as the H-burning shell is active (e.g. Vassiliadis & Wood 1994); and (iv) Nova systems during their stationary nuclear burning phases (e.g. Tuchman & Truran 1998).

These relations are known to be quite different for RGB and TP-AGB stars, and also vary among stars in the same evolutionary stage but with different envelope chemical composition. In view of interpreting these differences as the effect of different physical conditions, it would be advisable to derive quantitatively the dependence of the quiescent luminosity of a giant (RGB and TP-AGB) star on basic quantities, namely: core mass M_c , core radius R_c , and chemical composition.

To this aim, we adopt the same formalism as fully described in Refsdal & Weigert (1970), which is based on the use of homology relations applied to the case of stars with high-density cores surrounded by nuclear burning shell(s). The authors demonstrated, for instance, that for low values of the core mass (i.e. $M_c \lesssim 0.45 M_\odot$) and negligible radiation pressure (i.e. $\beta \sim 1$) the luminosity is expected to depend on the envelope mean molecular weight as $L \propto \mu^{7-8}$. Such theoretical prediction is found to describe extremely well the significant composition dependence of the luminosity of low-mass stars evolving along the RGB, as shown by calculations of full stellar models (e.g. Boothroyd & Sackmann 1988; see also Sect. 3.1). However, the results of Refsdal & Weigert (1970) cannot be straightforwardly extended to the case of stars evolving through the AGB phase, since the condition $\beta \sim 1$ is generally not fulfilled, due to the increasing importance of radiation pressure.

Therefore, our target is to generalize the formalism developed by Refsdal & Weigert (1970) for any value of β in the range $(0, 1)$. To do this, we follow the indications suggested by Kippenhahn (1981), who first pointed out that the quite different $M_c - L$ relations for RGB and AGB stars are indeed the

expression of a unique relation modulated by the variation of β . However, as Kippenhahn (1981) explicitly derived the dependence of the luminosity only on M_c and R_c , in this work we will extend the same kind of analysis to the composition dependence as a function of β . In this way, we can predict to which extent the chemical properties of AGB stars (with $\beta < 1$) may affect their $M_c - L$ relation (see Sects. 3.2, and 3.3).

Moreover, given the generality of the method, we can also investigate the sensitiveness of the $M_c - L$ relation to the dominant nuclear energy source operating the H-burning shell, i.e. the degree of temperature dependence of the relevant reaction rates. This will turn out to be significant in view of interpreting the peculiar luminosity evolution of RGB stars with initial zero-metallicity, where the p-p chain (not the CNO cycle as is usually the case) provides most of the stellar energy (see Sect. 3.1).

Finally, we invite the reader to refer to the works by Refsdal & Weigert (1970) and Kippenhahn (1981) for a full understanding of the analytical derivation, since many details will be omitted here to avoid redundant repetitions and lengthy demonstrations. However, the basic steps will be indicated in the next sections.

2. The analytical method based on homology relations

We will describe in the following the analytical procedure adopted to derive the dependence of the luminosity on M_c , R_c , and chemical composition.

Concerning this latter it can be seen, from the basic stellar structure equations, that the effect of the composition enters the $M_c - L$ relation via three parameters (Refsdal & Weigert 1970; Tuchman et al. 1983):

1. the mean molecular weight μ ;
2. the product of the abundances of the interacting nuclei involved in the nuclear burning of hydrogen, $\epsilon_0 = X^2$ (for the p-p chain; X is the hydrogen abundance in mass fraction) or $\epsilon_0 = XZ_{\text{CNO}}$ (for the CNO-cycle; Z_{CNO} is the total abundance of CNO elements);
3. a factor κ_0 expressing the composition dependence of the opacity κ .

The mean molecular weight μ explicitly appears only in the equation of state:

$$P = P_G + P_R \propto \frac{\rho T}{\beta \mu} \quad (1)$$

assuming that the total pressure, P , is the sum of the contributions of the gas (supposed perfect), P_G , and of the radiation, P_R . The quantity β is defined as the ratio P_G/P . For a fully ionized gas $\mu = 4/(5X + 3 - Z)$.

The parameter ϵ_0 is related to the rate of energy generation by nuclear burning ϵ , which can be conveniently approximated as:

$$\epsilon = \epsilon_0 \rho^{n-1} T^\nu \quad (2)$$

with n and ν being determined by the rates of the nuclear reactions under consideration. Typical values are: ($n = 2$, $\nu \sim 4$)

for the p-p chain, ($n = 2$, $\nu \sim 14 \div 20$) for the CNO-cycle, ($n = 3$, $\nu \sim 22$) for the triple α -reaction.

The parameter κ_0 is related to the opacity, which can be expressed:

$$\kappa = \kappa_0 P^a T^b \quad (3)$$

with the exponents a and b depending on the dominating opacity source. Note that in the case κ is mostly due to the Thomson electron scattering ($\kappa = 0.2(1 + X)$ without any dependence on pressure and temperature), we get $\kappa_0 = 1 + X$, and $a = b = 0$.

Then, adopting the homology relations presented in Refsdal & Weigert (1970) we can express the luminosity as a power-law relation:

$$L \propto M_c^{\delta_1} R_c^{\delta_2} \mu^{\delta_3} \kappa_0^{\delta_4} \epsilon_0^{\delta_5} \quad (4)$$

which, under the assumptions of a fully ionized gas, with electron scattering opacity, can be written:

$$L \propto M_c^{\delta_1} R_c^{\delta_2} \left(\frac{4}{5X + 3 - Z} \right)^{\delta_3} (1 + X)^{\delta_4} (XZ_{\text{CNO}})^{\delta_5} \quad (5)$$

for dominating CNO-cycle, or

$$L \propto M_c^{\delta_1} R_c^{\delta_2} \left(\frac{4}{5X + 3 - Z} \right)^{\delta_3} (1 + X)^{\delta_4} (X^2)^{\delta_5} \quad (6)$$

for dominating p-p chain.

The exponents (δ_i , $i = 1, 5$) are the unknown quantities to be determined, as a function of β , of the opacity parameters (a , b), and of the energy parameters (n , ν).

The composition dependence is the first being considered here, for its particular relevance to the applications discussed in the second part of this work (Sect. 3). For the sake of completeness, the results for the dependences on M_c and R_c are then briefly presented (Sect. 2.3).

2.1. The dependence on μ

The dependence on μ can be derived treating it as an independent parameter to express the homology relations:

$$\begin{aligned} \rho(r/R_c) &\propto \mu^{\alpha_3} \\ T(r/R_c) &\propto \mu^{\beta_3} \\ P(r/R_c) &\propto \mu^{\gamma_3} \\ L(r/R_c) &\propto \mu^{\delta_3}. \end{aligned} \quad (7)$$

Such expressions imply the assumption that $\mu/\mu' = \text{const.}$ at each corresponding point (i.e. $r/R_c = r'/R'_c$) of models with different core radii (i.e. R_c and R'_c). Here r denotes the radial coordinate of any point inside the region extending from the bottom of the burning shell, $r = R_c$, up to a point, $r = r_0$, where the variables ρ , P , and T have already significantly decreased, and where $L_r = L$. Moreover, it is assumed that the other quantities (i.e. ρ , T , P , L , ϵ_0 , and κ_0) do not vary among models at corresponding points.

The exponents α_3 , β_3 , γ_3 , and δ_3 are the unknown parameters to be singled out. To this aim, we need to integrate the

equations of hydrostatic equilibrium, radiative transport, and energy generation, basing on the proportionality relations given in Eqs. (7), and using the expressions of Eqs. (2) and (3) for the rate of nuclear energy generation and opacity, respectively. We then derive:

$$\begin{aligned} P(r/R_c) &\propto \mu^{\alpha_3} \\ T^{4-b}(r/R_c) &\propto \mu^{\alpha_3+a\gamma_3+\delta_3} \\ L(r/R_c) &\propto \mu^{n\alpha_3+\nu\beta_3}. \end{aligned} \quad (8)$$

Since there are four unknown quantities, one more relation is needed in order to close the system of Eqs. (8). This is given by the equation of state (Eq. (1)), which should be expressed in a more suitable form fully explicating its dependence on β . For this purpose, we must consider that β is a function of density, temperature, and molecular weight. We can then derive the three dependences as follows. As indicated by Kippenhahn (1981) the logarithmic derivatives of β with respect to the density and temperature are:

$$\left(\frac{\partial \ln \beta}{\partial \ln \rho}\right)_{T,\mu} = 1 - \beta, \quad \left(\frac{\partial \ln \beta}{\partial \ln T}\right)_{\rho,\mu} = -3(1 - \beta). \quad (9)$$

Similarly, we can make a step ahead and derive also the logarithmic derivative of β with respect to the mean molecular weight:

$$\left(\frac{\partial \ln \beta}{\partial \ln \mu}\right)_{T,\rho} = \beta - 1. \quad (10)$$

Hence, the equation of state can be re-written:

$$P \propto \mu^{-\beta} \rho^\beta T^{4-3\beta}. \quad (11)$$

Note that in the limiting cases $\beta = 1$ and $\beta = 0$, we obtain the right thermodynamical dependence of the total pressure when due to the sole contribution of gas and radiation, respectively. It should also be remarked that Eq. (11) is expected to apply for all r/R_c . However, since pressure changes by several order of magnitudes throughout the shell and radiative buffer above, it is far from obvious that Eq. (11) does apply throughout the region. In fact it does, for two reasons: (1) in RGB stars because $\beta \sim 1$, and (2) in AGB stars because β is constant throughout the shell and sub-convective layers (“radiative zero” condition).

At this point, all the necessary information is available. Comparing the exponents of Eqs. (7), (8), (11), we can solve the system of 4 algebraic equations in the unknowns α_3 , β_3 , γ_3 , and δ_3 , yielding:

$$\alpha_3 = -\frac{\beta(\nu + b - 4)}{(\nu + b - 4)(1 - \beta) + (1 + a + n)(4 - 3\beta)} \quad (12)$$

$$\beta_3 = \frac{\beta(1 + a + n)}{(\nu + b - 4)(1 - \beta) + (1 + a + n)(4 - 3\beta)} \quad (13)$$

$$\gamma_3 = \alpha_3 \quad (14)$$

$$\delta_3 = \frac{\beta[\nu(1 + a + n) - n(\nu + b - 4)]}{(\nu + b - 4)(1 - \beta) + (1 + a + n)(4 - 3\beta)}. \quad (15)$$

It is worth noticing that for $\beta = 1$ we get exactly the same results as given by Refsdal & Weigert (1970) (see their Eqs. (27) and Sect. II d)), as expected.

More generally, Eq. (15) allows us to investigate the μ -dependence of the luminosity in the whole range $0 \leq \beta \leq 1$, so that it is possible to set quantitative predictions both for RGB stars (with $\beta \sim 1$) and AGB stars (with $\beta < 1$). The results are displayed in Fig. 1, assuming $a = b = 0$, and characteristic values of the parameters (n , ν) corresponding to relevant kinds of energy sources, as already indicated in Sect. 2.

From the inspection of the bottom-left panel of Fig. 1 the following features should be noticed:

- In the case $\beta = 1$ and dominant CNO cycle, the exponent $\delta_3 \sim 7$, which well reproduces the μ -dependence of the luminosity for RGB stars as indicated by calculations of stellar models (Boothroyd & Sackmann 1988);
- In the case $\beta = 1$ and dominant p-p chain, we get $\delta_3 \sim 4$, i.e. a weaker μ -dependence;
- In any case, for $\beta \sim 0$ the μ -dependence vanishes as expected when the gas pressure goes to zero;
- In the case $\beta \sim 0.5 - 0.8$, which are typical values found in low-mass AGB stars, and dominating CNO-cycle, the μ -dependence is not at all negligible. Specifically, it results $\delta_3 \sim 1 \div 3$, a range which is consistent with the finding by Boothroyd & Sackmann (1988) (i.e. $L \propto \mu^3$) in their analysis of low-mass (hence with larger β) AGB stars.

2.2. The dependence on ϵ_0 and κ_0

In an analogous way as described in the previous section, we aim now at deriving the dependence of the luminosity on both ϵ_0 and κ_0 , already defined in Sect. 2. We first write the homology relations:

$$\begin{aligned} \rho(r/R_c) &\propto \kappa_0^{\alpha_4} \epsilon_0^{\alpha_5} \\ T(r/R_c) &\propto \kappa_0^{\beta_4} \epsilon_0^{\beta_5} \\ P(r/R_c) &\propto \kappa_0^{\gamma_4} \epsilon_0^{\gamma_5} \\ L(r/R_c) &\propto \kappa_0^{\delta_4} \epsilon_0^{\delta_5} \end{aligned} \quad (16)$$

assuming that the functions $\kappa_0(r/R_c)$ and $\epsilon_0(r/R_c)$ scale up by a constant factor at corresponding points of models with different core radii.

Then, integrating the basic stellar equations, expressing the total pressure P as in Eq. (11), comparing the exponents relative to the four variables ρ , T , P and L , and solving the system of 8 algebraic equations, we finally derive the 8 parameters:

$$\alpha_4 = \frac{(4 - 3\beta)}{(4 - b - \nu)(1 - \beta) - (1 + a + n)(4 - 3\beta)} \quad (17)$$

$$\alpha_5 = \gamma_4 = \gamma_5 = \alpha_4 \quad (18)$$

$$\beta_4 = \beta_5 = \frac{(1 - \beta)}{(4 - b - \nu)(1 - \beta) - (1 + a + n)(4 - 3\beta)} \quad (19)$$

$$\delta_4 = \frac{n(4 - 3\beta) + \nu(1 - \beta)}{(4 - b - \nu)(1 - \beta) - (1 + a + n)(4 - 3\beta)} \quad (20)$$

$$\delta_5 = 1 + \frac{n(4 - 3\beta) + \nu(1 - \beta)}{(4 - b - \nu)(1 - \beta) - (1 + a + n)(4 - 3\beta)}. \quad (21)$$

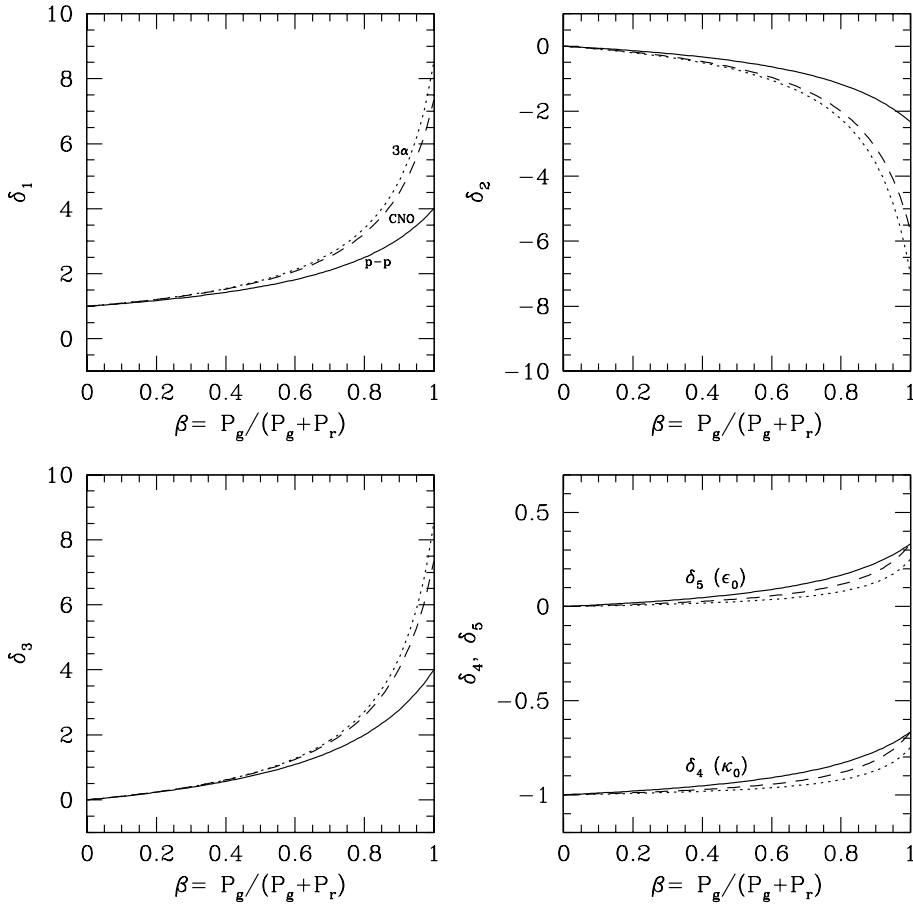


Fig. 1. Expected behaviour, as a function of β , of the exponents in Eq. (4): δ_1 (related to the core mass), δ_2 (related to the core radius), δ_3 (related to the mean molecular weight), δ_4 (related to the opacity), and δ_5 (related to the nuclear reaction rates). We adopt $a = b = 0$ for all cases, with $(n = 2, \nu = 4)$ for the p-p chain (solid line), $(n = 2, \nu = 14)$ for the CNO-cycle (dashed line), and $(n = 3, \nu = 22)$ for the triple α -reaction (dotted line).

Again, for $\beta = 1$ the above equations yield the same results as in Refsdal & Weigert (1970), but now accounting for any possible value of β .

Concerning the exponents related to the luminosity, δ_4 and δ_5 , their predicted trends are illustrated in the bottom-right panel of Fig. 1. We can notice that:

- their effect on the luminosity is usually much weaker than that produced by μ , except for values of β very close to zero;
- in the relevant range of β for AGB stars, the predicted values for δ_5 are in a rather good agreement with the results by Boothroyd & Sackmann (1988), who quoted $L \propto Z_{\text{cno}}^{0.04}$ basing on their evolutionary calculations of AGB models;
- in the case $\beta \sim 0$ and $a = b = 0$ we get $\delta_4 \sim -1$ and $\delta_5 \sim 0$;
- both δ_4 and δ_5 do not vary significantly adopting different values of the parameters a, b, ν , and n , within a reasonable range.

2.3. The dependences on M_c and R_c

In a similar fashion, we can also derive the dependences of the luminosity on M_c and R_c

$$L \propto M_c^{\delta_1} R_c^{\delta_2} \dots \quad (22)$$

For the sake of conciseness, we report here only the results for δ_1 , and δ_2 (see also Fig. 1, top-left and top-right panels, respectively):

$$\delta_1 = \frac{na(4 - 3\beta) - n(4 - b) - \nu(1 + a\beta)}{(4 - b - \nu)(1 - \beta) - (1 + a + n)(4 - 3\beta)}, \quad (23)$$

and

$$\delta_2 = \frac{\beta(3n + 3b + \nu - 3) - (3 + n)[b + a(4 - 3\beta)]}{(4 - b - \nu)(1 - \beta) - (1 + a + n)(4 - 3\beta)} \quad (24)$$

omitting the formulas for $(\alpha_i, \beta_i, \gamma_i; i = 1, 2)$. The above equations for δ_1 and δ_2 recover exactly those presented by Kippenhahn (1981) setting $a = b = 0$, and those derived by Refsdal & Weigert (1970) with $\beta = 1$.

In brief, we can outline the following:

- δ_1 and δ_2 show opposite trends with β . However, it should be recalled that, though M_c and R_c are formally treated as independent parameters, an $M_c - R_c$ relation is expected to exist for highly condensed cores which may be assimilated to white dwarf structures (Chandrasekhar 1939; Refsdal & Weigert 1970; Tuchman et al. 1983). Considering the homology dependences of the luminosity only on M_c and R_c , we get the differential equation (see Eq. 15 in Kippenhahn 1981):

$$\frac{d \ln L}{d \ln M_c} = \delta_1 + \delta_2 \times \frac{d \ln R_c}{d \ln M_c} \quad (25)$$

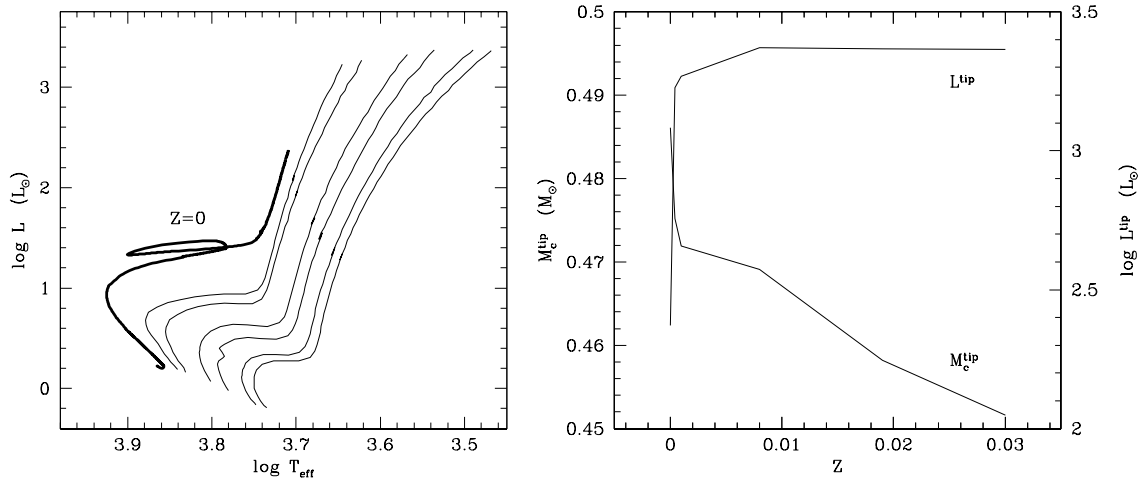


Fig. 2. Left-hand side panel: Evolutionary tracks up to the He-flash of $1 M_{\odot}$ models with different initial metallicities (i.e. $Z = 0.030, 0.019, 0.008, 0.004, 0.001, 0.0004$ from right to left; taken from Girardi et al. 2000) (thin lines), and zero metallicity (thick line; Marigo et al. 2000, in preparation). Right-hand side panel: luminosity and core mass on the RGB tip of $1 M_{\odot}$ models as a function of metallicity.

where $d \ln R_c / d \ln M_c$ is the slope of the $M_c - R_c$ relation. In other words, the quantity $\delta_2 \times d \ln R_c / d \ln M_c$ represents the dependence on the core mass via $R_c(M_c)$. This can be estimated from Eq. (25), e.g. taking the right-hand side from the results of stellar models, and evaluating δ_1 and δ_2 from the homology relations.

- As for μ , the dependence on M_c grows at increasing β . For $\beta \sim 1$, we get typical relations $L \propto M_c^7$ for dominant CNO-cycle, and $L \propto M_c^4$ for dominant p-p chain. Again, the former prediction very well agrees with the results of evolutionary stellar calculations of RGB stars (Boothroyd & Sackmann 1988). The effect of a weaker M_c -dependence predicted in the latter case will be investigated in Sect. 3.1. In both cases ($\delta_1 \sim 7$ and $\delta_2 \sim 4$), the luminosity should be just moderately affected by changes in the core radius, i.e. the product $|\delta_2 \times d \ln R_c / d \ln M_c|$ (calculated with Eq. (25)) is typically less than unity for $\beta \sim 1$.
- At decreasing β the dependence on M_c quickly approaches a linear one ($\delta_1 = 1$ for $\beta = 0$), whereas the dependence on R_c tends to vanish ($\delta_2 = 0$ for $\beta = 0$). This result is consistent with the typical flatter slopes ($\sim 1 - 2$) of the fitting $M_c - L$ relations derived from evolutionary calculations of AGB stars.

3. Some applications

3.1. The $M_c - L$ relation on the RGB: the zero-metallicity case

Evolutionary calculations of low-mass models (see Figs. 2 and 3) indicate that

1. towards lower metallicities and for given stellar mass the luminosity at the tip of the RGB decreases, whereas the core mass increases;
2. at decreasing metallicity the luminosity on the RGB is lower for given core mass; and

3. the slope of the $M_c - L$ relation on the RGB is flatter for $Z = 0$ models than for $Z \neq 0$ models.

The first point clearly confirms that the luminosity is not solely a function of the core mass. To better interpret the above trends we can make use of the analytical relations obtained in the previous sections.

Let us consider the $M_c - L$ relation presented by Boothroyd & Sackmann (1988)

$$L = (11.6 M_c \mu)^7 Z_{\text{cno}}^{1/12} \quad (26)$$

which is a fitting formula of evolutionary calculations for RGB stars with different metallicities (masses and luminosities are in solar units).

To check the reliability of homology predictions, we first compute the exponents δ_1 and δ_3 with the aid of Eqs. (23) and (15), and compare them with those given in Eq. (26). To do this, β , the opacity parameters, and the energy parameters should be specified. We set $\beta = 1$ – which is proved to be a good approximation for all RGB models here considered –, and $a = b = 0$ – for the sake of simplicity. The adoption of more proper values for a and b would result in very small corrections, as pointed out by Refsdal & Weigert (1970).

The choice of the energy parameters requires some comments. It turns out that the bulk of energy produced in the H-burning shell is provided by the CNO-cycle for both $1 M_{\odot}$ RGB models with $Z = 0.019$ and $Z = 0.0004$, whereas energy production is dominated by the p-p chain in the $1 M_{\odot}$ RGB model with $Z = 0$. It follows that typical values ($n = 2, \nu = 14 - 16$) and ($n = 2, \nu = 4 - 6$) should be adopted in the two cases, yielding $\delta_1 \sim \delta_3 \sim 7 - 8$ and $\delta_1 \sim \delta_3 \sim 4.0 - 4.7$ for RGB models in which the CNO cycle and p-p chain dominates, respectively.

The predictions for the former case (CNO dominated) are in excellent agreement with Eq. (26), thus reproducing the results of complete stellar calculations of RGB stars with dominating CNO cycle. Moreover, it is worth remarking that points (1) and (2) – mentioned at the beginning of this section – are explained

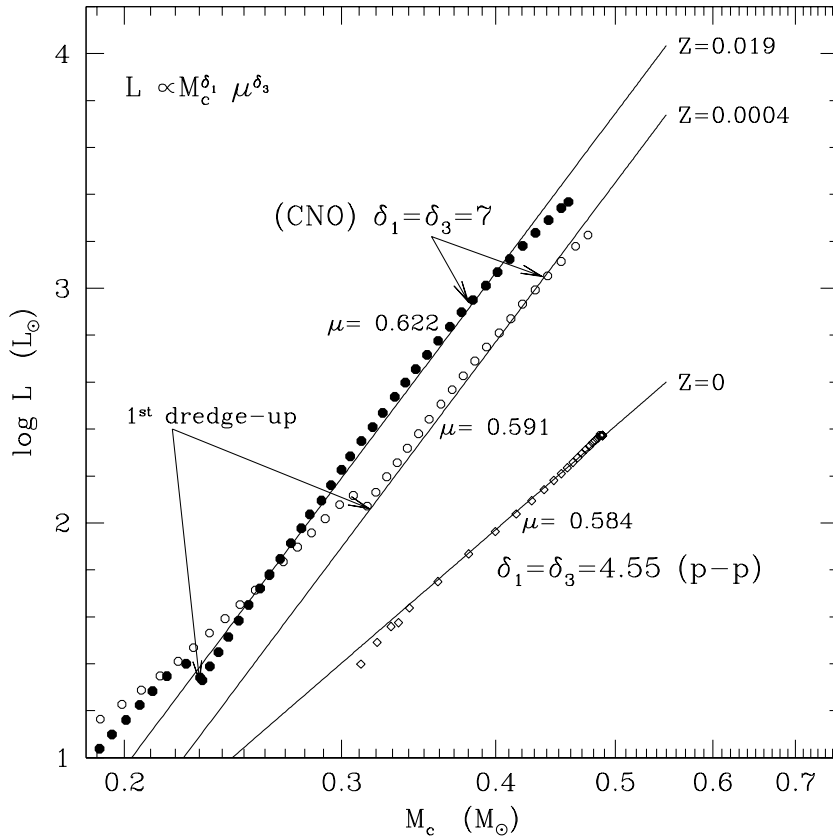


Fig. 3. $M_c - L$ relation on the RGB. Symbols correspond to selected models for $1 M_\odot$ stars with different metallicities, and mean molecular weights after the first dredge-up as indicated. Predictions from homology relations are shown as solid lines. See text for further explanation.

as the effect of differences in the mean molecular weight, i.e. for given M_c and lower μ , L is lower.

The predictions for the latter case (p-p dominated) fully explain point (3). In fact, the weaker temperature dependence of p-p reactions results in a flatter slope of the $\log M_c - \log L$ relation followed by the $1 M_\odot$ model with initial zero metallicity. A good fit is obtained adopting

$$L = (11.6 M_c \mu)^{4.55} \quad (27)$$

with L and M_c in solar units. This power-law relation has the same base as in Eq. (26), but a different exponent, derived under the assumption of dominating p-p reactions with energy parameters ($n = 2$, $\nu \sim 5.6$). Taking for μ the value after the first dredge-up as indicated by the calculations by Girardi et al. (2000), and letting M_c vary over the relevant range, we finally obtain the relation shown in Fig. 3, which remarkably well matches the evolutionary results at initial zero metallicity.

To summarise the conclusions, we can notice that, for $\beta \sim 1$, the *mean molecular weight* determines the luminosity level of the $M_c - L$ relation (i.e. the *intercept* in logarithmic plot) for RGB models with similar energy properties, whereas the kind of *nuclear energy source* affects the rate of the geometric increment of the luminosity with the core mass (i.e. the *slope* of the $\log M_c - \log L$ relation). The latter point explains the fact that the $\log M_c - \log L$ relations of the $1 M_\odot$ models with $Z = 0.019$ and $Z = 0.0004$ are almost parallel, whereas the relation for $Z = 0$ runs flatter.

3.2. The composition dependence of $M_c - L$ relation on the TP-AGB phase

Let us now apply the relations derived in Sects. 2.1 and 2.2 to TP-AGB stars, in order to estimate the degree of dependence of the quiescent luminosity on envelope composition.

First of all, we need to know how β varies during the TP-AGB evolution of a stellar model, so that we can directly obtain δ_3 from Eq. (15). To this aim, we make use of the definition of radiative gradient:

$$\nabla_r = \frac{3}{16\pi acG} \kappa \frac{L_r}{M_r} \frac{P_r}{T_r^4} \quad (28)$$

with usual meaning of the quantities. In the context of the present study, the radial coordinate r refers to the top of the H-burning shell or, equivalently, to some point near the bottom of the radiative inert region (extending up to the base of the convective envelope). Here L_r/M_r is nearly constant and equal to L/M_c , the opacity is dominated by Thomson electron scattering, i.e. $\kappa = 0.2(X + 1)$, and the radiative gradient, ∇_r , approaches the “radiative zero” value of 0.25 (see, for instance, Scalo et al. 1975). Under these conditions, and reminding that $P_r/T_r^4 \propto (1 - \beta)^{-1}$, we can express β as a function of the core mass M_c , surface luminosity L and hydrogen abundance X in the envelope:

$$\beta = 1 - 7.956 \times 10^{-6} (1 + X) \frac{L/L_\odot}{M_c/M_\odot}. \quad (29)$$

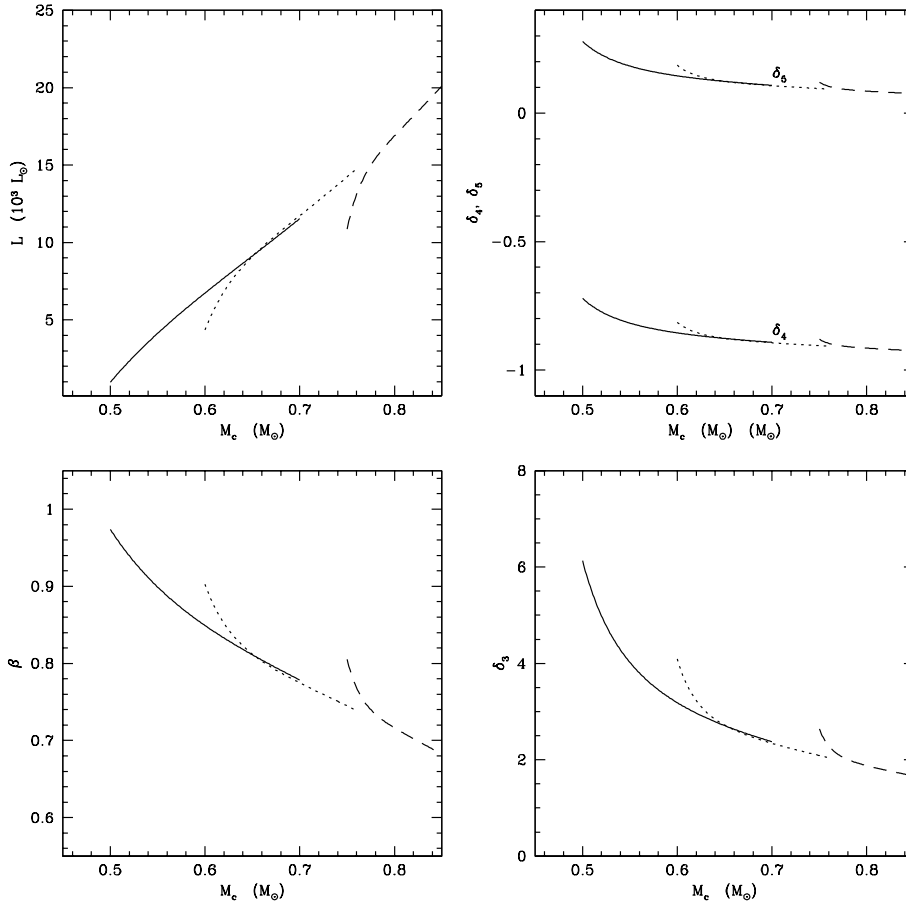


Fig. 4. Predicted evolution as a function of the core mass of the pre-flash maximum luminosity L , ratio of the gas pressure over total pressure β estimated with Eq. (29), and exponents δ_3 , δ_4 , and δ_5 , from the first thermal pulse of three TP-AGB models.

Fig. 4 illustrates the results of synthetic calculations for three TP-AGB models with original solar composition. The evolution of the quiescent luminosity (top-left panel) is computed according to the prescription presented by Wagenhuber & Groenewegen (1998). This gives an accurate fit to the results of extensive full evolutionary calculations of the TP-AGB phase (Wagenhuber 1996), expressing the quiescent luminosity as a function of the core mass, from the first thermal pulse.

Then, once β is evaluated with the aid of Eq. (29) for current values of L and M_c (top-left panel), the composition parameters δ_3 (Eq. 15), δ_4 (Eq. 20), and δ_5 (Eq. 21) can be calculated (right panels). In other words, we are able to predict the current composition dependence of the quiescent luminosity as the star evolves on the TP-AGB.

These simple synthetic calculations are meant to be indicative examples, showing the expected sensitiveness of the luminosity to possible changes in the surface chemical composition during the evolution caused, for instance, by convective dredge-up episodes. For the sake of simplicity, these events are assumed not to occur in the cases under consideration, to avoid the complication due to feedback effects. We will discuss this point below in this section.

An interesting point to be noticed in Fig. 4 is that the composition dependence is quite strong during the initial (and fainter) part of the TP-AGB phase (e.g. $\delta_3 \sim 5 - 6$ in the first stages of the model with the lowest core mass), then becoming weaker as

the full-amplitude regime is attained. This trend simply reflects the rate of increase of the radiation pressure (i.e. β decreases) during the evolution (bottom-left panel of Fig. 4). Therefore, we would expect that, if an efficient dredge-up occurs quite early during the TP-AGB evolution, the increase of the mean molecular weight could alter the asymptotic approach towards the full amplitude regime, as otherwise expected for unchanged chemical composition. As already suggested by Marigo et al. (1999), this prediction, in combination with additional effects, may concur to explain the recent results by Herwig et al. (1998), who find a steeper increase of the luminosity in stellar models with extremely efficient dredge-up (with $\lambda \sim 1$ and larger) already from the first thermal pulses.

Another point to be remarked is that any change in the envelope composition is expected to produce a certain feedback on the luminosity, in addition to the direct effect already discussed. In fact, a variation $\Delta\mu > 0$ produces $\Delta L > 0$, since δ_3 is always positive (we do not consider here the extreme case $\beta = 0$ for which $\delta_3 = 0$). At the same time, we get $\delta T > 0$ and $\delta P < 0$ (as $\beta_3 > 0$ and $\gamma_3 < 0$), both causing $\delta\beta < 0$ and $\Delta L < 0$. In other words, the effective increase of L due to an increment of μ is somewhat reduced with respect to that directly predicted by Eq. (15). Similar effects are produced by variations of the other two composition parameters, ϵ_0 , and κ_0 (see also Sect. II g) in Refsdal & Weigert 1970).

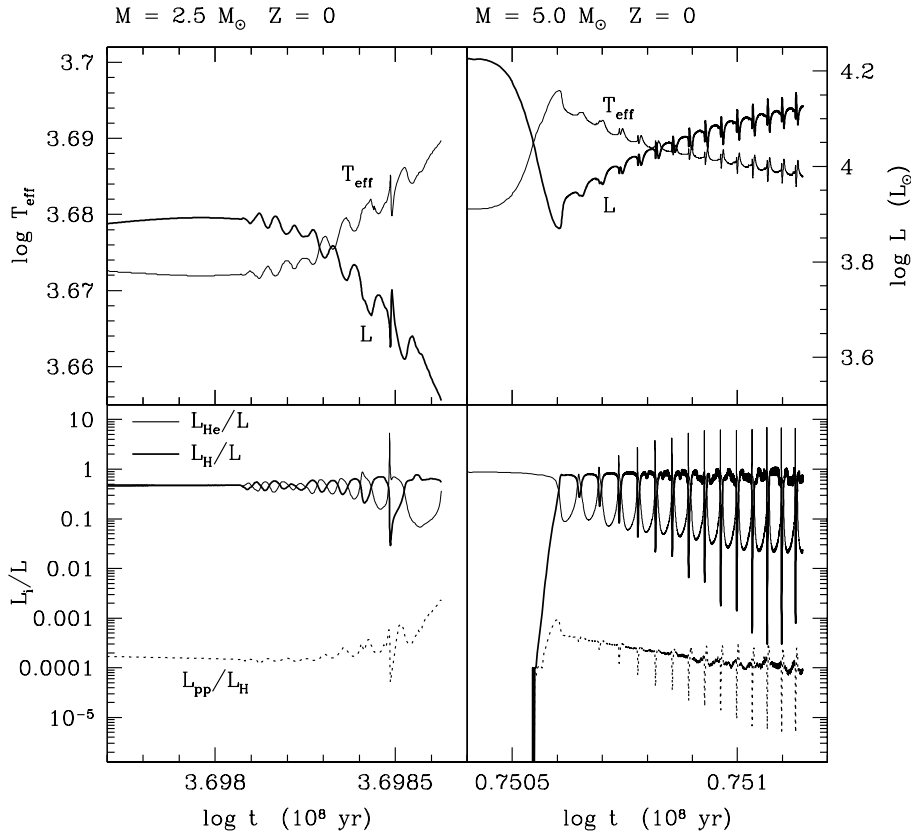


Fig. 5. Surface and energy properties of AGB models with initial zero metallicity and masses of $2.5 M_{\odot}$ (left-hand side panels), and $5.0 M_{\odot}$ (right-hand side panels), since the first appearance of the He-shell thermal instabilities. The fractional contributions of shell nuclear burnings to the surface luminosity (L_{H}/L and L_{He}/L) are displayed, together with that provided by p-p chain relative to the integrated energy generation from shell H-burning ($L_{\text{pp}}/L_{\text{H}}$).

In conclusion, as already suggested by Kippenhahn (1981) and confirmed by stellar evolutionary calculations (see, for instance Boothroyd & Sackmann 1988), we quantitatively demonstrate that the composition dependence of the luminosity of TP-AGB stars is weaker than for RGB stars (due to the increasing importance of the radiation pressure). However, non-negligible effects on the luminosity can be driven by significant variations of the mean molecular weight, either related to originally different chemical compositions, or caused by the third dredge-up. The occurrence of the latter process already since the first thermal pulses could significantly affect the luminosity evolution, in particular, of low-mass TP-AGB stars (with large β). Finally, we remark that the relations presented in Sects. 2.1 and 2.2 can be usefully employed to improve the analytical description of the luminosity evolution in synthetic AGB models.

3.3. The case of zero-metallicity AGB stars

The last application refers to the TP-AGB phase of stars with initial zero metallicity. In Fig. 5 we show the time evolution of the surface properties (L and T_{eff}) and of the contributions of the nuclear energy sources (He- and H-burning shells) for two AGB models of different masses ($2.5 M_{\odot}$, and $5.0 M_{\odot}$), taken from Marigo et al. (2000, in preparation).

As we can see from Fig. 5, the $2.5 M_{\odot}$ experiences weak luminosity fluctuations instead of “normal” thermal pulses, whereas the $5 M_{\odot}$ model shows the occurrence of rather strong He-shell flashes. Actually, it has already been pointed out by

Sujimoto et al. (1984; see also Chieffi & Tornambè 1984, and Domínguez et al. 1999) that the occurrence of thermal pulses in zero-metallicity AGB stars is critically dependent on the core mass and abundance of CNO elements in the envelope. To this respect, a detailed discussion of our zero-metallicity models is given in Marigo et al. (2000, in preparation) and will not be repeated here.

What we simply aim to do in this work is to test whether the homology predictions presented in Sect. 2 are able to account for the quite different trends in the surface properties of the two AGB models here considered. In fact, as we can notice from the top panels of Fig. 5, the $5 M_{\odot}$ model is climbing the AGB at increasing luminosities (and decreasing effective temperatures), whereas the $2.5 M_{\odot}$ model is clearly evolving downward on its Hayashi track. Moreover, in both cases the p-p chain is negligibly contributing to the nuclear energy generated within the H-shell ($L_{\text{pp}}/L_{\text{H}} \lesssim 10^{-4}$), i.e. the CNO-cycle is the dominant energy source. We also report that the third dredge-up is never found to occur in these models up to the moment at which the calculations were stopped.

Once β is estimated with the aid of Eq. (29), the exponents δ_3 , δ_4 , and δ_5 can be calculated assuming $a = b = 0$, and ($n = 2$, $\nu = 14$), the latter being suitable for dominant CNO cycle. The luminosity is then derived from Eq. (5), taking the values of the core mass and composition factors (μ , X , and Z_{CNO}) from selected models of the $2.5 M_{\odot}$ and $5 M_{\odot}$ stars (see Figs. 6 and 7). It turns out that in both cases the trend in the

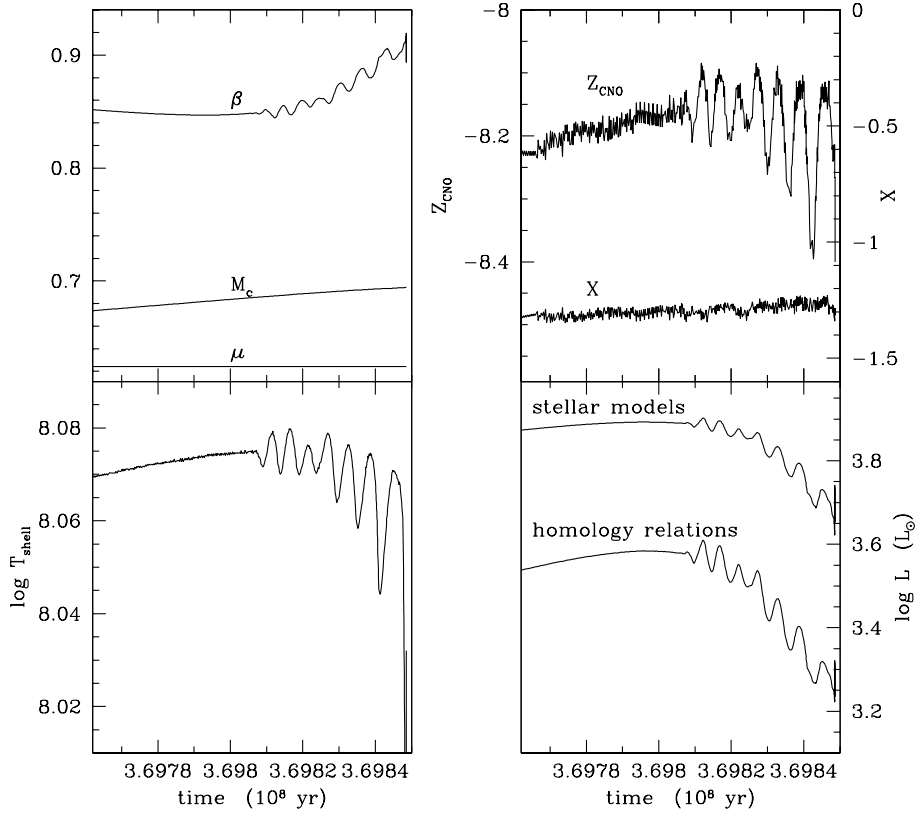


Fig. 6. Time behaviour of relevant quantities characterising the evolution of $2.5 M_{\odot}$ AGB model, namely: the gas pressure over total pressure ratio (β), and the mass coordinate (M_c) at the top of the H-burning shell; the mean molecular weight in the envelope (μ); the abundances (by number) of CNO elements (Z_{CNO}) and hydrogen (X), and the temperature (T_{shell}) at the point of maximum energy production in the H-burning shell. The “true” stellar luminosity predicted by full stellar models is compared to that obtained from homology relations (arbitrarily shifted by a constant on the logarithmic axis).

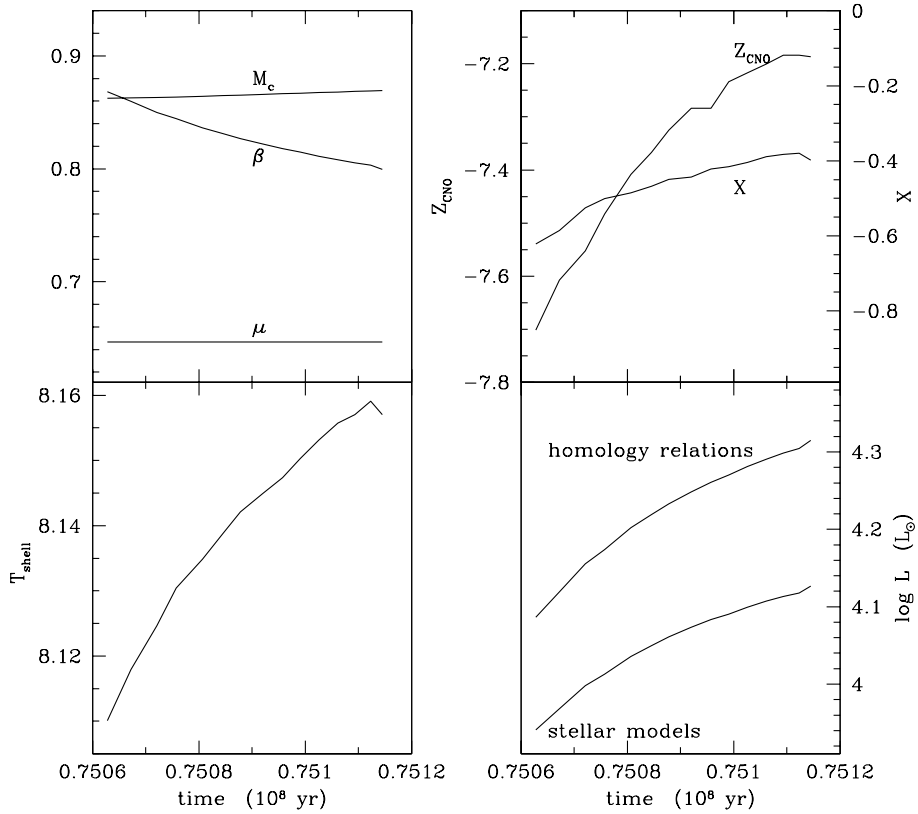


Fig. 7. The same as in Fig. 6, but for the ($5 M_{\odot}$, $Z = 0$) model. The quantities are shown at the stage of the maximum quiescent luminosity immediately preceding a thermal pulse.

luminosity evolution predicted by stellar evolutionary calculations is satisfactorily reproduced by homology relations.

In particular, the oscillating behaviour in the surface luminosity of the $2.5 M_{\odot}$ model shows up in correspondence with that of β , which presents a mirror-like trend. The average increase of β actually determines the long-term decrease of the luminosity, despite of the progressive increment of the core mass. Moreover, it is interesting to notice that, in the regime of oscillating luminosity, the total abundance of CNO elements – at the point of maximum nuclear energy efficiency in the H-burning shell – also presents clear fluctuations, reflecting a similar trend in the temperature at the same mesh-point. To this regard, it should be remarked that although for this model the CNO abundance in the envelope is zero (as it is not changed by any dredge-up episode), the CNO catalysts in the H-burning shell are self-produced, starting from the synthesis of primary ^{12}C via the triple α -reaction, operating at typical shell temperatures.

The increase with time of the pre-flash maximum luminosity in the $5 M_{\odot}$ model is also well reproduced by homology predictions, being essentially determined by the rate of increase of the core mass. Finally, we can notice that, contrary to the $2.5 M_{\odot}$ model, in this case β is decreasing.

4. Concluding remarks

A *general formalism* based on homology relations is presented to derive the structural dependences of the quiescent luminosity of shell-burning stars in different evolutionary phases (RGB and AGB), with different chemical compositions, and different nuclear energy sources.

The reliability of this formalism is tested through several consistency checks. First, we are able to get exactly the same formulas as in earlier similar works for specific choices of the parameters, e.g. Refsdal & Weigert (1970) for $\beta = 1$. Second, our predictions are found to be in good agreement with some basic results of complete stellar calculations. In particular, as far as the composition dependence of the $M_c - L$ relation is concerned, we recover the finding of evolutionary calculations that $L \propto \mu^7$ for RGB stars with $Z_{\text{cno}} > 0$, and $L \propto \mu^3$ for AGB stars with $0.5M_{\odot} \lesssim M_c \lesssim 0.7M_{\odot}$ (e.g. Boothroyd & Sackmann 1988). Moreover, according to our results, the effect on the luminosity of TP-AGB stars produced by significant composition changes should be larger in the case of low-mass stars that experience the third dredge-up after the first thermal pulses.

The case of zero-metallicity giant stars is also investigated. We show that the particular luminosity evolution of RGB stars with $Z = 0$ can be very well explained by considering not only the dependence on the mean molecular weight, but also the kind of dominant nuclear energy source (i.e. the p-p reactions). Moreover, a good reproduction of the luminosity trend of AGB models with initial zero metallicity is obtained.

Finally, it is worth remarking that the analytical prescriptions presented in this work could be usefully employed in synthetic evolution models to improve them in accuracy, and to test the effects of different physical conditions, i.e. dominant nuclear reaction rates, opacity source, relative contributions of gas/radiation to the total pressure, and chemical composition.

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