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## A generalization of the Hopf–Cole transformation for stationary Mean-Field Games systems



*Une généralisation de la transformation de Hopf–Cole pour des systèmes de jeux à champ moyen stationnaires*

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## ABSTRACT

In this note we propose a transformation that decouples stationary Mean-Field Games systems with superlinear Hamiltonians of the form  $|p|^{r'}$ ,  $r' > 1$ , and turns the Hamilton–Jacobi–Bellman equation into a quasi-linear equation involving the  $r$ -Laplace operator. Such a transformation requires an assumption on solutions to the system, which is satisfied for example in space dimension one or if solutions are radial.

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## R É S U M É

On propose dans cette Note une transformation qui découple les systèmes de jeux à champ moyen stationnaires pour des hamiltoniens superlinéaires de la forme  $|p|^{r'}$ ,  $r' > 1$ , et qui transforme l'équation de Hamilton–Jacobi–Bellman en une équation quasi linéaire introduisant le  $r$ -laplacien. Une telle transformaton nécessite une hypothèse sur la solution : cette hypothèse est satisfaite, par exemple, dans le cas unidimensionnel ou dans le cas où la solution est radiale.

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## 1. Introduction

Mean-Field Games (briefly MFG) is a branch of Dynamic Games that has been proposed independently by Lasry, Lions [7–9,11] and Caines, Huang, Malhamé [6], and aims at modeling and analyzing decision processes involving a very large number of indistinguishable rational agents. In MFG, every agent belonging to a population of infinite individuals has the goal of minimizing some cost that depends on his own state and on the average distribution of the other players. Suppose that the state of a typical player is driven by the stochastic differential equation

$$dX_s = -\alpha_s ds + \sqrt{2\nu} dB_s \in \Omega,$$

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where  $\alpha_s$  is the control,  $B_s$  is a Brownian motion,  $\nu > 0$  and the domain  $\Omega \subseteq \mathbb{R}^d$ ,  $d \geq 1$ , is the so-called state space. Suppose also that the cost functional has the long-time-average form

$$\mathcal{J}(X_0, \alpha) = \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{E}[L(\alpha_s) + f(X_s, \hat{m}_s)] ds,$$

where the (convex) Lagrangian function  $L(\alpha)$  is associated with the cost paid by the player to change his own state, and the term involving  $f$ , that we assume to be a  $C^1(\Omega \times [0, \infty))$  function, is the cost paid for being at state  $x \in \Omega$ , and it depends on the empirical density  $\hat{m}_s$  of the other players. Then, under the assumption that players are indistinguishable, it has been shown that equilibria of the game (in the sense of Nash) are captured by the following system of non-linear elliptic equations

$$\begin{cases} \text{(HJB)} & -\nu \Delta u(x) + H(Du(x)) + \lambda = f(x, m(x)) & \text{in } \Omega \\ \text{(K)} & -\nu \Delta m(x) - \operatorname{div}(DH(Du(x))m(x)) = 0 & \text{in } \Omega \\ & \int_{\Omega} m(x) dx = 1, m \geq 0. \end{cases} \tag{1}$$

Here,  $H$  denotes the Legendre transform of  $L$ . The two unknowns  $u, \lambda$  in the Hamilton–Jacobi–Bellman equation (1)-(HJB) provide respectively the optimal control of a typical player, given in feedback form by  $\alpha^* : x \mapsto -DH(Du(x))$ , and the average cost  $\mathcal{J}(X_0, \alpha^*)$ . On the other hand, the solution  $m$  of the Kolmogorov equation (1)-(K) is the stationary distribution of players implementing the optimal strategy, that is the long-time behavior of the whole population playing in an optimal way. The two equations are coupled via the cost function  $f$ . Note that a set of boundary conditions is usually associated with (1), for example  $u, m$  can be assumed to be periodic if  $\Omega = (0, 1)^d$  or Neumann conditions are imposed if  $\Omega$  is bounded and  $X_s$  is subject to reflection at  $\partial\Omega$ . In some models,  $\Omega$  is the whole  $\mathbb{R}^d$ .

A relevant class of MFG models assumes that the Lagrangian function  $L$  has the form  $L(\alpha) = \frac{l_0}{r} |\alpha|^r$ ,  $l_0 > 0, r > 1$ . Consequently, the Hamiltonian function  $H$  becomes

$$H(p) = \frac{h_0}{r'} |p|^{r'}, \quad h_0 = l_0^{1-r'} > 0, r' = \frac{r}{r-1} > 1. \tag{2}$$

In the particular situation where  $L$  and  $H$  are quadratic (namely  $r = r' = 2$ ), it has been pointed out (see [7,11]) that the so-called Hopf–Cole transformation decouples (1), and reduces it to a single elliptic semilinear equation of generalized Hartree type. Precisely, let  $\varphi := ce^{-\frac{u}{2\nu}}$ ,  $c > 0$ , then (1)-(HJB) reads (setting for simplicity  $h_0 = 1$ )

$$-2\nu^2 \Delta \varphi + (f(x, m) - \lambda) \varphi = 0 \quad \text{in } \Omega \tag{3}$$

for all  $c > 0$ . Moreover, if we set  $c^2 = \left(\int_{\Omega} e^{-\frac{u}{\nu}}\right)^{-1}$ , an easy computation shows that  $\varphi^2$  is also a solution to (1)-(K), so if uniqueness of solutions for such equation holds (that is true, for example, if suitable boundary conditions are imposed), then  $m = \varphi^2$ , and therefore  $\varphi$  becomes the only unknown in (3). This transformation can be exploited to study quadratic MFG systems, both from the theoretical and numerical point of view. This strategy is adopted, for example, in the works [1,3–5].

The aim of this note is to show that, if  $r' \neq 2$ , there exists a similar change of variables, involving a suitable power of  $m$ , that in some cases decouples (1) and turns the Hamilton–Jacobi–Bellman equation (1)-(HJB) into a quasi-linear equation of the form

$$\begin{cases} -\mu \Delta_r \varphi + (f(x, \varphi^r) - \lambda) \varphi^{r-1} = 0 & \text{in } \Omega, \\ \int_{\Omega} \varphi^r dx = 1, \varphi > 0, \mu = \nu \left(\frac{\nu r}{h_0}\right)^{r-1}, \end{cases} \tag{4}$$

where  $\Delta_r \varphi = \operatorname{div}(|D\varphi|^{r-2} D\varphi)$  is the standard  $r$ -Laplace operator ( $r$  is the conjugate exponent of  $r'$ ). Such a transformation is in particular possible if the vector field  $\nu Dm + DH(Du)m$ , which is divergence-free because of (1)-(K), is identically zero on  $\Omega$ .

Let us briefly recall in which sense  $(u, m, \lambda)$  solves (1).

**Definition 1.1.** We say that a triple  $(u, m, \lambda) \in C^2(\Omega) \times W_{\text{loc}}^{1,2}(\Omega) \times \mathbb{R}$  is a (local) solution to (1) if  $u, \lambda$  solve pointwise (1)-(HJB) and  $m$  solves (1)-(K) in the weak sense, namely

$$\nu \int_{\Omega} Dm \cdot D\xi + \int_{\Omega} m DH(Du) \cdot D\xi = 0 \quad \forall \xi \in C_0^\infty(\Omega). \tag{5}$$

We say that a couple  $(\varphi, \lambda) \in (W_{\text{loc}}^{1,r}(\Omega) \cap L^\infty(\Omega)) \times \mathbb{R}$  is a solution to (4) if

$$\mu \int_{\Omega} |D\varphi|^{r-2} D\varphi \cdot D\xi + \int_{\Omega} (f(x, \varphi^r) - \lambda) \varphi^{r-1} \xi = 0 \quad \forall \xi \in C_0^\infty(\Omega).$$

Then, the transformation can be stated as follows.

**Theorem 1.2.** *Suppose that  $H$  satisfies (2).*

a) *Let  $(u, m, \lambda)$  be a solution to (1). If the following equality holds,*

$$\nu Dm + h_0 m |Du|^{r'-2} Du = 0 \quad \text{a.e. in } \Omega, \tag{6}$$

then

$$\varphi := m^{\frac{1}{r}} \quad \text{in } \Omega \tag{7}$$

is a solution to (4).

b) *Let  $(\varphi, \lambda)$  be a solution to (4), and suppose that there exists  $u \in C^1(\Omega)$  such that*

$$h_0 \varphi |Du|^{r'-2} Du + \nu r D\varphi = 0 \quad \text{in } \Omega. \tag{8}$$

Then,  $(u, m, \lambda)$  is a solution to (1), where  $m := \varphi^r$  in  $\Omega$ .

The proposed transformation reveals a connection between some (stationary) MFG systems with non-quadratic Hamiltonians of the form (2) and  $r$ -Laplace equations, which have been widely studied in the literature and appear in many other areas of interest. Apart from existence and uniqueness issues, this link might shed some light on MFG problems in general, which can be translated into problems involving the  $r$ -Laplacian (e.g., qualitative properties of solutions, MFG in unbounded domains, vanishing viscosity limit  $\nu \rightarrow 0$ ).

In the quadratic case  $r' = 2$ , we have mentioned that if solutions to (1)-(K) are unique, then  $m = e^{-\frac{h_0 u}{\nu}} / \left( \int_{\Omega} e^{-\frac{h_0 u}{\nu}} \right)^{-1}$ .

In this case condition, (6) is easily verified, and the assertion of Theorem 1.2 is that  $\varphi = m^{1/2}$  solves (4) with  $r = 2$ , which is precisely (3). In this sense, our transformation can be seen as a generalization of the standard Hopf–Cole case.

Note that the change of variables  $m = \varphi^r$  is local, in particular it is independent of boundary conditions that might be added to (1). However, in order to verify (6), (8) and therefore to apply Theorem 1.2, it is necessary to specify additional information on the problem. Space dimension  $d = 1$  and Neumann conditions at the boundary, or  $u, m, \varphi$  enjoying radial symmetry are two possible scenarios where (6), (8) hold.

**Corollary 1.3.** *Suppose that  $H$  satisfies (2) and  $\Omega = \{x \in \mathbb{R}^d : |x| < R\}$  for some  $R \in (0, \infty]$ . Then,  $(u, m, \lambda)$  is a radial solution to (1) if and only if  $(\varphi, \lambda)$  is a radial solution to (4), where  $\varphi = m^{1/r}$ .*

**Corollary 1.4.** *Suppose that  $H$  satisfies (2) and  $\Omega = (a, b)$  for some  $-\infty < a < b < \infty$ . Then,  $(u, m, \lambda) \in C^2(\overline{\Omega}) \times W^{1,2}(\Omega) \times \mathbb{R}$  is a solution to (1) satisfying the Neumann boundary conditions<sup>2</sup>  $u'(a) = u'(b) = m'(a) = m'(b) = 0$  if and only if  $(\varphi, \lambda) \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$  is a solution to (4) satisfying the Neumann boundary conditions  $\varphi'(a) = \varphi'(b) = 0$ , where  $\varphi = m^{1/r}$ .*

## 2. Proofs

**Remark 1.** We point out that  $m \in W_{loc}^{1,q}(\Omega)$  for all  $q \geq 1$  by standard regularity results on weak solutions to Kolmogorov equations. Moreover, the Harnack inequality guarantees that  $m > 0$  on  $\Omega$ . However, even if  $u \in C^2(\Omega)$ , we do not expect in general the same regularity for  $m$ , since  $DH(Du)$  might lack the desired smoothness if  $1 < r' < 2$ . If  $r' \geq 2$ , it is possible to conclude that  $m$  is twice differentiable on  $\Omega$  and it solves (1)-(K) in the classical sense.

On the other hand, it is known that a solution  $\varphi$  of (4) enjoys local  $C^{1,\alpha}$  regularity (see, for example, [2,10]).

**Proof of Theorem 1.2.** (a). Note that  $m \in W_{loc}^{1,q}(\Omega)$  for all  $q \geq 1$  (see Remark 1), and therefore  $\varphi^{r-1}$  has the same regularity. Moreover,  $m > 0$  on  $\Omega$ , hence  $\varphi > 0$  as well. Equalities (6) and (7) imply that

$$\nu r \frac{D\varphi}{\varphi} = -h_0 |Du|^{r'-2} Du \quad \text{a.e. in } \Omega. \tag{9}$$

We multiply the Hamilton–Jacobi–Bellman equation (1)-(HJB) by  $\xi \varphi^{r-1}$ , where  $\xi \in C_0^\infty(\Omega)$  is a generic test function. Integrating by parts,

$$\nu \int_{\Omega} Du \cdot D(\xi \varphi^{r-1}) + \int_{\Omega} \frac{h_0}{r'} |Du(x)|^{r'} \xi \varphi^{r-1} + \int_{\Omega} (\lambda - f) \xi \varphi^{r-1} = 0. \tag{10}$$

<sup>2</sup> For equations (1)-(K) and (4), Neumann boundary conditions are intended in the weak sense, namely the space of test functions  $\xi$  is set to be  $C^\infty(\overline{\Omega})$ .

We note that by (9)

$$\nu \int_{\Omega} \xi Du \cdot D(\varphi^{r-1}) = \nu(r-1) \int_{\Omega} \xi Du \cdot D\varphi \varphi^{r-2} = -h_0 \frac{r-1}{r} \int_{\Omega} \xi |Du|^{r'} \varphi^{r-1},$$

and  $(r')^{-1} = (r-1)r^{-1}$ , so (10) becomes

$$\nu \int_{\Omega} Du \cdot D\xi \varphi^{r-1} + \int_{\Omega} (\lambda - f)\xi \varphi^{r-1} = 0. \tag{11}$$

Again using (9), we obtain

$$\begin{aligned} \int_{\Omega} |D\varphi|^{r-2} D\varphi \cdot D\xi &= - \left(\frac{h_0}{\nu r}\right)^{r-1} \int_{\Omega} |Du|^{(r'-1)(r-2)+r'-2} \varphi^{r-1} Du \cdot D\xi \\ &= \frac{1}{\nu} \left(\frac{h_0}{\nu r}\right)^{r-1} \int_{\Omega} (\lambda - f)\xi \varphi^{r-1}, \end{aligned} \tag{12}$$

since  $(r'-1)(r-2) + r' - 2 = 0$ , and by virtue of (11). Equality (12), which holds for all test functions  $\xi$ , is precisely the weak formulation of (4), hence we are done. We finally observe that  $\varphi$  enjoys local  $C^{1,\alpha}$  regularity (see Remark 1), which is inherited by  $m$ .

b). It is easily verified, in view of (8), that  $m$  solves (1)-(K). Since  $\varphi = m^{1/r}$  is positive (see Remark 1),

$$\nu r \frac{D\varphi}{\varphi} = -h_0 |Du|^{r'-2} Du \quad \text{in } \Omega.$$

By carrying out backwards computations (10)–(12) of part a), it follows that  $u, \lambda$  is a weak solution to (1)-(HJB). Standard regularity results for the Poisson equation guarantee that  $u \in C^2(\Omega)$  and (1)-(HJB) is satisfied in the classical sense.  $\square$

**Proof of Corollary 1.3.** In the following,  $\rho := |x|$  and  $\mathbf{e}_\rho = \mathbf{e}_\rho(x) = x/\rho$  will denote the standard unit vector in the radial direction.

Let  $(u, m, \lambda)$  be a radial solution to (1). We write  $u(x) = u(\rho)$ ,  $m(x) = m(\rho)$ ,  $Du(x) = u'(\rho)\mathbf{e}_\rho$ ,  $Dm(x) = m'(\rho)\mathbf{e}_\rho$ . The Kolmogorov equation (1)-(K) reads

$$\int_{\Omega} (\nu Dm + h_0 |Du|^{r'-2} Du m) \cdot D\xi = 0 \tag{13}$$

for all test functions  $\xi \in C_0^\infty(\Omega)$ , and

$$\nu Dm(x) + h_0 |Du(x)|^{r'-2} Du(x) m(x) = F(\rho)\mathbf{e}_\rho$$

for all  $x \in \Omega$ , where  $F(\rho)\rho^{(d-1)/q} \in L^q((0, R'))$ , for all  $q \geq 1$ ,  $R' < R$ , so (13) becomes

$$\int_0^R F(\rho)\xi'(\rho)\rho^{d-1} d\rho = 0$$

for all radial test functions  $\xi$ . It is then possible to conclude that  $F(\rho)\rho^{d-1} = 0$  a.e. in  $(0, R)$ , so (6) holds and  $(\varphi, \lambda)$  is a solution to (4) because of Theorem 1.2, a).

If  $(\varphi, \lambda)$  is a radial solution to (4), say  $\varphi(x) = \varphi(\rho)$ ,  $D\varphi(x) = \varphi'(\rho)\mathbf{e}_\rho$ , then  $\varphi'(0) = 0$ . Setting

$$b(\rho) := \frac{-\nu r \varphi'(\rho)}{h_0 \varphi(\rho)} \quad \forall \rho \in [0, R)$$

we have  $b \in C^{0,\alpha}([0, R'])$  for all  $0 < R' < R$  and  $b(0) = 0$ , so

$$u(\rho) := \int_0^\rho |b(\sigma)|^{\frac{2-r'}{r'-1}} b(\sigma) d\sigma$$

defines a radial function  $u(x) = u(\rho)$  that belongs to  $C^1(\Omega)$ . An easy computation shows that  $u$  satisfies (8), hence  $(u, m, \lambda)$  is a solution to (1) as a consequence of Theorem 1.2, b).  $\square$

**Proof of Corollary 1.4.** We proceed as in the proof of Corollary 1.3. Let  $(u, m, \lambda)$  be a solution to (1) and set

$$vm'(x) + h_0|u'(x)|^{r'-2}u'(x)m(x) =: F(x)$$

in  $[a, b]$ , and  $F \in L^q((a, b))$ . Therefore,

$$\int_a^b F(x)\xi'(x) dx = 0$$

for all test functions  $\xi \in C^\infty([a, b])$ , so  $F(x) = 0$  for a.e.  $x \in [a, b]$ . Hence (6) holds and one implication stated by the corollary follows by Theorem 1.2, a). Vice-versa, (8) holds if we choose  $u(x) := \int_a^x |b(y)|^{(2-r')/(r'-1)} b(y) dy$ , where  $b(y) = (-vr\varphi'(y))/(h_0\varphi(y))$  for all  $y \in [a, b]$ , and Theorem 1.2, b) applies.  $\square$

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