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RINGS OF PURE GLOBAL DIMENSION ZERO AND MITTAG–LEFFLER MODULES

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We investigate the rings over which every countably generated module is pure-projective and generalize the theory of rings of pure global dimension zero. This class of rings is studied in connection with Mittag–Leffler modules. We also give a characterization of Mittag–Leffler abelian groups.

Introduction

If M is a left module over a ring R and $\{A_i\}$ is a family of right R-modules, it is possible to define a canonical homomorphism $(\prod A_i)\otimes_R M \to \prod (A_i \otimes_R M)$. Lenzing proved that this homomorphism is epic for every family of right R-modules $\{A_i\}$ if and only if M is finitely generated, and that it is an isomorphism for every family of right R-modules $\{A_i\}$ if and only if M is finitely presented [11]. Goodearl began an investigation of the conditions on M under which this canonical homomorphism $(\prod A_i)\otimes_R M \to \prod (A_i \otimes_R M)$ is monic and proved that it is monic for every family of *flat* right R-modules $\{A_i\}$ if and only if for any finitely generated submodule N of M the inclusion $N \to M$ factors through a finitely presented module [9]. Raynaud and Gruson gave a significant contribution to the study of the modules M for which the canonical homomorphism in question is monic for every family of right R-modules $\{A_i\}$ [12]. They called the modules M with this property *Mittag–Leffler modules* because they can be described as those modules that are direct limits $\varinjlim F_i$ of finitely presented left R-modules F_i such that the inverse system of abelian groups $\mathrm{Hom}_R(F_i, N)$ is a Mittag–Leffler inverse system in the sense of Grothendieck [10, Chapter 0_{III}] for every left R-module N. Raynaud and Gruson gave several other characterizations of Mittag–Leffler modules [12]. (We have summarized most of these characterizations in Theorem 6). In this paper we continue the study of Mittag–Leffler modules particularly in connection with the rings of left pure global dimension zero, that is, the rings for which every left R-module is pure-projective.

In the first section we consider *pure-split* left modules, that is, the modules in which every pure submodule is a direct summand, and prove that they are exactly the modules M such that every pure epimorphism of M onto any strictly indecomposable module N splits (Proposition 1). Here *strictly indecomposable* means that N is a module for which the intersection of all non-zero pure submodules is non-zero. As a corollary we obtain a characterization of left perfect rings as the rings over which every countably generated strictly indecomposable flat left module is projective (Corollary 2).

In the second section we study how the various well-known properties of modules over the rings of pure global dimension zero behave when we consider these properties not on all modules but only on countably generated modules. Actually, the result we obtain (Proposition 4) can be applied both to the rings for which every countably generated left R-module is pure-projective and to the rings for which every left R-module is pure-projective, that is, the rings of left pure global dimension zero (Proposition 5).

In the last section we proceed to the study of Mittag–Leffler modules. By applying Raynaud and Gruson's results we succeed in characterizing the Mittag–Leffler abelian groups, that is, the Mittag–Leffler modules over the ring $\mathbb{Z}$ of integers (Proposition 7). It is easy to see that every pure-projective module is a Mittag–Leffler module, and it is shown in [12] that the converse is true for countably generated modules. But the converse is not true in general for uncountably generated modules; indeed, by a theorem of Zimmermann-Huisgen [15] we can see that if every Mittag–Leffler left R-module is pure-projective, then R must be left perfect and of left pure global dimension ≤ 1. However we proved (Theorem 8) that the assumption that every strictly indecomposable left R-module is a Mittag–Leffler module – and, a fortiori, the stronger assumption that *every* left R-module is a Mittag–Leffler module – guarantees that R is of left pure global dimension zero. This immediately yields another characterization of the rings of finite representation type as the rings for which the canonical homomorphism

$$\left(\prod_{i\in I} A_i\right) \underset{R}{\otimes} \left(\prod_{j\in J} B_j\right) \to \prod_{(i,j)\in I\times J} \left(A_i \underset{R}{\otimes} B_j\right)$$

is monic for every family of right R-modules $\{A_i\}_{i\in I}$ and every family of left R-modules $\{B_j\}_{j\in J}$ (Corollary 9).

Throughout this paper R is an associative ring with identity. If M, N are left R-modules, a *pure epimorphism* $M \to N$ is an epimorphism of R-modules whose kernel is a pure submodule of N, and an epimorphism $M \to N$ *splits* if its kernel is a direct summand of M.

1. Pure-split, strictly indecomposable and pure-simple modules

Let M be a left R-module. We shall say that M is *pure-split* if every pure submodule of M is a direct summand of M. Every pure submodule of a pure-split module is pure-split, because if N is a pure submodule of M, every pure submodule of N is a pure submodule of M and every direct summand of M contained in N is a direct summand of N.

We call a non-zero module M *strictly indecomposable* if the intersection of all non-zero pure submodules of M is non-zero. Since every direct summand of M is a pure submodule, every strictly indecomposable module is indecomposable. Moreover, every non-zero pure submodule of a strictly indecomposable module is strictly indecomposable. Finally, we say that a non-zero module M is *pure-simple* if it has no pure submodules other than M and 0. It is clear that M is pure-simple if and only if it is pure-split and indecomposable, and every pure-simple module is strictly indecomposable.

Proposition 1. *A left module M is pure-split if and only if every pure epimorphism from M onto a strictly indecomposable module splits. Every pure-split module is a direct sum of pure-simple submodules.*

Proof. We claim that if M is a left module such that every pure epimorphism from M onto a strictly indecomposable module splits, then for every proper pure submodule P of M there exists a strictly indecomposable submodule U of M such that $U \cap P = 0$ and $U + P = U \oplus P$ is pure in M. If we prove the claim, M is a module with the property that every pure epimorphism from M onto a strictly indecomposable module splits, and N is a pure submodule of M, then by Zorn's Lemma it is easy to show that there exists a family $\{U_i\}_i$ of strictly indecomposable submodules of M, which is maximal with respect to the property that the sum $N + \sum U_i$ is direct and $N + \sum U_i = N \oplus \bigoplus U_i$ is pure in M. Then $N + \sum U_i = M$, because if $N + \sum U_i$ is a proper submodule of M, then by our claim there exists a strictly indecomposable submodule U of M such that $U \cap (N + \sum U_i) = 0$ and $U + (N + \sum U_i) = U \oplus N \oplus \bigoplus U_i$ is pure in M; this contradicts the maximality of the family $\{U_i\}_i$. Therefore we have that $N + \sum U_i = N \oplus \bigoplus U_i = M$, so that N is a direct summand of M. This proves that M is pure-split. In particular, for $N = 0$ we get $\sum U_i = \bigoplus U_i = M$ and M is a direct sum of strictly indecomposable submodules U_i. Since each U_i is a direct summand, whence a pure submodule, of the pure-split module M, each U_i is also pure-split and indecomposable. Therefore, each U_i is pure-simple and $M = \bigoplus U_i$ is a direct sum of pure-simple submodules. This concludes the proof of the proposition, if we prove the claim.

In order to prove the claim, let M be a left module such that every pure epimorphism from M onto a strictly indecomposable module splits and let P be a proper pure submodule of M. Choose an element $x \in M$ such that $x \notin P$. Since the union of a chain of pure submodules of M is a pure submodule of M, by Zorn's Lemma there exists a submodule M_0 of M maximal with respect to the property of being a pure submodule of M containing P and such that $x \notin M_0$. It follows that x belongs to every pure submodule of M properly containing M_0. Since there is a one-to-one correspondence between the pure submodules of M containing M_0 and the pure submodules of M/M_0, the module M/M_0 must be strictly indecomposable. By our assumption the canonical projection of M onto M/M_0 splits, i.e., M_0 is a direct summand of M. Let U be a submodule of M such that $M = U \oplus M_0$; then $U \cong M/M_0$ is strictly indecomposable. Since $P \subseteq M_0$, it follows that $U \cap P = 0$, so that $U + P = U \oplus P$. Since P is pure in M, P is pure in M_0 as well, so that $U \oplus P$ is pure in $U \oplus M_0 = M$. This proves the claim. □

It should be noted that our proof of Proposition 1 is essentially a refinement and a modification of an argument due to Zimmermann [16].

The second part of Proposition 1 cannot be inverted, that is, not every direct sum of pure-simple modules is pure-split. For instance, the ring $\mathbb{Z}$ of integers is pure-simple as a module over itself, but the following corollary shows that not all the free $\mathbb{Z}$-modules are pure-split because $\mathbb{Z}$ is not a perfect ring.

Corollary 2. *Let R be a ring with identity. The following conditions are equivalent*:

(1) *The ring R is left perfect.*

(2) *Every free left R-module is pure-split.*

(3) *Every countably generated flat left R-module is pure-projective.*

(4) *Every countably generated strictly indecomposable flat left R-module is pure-projective.*

Proof. (1) ⇒ (2). Let R be a left perfect ring and let F be a free left R-module. If N is a left R-module and $\varphi : F \to N$ is a pure epimorphism, then the module N is flat. Since R is left perfect, every flat left R-module is projective [5, Theorem P], so that N is projective and φ splits. This proves (2).

(2) ⇒ (3). Let N be a countably generated flat left R-module. If $\varphi : R^{(\aleph_0)} \to N$ is an epimorphism, φ is a pure epimorphism because N is flat. But $R^{(\aleph_0)}$ is pure-split, so that φ splits. Therefore N is isomorphic to a direct summand of $R^{(\aleph_0)}$, i.e., it is pure-projective.

(3) ⇒ (4). Obvious.

(4) ⇒ (1). In order to prove that R is left perfect it is sufficient to prove that R satisfies the descending chain condition on principal right ideals [5, Theorem P]. Equivalently, we must prove that if $\{a_n\}_{n=1,2,\dots}$ is any sequence of elements in the ring R, the chain of right ideals $a_1R \supseteq a_1a_2R \supseteq a_1a_2a_3R \supseteq \cdots$ terminates. Let F be the free left R-module with countable basis $x_1, x_2, \dots$. Then for any strictly indecomposable left R module N and every pure epimorphism $\varphi : F \to N$, N is countably generated because φ is onto, and N is flat because F is free and φ is a pure epimorphism. By our assumption (6) the module N is pure-projective, so that the pure epimorphism φ splits. This proves that F is pure-split by Proposition 1. Now

let G be the submodule of F generated by $\{x_n - a_n x_{n+1}\}_{n=1,2,\dots}$. By [5, Lemma 1.1], F/G is a flat R-module, or equivalently G is a pure submodule of F, so that G is a direct summand of F because F is pure-split. By [5, Lemma 1.3] the chain $a_1 \supseteq a_1 a_2 R \supseteq a_1 a_2 a_3 R \supseteq \cdots$ terminates, and therefore R is left perfect. □

Recall that a *decomposition that complements direct summands* is a direct decomposition $M = \bigoplus_{i \in I} U_i$ of a module M with the following property: for each direct summand N of M there is a subset I_0 of I such that $M = N \oplus (\bigoplus_{i \in I_0} U_i)$ [1].

Proposition 3. *Let M be a pure-split left R-module and let the endomorphism ring of every indecomposable (whence pure-simple) direct summand of M be a local ring. Then M has a direct decomposition that complements direct summands.*

Proof. Let $M = \bigoplus_{i \in I} U_i$ be a direct decomposition of M into indecomposable summands U_i. Let N be a proper direct summand of M. Then $M = N \oplus N'$ for a nonzero submodule N' of M. In particular N' is also pure-split and so it is a direct sum $N' = \bigoplus V_j$ of indecomposable submodules V_j. Thus we have the direct decomposition $M = N \oplus \bigoplus V_j$. For each j let $\varepsilon_j : M \to V_j$ be the projection with respect to this decomposition. Choose an index j_0. Since the endomorphism ring of V_j is a local ring for every j, by [3, Theorem 2] there exists an index $i_0 \in I$ such that U_{i_0} is mapped by ε_{j_0} isomorphically onto V_{j_0}. Since the kernel of ε_{j_0} is $N \oplus \bigoplus_{j \neq j_0} V_j$, this implies that $M = N \oplus U_{i_0} \oplus \bigoplus_{j \neq j_0} V_j$ and in particular $N \oplus U_{i_0}$ is a direct summand of M. By Zorn's Lemma we can find a maximal subset I_0 of I such that the sum $N + \sum_{i \in I_0} U_i$ is direct and is pure in M. Suppose $N + \sum_{i \in I_0} U_i = N \oplus \bigoplus_{i \in I_0} U_i$ were a proper submodule of M. This is a pure submodule of M, whence a direct summand of M. By applying the above argument to $N \oplus \bigoplus_{i \in I_0} U_i$ instead of N, we know that there would be an $i_1 \in I$ such that $i_1 \notin I_0$ and $N \oplus \bigoplus_{i \in I_0} U_i + U_{i_1} = N \oplus \bigoplus_{i \in I_0} U_i \oplus U_{i_1}$ is a direct summand, whence a pure submodule of M. But this clearly contradicts the maximality of I_0. Thus we have $M = N \oplus \bigoplus_{i \in I_0} U_i$, which shows that the decomposition $M = \bigoplus_{i \in I} U_i$ complements direct summands. □

2. Classes of pure-projective modules

We shall now consider the classes $\mathscr{C}$ of left R-modules that satisfy the following two conditions:

(a) Every countably generated left R-module is in $\mathscr{C}$,

(b) If M is in $\mathscr{C}$, then every pure epimorphic image of M is in $\mathscr{C}$.

It is clear that the class of all countably generated left R-modules as well as the class of all left R-modules satisfy the above conditions.

Proposition 4. *Let $\mathscr{C}$ be a class of left R-modules that satisfies conditions* (a) *and* (b). *Then the following conditions are equivalent*:

(1) *Every module in $\mathscr{C}$ is pure-projective.*
(2) *Every module in $\mathscr{C}$ is a direct sum of finitely presented submodules.*
(3) *Every indecomposable module in $\mathscr{C}$ is finitely presented.*
(4) *Every strictly indecomposable module in $\mathscr{C}$ is pure-projective.*

Moreover, if there exists a class $\mathscr{C}$ of left R-modules satisfying the conditions (a) *and* (b) *and the equivalent conditions* (1)–(4), *then R is a left Artinian ring and every module in $\mathscr{C}$ is pure-split.*

Proof. Assume that (1) holds. Let us prove that R is a left perfect ring. By Corollary 2 it is sufficient to prove that every countably generated strictly indecomposable flat left R-module is pure-projective. Such a module M is in $\mathscr{C}$ by condition (a), so that M is pure-projective by condition (1). This proves that R is left perfect.

Let us show that R is left Artinian. If I is a left ideal of R, then R/I is in $\mathscr{C}$ by the condition (a). Therefore R/I is pure-projective by (1), that is, R/I is a direct summand of a direct sum of finitely presented modules. Since R/I is cyclic, R/I is a direct summand of a *finite* direct sum of finitely presented modules. It follows that R/I itself is finitely presented, so that I is finitely generated (Schanuel's Lemma). This proves that R is left Noetherian. Since a left perfect, left Noetherian ring is left Artinian, R must be a left Artinian ring.

Let us show that every module in $\mathscr{C}$ is pure-split. We have to prove that every pure epimorphism $M \to N$ with M in $\mathscr{C}$ and N any left R-module splits. But N is in $\mathscr{C}$ by (b), so that N is pure-projective by (1). Therefore $M \to N$ splits.

We are now ready to show the equivalence of (1)–(4).

(1) $\Rightarrow$ (2). We have proved that if (1) holds, R is left Artinian and every module in $\mathscr{C}$ is pure-split. By Proposition 1, every module in $\mathscr{C}$ is a direct sum of pure-simple modules. Since all direct summands of modules in $\mathscr{C}$ are in $\mathscr{C}$ and every module in $\mathscr{C}$ is pure-projective by (1), in order to prove (2) it is sufficient to show that every pure-simple, pure-projective left module over a left Artinian ring is finitely presented. If M is a pure-simple, pure-projective left module over a left Artinian ring R, M is an indecomposable module isomorphic to a direct summand of a direct sum of finitely presented modules. Since every finitely presented module over an Artinian ring is a finite direct sum of finitely generated indecomposable modules, M is an indecomposable module isomorphic to a direct summand of a direct sum of finitely generated indecomposable modules. Moreover the endomorphism ring of each finitely generated indecomposable left R-module is local. It follows that M is isomorphic to one of the finitely generated indecomposable direct summands [3, Theorem 1]. Since R is left Artinian, M is finitely presented.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). Trivial.

(4) $\Rightarrow$ (1). Suppose (4) holds. Let us show that every module M in $\mathscr{C}$ is pure-split. By Proposition 1 it is sufficient to show that any pure epimorphism $\varphi : M \to N$ splits whenever N is a strictly indecomposable left R-module. Now N is in $\mathscr{C}$ by (b), so that N is pure-projective by (4), and thus φ splits. This proves that every module in $\mathscr{C}$ is pure-split, so that it is a direct sum of pure-simple submodules by Proposition

1. Each of these direct summands is in $\mathscr{C}$ by (b), and is strictly indecomposable because it is pure-simple. By (4), each summand is pure-projective. Therefore, every M in $\mathscr{C}$ is a direct sum of pure-projective modules, i.e., M itself is pure-projective. □

If $\mathscr{C}$ is the class of all left R-modules, we have the following known result as a particular case of Proposition 4:

Proposition 5. *Let R be a ring. The following statements are equivalent:*

(1) *Every left R-module is pure-projective, i.e., the ring R has left pure global dimension zero.*

(2) *Every left R-module is pure-split.*

(3) *Every pure-projective left R-module is pure-split.*

(4) *Every left R-module is pure-injective.*

(5) *Every left R-module is a direct sum of finitely presented submodules.*

(6) *Every indecomposable left R-module is finitely presented.*

(7) *Every strictly indecomposable left R-module is pure-projective.*

In this case R is necessarily a left Artinian ring.

Note that condition (5) is also equivalent to the following weaker condition:

(8) *Every left R-module is a direct sum of finitely generated submodules.*

Conditions (5) and (8) are equivalent because if (8) holds, every injective left R-module is a direct sum of finitely generated submodules, so that R is left Noetherian by the Faith–Walker Theorem [2, Theorem 25.8]. Since over a left Noetherian ring finitely generated and finitely presented left modules coincide, (8) implies (5).

We point out that the equivalent conditions in Proposition 5 are also equivalent to the following condition obtained via Σ-pure-injective modules by Zimmermann-Huisgen [14]:

(9) *Every left R-module is a direct sum of indecomposable submodules.*

3. Mittag–Leffler modules

A left R-module M satisfying the equivalent conditions stated in the next theorem is said to be a *Mittag–Leffler module.* The equivalence of these conditions, which we have collected in a unique theorem, was essentially proved by Raynaud and Gruson in [12].

Theorem 6 (Raynaud and Gruson [12]). *Let M be a left R-module. The following conditions are equivalent:*

(1) *For every family* $\{A_i\}$ *of right R-modules, the canonical homomorphism*

$$(\prod A_i) \otimes_R M \to \prod (A_i \otimes_R M)$$

is a monomorphism.

(2) *For every finitely presented left R-module F and every homomorphism* $\varphi: F \to M$ *there exist a finitely presented left R-module G, a homomorphism* $\psi: F \to G$ *and a homomorphism* $\chi: G \to M$ *such that* $\varphi = \chi\psi$ *and* $\ker(A \otimes \varphi) = \ker(A \otimes \psi)$ *for every right R-module A.*

(3) *For every finitely generated submodule* M_0 *of M there exist a finitely presented left R-module G, a homomorphism* $\psi: M_0 \to G$ *and a homomorphism* $\chi: G \to M$ *such that* $\chi\psi$ *is the inclusion* $\varepsilon: M_0 \to M$ *and* $\ker(A \otimes \varepsilon) = \ker(A \otimes \psi)$ *for every right R-module A.*

(4) *Every countable subset of M is contained in a pure-projective countably generated pure submodule of M.*

(5) *Every finite subset of M is contained in a pure-projective pure submodule of M.*

Remark. It is easy to see that the class of all left R-modules M that satisfy condition (1) of the theorem contains all finitely presented modules and is closed for pure submodules, pure extensions and (possibly infinite) direct sums. It follows that every pure-projective module is Mittag-Leffler, but the converse is not true, as Proposition 7 will show. In that proposition we shall give a characterization of the Mittag-Leffler abelian groups, that is, the Mittag-Leffler modules over the ring $\mathbb{Z}$ of integers. Note that by the condition 4 of Theorem 6 every countably generated Mittag-Leffler module is pure-projective.

Proof of Theorem 6. The equivalence (1)$\Leftrightarrow$(2) is proved in [12, Seconde partie, Propositions 2.1.1 and 2.1.5].

(2)$\Rightarrow$(3). If M_0 is a finitely generated submodule of M, there exist a finitely generated free module F and an epimorphism $\varphi_0: F \to M_0$. Let $\varepsilon: M_0 \to M$ be the inclusion. If condition (2) holds, there exist a finitely presented left R-module G, a homomorphism $\psi_0: F \to G$ and a homomorphism $\chi: G \to M$ such that $\varepsilon\varphi_0 = \chi\psi_0$ and $\ker(A \otimes \varepsilon\varphi_0) = \ker(A \otimes \psi_0)$ for every right R-module A. In particular, $\ker \varphi_0 = \ker \varepsilon\varphi_0 = \ker \psi_0$, so that ψ_0 factors through φ_0, i.e., there exists a homomorphism $\psi: M_0 \to G$ such that $\psi\varphi_0 = \psi_0$. Then $\varepsilon\varphi_0 = \chi\psi_0 = \chi\psi\varphi_0$, so that $\varepsilon = \chi\psi$. Since $\ker(A \otimes \varepsilon\varphi_0) = \ker(A \otimes \psi_0) = \ker(A \otimes \psi\varphi_0)$ and $A \otimes \varphi_0$ is onto for every right R-module A, it follows that $\ker(A \otimes \varepsilon) = \ker(A \otimes \psi)$ for every A. This proves (3).

(3)$\Rightarrow$(2). Let $\varphi: F \to M$ be a homomorphism of a finitely presented left R-module F into M. Set $M_0 = \varphi(F)$. If condition (3) holds, there exist a finitely presented left R-module G, a homomorphism $\psi_0: M_0 \to G$ and a homomorphism $\chi: G \to M$ such that $\chi\psi_0$ is the inclusion $\varepsilon: M_0 \to M$ and $\ker(A \otimes \varepsilon) = \ker(A \otimes \psi_0)$ for every right R-module A. If $\varphi': F \to M_0$ is the epimorphism obtained from φ by restricting its codomain to $\varphi(F) = M_0$, then $\varphi = \varepsilon\varphi' = \chi\psi_0\varphi'$; moreover, $\ker(A \otimes \varepsilon) = \ker(A \otimes \psi_0)$

for every right R-module A implies that $\ker(A \otimes \varepsilon\varphi') = \ker(A \otimes \psi_0\varphi')$, that is, $\ker(A \otimes \varphi) = \ker(A \otimes \psi_0\varphi')$. Therefore $\psi = \psi_0\varphi' : F \to G$ has the properties required in (2).

(2) ⇒ (4). Proved in [12, Seconde partie, Théorème 2.2.1].

(4) ⇒ (5). Trivial.

(5) ⇒ (1). If x is in the kernel of the mapping $(\prod A_i) \otimes_R M \to \prod (A_i \otimes_R M)$, then by (5) there exists a pure-projective pure submodule P of M such that x is in the image of the canonical mapping $(\prod A_i) \otimes_R P \to (\prod A_i) \otimes_R M$. In the commutative diagram

$$\begin{array}{ccc} (\prod A_i) \otimes_R P & \longrightarrow & \prod (A_i \otimes_R P) \\ \downarrow & & \downarrow \\ (\prod A_i) \otimes_R M & \longrightarrow & \prod (A_i \otimes_R M) \end{array}$$

the upper arrow is a monomorphism because P satisfies condition (1) as remarked immediately after the statement of Theorem 6, and the vertical arrow on the right is a monomorphism because P is pure in M; it follows that $x = 0$. □

Condition (3) in the statement of Theorem 6 is the natural extension of condition (c) in the statement of [9, Theorem 1].

In the next proposition we characterize the Mittag-Leffler modules over the ring $\mathbb{Z}$ of integers. For any abelian group G, let $t(G)$ denote the torsion subgroup of G, and let G^1 denote the so-called first Ulm subgroup of G defined by $G^1 = \bigcap_{n>0} nG$. Recall that an abelian group H is $\aleph_1$-free if all countable subgroups of H are free [8, §19]. For instance, all direct products $\mathbb{Z}^X$ of copies of $\mathbb{Z}$ are $\aleph_1$-free.

Proposition 7. *Let G be an abelian group. The following statements are equivalent:*

(1) *G is a Mittag-Leffler $\mathbb{Z}$-module.*

(2) *$G^1 = 0$ and $G/t(G)$ is $\aleph_1$-free.*

Proof. (1) ⇒ (2). Let G be a Mittag-Leffler $\mathbb{Z}$-module and let $x \in G^1$. Then there exists a pure-projective pure subgroup H of G such that $x \in H$. Since H is pure in G, $H^1 = G^1 \cap H$. Therefore $x \in H^1$. But H is pure-projective, i.e., it is a direct sum of cyclic groups [8, Theorem 30.2]. Therefore $H^1 = 0$. This proves that $x = 0$ and $G^1 = 0$.

Let us show that if G is a Mittag-Leffler $\mathbb{Z}$-module, then $G/t(G)$ is $\aleph_1$-free. Let H' be a countable subgroup of $G/t(G)$. Then there exists a countably generated subgroup H of G such that $H' = H + t(G)/t(G)$. Since G is Mittag-Leffler, there exists a pure-projective pure subgroup L of G containing H (Theorem 6). Then L is a direct sum of cyclic groups [8, Theorem 30.2], so that $L/t(L)$ is free. It follows that $H' = H + t(G)/t(G)$ is a subgroup of $L + t(G)/t(G) \cong L/L \cap t(G) = L/t(L)$, which is free. This shows that H' is free, i.e., $G/t(G)$ is $\aleph_1$-free.

(2) ⇒ (1). Since $t(G)$ is pure in G, in order to show that G is a Mittag–Leffler module it is sufficient to show that both $t(G)$ and $G/t(G)$ are Mittag–Leffler modules. Since $G^1=0$ implies $t(G)^1=0$, the implication (2) ⇒ (1) will be proved if we show that: (a) if G is a torsion abelian group and $G^1=0$, then G is a Mittag–Leffler module, and (b) if G is an $\aleph_1$-free abelian group, then G is a Mittag–Leffler module.

(a) Let G be a torsion abelian group with $G^1=0$. Without loss of generality we can suppose that G is a p-group for some prime p. In order to prove that G is a Mittag–Leffler module it is sufficient to show that every countable subset of G is contained in a pure-projective countably generated pure subgroup of G. By [8, Proposition 26.2] every countable subset of G is contained in a countably generated pure subgroup H of G. Since $G^1=0$, H is a countable p-group with no nonzero element of infinite height. Therefore, H is a direct sum of cyclic groups by [8, Theorem 17.3]. In particular, H is pure-projective.

(b) Let G be an $\aleph_1$-free abelian group. In order to prove that G is a Mittag–Leffler module it is sufficient to show that every countable subset of G is contained in a pure-projective countably generated pure subgroup of G. By [8, Proposition 26.2] every countable subset of G is contained in a countably generated pure subgroup H of G. Since G is $\aleph_1$-free and H is a countable subgroup, H is free. In particular H is pure-projective. □

Remark. We have already observed that the abelian group $\mathbb{Z}^X$ is a Mittag–Leffler $\mathbb{Z}$-module, which is not free if X is infinite. This can be generalized as follows: "Let R be a right Noetherian ring. Then: (1) for every set X, the left R-module R^X is a Mittag–Leffler module; (2) the left R-module R^X is pure-projective for every set X if and only if R is a left perfect ring." *Proof.* In [12, Seconde partie, 2.4.2] it is proved that every R^X is a flat Mittag–Leffler left R-module if R is right Noetherian. In particular, because of flatness, all the R^X are pure-projective if and only if they are all projective, that is, if and only if R is left perfect and right coherent [6, Theorem 3.3]. But R is right Noetherian and therefore right coherent.

In the following theorem we give some further characterizations of the class of rings described in Proposition 5 via Mittag–Leffler modules:

Theorem 8. *Let R be a ring. The following statements are equivalent*:

(1) *Every left R-module is pure-projective, i.e., R has left pure global dimension zero.*

(2) *Every left R-module is a Mittag–Leffler module.*

(3) *Every pure epimorphic image of a Mittag–Leffler left R-module is a Mittag–Leffler module.*

(4) *Every strictly indecomposable left R-module is a Mittag–Leffler module.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial.

(3) ⇒ (4). If M is a strictly indecomposable left R-module, M is a pure epimorphic image of a pure-projective module P. In particular, P is a Mittag-Leffler module. By condition (3), M itself must be a Mittag-Leffler module.

(4) ⇒ (1). Since every countably generated Mittag-Leffler module is pure-projective (see our remark after the statement of Theorem 6), we can apply Proposition 4 to the class $\mathscr{C}$ of all countably generated left R-modules. It follows that R is left Artinian and every countably generated indecomposable left R-module is finitely generated. Let M be a strictly indecomposable left R-module that is not pure-projective. Then M is not finitely presented, so that M is not finitely generated because R is left Artinian. Then it is easy to see that M has a countably generated submodule M_0 that is not finitely generated. Since M is a Mittag-Leffler module, by condition (4) of Theorem 6 there exists a countably generated pure submodule N of M such that $M_0 \subseteq N$. Since M is strictly indecomposable, its pure submodule N is also strictly indecomposable and therefore N is finitely generated as we saw above. Since R is left Artinian, the submodule M_0 of the finitely generated module N is also finitely generated. This contradiction shows that every strictly indecomposable left R-module is pure-projective. By Proposition 5 it follows that every left R-module is pure-projective. □

Recall that a ring R is said to be *of finite representation type* in case it is left Artinian and has only finitely many indecomposable modules up to isomorphism.

Corollary 9. *Let R be a ring with identity. The following statements are equivalent*:

(1) *The ring R is of finite representation type.*

(2) *For every family $\{A_i\}_{i\in I}$ of right R-modules and every family $\{B_j\}_{j\in J}$ of left R-modules, the canonical homomorphism*

$$\left(\prod_{i\in I} A_i\right) \otimes_R \left(\prod_{j\in J} B_j\right) \to \prod_{(i,j)\in I\times J} \left(A_i \otimes_R B_j\right)$$

is a monomorphism.

Proof. It is well known that R is of finite representation type if and only if every left and every right R-module is pure-projective [7]. Condition (2) restricted to the case of a family $\{B\}$ consisting of a single left R-module B states that every left R-module B is a Mittag-Leffler module. By Theorem 8, every left module B is pure-projective. Similarly, every right R-module A is pure-projective. Therefore, R is of finite representation type.

Conversely, if R is a ring of finite representation type, every right and every left R-module is pure-projective, so they all are Mittag-Leffler modules. Therefore, if $\{A_i\}_{i\in I}$ is a family of right R-modules and $\{B_j\}_{j\in J}$ is a family of left R-modules, the canonical homomorphisms

$$\left(\prod_{i\in I} A_i\right) \otimes_R \left(\prod_{j\in J} B_j\right) \to \prod_{i\in I}\left(A_i \otimes_R \left(\prod_{j\in J} B_j\right)\right)$$

and

$$A_i \otimes_R \left(\prod_{j\in J} B_j\right) \to \prod_{j\in J}\left(A_i \otimes_R B_j\right)$$

are monomorphisms for each $i\in I$. It follows immediately that the homomorphism in the statement of the corollary is a monomorphism. □

We conclude by considering the case in which the class of Mittag–Leffler modules coincides with the class of pure-projective modules.

Proposition 10. *Let R be a ring with identity. Suppose that every Mittag–Leffler left R-module is pure-projective. Then R is a left perfect ring of left pure global dimension* ≤ 1.

Proof. Recall that a left R-module M is said to be *locally projective* [13] (or a *flat strictly Mittag–Leffler module* [12]) if the following property holds: for any epimorphism $f: A\to C$, any homomorphism $h: M\to C$ and any finitely generated submodule M_0 of M, there exists a homomorphism $g: M\to A$ such that the restrictions of fg and h to M_0 coincide. It is easy to see that every locally projective module is a flat Mittag–Leffler module. Therefore, if every Mittag–Leffler left R-module is pure-projective, then every locally projective left R-module is a pure-projective flat module, that is, a projective module. It follows from [15, Proposition 33, p. 61] that R is left perfect.

Let M be a left R-module. Then there exists a pure exact sequence of left R-modules $0\to K\to P\to M\to 0$ with P pure-projective. Since the sequence is pure and P is a Mittag–Leffler module, it follows that K also is a Mittag–Leffler module. By our assumption, K is pure-projective. This shows that the pure-projective dimension of M is ≤ 1, so that the left pure global dimension of R is ≤ 1. □

Corollary 11. *Let R denote an algebra over an uncountable algebraically closed field k. Suppose that either*

(a) *R is a hereditary finite-dimensional k-algebra, or*

(b) *R is a radical-squared zero finite-dimensional k-algebra, or*

(c) $R=k[\Gamma]$ *for a connected quiver* Γ *(with or without cycles), or*

(d) *R is a finite-dimensional local k-algebra, or*

(e) *R is a finite-dimensional commutative k-algebra.*

Then if every Mittag–Leffler left R-module is pure-projective, R must be of finite representation type.

Proof. By Proposition 10, the left pure global dimension of R is ≤ 1. If conditions (a) or (b) hold, then R is of finite representation type by [4, Theorem 3.4]. If condition (c) holds, then either Γ is a Dynkin diagram or Γ is an oriented cycle by [4, Corollary 3.5]. If Γ is an oriented cycle, then $R=k[\Gamma]$ is two-sided Noetherian. Then R^X is a Mittag-Leffler R-module for every set X by the remark after the proof of Proposition 7. Therefore, R^X is pure-projective, so that R is a left perfect ring by the same remark. Since a left Noetherian left perfect ring is left Artinian, R must be a left Artinian ring and this is a contradiction. Therefore, Γ cannot be an oriented cycle and must be a Dynkin diagram, so that $R=k[\Gamma]$ is of finite representation type.

If condition (d) holds, then either R is of finite representation type or left p.gl.dim $R\geq 2$ [4, Proposition 5.3], so that we conclude by Proposition 10. Finally, if condition (e) holds, R is the direct product of a finite number of local k-algebras, so that we are led again to the case (d). □

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