Integration on manifolds by mapped low-discrepancy points and greedy minimal k_s-energy points

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1. Low-Discrepancy Points

For the QMC method, because of Koksma-Hawka inequality (cf. [3]) it is natural to use low-discrepancy sequences to integrate functions in $[0,1)^d$.
Low-discrepancy sequences are those whose star discrepancy has order $\log(N)^d/N$.
Some examples of low-discrepancy sequences are the Halton sequence, Hammersley point sets, Sobol sequences or the Fibonacci lattice (see e.g. [4]).

2. Preserving Measure Maps

Let us consider the measure $\mathcal{H}_2$ on a manifold $\mathcal{M}$ of dimension 2 which, by means of the area formula [1] is, with respect to the Lebesgue measure λ_2 on a rectangle $\mathcal{U} \subset \mathbb{R}^2$, of the type

$$\int_{\mathcal{U}} g(\boldsymbol{x}) \mathrm{d}\lambda_2(\boldsymbol{x}),$$

with g a density function that depends on the parametrization Φ of $\mathcal{M}$.
Then, we look for a *change of variables*

$$\begin{array}{cccc} \Psi: & \mathcal{U}' & \longrightarrow & \mathcal{U} \\ & \boldsymbol{x}' & \longrightarrow & \Psi(\boldsymbol{x}') = \boldsymbol{x}, \end{array} \qquad (1)$$

with $\mathcal{U}'$ a rectangle of $\mathbb{R}^2$. And we require that

$$g(\Psi(\boldsymbol{x}'))|J\Psi(\boldsymbol{x}')| = g(\boldsymbol{x}) = 1, \qquad (2)$$

in order to have

$$\mathcal{H}_2(\mathcal{M}) = \mu(\mathcal{M}),$$

where $\mu(A) := \lambda_2(\Phi^{-1}(A)) = \int_{\Phi^{-1}(A)} \mathrm{d}\lambda_2$.
The points $\Phi(\mathcal{S})$, where $\mathcal{S}$ are uniformly distributed in the rectangle $\mathcal{U}$, are naturally uniformly distributed with respect to the measure μ on the manifold $\mathcal{M}$.
Then, by using the change of variables (1) we will have that the sequence $\Phi(\Psi(\mathcal{S}'))$, where $\mathcal{S}'$ is a sequence uniformly distributed on $\mathcal{U}'$ with respect the Lebesgue measure, will be **uniformly distributed with respect to the measure $\mathcal{H}_2$ on $\mathcal{M}$.**

3. Greedy k_s-energy Points

Using a greedy algorithm we could have an approximation of the minimal Riesz s-energy points.
Let $k : X \times X \to \mathbb{R} \cup \{\infty\}$ be a simmetric kernel on a locally compact Hausdorff space X, and let $A \subset X$ be a compact set. A sequence $(a_n)_{n=1}^{\infty} \subset A$ is called a greedy minimal k-energy sequence on A if it is generated in the following way:
(i). a_1 is selected arbitrarily on A.
(ii). Assuming that $a_1, \dots, a_n$ are already selected, then a_{n+1} is chosen in such way that it satisfies
$\sum_{i=1}^n k(a_{n+1}, a_i) = \inf_{x \in A} \sum_{i=1}^n k(x, a_i)$, for every $n \geq 1$.
For the M. Riesz kernel in $X = \mathbb{R}^{d'}$ which depends on a parameter s in $[0, +\infty)$ we set

$$k_s(\boldsymbol{x}, \boldsymbol{y}) := K_s(|\boldsymbol{x} - \boldsymbol{y}|),\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{d'},$$

where $|\cdot|$ is the Euclidean norm and

$$K_s(t) := \begin{cases} t^{-s} & \text{if } s > 0 \\ -\log(t) & \text{if } s = 0, \end{cases}$$

For $k = k_s$ we obtain the **greedy k_s-energy points**

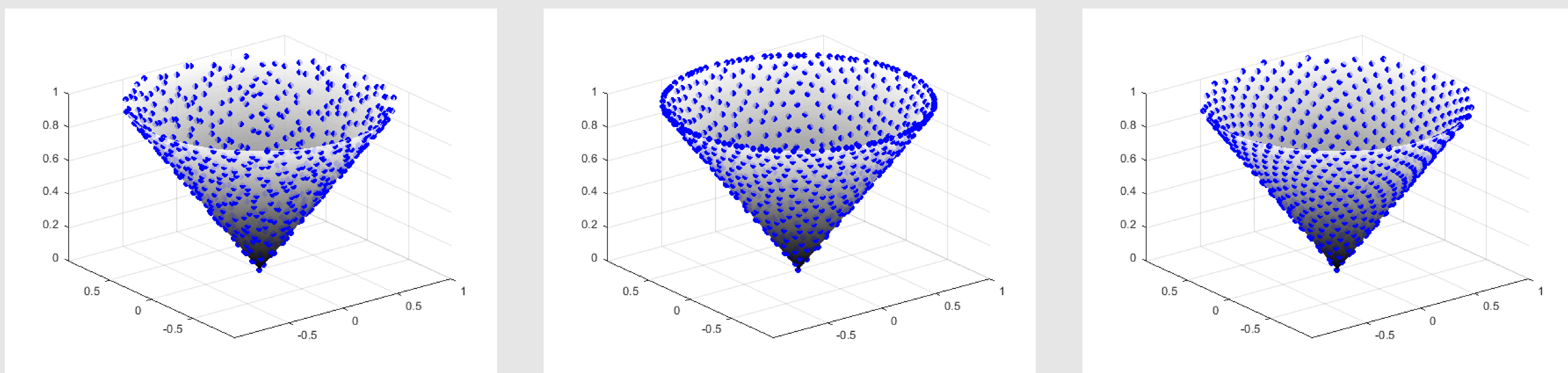
4. Numerical Experiments

Let be.

$$f_1(x,y,z) := \begin{cases} \cos(30xyz) & \text{if } z < \frac{1}{2} \\ (x^2+y^2+z^2)^{3/2} & \text{if } z \geq \frac{1}{2}, \end{cases}$$

$$f_2(x,y,z) := e^{-\sin(2x^2+3y^2+5z^2)}.$$

To compute the integrals
$\frac{1}{\mathcal{H}_d(\mathcal{M})} \int_{\mathcal{M}} f_i(\boldsymbol{x}) \mathrm{d}\mathcal{H}_d(\boldsymbol{x}), \quad i = 1, 2$
we use a QMC method with
(a). low discrepancy points mapped on the manifolds,
(b). greedy minimal k_s-energy points.



(a) Halton points (b) Greedy minimal k_2-energy points (c) Fibonacci points

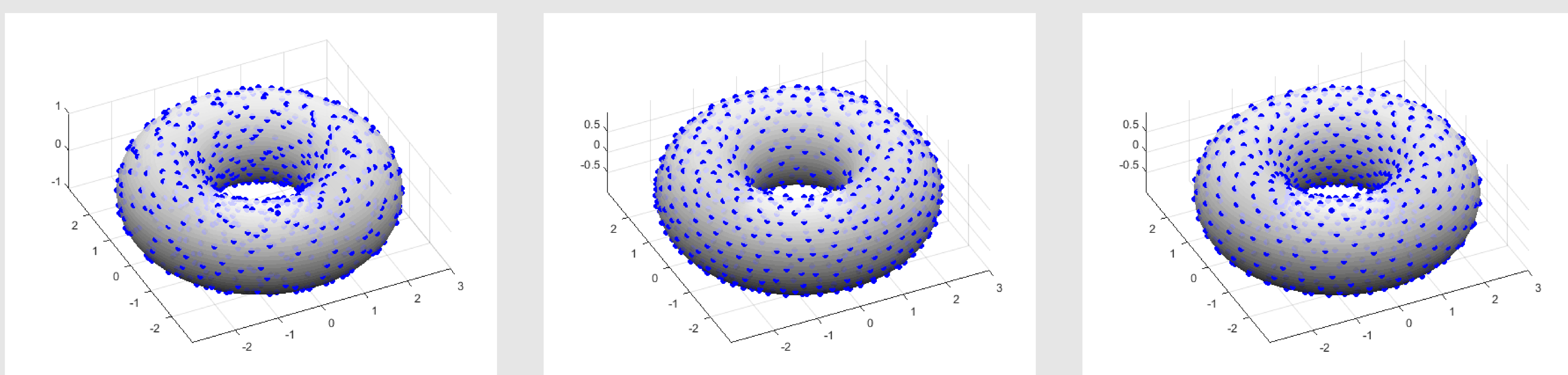
Figure 1: 610 points on the cone

N	Halton	Fibonacci	GM k_2
144	1.476e-01	1.089e-02	8.406e-02
610	4.415e-02	8.045e-05	4.872e-03
2584	6.847e-04	3.115e-06	5.177e-03

Table 1: Relative errors for f_1 on the cone with Fibonacci, Halton and Greedy Minimal k_2-energy points

N	Halton	Fibonacci	GM k_2
144	3.282e-03	1.025e-04	2.834e-03
610	1.755e-03	5.727e-06	2.251e-03
2584	7.294e-05	3.190e-07	1.325e-03

Table 2: Relative errors for f_2 on the cone with Fibonacci, Halton and Greedy Minimal k_2-energy points



(a) Halton points (b) Greedy minimal k_2-energy points (c) Fibonacci points

Figure 2: 610 points on the torus

N	Halton	Fibonacci	GM k_2
144	1.218e-01	1.690e-01	3.081e-02
610	1.453e-01	1.410e-01	4.728e-02
2584	1.414e-01	1.411e-01	2.297e-02

Table 3: Relative errors for f_1 on the torus with Fibonacci, Halton and Greedy Minimal k_2-energy points

N	Halton	Fibonacci	GM k_2
144	3.033e-02	3.426e-02	4.949e-03
610	2.716e-03	8.821e-03	1.349e-02
2584	8.763e-03	6.453e-03	1.673e-03

Table 4: Relative errors for f_2 on the torus with Fibonacci, Halton and Greedy Minimal k_2-energy points

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