A MODEL PROBLEM FOR MEAN FIELD GAMES
ON NETWORKS

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Abstract. In [14], Guéant, Lasry and Lions considered the model problem
"What time does meeting start?" as a prototype for a general class of op-
timization problems with a continuum of players, called Mean Field Games
problems. In this paper we consider a similar model, but with the dynamics of
the agents defined on a network. We discuss appropriate transition conditions
at the vertices which give a well posed problem and we present some numerical
results.

1. **Introduction.** The study of pedestrian flow in a crowd environment is attract-
ing an increasing interest and some models based on optimization principles have
been recently proposed, see for example [4, 8, 20]. In some applications (crowd
motion in shopping centers, stations, airports) the dynamics of the population is
defined on a network rather than in an Euclidean domain. There is a large literature
concerning vehicular traffic on road networks (see [11] and reference therein). These
models are based on a fluid-dynamical approach with the dynamics described by
some nonlinear conservation law and appropriate transition conditions at the junc-
tions modeling the interactions of the cars coming from different roads.

Vehicular traffic models do not seem to be adequate to reproduce the pedestrian
flow since they do not take into account the interactions and the goal-directed
decisions of the agents.

The aim of this paper is to study a simple optimization model for the evolution
of a large number of agents moving on a network. The model is based on the
one described by Guéant, Lasry and Lions in [14], titled "What time does meeting

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start?", and consists in finding the optimal arrival time at a place where the meeting is being held with the starting time defined by means of a quorum rule. In other words, we consider the following problem: a meeting is scheduled in a point v_0 of a network at time t_0 and it cannot start before that a certain percentage of the numerous participants is really present. The dynamics of the participants are restricted to the network and they are affected by some random noise. The purpose of each participant is to minimize the sum of two costs: the cost of waiting the starting of the meeting and the reputation cost of being late. This problem can be considered as a prototype for a large class of optimization problems based on the Mean Field Game (MFG) theory. This theory has been introduced by Lasry and Lions [17] (see also [1], [6], [12]) with the aim of describing the behavior of very large number of agents who take decisions in a context of strategic interactions.

The main novelty of our problem is due to the presence of transition vertices (namely, points of the network where several edges meet each other); indeed, the issue is to impose transition conditions both to the Hamilton-Jacobi-Bellmann equation and to the Fokker-Planck equation in order to obtain a well posed MFG problem. It is known that a parabolic equation on a network has to be complemented with the usual initial-boundary conditions and some transition conditions at the internal vertices (see [3, 18]). In fact, in our model the stochastic differential equation describing the motion of the agent inside the arcs is coupled with a condition prescribing the probability that it enter in a given edge when it occupies a transition vertex; this fact give rise to a Kirchhoff type condition (see [9]). Using an appropriate change of variable we transform the original MFG system in a forward-backward system of two heat equations coupled via the initial datum. Relying on classical results for the heat equation on networks and some appropriate estimates for the specific problem, we prove the well-posedness of the heat system and the existence of a mean field for the quorum problem. Going back to the original MFG problem we obtain existence and uniqueness of the solution to a system composed by a *backward* Hamilton-Jacobi-Bellman equation and a *forward* Fokker-Planck on the arcs with transitions conditions expressing respectively the probability that a single agent enters a given arc and the conservation of the density of the agents through a vertex.

Another issue of this paper is to present some numerical methods for computing the mean field; our scheme is based on a finite difference technique for both the equations and on a discretization of the two transition conditions.

As a matter of fact, the approach of this paper strongly relies on the aforementioned change of variables: we refer the reader to future works for more general problems, possibly also the congestion one. Moreover, we believe that a convergence analysis of the numerical scheme could be carried out using some arguments of [15] and monotonicity property. However, since some technicalities have to be checked, this issue will be the subject of a separate paper.

The paper is organized as follows. In Section 2 we describe the model problem. In Section 3 we prove some technical results concerning the heat equation on the network which are used in Section 4 to show the existence of the mean field. in Section 5, we illustrate the problem with some numerical examples. Finally, the Appendix contains some technical proofs.

Notations. A network $\Gamma = (V, E)$ is a finite collection of points $V := \{v_i\}_{i \in I}$ in $\mathbb{R}^n$ connected by continuous, non self-intersecting arcs $E := \{e_j\}_{j \in J}$. Each arc e_j is parametrized by a smooth function $\pi_j : [0, l_j] \to \mathbb{R}^n$, $l_j > 0$.

For $i \in I$ we set $Inc_i := \{j \in J \mid e_j \text{ is incident to } v_i\}$. We denote by $I_B := \{i \in I \mid \#Inc_i = 1\}$, $I_T := I \setminus I_B$, by $\partial\Gamma := \{v_i \in V \mid i \in I_B\}$, the set of boundary vertices of Γ, and by $\Gamma_T := \{v_i \mid i \in I_T\}$, the set of transition vertices.

The network is not oriented, but the parametrization of the arcs induces an orientation which can be expressed by the *signed incidence matrix* $A = \{a_{ij}\}$ with

$$a_{ij} := \left\{ \begin{array}{rl} 1 & \text{if } v_i \in e_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in e_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & \text{otherwise.} \end{array} \right.$$

In the following we always identify $x \in e_j$ with $y = \pi_j^{-1}(x) \in [0, l_j]$. For any function $u : \Gamma \to \mathbb{R}$ and each $j \in J$ we denote by $u_j : [0, l_j] \to \mathbb{R}$ the restriction of u to e_j, i.e. $u_j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$. For $\gamma \in \mathbb{N}$, we define differentiation along an edge e_j by

$$\partial_j^\gamma u(x) := \frac{d^\gamma u^j}{dy^\gamma}(y), \qquad \text{for } y = \pi_j^{-1}(x),\, x \in e_j$$

and at a vertex v_i by

$$\partial_j^\gamma u(v_i) := \frac{d^\gamma u^j}{dy^\gamma}(y) \qquad \text{for } y = \pi_j^{-1}(v_i),\, j \in Inc_i.$$

2. **The model problem.** Following [14], we describe the model "What time does meeting start?" with the variant that the dynamics of the agents are defined on a network Γ. For the sake of simplicity, we assume that the place where the meeting is being held is the unique boundary vertex, namely $\partial\Gamma = \{v_0\}$; the general case can be dealt with by using easy adaptations. The meeting is scheduled at a certain time t_0 but common experience says that in general it starts at a time T *greater than* t_0, when a certain rule is reached, for example the presence of a certain percentage of participants.

At the initial time there is a continuum of indistinguishable players distributed according to a distribution function $m_0 : \Gamma \to \mathbb{R}$. The player's dynamics is subject to random perturbations. We assume that, inside each edge e_j, the generic agent moves according to the process

$$dX(t) = a(t)dt + \sigma dW(t) \tag{1}$$

where the drift a is the control variable (and it coincides with the speed), $\sigma = (\sigma_j)_{j \in J}$ with $\sigma_j > 0$ and W is a Brownian process, which is an independent disturbance for each player. Moreover we assume that, at each transition vertex v_i, it spends zero time a.s. and it enters in one of the incident edges with the same probability $1/\#(Inc_i)$ (see [9, 10] for stochastic differential equations on networks). We denote by τ the random time the agent reaches v_0, i.e.

$$\tau := \inf\{t > 0 : X(t) \in \partial\Gamma\}.$$

Moreover each player wants to optimize its arrival time τ taking into account various parameters, which are encoded in the cost functional

$$J(x, t, a(\cdot)) = \int_t^{\tau \wedge T_{max}} \frac{1}{2} a^2(t)dt + c(\tau \wedge T_{max}) \tag{2}$$

where $\frac{1}{2}a^2(t)$ is the actual cost of moving along the network at the velocity a while c is the final cost and $T_{max} \in \mathbb{R}$ is a time which cannot be exceeded for the end of

the meeting. The cost function $c : [0, T_{max}] \to \mathbb{R}$ is given by

$$c(s) = c_1(s - t_0) + c_2(s - T) + c_3(T - s), \qquad s \in [0, T_{max}] \qquad (3)$$

where $c_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$ are smooth functions such that $c_i(s) = 0$ for $s \leq 0$ and $c_i(s) > 0$ for $s > 0$. The term $c_1(s - t_0)$ represents a reputation cost of lateness in relation to the scheduled time t_0; the term $c_2(s - T)$ a cost of lateness in relation to the actual starting time of the meeting T; $c_3(T - s)$ a waiting time cost which corresponds to the time lost waiting the starting of the meeting. It is worth noticing that the cost c depends on T via the cost of lateness and the cost of waiting; hence, in order to display this dependence, from now on we write c_T.

Nash equilibrium theory assumes that each player optimizes the arrival time provided that he knows the actual time T of the starting of the meeting. Hence each agent has to solve the optimization problem

$$u(x, t) = \min_{a(\cdot)} J(x, t, a(\cdot)) \qquad (4)$$

where $(x, t) \in \Gamma \times [0, T_{max}]$. Note that $\max_{a \in \mathbb{R}} \{ -ap + \frac{1}{2} |a|^2 \} = -\frac{1}{2} |p|^2$ for $a = -p$ and the optimal control in feed-back form is given by $a^*(x, t) = -\partial_x u(x, t)$. By an application of the Dynamic Programming Principle the value function, if it is assumed to be regular, formally solves the Hamilton-Jacobi-Bellman equation

$$\partial_t u + \nu \partial_x^2 u + \frac{1}{2} |\partial_x u|^2 = 0 \quad (x, t) \in \Gamma \times (0, T_{max}),$$

where $\nu = \sigma^2 / 2$ (i.e., $\nu_j = \sigma_j^2 / 2 \quad \forall j \in J$), with final-boundary conditions and transition on internal vertices (Kirchhoff condition)

$$u(x, T_{max}) = c_T(T_{max}) \quad x \in \Gamma, \qquad u(v_0, t) = c_T(t) \quad s \in [0, T_{max}],$$

$$\sum_{j \in Inc_i} a_{ij} \partial_j u(v_i, s) = 0 \qquad (v_i, s) \in \Gamma_T \times (0, T_{max}).$$

On the other hand, we assume that the evolution of the initial distribution m_0 evolves by duality; therefore, multiplying the Hamilton-Jacobi-Bellman equation by a test function and integrating by parts, we obtain the Fokker-Planck equation inside each edge

$$\partial_t m - \nu \partial_x^2 m - \partial_x((-\partial_x u) m) = 0 \quad (x, t) \in \Gamma \times (0, T_{max})$$

while, taking into account the Kirchhoff condition for u, we infer a Kirchhoff condition on internal vertices

$$\sum_{j \in Inc_i} a_{ij} \nu_j [\partial_j m - m \partial_j u](v_i, s) = 0 \qquad (v_i, s) \in \Gamma_T \times (0, T_{max}).$$

Observe that the previous Kirchhoff condition implies that the parabolic flux of the agents is null at the junctions, giving the conservation of the total mass (see [7] for similar assumptions). Moreover, we assume that m satisfies the initial-boundary condition (with a "smooth fit")

$$m(v_0, s) = 0 \qquad s \in [0, T], \qquad m(x, 0) = m_0(x) \qquad x \in \Gamma.$$

The flow of participants reaching v_0 is given by $s \mapsto \partial_x m(v_0, s)$, hence the cumulative distribution F of the arrival times is

$$F(s) = \int_0^s \nu \partial_x m(v_0, r) dr.$$

The actual starting time T is fixed by a quorum rule, which means that the meeting starts when a given percentage θ of the participants has reached the meeting place v_0. Given m, we set

$$
T = \begin{cases} t_0, & F^{-1}(\theta) \leq t_0 \\ F^{-1}(\theta), & t_0 < F^{-1}(\theta) < T_{max} \\ T_{max}, & F^{-1}(\theta) \geq T_{max}. \end{cases} \tag{5}
$$

Note that T is the mean field, i.e. the information that the single agent has about the behavior of the other agents: the starting rule induces a strategic interactions among the participants and T influences as an external field the decisions of the agents. The main point is to prove the existence and the uniqueness of a time T which is coherent with the expectations of the participants. As in [14], this can be done by proving that the scheme:

$$
T \to u \to m \to T^* \tag{6}
$$

with T^* defined by (5), has a fixed point in $[t_0, T_{max}]$. To this end, it is important to study existence and uniqueness of a solution to the forward-backward system

$$
\begin{cases} \partial_t u + \nu \partial_x^2 u + \frac{1}{2}|\partial_x u|^2 = 0 & (x,s) \in \Gamma \times (0, T_{max}) \\ \partial_t m - \nu \partial_x^2 m + \partial_x(\partial_x u\, m) = 0 & (x,s) \in \Gamma \times (0, T_{max}) \\ \sum_{j \in Inc_i} a_{ij} \partial_j u(v_i, s) = 0 & (v_i, s) \in \Gamma_T \times (0, T_{max}) \\ \sum_{j \in Inc_i} a_{ij} \nu_j [\partial_j m - m \partial_j u](v_i, s) = 0 & (v_i, s) \in \Gamma_T \times (0, T_{max}) \\ m(x,0) = m_0(x),\ u(x, T_{max}) = c_T(T_{max}) & x \in \Gamma \\ m(v_0, s) = 0,\ u(v_0, s) = c_T(s) & s \in [0, T_{max}] \end{cases} \tag{7}
$$

where T is a fixed constant (known by each agent). For the sake of simplicity, from now on we assume

$$
\nu_j = 1 \qquad \forall j \in Inc_i. \tag{8}
$$

As in [14, 15] we apply a change of variable which transforms system (7) into a forward-backward system of heat equations coupled through the initial conditions.

Proposition 1. *If (ϕ, ψ) is a smooth solution of the system*

$$
\begin{cases} -\partial_t \phi - \partial_x^2 \phi = 0 & (x,s) \in \Gamma \times (0, T_{max}), \\ \partial_t \psi - \partial_x^2 \psi = 0 & (x,s) \in \Gamma \times (0, T_{max}) \\ \sum_{j \in Inc_i} a_{ij} \partial_j \phi(v_i, s) = 0 & (v_i, s) \in \Gamma_T \times (0, T_{max}) \\ \sum_{j \in Inc_i} a_{ij} \partial_j \psi(v_i, s) = 0 & (v_i, s) \in \Gamma_T \times (0, T_{max}) \\ \psi(x,0) = \frac{m_0(x)}{\phi(x,0)},\ \phi(x, T_{max}) = e^{c_T(T_{max})} & x \in \Gamma \\ \psi(v_0, s) = 0,\ \phi(v_0, s) = e^{c_T(s)} & s \in [0, T_{max}] \end{cases} \tag{9}
$$

with $\phi > 0$, then

$$
(u, m) = (2\ln(\phi), \phi\,\psi) \tag{10}
$$

is a smooth solution of system (7).

Proof. Let (ϕ, ψ) and (u, m) be defined as in the statement. The proofs that (u, m) is a solution to the PDEs and to initial-final-boundary conditions of (7) follow by

easy calculations; hence, we shall omit them. Let us prove that (u, m) verifies the transitions condition of (7). Since $\phi = e^u$, we get

$$0 = \sum_{j \in Inc_i} a_{ij} \partial_j \phi = e^u \sum_{j \in Inc_i} a_{ij} \partial_j u$$

which amounts to the first transition condition in (7). On the other hand, since $\psi = me^{-u}$, we have

$$0 = \sum_{j \in Inc_i} a_{ij} \partial_j \psi = e^{-u} \sum_{j \in Inc_i} a_{ij}(\partial_j m - m \partial_j u).$$

Taking into account the previous relation, we obtain the second transition condition in (7). □

Remark 1. It is worth to observe that, by similar arguments, one can linearize a more general class of MFG systems (see [15]). Actually, assume that ν_j are positive constants and that the cost J in (2) includes a potential term depending on the distribution of other players, i.e.

$$J(x, t, a(\cdot)) = \int_t^{\tau \wedge T_{max}} \left[\frac{1}{2} a^2(t) + f(X(t), m(t)) \right] dt + c(\tau \wedge T_{max}).$$

In this case, in the system (7) the Hamilton-Jacobi-Bellman equation is

$$\partial_t u + \nu \partial_x^2 u + \frac{1}{2} |\partial_x u|^2 = -f(x, m) \quad (x, s) \in \Gamma \times (0, T_{max}),$$

while the Fokker-Planck equation and the boundary-transition conditions are left unchanged. Now, $(\phi, \psi) = (e^{u/\sigma^2}, me^{-u/\sigma^2})$ solve

$$\begin{cases} -\partial_t \phi - \nu \partial_x^2 \phi = -\frac{\phi}{2\nu} f(x, \phi\psi), \quad \partial_t \psi - \nu \partial_x^2 \psi = \frac{\psi}{2\nu} f(x, \phi\psi) \quad \text{in } \Gamma \times (0, T_{max}), \\ \sum_{j \in Inc_i} a_{ij} \partial_j \phi(v_i, s) = \sum_{j \in Inc_i} a_{ij} \nu_j (\phi \partial_j \psi)(v_i, s) = 0 \quad \text{in } \Gamma_T \times (0, T_{max}) \\ \psi(\cdot, 0) = \frac{m_0(\cdot)}{\phi(\cdot, 0)}, \ \phi(\cdot, T_{max}) = e^{\frac{c_T(T_{max})}{\sigma^2}}, \ \psi(v_0, \cdot) = 0, \ \phi(v_0, \cdot) = e^{\frac{c_T(\cdot)}{\sigma^2}}. \end{cases}$$

3. **The heat equation on a network.** In this section we collect some technical results about existence, uniqueness and a priori estimates for classical solutions to (9). These results will be used in the next section to prove the existence of the mean field T. In this section, $T \in (0, T_{max}]$ is a fixed constant known by all the agents.

We introduce some functional spaces on the network. We recall that the Sobolev space $W_{q,(a,b) \times (0,T)}^{2,1}$ (with $q \geq 1$) consists of the elements of $L^q((a, b) \times (0, T))$ having generalized derivatives of the form $\partial_t^r \partial_x^s$ with $2r + s = 2$ and it is endowed with its usual norm (see [16]). For $q \in \mathbb{N}$ and $\alpha \in (0, 1)$, $C^{(q+\alpha)}([a, b])$ stands for the Banach space of q times differentiable functions on $[a, b]$, whose q-th derivative is Hölder continuous with exponent α and it is endowed with the usual Hölder norm $|\cdot|_{[a,b]}^{(q+\alpha)}$. For $\alpha \in (0, 1)$, $C^{(2+\alpha, 1+\alpha/2)}([a, b] \times [0, T])$, with the norm $|\cdot|_{[a,b] \times [0,T]}^{(2+\alpha, 1+\alpha/2)}$, denotes the Banach space of functions $f : [a, b] \times [0, T] \to \mathbb{R}$ whose derivatives $\partial_x^2 f$ and $\partial_t f$ are Holder continuous with exponent α and respectively $\alpha/2$.

Definition 3.1.   i) For $q \in \mathbb{N}$ and $\alpha \in (0, 1)$, we set

$$C^{(q+\alpha)}(\Gamma) := \{ u \in C(\Gamma) \mid \forall j \in J, \ u_j \in C^{(q+\alpha)}([0, l_j]) \}$$

which is a Banach space with respect to its norm $|u|_\Gamma^{(q+\alpha)} := \sup_{j \in J} |u_j|_{[0,l_j]}^{(q+\alpha)}$.

ii) For $\alpha \in (0,1)$, we set

$$C^{(2+\alpha,1+\alpha/2)}(\Gamma \times [0,T]) :=$$

$$\{u \in C(\Gamma \times [0,T]) \mid \forall j \in J,\ u_j \in C^{(2+\alpha,1+\alpha/2)}([0,l_j] \times [0,T])\}$$

which is a Banach space with respect to its norm $|u|^{(2+\alpha,1+\alpha/2)}(\Gamma \times [0,T]) := \sup_{j \in J} |u_j|^{(2+\alpha,1+\alpha/2)}_{[0,l_j] \times [0,T]}$.

In the next proposition we establish the well-posedness of the initial-boundary problem for the heat equation obtained by the Hamilton-Jacobi-Bellman equation of (7) via the change of variable (10).

Proposition 2. *Assume that $w_0 \in C^{(1+\alpha/2)}([0,T_{max}])$, for some $\alpha \in (0,1)$. Then there exists a unique solution $w \in C^{(2+\alpha,1+\alpha/2)}(\Gamma \times [0,T_{max}])$ of the problem*

$$\begin{cases} -\partial_t w - \partial_x^2 w = 0 & (x,s) \in \Gamma \times (0,T_{max}) \\ \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i,s) = 0 & (v_i,s) \in \Gamma_T \times (0,T_{max}) \\ w(v_0,s) = w_0(s) & s \in [0,T_{max}] \\ w(x,T_{max}) = w_0(T_{max}) & x \in \Gamma. \end{cases} \tag{11}$$

Moreover, the following estimate holds

$$|w|^{(2+\alpha,1+\alpha/2)}_{\Gamma \times [0,T_{max}]} \leq K_0 |w_0|^{(1+\alpha/2)}_{[0,T_{max}]} \tag{12}$$

where K_0 is a constant independent of w_0. Finally, for $w_0 > 0$, we have $w \geq \min w_0$ in $\Gamma \times [0,T_{max}]$.

Proof. The statement is an immediate consequence of the result in [2]. Let us just note that the compatibility conditions in [2] are obviously satisfied because the terminal condition is constant and the right-hand side of the Kirchhoff condition is null. Moreover the strict positivity of w is a consequence of the comparison principle for classical solution of the heat equation (see [3]). We observe that it can be proved using the same arguments of [15, Proposition 2]. □

Since v_0 is a boundary vertex, there exists a unique edge, say e_0 incident to it. Without any loss of generality, we denote v_1 the other endpoint of e_0 and we assume that the parametrization of e_0 fulfills:

$$\pi_0(0) = v_0 \quad \text{and} \quad \pi(l_0) = v_1. \tag{13}$$

For $\lambda \in (0,1)$, we set

$$e_{0,\lambda} := \pi_0([0,\lambda l_0]), \qquad v'_\lambda := \pi_0(\lambda l_0) \tag{14}$$

namely, v'_λ is a point in the edge e_0 while $e_{0,\lambda}$ is the part of e_0 between v_0 and v'_λ.

In the next proposition, we establish existence and uniqueness of a classical solution to the heat equation obtained by the Fokker-Planck equation of (7) via (10). Moreover we show a "weak" continuous dependence estimate in the sub-edge $e_{0,1/2}$ with respect to the initial datum $\mu(\cdot)/w(\cdot,0)$ where w is the solution of (11).

Proposition 3. *Let w be the solution of problem (11) and assume*

$$\mu_0 \in C^{(2+\alpha)}(\Gamma), \quad \text{with } \mu_0(v_0) = 0. \tag{15}$$

Then there exists a unique solution $\mu \in C^{2,1}(\Gamma \times (0, T_{max})) \cap C^0(\Gamma \times [0, T_{max}])$ of the problem

$$
\begin{cases}
\partial_t \mu - \partial_x^2 \mu = 0 & (x, s) \in \Gamma \times (0, T_{max}) \\
\sum_{j \in Inc_i} a_{ij} \partial_j \mu(v_i, s) = 0 & (v_i, s) \in \Gamma_T \times (0, T_{max}) \\
\mu(v_0, s) = 0 & s \in [0, T_{max}] \\
\mu(x, 0) = \frac{\mu_0(x)}{w(x,0)} & x \in \Gamma.
\end{cases}
\tag{16}
$$

Moreover, for every $q \geq 1$, the following estimate holds

$$
|\mu|_{q, e_{0,1/2} \times [0, T_{max}]}^{2,1} \leq K_1 |\mu_0 / w(\cdot, 0)|_{\Gamma}^{(2+\alpha)}
\tag{17}
$$

where K_1 is a constant independent of μ_0 and w.

The proof is postponed to the Appendix.

In the next proposition, we establish two continuous dependence estimates for the solution of problem (16) with respect to the initial datum: the former is a "strong" estimate in the sub-edge $e_{0,1/2}$ while the latter is the classical estimate in the whole network.

Proposition 4. *Let μ be the solution to* (16). *Besides the hypotheses of Proposition 3, assume*

$$
\partial_x \mu_0(v_0) = \partial_x^2 \mu_0(v_0) = 0.
\tag{18}
$$

i) *There holds*

$$
|\mu|_{e_{0,1/2} \times [0, T_{max}]}^{(2+\alpha, 1+\alpha/2)} \leq K_2 |\mu_0 / w(\cdot, 0)|_{\Gamma}^{(2+\alpha)}
\tag{19}
$$

where K_2 is a constant independent of μ_0 and w.

ii) *Under the further assumption*

$$
\partial_j \mu_0(v_i) = \partial_j^2 \mu_0(v_i) = 0 \qquad \forall i \in I_T, \, j \in Inc_i,
\tag{20}
$$

the function μ belongs to $C^{(2+\alpha, 1+\alpha/2)}(\Gamma \times [0, T_{max}])$ and verifies

$$
|\mu|_{\Gamma \times [0, T_{max}]}^{(2+\alpha, 1+\alpha/2)} \leq K_3 |\mu_0 / w(\cdot, 0)|_{\Gamma}^{(2+\alpha)}
$$

where K_3 is a constant independent of μ_0 and w.

The proof is postponed to the Appendix. Let us now establish a well-posedness result for the system (9).

Theorem 3.2. *Assume that, for some $\alpha \in (0, 1)$, there holds*

$$
c_T \in C^{(1+\alpha/2)}([0, T_{max}]), \quad c_T \geq 0, \qquad m_0 \in C^{(2+\alpha)}(\Gamma) \text{ with } m_0(v_0) = 0.
\tag{21}
$$

Then, there exists a unique classical solution (ϕ, ψ) to the system (9) with $\phi > 0$. Moreover, the following estimates hold

(i) $\phi \geq 1$, $|\phi|_{\Gamma \times [0, T_{max}]}^{(2+\alpha, 1+\alpha/2)} \leq K |c_T|_{[0, T_{max}]}^{(1+\alpha/2)}$, $|\psi|_{q, e_{0,1/2} \times [0, T_{max}]}^{2,1} \leq K |m_0 / \phi(\cdot, 0)|_{\Gamma}^{(2+\alpha)}$;

(ii) *if m_0 fulfills* (18): $|\psi|_{e_{0,1/2} \times [0, T_{max}]}^{(2+\alpha, 1+\alpha/2)} \leq K |m_0 / \phi(\cdot, 0)|_{\Gamma}^{(2+\alpha)}$;

(iii) *if m_0 fulfills* (18) *and* (20): $|\psi|_{\Gamma \times [0, T_{max}]}^{(2+\alpha, 1+\alpha/2)} \leq K |m_0 / \phi(\cdot, 0)|_{\Gamma}^{(2+\alpha)}$;

where K is a constant independent of m_0 and c_T.

Proof. Proposition 2 ensures all the part of the statement concerning the function ϕ. Invoking Proposition 3 (respectively, Proposition 4-(i) and -(ii)), by the regularity and the lower bound of ϕ, we deduce the part of the statement concerning the function ψ in point (i) (respectively, in point (ii) and in point (iii)). $\square$

We also have existence and uniqueness for the solution to (7):

Corollary 1. *Under the hypotheses of Theorem 3.2, there exist a unique classical solution to the system* (7).

Being a straightforward consequence of the previous theorem, the proof of this result is omitted.

4. **The Mean Field Game result.** We prove the existence of a starting time T consistent with the corresponding flux of participants $\partial_x m$. To this end we show that the map from $[t_0, T_{max}]$ into itself, defined by the scheme (6) is continuous and therefore it admits a fixed point by the Brouwer's Theorem. For simplicity, we shall recast it in terms of couple (ϕ, ψ) solution of (9). Consider the function $\Psi : [t_0, T_{max}] \to [t_0, T_{max}]$ defined as

$$T \to c_T \to \phi \to \psi \to T^* =: \Psi(T) \tag{22}$$

where T^* is defined as in (5) with

$$F(s) = \int_0^s e^{c_T(r)} \, \partial_x \psi(v_0, r) \, dr =: \int_0^s \tilde{\psi}_T(r) \, dr. \tag{23}$$

In this section we assume the hypotheses of Theorem 3.2 and that the map

$$T \in [0, T_{max}] \mapsto c_T \in C^{(1+\alpha/2)}([0, T_{max}]) \tag{24}$$

is continuous. A crucial step to prove the existence of the mean field T is to establish some bounds for $\partial_x \psi(v_0, \cdot)$. In order to get such an estimate, we consider in the next Lemma two complementary assumptions.

Lemma 4.1. *Let* (ϕ, ψ) *be the solution to system* (9).
  (a) *If*
$$\partial_x m_0(v_0) > 0, \tag{25}$$
  then, there exists a value $\varepsilon > 0$, *independent of* T, *such that*
$$|\partial_x \psi(v_0, t)| > \varepsilon \qquad \forall t \in [0, T_{max}].$$
  (b) *If* m_0 *fulfills* (18), *then there holds:*
$$\partial_x \psi(v_0, t) > 0 \qquad \forall t \in (0, T_{max}].$$
  In particular, there exists a constant ε_T *such that*
$$|\partial_x \psi(v_0, t)| > \varepsilon_T \qquad \forall t \in [t_0, T_{max}].$$

Proof. (a). Owing to (21), the function m_0 satisfies: $m_0(v_0) = 0$ and $\partial_x m_0(v_0) > 0$. Moreover, Proposition 2 ensures that $|\frac{m_0}{\phi(\cdot,0)}|_{[0,l_0]}^{(2+\alpha)}$ is bounded independently of T. We infer that there exist $\xi_0 \in (0, l_0)$ and a sufficiently small $a > 0$ such that, for every $T \in [0, T_{max}]$ there holds

$$\frac{m_0(x)}{\phi(x, 0)} \geq a \sin\left(\frac{x\pi}{\xi_0}\right) \qquad \forall x \in [0, \xi_0].$$

One can easily check that the function

$$v(x, t) := a e^{bt} \sin(x\pi/\xi_0), \qquad \text{with} \qquad b := -\pi^2/\xi_0^2$$

solves the initial-boundary value problem

$$\begin{cases} \partial_t v - \partial_x^2 v = 0 & (x, t) \in (0, \xi_0) \times (0, T_{max}) \\ v(0, t) = v(\xi_0, t) = 0 & t \in (0, T_{max}) \\ v(x, 0) = a \sin(x\pi/\xi_0) & x \in (0, \xi_0) \end{cases}$$

while the function ψ is a supersolution to this problem. By the standard comparison principle, we infer: $\psi \geq v$ in $[0, \xi_0] \times [0, T_{max}]$. Since $\psi(0, \cdot) = v(0, \cdot)$ on $[0, T_{max}]$, we get $\partial_x \psi(0, t) \geq \partial_x v(0, t) = ae^{bt}\pi/\xi_0$. In particular, we deduce

$$|\partial_x \psi(0, t)| \geq ae^{bT_{max}}\pi/\xi_0 \qquad \forall t \in [0, T_{max}]$$

where all the constants are independent of T.

(*b*) . Being nonnegative, the function ψ attains a global minimum at each point (v_0, t) with $t \in (0, T_{max}]$. The Hopf Lemma prevents that $\partial_x \psi(v_0, t_0) \leq 0$ in these points. Hence, there holds: $\partial_x \psi(v_0, t) > 0$ in $(0, T_{max}]$. The second part of the statement follows by continuity. □

We shall establish the existence of a fixed point provided that m_0 fulfills either (25) or (18). We cope with these two cases separately in the next two statements.

Theorem 4.2. *Assume the hypotheses of Theorem 3.2-(i) and inequality (25). Then the map $\Psi : [0, T_{max}] \to [0, T_{max}]$ defined by (22) admits a fixed point.*

Proof. We shall follow the arguments of [14, Lemma 2.6]. In order to apply the Brouwer fixed point Theorem, we need to prove that the function Ψ defined in (22) is continuous. We consider two admissible flows $\tilde{\psi}_{T_1}, \tilde{\psi}_{T_2}$ (see equation (23) for their definition) and, without any loss of generality, we assume $\Psi(T_1) \leq \Psi(T_2)$. If $\Psi(T_1), \Psi(T_2) \in (t_0, T_{max})$, we have

$$0 = \int_0^{\Psi(T_1)} \tilde{\psi}_{T_1}(t)dt - \int_0^{\Psi(T_2)} \tilde{\psi}_{T_2}(t)dt$$
$$= \int_0^{\Psi(T_1)} (\tilde{\psi}_{T_1}(t) - \tilde{\psi}_{T_2}(t))dt - \int_{\Psi(T_1)}^{\Psi(T_2)} \tilde{\psi}_{T_2}(t)dt$$

(where the first equality is due to the fact that both integrals are equal to θ). Taking into account Lemma 4.1-(*a*), we obtain

$$\varepsilon(\Psi(T_2) - \Psi(T_1)) \leq \int_0^{\Psi(T_1)} (\tilde{\psi}_{T_1}(t) - \tilde{\psi}_{T_2}(t))dt \leq |\tilde{\psi}_{T_1} - \tilde{\psi}_{T_2}|_{L^1(0,T_{max})}.$$

The estimates in Theorem 3.2-(*i*) and the trace theorem (for instance, see [16, Theorem II.2.3]) yield

$$\Psi(T_2) - \Psi(T_1) \leq \text{const.} |c_{T_1} - c_{T_2}|_{[0,T_{max}]}^{(1+\alpha/2)}.$$

Taking into account assumption (24), we obtain that in this case the function Ψ is continuous.

When $\Psi(T_1) = t_0$ (respectively, $\Psi(T_2) = T_{max}$), we have

$$\int_0^{\Psi(T_1)} \tilde{\psi}_{T_1} - \int_0^{\Psi(T_2)} \tilde{\psi}_{T_2} \geq 0;$$

indeed, either $\tilde{\psi}_{T_1}$ is a flux which reaches θ at most at time $\Psi(T_1)$ or $\tilde{\psi}_{T_2}$ is a flux which does not reach the value θ before time T_{max} ; in other words, the former integral is $\geq \theta$ (respectively, the latter one is $\leq \theta$). Hence we can conclude by the same arguments as before. Therefore, the continuity of Ψ is achieved. □

Theorem 4.3. *Assume the hypotheses of Theorem 3.2-(ii). Then the map Ψ defined by (22) admits a fixed point.*

Proof. We shall argue adapting the arguments of Theorem 4.2: hence, our purpose is to prove that Ψ is continuous on $[t_0, T_{max}]$. To this end, let us fix $T \in [t_0, T_{max}]$. For every $T_1 \in [t_0, T_{max}]$ such that $\Psi(T) = \Psi(T_1)$, there is nothing to prove. We split the arguments according to the fact that $\Psi(T)$ belongs to (t_0, T_{max}), to $\{t_0\}$ or to $\{T_{max}\}$.

Case 1. $\Psi(T) \in (t_0, T_{max})$. Consider $T_1 \in [t_0, T_{max}]$ with $\psi(T_1) < \Psi(T)$; set

$$\tau := \inf\{t \in (0, T_{max}) \mid \int_0^t \tilde{\psi}_{T_1} = \theta\} \qquad (26)$$

and observe that $\Psi(T_1) = \max\{t_0, \tau\}$. Then, we have

$$0 = \int_0^\tau \tilde{\psi}_{T_1} - \int_0^{\Psi(T)} \tilde{\psi}_T = \int_0^\tau \left(\tilde{\psi}_{T_1} - \tilde{\psi}_T \right) - \int_\tau^{\Psi(T)} \tilde{\psi}_T$$

(the first equality is due to the fact that both the integrals are equal to θ). By Lemma 4.1-(b), we infer

$$\varepsilon_T \left(\Psi(T) - \Psi(T_1) \right) \leq \int_{\Psi(T_1)}^{\Psi(T)} \tilde{\psi}_T \leq \int_\tau^{\Psi(T)} \tilde{\psi}_T = \int_0^\tau \left(\tilde{\psi}_{T_1} - \tilde{\psi}_T \right)$$

$$\leq |\tilde{\psi}_{T_1} - \tilde{\psi}_T|_{L^1(0, T_{max})}$$

Arguing as before, we deduce that there exists a constant $\tilde{K}$ (depending on T) such that

$$\Psi(T) - \Psi(T_1) \leq \tilde{K}|T_1 - T|. \qquad (27)$$

Consider now a point $T_1 \in [t_0, T_{max}]$ with $\psi(T_1) > \Psi(T)$. Then, we have

$$0 \leq \int_0^{\Psi(T)} \tilde{\psi}_T - \int_0^{\Psi(T_1)} \tilde{\psi}_{T_1} = \int_0^{\Psi(T_1)} \left(\tilde{\psi}_T - \tilde{\psi}_{T_1} \right) + \int_{\Psi(T_1)}^{\Psi(T)} \tilde{\psi}_T$$

where the inequality is due to the fact that the first integral is equal to θ while the second one is less or equal to θ. Again by Lemma 4.1-(b), we infer

$$\varepsilon_T \left(\Psi(T_1) - \Psi(T) \right) \leq \int_{\Psi(T)}^{\Psi(T_1)} \tilde{\psi}_T \leq \int_0^{\Psi(T_1)} \left(\tilde{\psi}_T - \tilde{\psi}_{T_1} \right) \leq |\tilde{\psi}_{T_1} - \tilde{\psi}_T|_{L^1(0, T_{max})}.$$

Arguing as before, for some constant $\tilde{K}'$ (depending on T), we get

$$\Psi(T_1) - \Psi(T) \leq \tilde{K}'|T_1 - T|.$$

By this relation and (27), the proof of the continuity of Ψ in T is accomplished.

Case 2. $\Psi(T) = T_{max}$. For $T_1 \in [t_0, T_{max}]$ with $\Psi(T_1) = T_{max}$, there is nothing to prove; hence, without any loss of generality, we assume that $\Psi(T_1) < T_{max}$. We have

$$0 \leq \int_0^{\Psi(T_1)} \tilde{\psi}_{T_1} - \int_0^{T_{max}} \tilde{\psi}_T = \int_0^{\Psi(T_1)} \left(\tilde{\psi}_{T_1} - \tilde{\psi}_T \right) - \int_{\Psi(T_1)}^{T_{max}} \tilde{\psi}_T$$

Arguing as before, we accomplish the proof in this case.

Case 3. $\Psi(T) = t_0$. For $T_1 \in [t_0, T_{max}]$ with $\Psi(T_1) = t_0$, there is nothing to prove; hence, without any loss of generality, we assume that $\Psi(T_1) > t_0$. We have

$$0 \leq \int_0^{t_0} \tilde{\psi}_T - \int_0^{\Psi(T_1)} \tilde{\psi}_{T_1} = \int_0^{\Psi(T_1)} \left(\tilde{\psi}_T - \tilde{\psi}_{T_1} \right) + \int_{\Psi(T_1)}^{t_0} \tilde{\psi}_T.$$

By the same arguments as those used before, we accomplish the proof. $\qquad\square$

Corollary 2. *Under the hypotheses of either Theorem 4.2 or Theorem 4.3, there exists a value T which is coherent with the expectation of the participants to the meeting.*

We conclude with a uniqueness result for the fixed point under some monotonicity condition on the cost c_T.

Proposition 5. *Assume that the cost c_T does not depend on the term c_2, then the map Ψ defined by (22) admits a unique fixed point.*

Proof. Existence of a fixed point is proved in either Theorem 4.2 or Theorem 4.3. Assume by contradiction that there exist $T_1, T_2 \in [0, T_{max}]$ with $T_1 > T_2$ such that $T_i = \Psi(T_i)$. Let c_{T_i} and (ϕ_i, ψ_i) be the costs and the solutions of (9) corresponding to T_i, $i = 1, 2$. Then, $(\phi, \psi) := (\phi_1 - \phi_2, \psi_1 - \psi_2)$ satisfies (9) with $m_0/\phi(\cdot, 0)$, $e^{c_T(T_{max})}$ and $e^{c_T(\cdot)}$ replaced respectively by $m_0/\phi_1(\cdot, 0) - m_0/\phi_2(\cdot, 0)$, $e^{c_{T_1}(T_{max})} - e^{c_{T_2}(T_{max})}$ and $e^{c_{T_1}(\cdot)} - e^{c_{T_2}(\cdot)}$. We have

$$0 = \int_0^{T_{max}} \int_\Gamma [-\partial_t \phi - \partial_x^2 \phi] \psi \, dx \, dt = \int_0^{T_{max}} \int_\Gamma [\partial_t \psi \, \phi + \partial_x \phi \, \partial_x \psi] dx \, dt$$
$$- \int_\Gamma \left[\psi(x, \cdot) \phi(x, \cdot) \right]_0^{T_{max}} dx - \sum_{i \in I} \sum_{j \in Inc_i} \int_0^{T_{max}} (-a_{ij}) \partial_j \phi(v_i, t) \psi(v_i, t) dt$$

(the term $-a_{ij}$ takes into account the orientation of the edge e_j). Similarly

$$0 = \int_0^{T_{max}} \int_\Gamma [\partial_t \psi - \partial_x^2 \psi] \phi \, dx \, dt = \int_0^{T_{max}} \int_\Gamma [\partial_t \psi \, \phi + \partial_x \phi \partial_x \psi] dx \, dt$$
$$- \sum_{i \in I} \sum_{j \in Inc_i} \int_0^{T_{max}} (-a_{ij}) \partial_j \psi(v_i, t) \phi(v_i, t) dt.$$

Subtracting the previous inequality and using the transition conditions at the internal nodes we get

$$0 = \int_\Gamma \left(\frac{m_0(x)}{\phi_1(x, 0)} - \frac{m_0(x)}{\phi_2(x, 0)} \right) (\phi_1(x, 0) - \phi_2(x, 0)) dx$$
$$- (e^{c_{T_1}(T_{max})} - e^{c_{T_2}(T_{max})}) \int_\Gamma (\psi_1(x, T_{max}) - \psi_2(x, T_{max})) dx$$
$$+ \int_0^{T_{max}} (e^{c_{T_1}(t)} - e^{c_{T_2}(t)})(\partial_0 \psi_1(v_0, t) - \partial_0 \psi_2(v_0, t)) dt$$

(recall that e_0 is the unique arc incident to v_0 parameterized in such a way that v_0 is the initial point).

The first term in the previous inequality is negative. By the assumption on c_T, the map $T \mapsto c_T$ is increasing in T and $c_{T_1}(T_{max}) = c_{T_2}(T_{max})$. Hence the second term is null. Moreover, since $T_1 > T_2$ and therefore $c_{T_1} > c_{T_2}$ on $[0, T_{max}]$, we have $\phi_1 \geq \phi_2$, hence $\psi_1 \leq \psi_2$ and, by $\psi_i(v_0, t) = 0$ for $i = 1, 2$, $\partial_0 \psi_1(v_0, t) \leq \partial_0 \psi_2(v_0, t)$. It follows that also the third term is negative, hence $\phi_1(x, 0) = \phi_2(x, 0)$ for $x \in \Gamma$ and therefore a contradiction to $c_{T_1} > c_{T_2}$. $\square$

5. **Numerical simulation.** In this section we propose a numerical method to compute the mean field T. The scheme is based on a finite difference approximation of the system (9) with an iterative procedure to solve the fixed point map (22).

On each interval $[0, l_j]$, $j \in J$, it is defined an uniform partition $y_k = k h_j$ with space step $h_j = \frac{l_j}{M_j}$ and $k = 0, \dots, M_j$. In this way a spatial grid $\mathcal{G}(\Gamma) = \{x_{j,k} = \pi_j(y_k), \ j \in J, \ k = 0, \dots, M_j\}$ is defined on the network Γ. A time step Δt is also introduced to obtain a uniform grid $t_n = n\Delta t$, $n = 0, 1, \dots, N_{max}$ with $N_{max} = [T_{max}/\Delta t]$, on the time interval $[0, T_{max}]$.

We will approximate the solution (ϕ, ψ) of (9) by two sequences $\{\phi^n\}_n$ and $\{\psi^n\}_n$, where, for each $n = 0, \dots, N_{max}$, $\phi^n, \psi^n : \mathcal{G}(\Gamma) \to \mathbb{R}$ and $\phi_{j,k}^n \simeq \phi(x_{j,k}, t_n)$, $\psi_{j,k}^n \simeq \psi(x_{j,k}, t_n)$. The discrete functions $\{\phi^n\}_n$ and $\{\psi^n\}_n$ are computed by the following forward-backward explicit finite difference scheme:

$$\begin{cases} \phi_{j,k}^n = \phi_{j,k}^{n+1} + \dfrac{\Delta t}{h_j^2} \left(\phi_{j,k+1}^{n+1} - 2\phi_{j,k}^{n+1} + \phi_{j,k-1}^{n+1} \right), & n = N_{max} - 1, \dots, 0 \\[2mm] \psi_{j,k}^{n+1} = \psi_{j,k}^n + \dfrac{\Delta t}{h_j^2} \left(\psi_{j,k+1}^n - 2\psi_{j,k}^n + \psi_{j,k-1}^n \right), & n = 0, \dots, N_{max} - 1 \\[2mm] \text{for } k = 1, \dots, M_j - 1 \text{ and } j \in J. \end{cases} \quad (28)$$

At each time iteration n, to compute $\{\phi^n\}_n$ and $\{\psi^n\}_n$ it is necessary to fix the values of these functions at the boundary of the arcs e_j, $j \in J$, i.e. at the transition vertices v_i, $i \in I_T$. We define an approximation of the Kirchhoff's condition which together with the continuity condition across the vertices will give the $\#(Inc_i)$ conditions necessary to determine in a unique way the value of the functions ϕ^n and ψ^n at v_i.

We introduce two sets of indices $Inc_i^+ = \{j \in J \,|a_{ij} = 1\}$ and $Inc_i^- = \{j \in J \,|a_{ij} = -1\}$. Moreover we denote by $\phi^n(v_i)$, $\psi^n(v_i)$ the values of the functions ϕ^n, ψ^n at $v_i \in V$. If $j \in Inc_i^+$, then $\phi(\pi_j(y_0), t_n) \simeq \phi_{j,0}^n = \phi^n(v_i)$ while if $j \in Inc_i^-$, then $\phi(\pi_j(y_{M_j}), t_n) \simeq \phi_{j,M_j}^n = \phi^n(v_i)$. We define the following finite differences approximations of the derivatives at v_i along an edge e_j:

$$\partial_j \phi(v_i, t_n) \simeq \frac{\phi_{j,1}^n - \phi^n(v_i)}{h_j}, \quad \partial_j \psi(v_i, t_n) \simeq \frac{\psi_{j,1}^n - \psi^n(v_i)}{h_j} \qquad j \in Inc_i^+,$$

$$\partial_j \phi(v_i, t_n) \simeq \frac{\phi_{j,M_j-1}^n - \phi^n(v_i)}{h_j}, \quad \partial_j \psi(v_i, t_n) \simeq \frac{\psi_{j,M_j-1}^n - \psi^n(v_i)}{h_j} \qquad j \in Inc_i^-.$$

We rewrite the transition conditions in (9) as

$$\sum_{j \in Inc_i^+} \partial_j \phi(v_i, s) - \sum_{j \in Inc_i^-} \partial_j \phi(v_i, s) = 0, \qquad (29)$$

$$\sum_{j \in Inc_i^+} \partial_j \psi(v_i, s) - \sum_{j \in Inc_i^-} \partial_j \psi(v_i, s) = 0 \qquad (30)$$

and we consider the following finite difference approximation

$$\sum_{j \in Inc_i^+} \frac{1}{h_j} (\phi_{j,1}^n - \phi^n(v_i)) - \sum_{j \in Inc_i^-} \frac{1}{h_j} (\phi^n(v_i) - \phi_{j,M_j-1}^n) = 0, \qquad (31)$$

$$\sum_{j \in Inc_i^+} \frac{1}{h_j} (\psi_{j,1}^n - \psi^n(v_i)) - \sum_{j \in Inc_i^-} \frac{1}{h_j} (\psi^n(v_i) - \psi_{j,M_j-1}^n) = 0. \qquad (32)$$

Given a discrete function $f : \mathcal{G}(\Gamma) \to \mathbb{R}$, we consider a continuous piecewise linear reconstruction $I[f] : \Gamma \to \mathbb{R}$ such that $I[f] \,|_{(x_{j,k}, x_{j,k+1})}$ is linear for all $j \in J$ and $k = 0, \dots, M_j - 1$ and $I[f](x_{j,k}) = f_{j,k}$. To guarantee the continuity on Γ of the

linear interpolation $I[\cdot]$ applied to the discrete functions ϕ^n and ψ^n, we need to impose the following continuity conditions:

$$\phi_{j,0}^n = \phi^n(v_i), \quad \psi_{j,0}^n = \psi^n(v_i) \qquad \qquad \text{if } i \in I_T, \, j \in Inc_i^+, \qquad (33)$$

$$\phi_{j,M_j}^n = \phi^n(v_i), \quad \psi_{j,M_j}^n = \psi^n(v_i) \qquad \text{if } i \in I_T, \, j \in Inc_i^-. \qquad (34)$$

At each time step t_n, the $\#(Inc_i) - 1$ conditions given by (33)-(34) coupled with (31)-(32) give $\#(Inc_i)$ relations which uniquely determine $\phi^n(v_i)$ and $\psi^n(v_i)$.

Summarizing, we approximate (9) by computing the couple of discrete functions $\{(\phi^n, \psi^n)\}_n$ which solve the finite difference scheme (28) together with

i: the conditions (31)-(34) at the vertices $v_i \in \Gamma_T$;
ii: the boundary condition

$$\phi^n(v_0) = e^{c_T(t_n)}, \quad \psi^n(v_0) = 0 \quad n = 0, \dots, N_{max};$$

iii: the initial and terminal conditions:

$$\phi_{j,k}^{N_{max}} = e^{c_T(T_{max})}, \quad \psi_{j,k}^0 = \frac{m_0(x_{j,k})}{\phi_{j,k}^0}, \quad k = 0, \dots, M_j - 1, \, j \in J.$$

Defined a function $\{\psi^n\}_n$ by means of the previous scheme, we consider the following approximation of the cumulative distribution (23)

$$\tilde{F}(t_n) = \frac{\Delta t}{h_0} \sum_{k=0}^{n} e^{c_T(k\Delta t)} \psi_{0,1}^k, \qquad (35)$$

wher e_0 denotes the edge incident v_0 with $\pi_0(0) = v_0$ (here, the boundary condition $\psi_{0,0}^k = \psi^k(v_0) = 0$ is taken into account).

To approximate the fixed point of the map Ψ defined in (22) we apply the following Algorithm 1. Given an initial guess T_1 and denoted by T_2 an initial value to enter the loop and by τ as threshold for the stopping criteria, we consider

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Algorithm 1: Fixed Point Iterations

Data: initial guess T_1, T_2, threshold value τ
Result: approximated mean field T_2
while $|T_1 - T_2| > \tau$ **do**
   set $T_1 \leftarrow T_2$;
   solve (28) with $T = T_1$ and conditions $i), ii), iii)$;
   compute
$$T_{N^*} = \min\{n\Delta t, n = 0, \dots, N_{max} | \tilde{F}(t_n) > \theta\}$$
   if $T_{N^*} < t_0$ **then**
    | set $T_2 \leftarrow t_0$;
   else
    └ set $T_2 \leftarrow T_{N^*}$;

5.1. **Example 1: A simple graph.** We consider a simple graph with four vertexes and four edges, as shown in Fig.1.

The initial mass distribution is given by
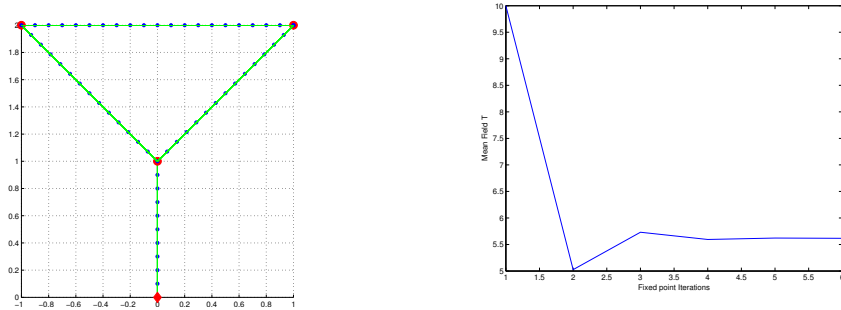
$$m_0(x) = \frac{g(x)}{\int_\Gamma g(y)dy},$$

FIGURE 1. Left: Graph configuration. Right: mean field approx-
imated time T_2 vs. fixed point iterations, computed by Algorithm
1

where $g(x)$ is the restriction to $\Gamma \subset \mathbb{R}^2$ of the function $|x|$. The scheduled time is $t_0 = 0.5$, the maximal waiting time is $T_{max} = 10$, the cost function is

$$c_T(s) = 0.1 \max(s - t_0, 0) + 0.1 \max(T - s, 0)$$

and the percentage value of the arrival players is $\theta = 0.5$. Let us note that a piecewise linear cost does *not* satisfy the assumptions of Theorem 3.2; in spite of this fact, we chose such a cost because it is more natural and it coincides with the one illustrated in the introduction of the paper [14] by Guéant, Lasry and Lions. Moreover, the algorithm performs well nonetheless.

For each arc $j \in J$, we consider the same space step $h_j = h$ and we run a series of numerical tests varying the space step according to the first column of Table 5.1. The time step has to verify the stability condition $\Delta t < h^2$ and then we choose $\Delta t = h^2/4$. For each test we compute the following error

$$E_h(T) = \left| 1 - \theta - \sum_j \sum_i \psi_{j,i}^N \phi_{j,i}^N h_j \right| \simeq \left| 1 - \theta - \int_\Gamma m(x,T)dx \right|, \qquad (36)$$

where N is such that $T_2 = N\Delta t$ in Algorithm 1. Since θ represents the percentage of player exited from the boundary vertex v_0, then $1 - \theta$ represents the percentage of the residual population and the term on the right side of (36) should be zero. This error is shown in the second column of Table 5.1. In the third and fourth columns we show the computed mean time T_2, and the number of iterations needed by the Algorithm 1 to converge when $\tau = 10^{-4}$ and $T_1 = 10$. Table 5.1 shows small values for $E_h(T)$ and, even if we do not observe a monotone behavior, the smallest value is attained with the finer space grid.

The graph on the right of Figure 1 shows the convergence of the approximated mean field time T_2, computed by Algorithm 1 with space step $h = 2.50 \cdot 10^{-2}$. On the horizontal axis are the iterations of the fixed point, while on vertical axis the corresponding approximated mean field time T_2. In Fig.2, we show the initial mass distribution (left), equilibrium mass distribution (center) and the corresponding value function (right).

5.2. **Example 2: A more general graph.** We consider a more general graph with 17 vertexes and 22 edges, see Fig.3. The initial mass distribution is given by

h	$E_h(T_2)$	T_2	iterations
$1.00 \cdot 10^{-1}$	$8.27 \cdot 10^{-4}$	5.687	6
$5.00 \cdot 10^{-2}$	$1.34 \cdot 10^{-3}$	5.639	7
$2.50 \cdot 10^{-2}$	$9.04 \cdot 10^{-4}$	5.617	8
$1.25 \cdot 10^{-2}$	$5.02 \cdot 10^{-4}$	5.622	6

TABLE 1. Space steps (first column), $E_h(T_2)$ defined in (36) (second column), approximated mean field T_2 (third column), number of fixed point iterations (last column)
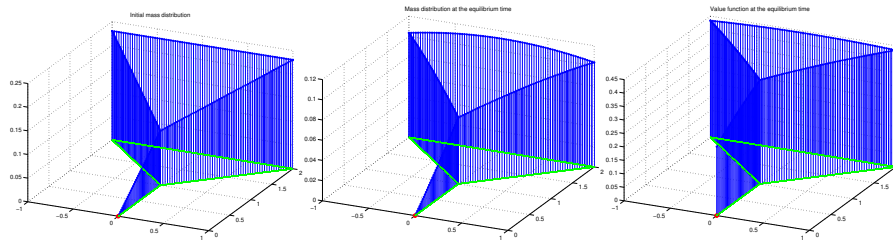

FIGURE 2.   Test 1: Initial mass distribution (left), distribution at the equilibrium time (center), value function (right)
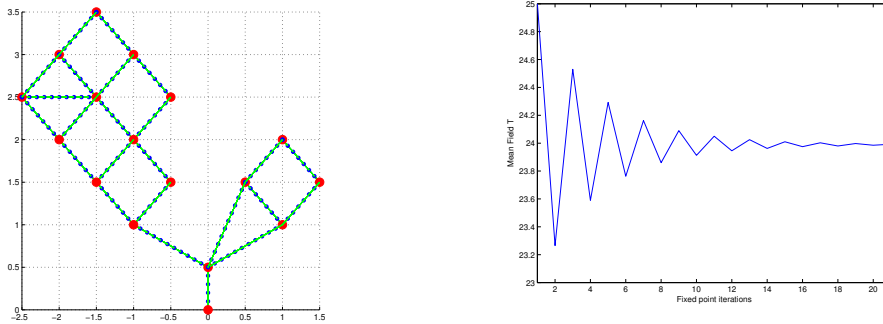

FIGURE 3.   Left: Graph configuration. Right: approximated mean field time T_2 vs. fixed point iterations, computed by Algorithm 1

$$m_0(x) = \frac{g(x)}{\int_\Gamma g(y)dy}, \quad g(x) = \max(\frac{1}{4} - |x - p_1|^2, 0) + \max(\frac{1}{4} - |x - p_2|^2, 0)$$

for every $x \in \Gamma$, with $p_1 = (1, 3/2)$ and $p_2 = (-3/2, 3)$. It describes the distribution of two populations, one concentrated around the point p_1, the other one around p_2.

The scheduled time is $t_0 = 0.5$, the maximum waiting time is $T_{max} = 25$, the cost function

$$c_T(s) = 0.1 \max(s - t_0, 0) + 0.1 \max(T - s, 0)$$

and the expected percentage of arrival players is $\theta = 0.7$. The Algorithm 1 is run with $h = 0.05$, $\Delta t = \frac{h^2}{4}$ and $\tau = 0.05$. We get $T = 23.99$ with error $E_h(T) = 2.35 \cdot 10^{-2}$. The graph on the right of Figure 3 shows the convergence of the

approximated mean field time T_2 computed by Algorithm 1: on the horizontal axis is the number of iterations of the fixed point algorithm, whereas on the vertical axis the corresponding mean field time.

Figure 4 shows the mass evolution at different times. It can be observed that at the initial time the diffusion spreads the population in all the directions on the graph, later the cost (2) favors the population closer to v_0 to reach the exit before of the population farther away.
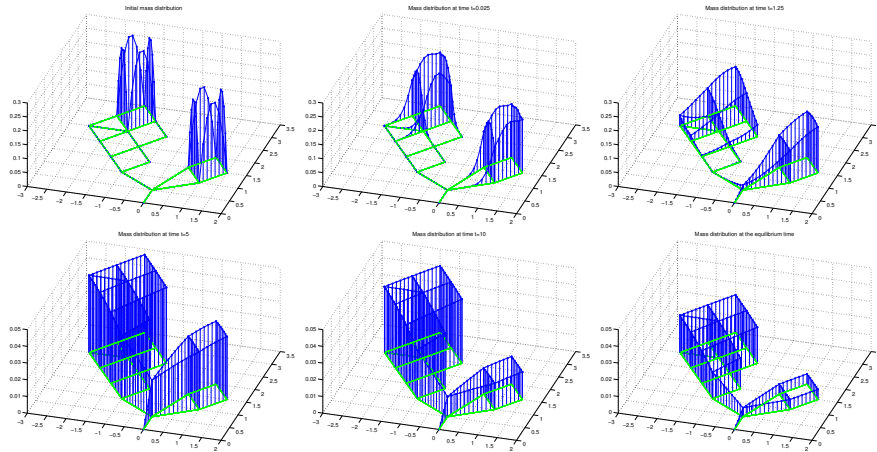

FIGURE 4.   Test 2:   Mass distribution at time: $t = 0, 0.025, 1.25, 5, 10, T = 24$

Appendix.

Proof of Prop. 3. For the sake of simplicity, K will denote a constant independent of μ_0 and w and it may change from line to line. Invoking [18, Theorem 5.4] (see also: [10, Theorem 3.2], [13, Theorem 3.6] or [19, Theorem 5.8]) we obtain that there exists a unique classical solution μ to problem (16) which fulfills the estimate

$$|\mu|_\infty \leq K|\mu_0/w(\cdot, 0)|_\infty. \tag{37}$$

For $\tilde{e} := \pi_0([l_0/4, 3l_0/4])$, we claim that μ belongs to $C^{(2+\alpha, 1+\alpha/2)}(\tilde{e} \times [0, T_{max}])$ with

$$|\mu|_{\tilde{e} \times [0, T_{max}]}^{(2+\alpha, 1+\alpha/2)} \leq K|\mu_0/w(\cdot, 0)|_\Gamma^{(2+\alpha)}. \tag{38}$$

In order to prove this estimate, we introduce two families of functions $\{\tilde{\mu}_{0,n}\}_n$ and $\{\tilde{\mu}_{1,n}\}_n$ such that

$$\tilde{\mu}_{0,n}, \tilde{\mu}_{1,n} \in C^1([0, T_{max}]), \qquad |\tilde{\mu}_{0,n}|_\infty + |\tilde{\mu}_{1,n} - \mu(v_1, \cdot)|_\infty \to 0 \quad \text{as } n \to +\infty,$$
$$\tilde{\mu}_{0,n}(0) = 0, \qquad \tilde{\mu}'_{0,n}(0) = D^2\left(\frac{\mu_0(\cdot)}{w(\cdot, 0)}\right)(v_0),$$
$$\tilde{\mu}_{1,n}(0) = \frac{\mu_0(v_1)}{w(v_1, 0)}, \qquad \tilde{\mu}'_{1,n}(0) = D^2\left(\frac{\mu_0(\cdot)}{w(\cdot, 0)}\right)(v_1),$$

By standard regularity theory for parabolic equations on domains in Euclidean spaces, the problem

$$\begin{cases} \partial_t \mu_n - \partial_x^2 \mu_n = 0 & (x, s) \in (0, l_0) \times (0, T_{max}) \\ \mu_n(0, s) = \tilde{\mu}_{0,n}(s), \quad \mu_n(l_0, s) = \tilde{\mu}_{1,n}(s) & s \in [0, T_{max}] \\ \mu_n(x, 0) = \frac{\mu_0(x)}{w(x, 0)} & x \in (0, l_0) \end{cases}$$

has a unique classical solution μ_n which belongs to $C^{(2+\alpha,1+\alpha/2)}((0,l_0)\times(0,T_{max}))$ for some α depending only on the features of the equation. By [16, Theorem IV.10.1], we deduce the following estimate in the domain $(l_0/4,3l_0/4)\times(0,T_{max})$

$$|\mu_n|_{(l_0/4,3l_0/4)\times(0,T_{max})}^{(2+\alpha,1+\alpha/2)} \le K\left(|\mu_0/w(\cdot,0)|_{e_0}^{(2+\alpha)} + |\mu_n|_\infty\right).$$

By Ascoli theorem, as $n\to+\infty$, (possibly, passing to a subsequence), the function μ_n converges uniformly to some function v and the same happens for $\partial_t\mu_n$, $\partial_x\mu_n$ and $\partial_x^2\mu_n$ with the corresponding derivatives of v. By the stability result we get $v=\mu$. Moreover, passing to the limit in the last estimate, we obtain

$$|\mu|_{(l_0/4,3l_0/4)\times(0,T_{max})}^{(2+\alpha,1+\alpha/2)} \le K\left(|\mu_0/w(\cdot,0)|_{e_0}^{(2+\alpha)} + |\mu|_\infty\right)$$

and, taking into account estimate (37) and the definition of the sub-edge $\tilde{e}$, we accomplish the proof of claim (38).

We observe that the function $\mu_{|e_{0,1/2}\times(0,T_{max})}$ is the unique classical solution to problem

$$\begin{cases} \partial_t\bar\mu - \partial_x^2\bar\mu = 0 & (x,s)\in e_{0,1/2}\times(0,T_{max}) \\ \bar\mu(v_0,s)=0, \quad \bar\mu(v'_{1/2},s)=\mu(v'_{1/2},s) & s\in[0,T_{max}] \\ \bar\mu(x,0)=\frac{\mu_0(x)}{w(x,0)} & x\in e_{0,1/2} \end{cases}$$

which is a standard initial-boundary value problem on an Euclidean domain. Invoking [16, Theorem IV.9.1], we infer that, for every $q\ge1$, μ belongs to $W_{q,e_{0,1/2}\times[0,T_{max}]}^{2,1}$ with

$$|\mu|_{q,e_{0,1/2}\times[0,T_{max}]}^{2,1} \le K\left(|\mu_0/w(\cdot,0)|_{e_{0,1/2}}^{(2)} + |\mu(v'_{1/2},\cdot)|_{(0,T_{max})}^{(1)}\right).$$

Owing to (38), estimate (17) is achieved. $\square$

Proof of Prop. 4. We shall improve some arguments of the proof of Proposition 3 taking advantage of the stronger compatibility condition given by (18). Here, the constant K is independent of μ_0 and w and it may change from line to line.

We consider the family of functions $\{\tilde\mu_{1,n}\}_n$ introduced in the proof of Proposition 3. By standard regularity theory for parabolic equations on domains in Euclidean spaces, the problem

$$\begin{cases} \partial_t\mu_n - \partial_x^2\mu_n = 0 & (x,s)\in(0,l_0)\times(0,T_{max}) \\ \mu_n(0,s)=0, \quad \mu_n(l_0,s)=\tilde\mu_{1,n}(s) & s\in[0,T_{max}] \\ \mu_n(x,0)=\frac{\mu_0(x)}{w(x,0)} & x\in(0,l_0) \end{cases}$$

has a unique classical solution μ_n which belongs to $C^{(2+\alpha,1+\alpha/2)}((0,l_0)\times(0,T_{max}))$ for some α depending only on the features of the equation. By [16, Theorem IV.10.1], we deduce the following estimate in the domain $(0,l_0/2)\times(0,T_{max})$

$$|\mu_n|_{[0,l_0/2]\times[0,T_{max}]}^{(2+\alpha,1+\alpha/2)} \le K\left(|\mu_0/w(\cdot,0)|_{[0,l_0]}^{(2+\alpha)} + |\mu_n|_\infty\right). \tag{39}$$

By Ascoli theorem, as $n\to+\infty$, (possibly, passing to a subsequence), the function μ_n converges to some function v uniformly and the same happens for $\partial_t\mu_n$, $\partial_x\mu_n$ and $\partial_x^2\mu_n$ with the corresponding derivatives of v. By the stability result we get $v=\mu$. Moreover, passing to the limit in the estimate (39), we obtain

$$|\mu|_{[0,l_0/2]\times[0,T_{max}]}^{(2+\alpha,1+\alpha/2)} \le K\left(|\mu_0/w(\cdot,0)|_{[0,l_0]}^{(2+\alpha)} + |\mu|_\infty\right).$$

Finally, taking into account estimate (37), we accomplish the proof.

The second part of the statement is a consequence of [2]; actually, in this case, the compatibility conditions are ensured by (20). Invoking [2], we obtain

$$|\mu|_{\Gamma \times [0,T_{max}]}^{(2+\alpha,1+\alpha/2)} \leq K_0 |\mu_0/w(\cdot,0)|_{\Gamma}^{(2+\alpha)}$$

where K_0 is the same constant as in Proposition 2. $\square$

Remark 2. As one can easily check, in the proof of previous Proposition 4, hypothesis (18) is needed only for guaranteeing the compatibility condition in v_0. As a matter of fact, it can be replaced by: $\partial_x^2(\mu_0(\cdot)/w(\cdot,0))(v_0) = 0$.

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