

Optimal Exercise of Swing Contracts in Energy Markets: An Integral Constrained Stochastic Optimal Control Problem*

Matteo Basei[†], Annalisa Cesaroni[†], and Tiziano Vargiolu[†]

Abstract. We characterize the value of swing contracts in continuous time as the unique viscosity solution of a Hamilton–Jacobi–Bellman (HJB) equation with suitable boundary conditions. The case of contracts with penalties is straightforward, and in that case only a terminal condition is needed. Conversely, the case of contracts with strict constraints gives rise to a stochastic control problem with a nonstandard state constraint. We approach this problem by a penalty method: we consider a general constrained problem and approximate the value function with a sequence of value functions of appropriate unconstrained problems with a penalization term in the objective functional. Coming back to the case of swing contracts with strict constraints, we finally characterize the value function as the unique viscosity solution with polynomial growth of the HJB equation subject to appropriate boundary conditions.

Key words. swing contracts, energy markets, stochastic control, state constraints, penalty methods, dynamic programming, HJB equation, viscosity solutions

AMS subject classifications. 91G20, 60H15, 49L25, 93E20

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1. Introduction. Energy is traded in financial markets, in its various forms (electricity, coal, gas, oil, etc.), mainly through two types of contracts, namely forwards and swings. Forward contracts are obligations between two parts to exchange some amount of energy in a specified form (electricity or some fuel) and for a prespecified amount of money: once settled, this contract is strictly binding for both parts, giving no flexibility to them. Conversely, swing contracts give a certain amount of flexibility to the buyer, while also giving the seller a certain guarantee that a minimum quantity of energy will be bought. This is due to the fact that energy storage is costly in the case of fuels and almost impossible in the case of electricity; moreover, energy markets are influenced by many elements (peaks in consumption related to sudden weather changes, breakdowns in power plants, financial crises, etc.). As a consequence, the price of energy is subject to remarkable fluctuations, so that flexibility is much welcomed by contract buyers.

The flexibility in swing contracts is implemented in this way (we here follow the approach in [3] and model the contract in continuous time): for a fixed contract maturity T (usually

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[†]Department of Mathematics, University of Padova, 35121 Padova, Italy (basei@math.unipd.it, acesar@math.unipd.it, vargiolu@math.unipd.it). The second author's research was partially supported by grant CPDA139108-2013 "Traffic Flow on Networks: Analysis and Control" of the University of Padova. The third author's research was partially supported by grant CPDA138873-2013 "Stochastic models with spatial structure and applications to new challenges in Mathematical Finance, with a focus on the post-2008 financial crisis environment and on energy markets" of the University of Padova.

one or several years), the buyer can choose, at each time $s \in [0, T]$, to buy a marginal amount of energy $u(s) \in [0, \bar{u}]$ at a prespecified strike price K , thus realizing a marginal profit (or loss) equal to $(P(s) - K)u(s)$, where $P(s)$ is the spot price of that kind of energy. This gives to the buyer the potential profit (or loss)

$$\int_0^T e^{-rs} (P(s) - K)u(s) ds,$$

with $r > 0$ the risk-free interest rate.

However, the energy seller usually wants the total amount of energy $Z(T) = \int_0^T u(s) ds$ to lie between a minimum and a maximum quantity, that is, $Z(T) \in [m, M]$. This is implemented in two main ways. The first way is to impose penalties when $Z(T) \notin [m, M]$, i.e., to make the buyer pay a penalty $\tilde{\Phi}(P(T), Z(T))$, where $\tilde{\Phi}(p, z)$ is a contractually fixed function, null for $z \in [m, M]$ and convex in z . The second way is to impose the constraint $Z(T) \in [m, M]$ to be satisfied strictly, i.e., to force the buyer to withdraw the minimum cumulative amount of energy m and to stop giving the energy when the maximum M has been reached.

We are interested in the problem of optimally exercising a swing contract in both cases. This problem can be modeled as a continuous time stochastic control problem: our aim is to study the corresponding value function and to characterize it as the unique viscosity solution of the related Hamilton–Jacobi–Bellman (HJB) equation. Swing contracts are treated either in discrete time [1, 2, 10, 15] via the dynamic programming principle and Bellman equations or in continuous time [3, 12] only by reporting a verification theorem for a smooth solution of the HJB equation, without reporting existence or uniqueness results for that. Besides, we also extend the approach in [3, 12], which treats only the case $m = 0$, to the case when $m > 0$, which is the most relevant case in practical applications (in fact, [17] reports that typically $m \in [0.8M, M]$). We also refer the reader to [7], where swing contracts in continuous time are treated with multiple stopping techniques, and [4], where swings are priced using a discrete time backward scheme for solving BSDEs with jumps.

In the case of swing contracts with penalties, we get a standard stochastic control problem, as the maximization of the final expected payoff for a buyer entering in the contract at a generic time $t \in [0, T]$ is given by

$$(1.1) \quad \tilde{V}(t, p, z) = \sup_{u \in \mathcal{A}_t} \mathbb{E}_{tpz} \left[\int_t^T e^{-r(s-t)} (P(s) - K)u(s) ds - e^{-r(T-t)} \tilde{\Phi}(P(T), Z(T)) \right],$$

with $(t, p, z) \in [0, T] \times \mathbb{R}^2$, where \mathcal{A}_t is the set of $[0, \bar{u}]$ -valued progressively measurable processes $u = \{u(s)\}_{s \in [t, T]}$. Thus, in this case classical theory (see [9, 11, 13]) can be applied: see section 2.

Conversely, swing contracts with strict constraints give rise to a stochastic control problem with integral constraints in the control:

$$(1.2) \quad V(t, p, z) = \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} \mathbb{E}_{tpz} \left[\int_t^T e^{-r(s-t)} (P(s) - K)u(s) ds \right],$$

with (t, p, z) in a suitable domain $\mathcal{D} \subseteq [0, T] \times \mathbb{R}^2$, where $\mathcal{A}_{tz}^{\text{adm}}$ is the set of processes $u \in \mathcal{A}_t$ such that \mathbb{P}_{tpz} -a.s. $Z^{t,z;u}(T) = z + \int_t^T u(s) ds \in [m, M]$. Due to the presence of the constraint on $Z^{t,z;u}(T)$, here classical theory does not apply.

This motivates us to consider in section 3 a more general class of integral constrained stochastic problems in the form

$$(1.3) \quad V(t, p, z) = \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} \mathbb{E}_{tpz} \left[\int_t^T e^{-r(s-t)} L(s, P(s), Z(s), u(s)) ds + e^{-r(T-t)} \Phi(P(T), Z(T)) \right].$$

with (t, p, z) in a suitable domain $\mathcal{D} \subseteq [0, T] \times \mathbb{R}^n \times \mathbb{R}$, where $\mathcal{A}_{tz}^{\text{adm}}$ is the set of processes $u \in \mathcal{A}_t$ such that, \mathbb{P}_{tpz} -a.s., $Z(T) = z + \int_t^T g(s, u(s)) ds \in [m, M]$.

Control problems with integral constraints are classical in control theory; for instance, they naturally arise in applications, e.g., control problems with bounded L^p norm of the controls, control problems with prescribed bounded total variation or total energy of the trajectories, and control systems with design uncertainties. However, the dynamic programming approach presents several technical difficulties. The main one relies on the fact that the dynamic programming principle is not satisfied directly by the value function, and the problem has to be attacked differently. As for the case of deterministic systems, we refer the reader to [19, 20] and references therein. As for the case of stochastic controls, the upper bound $Z(T) \leq M$ is analogous to the constraint of the so-called finite fuel problems, which are optimal control problems with an upper bound on the integral of the absolute value of the controls (see, e.g., [13, Chapter VIII] for an introduction to the problem and [18] and references therein). Instead, the lower bound $Z(T) \geq m$ is nonstandard. In the particular case of (1.2), and only with $m = 0$, such a bound has been studied (treated in [3] and generalized in [12], still with $m = 0$). However, we already said that this case is quite unrealistic, as the seller wants to be sure to sell some amount of energy, so typically $m > 0$.

Note that the control problem in (1.3) can be interpreted as a state constraint control problem in the following way. First of all, the integral constraint on the control can be written as a terminal constraint on the state variables $(P(s), Z(s))$: they have to satisfy a.s. $(P(T), Z(T)) \in G$, where G is a closed set in \mathbb{R}^{n+1} . In our case, $G = \mathbb{R}^n \times [m, M]$. Then, we introduce the set of points such that G is reachable from (t, p, z) , i.e.,

$$\mathcal{D} = \{(t, p, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} : \mathcal{A}_{tz}^{\text{adm}} \neq \emptyset\}.$$

Thus, we can redefine $\mathcal{A}_{tz}^{\text{adm}}$ in (1.3) as the set of processes $u \in \mathcal{A}_t$ such that, \mathbb{P}_{tpz} -a.s., $(s, P(s), Z(s)) \in \mathcal{D}$ for all $s \in [t, T]$.

Note that here \mathcal{D} is not given explicitly, but it is defined by a stochastic target problem. In [6], a similar stochastic control problem has been considered, with $G = \{(p, z) \in \mathbb{R}^n \times \mathbb{R} \mid g(p, z) \geq 0\}$ and $g(p, \cdot)$ increasing and right continuous. In this case, though, differently from our case, the set \mathcal{D} can be described as the epigraph of a continuous function, that is, $\mathcal{D} = \{(t, p, z) \mid w(t, p) \geq z\}$. We also refer the reader to [5], where reachable sets for state-constrained controlled stochastic systems have been studied.

To study our problem, we adopt a classical penalization method. We introduce the set $\tilde{\mathcal{D}} \subseteq \mathcal{D}$ (which is the set of points such that the interior of G is reachable from (t, p, z) ; for a precise definition we refer the reader to section 3.1), and we show that in $\tilde{\mathcal{D}}$ the function V in (1.3) is the limit of the value functions V^c of suitable unconstrained problems, where the constraint has been substituted by an appropriate penalization in the objective functional. This convergence result is obtained under a technical assumption (see Assumption 2), ensuring

that, roughly speaking, given a control in $\mathcal{A}_{tz}^{\text{adm}}$ we can modify it in order to steer the trajectory in the interior of G , not paying too much in the cost functional. This result is contained in Theorem 3.5 and Corollary 3.6. In Propositions 3.8 and 3.9 we prove that, under suitable assumptions, the function $V(t, \cdot, z)$ is Lipschitz continuous and a.e. twice differentiable. In section 3.4 we show that Assumption 2 is satisfied in the cases $g(s, v) = v$ and $g(s, v) = |v|^p$ ($p \geq 1$) if f and σ satisfy appropriate conditions.

In section 4 we apply these general results to the problem in (1.2). In this case stronger results will be achieved, since it can be proved directly that the value function is continuous not only in $\tilde{\mathcal{D}} \subsetneq \mathcal{D}$ but in the whole domain \mathcal{D} (Propositions 4.1 and 4.3). Thus, V can be characterized as the unique continuous viscosity solution with polynomial growth of the HJB equation under suitable boundary conditions (Theorem 4.4). As for the regularity of the value function, besides the above cited general results about the variable p (Proposition 4.5), we prove that $V(t, p, \cdot)$ is concave and study its monotonicity (Proposition 4.6).

The structure of the paper is as follows. In section 2 the evaluation problem for a swing contract with penalty is studied. Section 3 deals with a general class of constrained control problems, as in (1.3). Finally, in section 4 we deeply analyze the problem, outlined in (1.2), of the optimal exercise of swing contracts with strict constraints.

Notation. By $\|\cdot\|_\infty$ we denote the sup-norm. If $B \in M_{ij}(\mathbb{R})$ (i.e., a real $i \times j$ matrix), B^t denotes the transpose of B and $\text{tr}(B)$ denotes its trace. By $\bar{B}(x, R)$ we mean the closed ball in \mathbb{R}^n with center x and radius R . If $O \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}$, we denote by $C_b^k(O)$ (resp., $C_p^k(O)$) the set of functions of class $C^k(O)$ whose derivatives up to order k are bounded (resp., are polynomially growing). If ψ is a function from $(t, p, z) \in A \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ to \mathbb{R} , by ψ_t, ψ_z we mean the derivatives with respect to t and z , and by $D_p\psi, D_p^2\psi$ we mean the Jacobian and the Hessian matrix with respect to the variable p .

2. Swing contracts with penalties. In this section we consider the problem of the optimal exercise of swing contracts with penalties described in the introduction: to this purpose, we formalize a continuous time model to which we apply classical results in stochastic control.

Let $T > 0$, and fix a filtered probability space $(\Omega, \mathcal{F}_T, \{\mathcal{F}_s\}_{s \in [0, T]}, \mathbb{P})$ and a real $\{\mathcal{F}_s\}_s$ -adapted Brownian motion $W = \{W(s)\}_{s \in [0, T]}$. Let $t \in [0, T]$ and $p \geq 0$. We model the price of energy through a stochastic process $\{P^{t,p}(s)\}_{s \in [t, T]}$ which satisfies the SDE

$$(2.1) \quad dP^{t,p}(s) = f(s, P^{t,p}(s))ds + \sigma(s, P^{t,p}(s))dW(s), \quad s \in [t, T],$$

with initial condition $P^{t,p}(t) = p$. We assume

$$(2.2) \quad \begin{aligned} & f, \sigma \in C([0, T] \times \mathbb{R}; \mathbb{R}), \\ & |f(t, p) - f(t, q)| + |\sigma(t, p) - \sigma(t, q)| \leq \hat{C}|p - q| \quad \forall p, q \in \mathbb{R}, \quad \forall t \in [0, T], \end{aligned}$$

where $\hat{C} > 0$ is a constant.

In each $s \in [t, T]$, the holder can buy energy at a fixed unitary price $K > 0$ and with purchase intensity $u(s) \in [0, \bar{u}]$, where $\bar{u} > 0$ is a constant: this gives a net instantaneous profit (or loss) of $(P^{t,p}(s) - K)u(s)$. Let \mathcal{A}_t be the set of all $[0, \bar{u}]$ -valued progressively measurable processes $u = \{u(s)\}_{s \in [t, T]}$ (i.e., all the possible usage strategies of the contract). Let z be the

amount of energy purchased until time t , and let $u \in \mathcal{A}_t$ be an exercise strategy from time t on; for each $s \in [t, T]$ we denote by $Z^{t,z;u}(s)$ the energy bought up to time s :

$$Z^{t,z;u}(s) = z + \int_t^s u(\tau) d\tau, \quad s \in [t, T].$$

If the globally purchased energy $Z^{t,z;u}(T)$ does not fall within a fixed range $[m, M]$ ($m, M \in \mathbb{R}$, with $m \leq M$), the holder must pay a penalty $\tilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T))$, where $\tilde{\Phi}$ is a function from \mathbb{R}^2 to \mathbb{R} . In the typical case (see, for example, [1, 2, 11]) the penalty is directly proportional to $P^{t,p}(T)^+$ and to the entity of the overrunning or underrunning: this is obtained by setting

$$\tilde{\Phi}(p, z) = -Ap^+(z - M)^+ - Bp^+(m - z)^+$$

for all $(p, z) \in \mathbb{R}^2$, where $A, B > 0$ are suitable constants. In several practical cases, $A = B$. However, other kind of penalties are possible (see, e.g., [15]): typically p^+ , representing the spot price at the end T of the contract, is replaced either by an arithmetic mean of spot prices (thus requiring another state variable in the problem) or by a fixed (high) penalty. In light of the above discussion, we assume that, for all $p \in \mathbb{R}$,

$$(2.3) \quad \begin{aligned} \tilde{\Phi}(p, z) &= 0 \quad \forall z \in [m, M], \\ \tilde{\Phi}(p, \cdot) &\text{ is concave,} \\ |\tilde{\Phi}(p+h, z) - \tilde{\Phi}(p, z)| &\leq Ch(1 + |z|), \quad |\tilde{\Phi}(p, z+h) - \tilde{\Phi}(p, z)| \leq Ch(1 + |p|) \quad \forall z \in \mathbb{R}, h > 0, \end{aligned}$$

where $C > 0$ is a constant.

Let $r \geq 0$ be the risk-free rate. We get a stochastic optimal control problem with the following value function:

$$(2.4) \quad \tilde{V}(t, p, z) = \sup_{u \in \mathcal{A}_t} \tilde{J}(t, p, z; u)$$

for each $(t, p, z) \in [0, T] \times \mathbb{R}^2$, where

$$\tilde{J}(t, p, z; u) = \mathbb{E}_{t,p,z} \left[\int_t^T e^{-r(s-t)} (P^{t,p}(s) - K) u(s) ds + e^{-r(T-t)} \tilde{\Phi}(P^{t,p}(T), Z^{t,z;u}(T)) \right],$$

and by $\mathbb{E}_{t,p,z}$ we denote the mean value with respect to the probability \mathbb{P} (subscripts recall initial conditions).

Problem (2.4) belongs to a widely studied class of control problems: by well-known classical results, summarized in Theorem 2.1, the value function is the unique viscosity solution of the corresponding HJB equation subject to appropriate conditions (we refer the reader to [8] for the definition of viscosity solutions).

Theorem 2.1. *Under assumptions (2.2) and (2.3), the function \tilde{V} is the unique viscosity solution of*

$$(2.5) \quad \begin{aligned} -\tilde{V}_t(t, p, z) + r\tilde{V}(t, p, z) - f(t, p)\tilde{V}_p(t, p, z) - \frac{1}{2}\sigma^2(t, p)\tilde{V}_{pp}(t, p, z) \\ + \min_{v \in [0, \bar{u}]} [-v(\tilde{V}_z(t, p, z) + p - K)] = 0 \quad \forall (t, p, z) \in [0, T] \times \mathbb{R}^2, \end{aligned}$$

with final condition

$$(2.6) \quad \tilde{V}(T, p, z) = \tilde{\Phi}(p, z) \quad \forall (p, z) \in \mathbb{R}^2,$$

and such that

$$|\tilde{V}(t, p, z)| \leq \check{C}(1 + |p|^2 + |z|^2) \quad \forall (t, p, z) \in [0, T] \times \mathbb{R}^2$$

for some constant $\check{C} > 0$.

Proof. See Theorem 3.4, of which this theorem is a particular case. ■

We now list some properties of the function \tilde{V} with respect to the variables p and z . Let us start by proving regularity results with respect to the variable p .

Proposition 2.2. *Under assumptions (2.2) and (2.3), for each $(t, z) \in [0, T] \times \mathbb{R}$ the function $\tilde{V}(t, \cdot, z)$ is Lipschitz continuous, uniformly in t . Moreover, the derivative $\tilde{V}_p(t, p, z)$ exists for a.e. $(t, p, z) \in [0, T] \times \mathbb{R}^2$, and we have $|\tilde{V}_p(t, p, z)| \leq M_1(1 + |z|)$ for some constant $M_1 > 0$ depending only on \bar{u} , T , C and on the constants in (3.1) and (3.4).*

Proof. Let $(t, p, z) \in [0, T] \times \mathbb{R}^2$, $h > 0$, and $u \in \mathcal{A}_t$. By estimate (D.8) in [13, Appendix D] we have

$$\begin{aligned} & |\tilde{J}(t, p + h, z; u) - \tilde{J}(t, p, z; u)| \\ & \leq \mathbb{E}_{tpz} \left[\int_t^T |P^{t,p+h}(s) - P^{t,p}(s)| |u(s)| ds + C |P^{t,p+h}(T) - P^{t,p}(T)| (1 + |Z^{t,z;u}(T)|) \right] \\ & \leq T\bar{u} \mathbb{E}_{tpz} [\|P^{t,p+h}(\cdot) - P^{t,p}(\cdot)\|_\infty] + C \mathbb{E}_{tpz} [\|P^{t,p+h}(\cdot) - P^{t,p}(\cdot)\|_\infty] (1 + |z| + \bar{u}T) \\ (2.7) \quad & \leq M_1(1 + |z|)h, \end{aligned}$$

where $M_1 > 0$ is a constant. Since (2.7) holds for each $u \in \mathcal{A}_t$, we get

$$(2.8) \quad |\tilde{V}(t, p + h, z) - \tilde{V}(t, p, z)| \leq M_1(1 + |z|)h.$$

The function $\tilde{V}(t, \cdot, z)$ is therefore Lipschitz continuous, uniformly in t , and then a.e. differentiable by the Rademacher theorem (see [14]). By standard arguments (see [14]) it is possible to prove that the set of points where $\tilde{V}_p(t, p, z)$ does not exist is measurable in $[0, T] \times \mathbb{R}^2$; then by Fubini's theorem it follows that $\tilde{V}_p(t, p, z)$ exists for a.e. $(t, p, z) \in [0, T] \times \mathbb{R}^2$. The estimate on the derivative immediately follows by (2.8). ■

In the following proposition we collect some results about smoothness and monotonicity of the function \tilde{V} with respect to z .

Proposition 2.3. *Under assumptions (2.2) and (2.3), for each $(t, p) \in [0, T] \times \mathbb{R}$ the function $\tilde{V}(t, p, \cdot)$ has the following characteristics:*

- It is Lipschitz continuous, uniformly in t . Moreover, the derivative $\tilde{V}_z(t, p, z)$ exists for a.e. $(t, p, z) \in [0, T] \times \mathbb{R}^2$, and we have $|\tilde{V}_z(t, p, z)| \leq M_2(1 + |p|)$ for some constant $M_2 > 0$ depending only on \bar{u} , T , C and on the constants in (3.1) and (3.4).
- It is concave and a.e. twice differentiable.
- It is nondecreasing in $]-\infty, M - (T - t)\bar{u}]$ and nonincreasing in $[m, +\infty[$. In particular, if $M - (T - t)\bar{u} \geq m$, then the function $\tilde{V}(t, p, \cdot)$ is constant in $[m, M - (T - t)\bar{u}]$ (they all are maximum points).

Proof. *Item 1.* Let $(t, p, z) \in [0, T] \times \mathbb{R}^2$, $h > 0$, and $u \in \mathcal{A}_t$. Recall the following estimate from [13, Appendix D]: for each $k \geq 0$ there exists a constant $B_k \geq 0$, depending only on \bar{u} , T , C and on the constants in (3.1) and (3.4), such that

$$(2.9) \quad \mathbb{E}_{tpz} [\|P^{t,p;u}(\cdot)\|_\infty^k] \leq B_k(1 + |p|^k).$$

By this and the Lipschitzianity of $\tilde{\Phi}(P^{t,p}(T), \cdot)$ we have

$$\begin{aligned} |\tilde{J}(t, p, z + h; u) - \tilde{J}(t, p, z; u)| &\leq C \mathbb{E}_{tpz} [(1 + |P^{t,p}(T)|) |Z^{t,z+h;u}(T) - Z^{t,z;u}(T)|] \\ &\leq Ch(1 + \mathbb{E}_{tpz} [|P^{t,p}(T)|]) \leq M_2(1 + |p|)h, \end{aligned}$$

where $M_2 > 0$ is a constant. Then argue as in Proposition 2.2.

Item 2. Let $(t, p) \in [0, T] \times \mathbb{R}$, $z_1, z_2 \in \mathbb{R}$, and $u_1, u_2 \in \mathcal{A}_t$. Notice that $(u_1 + u_2)/2 \in \mathcal{A}_t$ and that

$$(2.10) \quad Z^{t, \frac{z_1+z_2}{2}; \frac{u_1+u_2}{2}}(T) = \frac{Z^{t, z_1; u_1}(T) + Z^{t, z_2; u_2}(T)}{2}.$$

By the concavity of the function $\tilde{\Phi}(P^{t,p}(T), \cdot)$ and by (2.10) we have

$$(2.11) \quad \frac{\tilde{J}(t, p, z_1; u_1) + \tilde{J}(t, p, z_2; u_2)}{2} \leq \tilde{J}\left(t, p, \frac{z_1 + z_2}{2}; \frac{u_1 + u_2}{2}\right) \leq V\left(t, p, \frac{z_1 + z_2}{2}\right).$$

Since (2.11) holds for each $u_1, u_2 \in \mathcal{A}_t$, we get

$$\frac{\tilde{V}(t, p, z_1) + \tilde{V}(t, p, z_2)}{2} \leq \tilde{V}\left(t, p, \frac{z_1 + z_2}{2}\right)$$

and then the concavity of the function $\tilde{V}(t, p, \cdot)$. The a.e. existence of the second derivative follows from the Alexandrov theorem.

Item 3. Let $(t, p) \in [0, T] \times \mathbb{R}$, $z_1 \leq z_2 \leq M - (T - t)\bar{u}$ (the case $m \leq z_1 \leq z_2$ is similar), and $u \in \mathcal{A}_t$. Since

$$Z^{t, z_1; u}(T) \leq Z^{t, z_2; u}(T) = z_2 + \int_t^T u(s) ds \leq z_2 + (T - t)\bar{u} \leq M$$

and since the function $\tilde{\Phi}(P^{t,p}(T), \cdot)$ is nondecreasing in $]-\infty, M]$ (as it is concave and null in $[m, M]$), we have that

$$(2.12) \quad \tilde{J}(t, p, z_1; u) \leq \tilde{J}(t, p, z_2; u).$$

As inequality (2.12) holds for each $u \in \mathcal{A}_t$, we get

$$\tilde{V}(t, p, z_1) \leq \tilde{V}(t, p, z_2).$$

The second part immediately follows, since $]-\infty, M - (T - t)\bar{u}] \cap [m, +\infty[= [m, M - (T - t)\bar{u}]$. ■

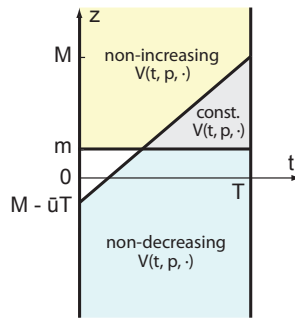


Figure 1. Monotonicity of $\tilde{V}(t, p, \cdot)$.

The monotonicity result in Proposition 2.3 is described in Figure 1.

The third part of Proposition 2.3 implies in particular that for suitable t and for all p , the function $V(t, p, \cdot)$ is constant in an interval. As a matter of fact, this was foreseeable: it is easy to check that if $M - (T - t)\bar{u} \geq m$ and $z \in [m, M - (T - t)\bar{u}]$, then $Z^{t,z;u}(T) \in [m, M]$ for each $u \in \mathcal{A}_t$, so that the penalization term in the objective functional vanishes and the initial value z does not influence the value function.

Remark 1. As observed in [3, equation (3.9)], by (2.5) a candidate optimal control policy is

$$(2.13) \quad \underline{u}(t, p, z) = \begin{cases} \bar{u} & \text{if } \tilde{V}_z(t, p, z) \geq p - K, \\ 0 & \text{if } \tilde{V}_z(t, p, z) < p - K. \end{cases}$$

Notice that by Proposition 2.3 the candidate in (2.13) is a.e. well defined. Moreover, since \tilde{V} is concave in z , for each fixed (t, p) there exists $\bar{z}(t, p) \in [-\infty, +\infty]$ such that $\tilde{V}_z(t, p, z) < p - K$ if and only if $z > \bar{z}(t, p)$: for t fixed, the function $\bar{z}(t, \cdot)$ (which in [3] is called exercise curve) can be used to write \underline{u} as

$$(2.14) \quad \underline{u}(t, p, z) = \begin{cases} \bar{u} & \text{if } z \leq \bar{z}(t, p), \\ 0 & \text{if } z > \bar{z}(t, p). \end{cases}$$

3. Integral constrained stochastic optimal control. Let us now consider the problem, outlined in the introduction, of optimally exercising swing contracts with strict constraints. Due to the presence of the constraint, in this case it is not possible to argue as in section 2 and use classical results in control theory. This motivates us to study a more general class of stochastic optimal control problems with integral constraints, of which swing contracts with strict constraints will be a particular case.

3.1. Formulation of the problem. Let $d, l, n \in \mathbb{N}$, $r \geq 0$, $T > 0$, and $m, M \in \mathbb{R}$, with $m < M$. Let $U \subseteq \mathbb{R}^l$ be nonempty and f, σ, g, L, Φ be functions satisfying the following assumption.

Assumption 1.

- (i) U is a compact subset of \mathbb{R}^l ;

- (ii) $f \in C([0, T] \times \mathbb{R}^n \times U; \mathbb{R}^n)$, $\sigma \in C([0, T] \times \mathbb{R}^n \times U; M_{nd}(\mathbb{R}))$, and there exists a constant $C > 0$ such that

$$(3.1) \quad \begin{aligned} |f(t, p, v) - f(t, q, v)| &\leq C|p - q| \quad \forall p, q \in \mathbb{R}^n, \quad \forall (t, v) \in [0, T] \times U, \\ |\sigma(t, p, v) - \sigma(t, q, v)| &\leq C|p - q| \quad \forall p, q \in \mathbb{R}^n, \quad \forall (t, v) \in [0, T] \times U; \end{aligned}$$

- (iii) $g \in C([0, T] \times U; \mathbb{R})$;

- (iv) $L \in C([0, T] \times \mathbb{R}^n \times \mathbb{R} \times U; \mathbb{R})$, $\Phi \in C(\mathbb{R}^n \times \mathbb{R}; \mathbb{R})$, and there exist constants $\tilde{C}, k > 1$ such that

$$(3.2) \quad \begin{aligned} |L(t, p, z, v)| &\leq \tilde{C}(1 + |p|^k + |z|^k) \quad \forall (t, p, z, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times U, \\ |\Phi(p, z)| &\leq \tilde{C}(1 + |p|^k + |z|^k) \quad \forall (p, z) \in \mathbb{R}^n \times \mathbb{R}. \end{aligned}$$

Moreover, for each compact subset $A \subseteq \mathbb{R}^{n+1}$ there exists a modulus of continuity ω_A such that

$$(3.3) \quad |L(t, p, z, v) - L(t, q, y, v)| \leq \omega_R(|p - q| + |z - y|)$$

for all $(t, v) \in [0, T] \times U$ and for all $(p, z), (q, y) \in A$.

Notice that conditions (3.1) imply that

$$(3.4) \quad \begin{aligned} |f(t, p, v)| &\leq \hat{C}(1 + |p|) \quad \forall (t, p, v) \in [0, T] \times \mathbb{R}^n \times U, \\ |\sigma(t, p, v)| &\leq \hat{C}(1 + |p|) \quad \forall (t, p, v) \in [0, T] \times \mathbb{R}^n \times U, \end{aligned}$$

where $\hat{C} > 0$ is a constant.

Let $(\Omega, \mathcal{F}_T, \{\mathcal{F}_s\}_{s \in [0, T]}, \mathbb{P}, W)$ be a fixed filtered probability space where a d -dimensional $\{\mathcal{F}_s\}_s$ -adapted Brownian motion $W = \{W(s)\}_{s \in [0, T]}$ is defined. If $t \in [0, T]$, let \mathcal{A}_t denote the set of all U -valued progressively measurable processes $u = \{u(s)\}_{s \in [t, T]}$ (*controls*) such that for each $p \in \mathbb{R}^n$ the n -dimensional SDE

$$(3.5) \quad dP^{t, p; u}(s) = f(s, P^{t, p; u}(s), u(s))ds + \sigma(s, P^{t, p; u}(s), u(s))dW(s), \quad s \in [t, T],$$

with initial condition

$$(3.6) \quad P^{t, p; u}(t) = p,$$

has a pathwise unique strong solution.

Let $t \in [0, T]$, $z \in \mathbb{R}$, $u \in \mathcal{A}_t$, and let

$$(3.7) \quad Z^{t, z; u}(s) = z + \int_t^s g(\tau, u(\tau))d\tau, \quad s \in [t, T].$$

A control $u \in \mathcal{A}_t$ is called *admissible* if the process $Z^{t, z; u}$ a.s. reaches the interval $[m, M]$ at the final time T :

$$\mathcal{A}_{tz}^{\text{adm}} = \{u \in \mathcal{A}_t : Z^{t, z; u}(T) \in [m, M] \quad \mathbb{P}_{tpz}\text{-a.s.}\}.$$

We will often write P^u and Z^u , in order to shorten the notation.

Given $(t, z) \in [0, T] \times \mathbb{R}$ and $A \subseteq \mathbb{R}$, we say that A is *reachable from* (t, z) if there exists a Borel measurable function u from $[t, T]$ to U (notice that then $u \in \mathcal{A}_t$) such that $Z^{t,z;u}(T) \in A$. Let $\mathcal{D}, \tilde{\mathcal{D}}, \mathcal{D}^\rho$ (for $0 < \rho < (M - m)/2$) denote the subsets of $[0, T] \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{D} &= \{(t, p, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} : [m, M] \text{ is reachable from } (t, z)\}, \\ \tilde{\mathcal{D}} &= \{(t, p, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} :]m, M[\text{ is reachable from } (t, z)\}, \\ \mathcal{D}^\rho &= \{(t, p, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} : [m + \rho, M - \rho] \text{ is reachable from } (t, z)\}. \end{aligned}$$

Notice that $\bigcup_\rho \mathcal{D}^\rho = \tilde{\mathcal{D}} \subseteq \mathcal{D}$. It is easy to prove that these sets are nonempty.

Lemma 3.1. *The sets $\mathcal{D}, \tilde{\mathcal{D}}, \mathcal{D}^\rho$ are nonempty.*

Proof. Let $0 < \rho < (M - m)/2$. As $\mathcal{D}^\rho \subseteq \tilde{\mathcal{D}} \subseteq \mathcal{D}$, it suffices to show that $\mathcal{D}^\rho \neq \emptyset$. Since $g([0, T] \times U) = [\xi_1, \xi_2]$, for suitable $(\tilde{t}, \tilde{z}) \in [0, T] \times \mathbb{R}$ we have that $Z^{\tilde{t}, \tilde{z}; u}(T) \in [\tilde{z} + \xi_1(T - \tilde{t}), \tilde{z} + \xi_2(T - \tilde{t})] \subseteq [m + \rho, M - \rho]$ for each Borel measurable function u from $[\tilde{t}, T]$ to U , and thus $(\tilde{t}, \tilde{p}, \tilde{z}) \in \mathcal{D}^\rho$ (arbitrary $\tilde{p} \in \mathbb{R}$). ■

If $(t, p, z) \in \mathcal{D}$, by \mathbb{E}_{tpz} we denote the mean value with respect to the probability $\mathbb{P}_{tpz} = \mathbb{P}$ (subscripts recall initial data). We can now define the *value function*.

Definition 3.2. *We set*

$$(3.8) \quad V(t, p, z) = \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} J(t, p, z; u)$$

for each $(t, p, z) \in \mathcal{D}$, where

$$\begin{aligned} J(t, p, z; u) = \mathbb{E}_{tpz} \left[\int_t^T e^{-r(s-t)} L(s, P^{t,p;u}(s), Z^{t,z;u}(s), u(s)) ds \right. \\ \left. + e^{-r(T-t)} \Phi(P^{t,p;u}(T), Z^{t,z;u}(T)) \right]. \end{aligned}$$

Let us prove that the value function (3.8) is well defined.

Lemma 3.3. *The expectations in (3.8) are well posed, $V(t, p, z) < \infty$, and*

$$(3.9) \quad |V(t, p, z)| \leq \Gamma(1 + |p|^k + |z|^k)$$

for each $(t, p, z) \in \mathcal{D}$, where k is as in (3.2) and $\Gamma \geq 0$ is a constant depending only on $U, T, C, \tilde{C}, \hat{C}$, and $\max g$. Moreover, \mathcal{D} is the maximal set in which expression (3.8) makes sense.

Proof. First of all, notice that $\mathcal{A}_{tz}^{\text{adm}} \neq \emptyset$ if and only if $(t, p, z) \in \mathcal{D}$ for each $p \in \mathbb{R}^n$. Recall estimate (2.9): it can be shown that B_k depends only on the set U and on constants T, C, \hat{C} (see [13, Appendix D]). By (3.2) and (2.9) we have

$$\begin{aligned} \mathbb{E}_{tpz} \left[\left| \int_t^T e^{-r(s-t)} L(s, P^{t,p;u}(s), Z^{t,z;u}(s), u(s)) ds + e^{-r(T-t)} \Phi(P^{t,p;u}(T), Z^{t,z;u}(T)) \right| \right] \\ \leq \tilde{C} \mathbb{E}_{tpz} \left[\int_t^T (1 + |P^{t,p;u}(s)|^k + |Z^{t,z;u}(s)|^k) ds + (1 + |P^{t,p;u}(T)|^k + |Z^{t,z;u}(T)|^k) \right] \\ \leq C_1 \mathbb{E}_{tpz} [1 + \|P^{t,p;u}(\cdot)\|^k + \|Z^{t,z;u}(\cdot)\|^k] \\ (3.10) \quad \leq C_2(1 + |p|^k + |z|^k) \end{aligned}$$

for suitable constants $C_1, C_2 > 0$. ■

We also require the following assumption to hold.

Assumption 2. Given $0 < \rho < (M - m)/2$ and a compact subset $A \subseteq \mathcal{D}^\rho$, there exist $\bar{\varepsilon} > 0$ and a function $0 < \eta \leq (M - m)/2$, both depending only on ρ, A, T , and U , with the following property: for each $0 < \varepsilon < \bar{\varepsilon}$, $(t, p, z) \in A$, $u \in \mathcal{A}_{tz}^{\text{adm}}$ there exists $\tilde{u} \in \mathcal{A}_t$ such that $|J(t, p, z; u) - J(t, p, z; \tilde{u})| \leq \varepsilon$ and such that a.s. $Z^{t,z;\tilde{u}}(T) \in [m + \eta(\varepsilon), M - \eta(\varepsilon)]$.

In section 3.4 we will give two examples of wide classes of problems satisfying Assumption 2.

3.2. Approximating problems. We would like to obtain for the problems of section 3.1 the standard results in unconstrained control theory: continuity of the value function and characterization of the value function by the HJB equation. A straightforward approach is not possible, since condition $u \in \mathcal{A}_{tz}^{\text{adm}}$ prevents us from simply adapting classical proofs.

The idea is then the following: to define suitable unconstrained problems which approximate our constrained problem and then to obtain the properties of the value function (3.8) through a limiting procedure. The construction of the approximating problems is based on the idea of penalizing the case $Z^{t,z;u}(T) \notin [m, M]$ by adding a suitable term in the objective functional.

Given $c > 0$, let Φ^c be the function from \mathbb{R} to \mathbb{R} defined by

$$(3.11) \quad \Phi^c(z) = -c \left[\left(z - \left(M - \frac{1}{\sqrt{c}} \right) \right)^+ + \left(\left(m + \frac{1}{\sqrt{c}} \right) - z \right)^+ \right]$$

for each $z \in \mathbb{R}$. This particular function has been chosen since we will need, as a key point in the proof of Theorem 3.5, that $\lim_{c \rightarrow +\infty} \Phi^c(m) = \lim_{c \rightarrow +\infty} \Phi^c(M) = -\infty$ and that definitely $\Phi^c(z) = 0$ for each $z \in]m, M[$.

Now let the assumptions of section 3.1 hold and consider the following unconstrained problem:

$$(3.12) \quad V^c(t, p, z) = \sup_{u \in \mathcal{A}_t} J^c(t, p, z; u),$$

where $(t, p, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$,

$$J^c(t, p, z; u) = \mathbb{E}_{t, p, z} \left[\int_t^T e^{-r(s-t)} L(s, P^{t,p;u}(s), Z^{t,z;u}(s), u(s)) ds + e^{-r(T-t)} \Phi(P^{t,p;u}(T), Z^{t,z;u}(T)) + e^{-r(T-t)} \Phi^c(Z^{t,z;u}(T)) \right],$$

and this time the maximization is performed over the set \mathcal{A}_t of all controls.

Problem (3.12) is a classical unconstrained stochastic control problem; therefore, by classical results, the function V^c is characterized by the HJB equation. Here is the precise statement.

Theorem 3.4. *Let the assumptions of section 3.1 hold, and let $c > 0$ and k as in (3.2).*

Then V^c is the unique continuous viscosity solution of

$$(3.13) \quad -V_t^c(t, p, z) + rV^c(t, p, z) + \min_{v \in U} \left[-f(t, p, v) \cdot D_p V^c(t, p, z) - g(t, v) V_z^c(t, p, z) - \frac{1}{2} \text{tr}(\sigma(t, p, v) \sigma^t(t, p, v) D_p^2 V^c(t, p, z)) - L(t, p, z, v) \right] = 0 \quad \forall (t, p, z) \in [0, T[\times \mathbb{R}^n \times \mathbb{R},$$

with final condition

$$(3.14) \quad V^c(T, p, z) = \Phi(p, z) + \Phi^c(z) \quad \forall (p, z) \in \mathbb{R}^{n+1},$$

and such that

$$|V^c(t, p, z)| \leq \check{C}(1 + |p|^k + |z|^k) \quad \forall (t, p, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}$$

for some constant $\check{C} > 0$.

Proof. The value function is a viscosity solution of (3.13) by a standard result in unconstrained control theory (the proof, for instance, can be achieved by slightly modifying the arguments in [13, Chapter IV]). As for uniqueness, see [9, Theorem 3.1]. ■

3.3. Properties of the value function. We now prove the central result of this paper: the value functions V^c in (3.12) converge, uniformly on the compact subsets of each \mathcal{D}^ρ , to the value function V in (3.8). On the one hand, this result provides an approximation of V (recall the characterization of the functions V^c in Theorem 3.4); on the other hand, in such a way V inherits continuity from the functions V^c .

Theorem 3.5. *Let the assumptions of section 3.1 hold. Then, as $c \rightarrow +\infty$, the functions V^c converge to V uniformly on compact subsets of \mathcal{D}^ρ for each $0 < \rho < (M - m)/2$.*

Proof. Let $0 < \rho < (M - m)/2$, A be a compact subset of \mathcal{D}^ρ , and let $R > 0$ be such that $B(0, R) \supseteq A$. For each $\varepsilon > 0$, we have to prove that there exists $\delta > 0$ such that

$$(3.15) \quad \left| \sup_{u \in \mathcal{A}_t} J^c(t, p, z; u) - \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} J(t, p, z; u) \right| \leq \varepsilon$$

for each $c \geq \delta$ and $(t, p, z) \in A$.

Step 1: Lower bound for V^c in A . By definition of \mathcal{D}^ρ , for each $(t, p, z) \in A$ let u_{tpz} be a Borel measurable function from $[t, T]$ to U such that $Z^{u_{tpz}}(T) \in [m + \rho, M - \rho]$. Since $[m + \rho, M - \rho] \subseteq [m + c^{-\frac{1}{2}}, M - c^{-\frac{1}{2}}]$ for $c \geq \rho^{-2}$, notice that $J^c(t, p, z; u_{tpz}) \equiv K_{tpz}$ for a suitable constant K_{tpz} for all $c \geq \rho^{-2}$. By estimates as in (3.10), it is easy to show that, for a constant $C_1 > 0$ and for $k \geq 0$ as in (3.2), we have $|K_{tpz}| \leq C_1(1 + |p|^k + |z|^k) \leq C_1(1 + 2R^k)$ for each $(t, p, z) \in A$, so that $K := \inf_{(t,p,z) \in A} K_{tpz} \in \mathbb{R}$. Therefore,

$$(3.16) \quad V^c(t, p, z) \geq J^c(t, p, z; u_{tpz}) = K_{tpz} \geq K$$

for each $(t, p, z) \in A$ and $c \geq \rho^{-2}$.

Step 2: New formulation of (3.15). Let $(t, p, z) \in A$. For each $n \in \mathbb{N}$ we set

$$B_n^{tpz} = \left\{ u \in \mathcal{A}_t : \frac{1}{n+1} < \mathbb{P}_{tpz}(Z^u(T) \notin [m, M]) \leq \frac{1}{n} \right\}.$$

Let $c \geq \rho^{-2}$, $n \in \mathbb{N}$, and $u \in B_n^{tpz}$. By noting that $\Phi^c \leq 0$ and that $\Phi^c(x) \leq -\sqrt{c}$ for $x \notin [m, M]$ and by estimates as in (3.10), we have

$$\begin{aligned}
 J^c(t, p, z; u) &= J(t, p, z; u) + \mathbb{E}_{tpz} \left[e^{-r(T-t)} \Phi^c(Z^u(T)) \right] \\
 &\leq J(t, p, z; u) + \mathbb{E}_{tpz} \left[e^{-r(T-t)} \Phi^c(Z^u(T)) 1_{\{Z^u(T) \notin [m, M]\}} \right] \\
 &\leq C_2(1 + |p|^k + |z|^k) - e^{-rT} \sqrt{c} \mathbb{P}_{tpz}(Z^u(T) \notin [m, M]) \\
 (3.17) \quad &< C_2(1 + 2R^k) - e^{-rT} \frac{\sqrt{c}}{n+1}
 \end{aligned}$$

for a suitable constant $C_2 > 0$. By (3.17) it follows that for each $n \in \mathbb{N}$ there exists $c(n) \geq \rho^{-2}$ such that

$$J^c(t, p, z; u) < K$$

for each $c \geq c(n)$ and $u \in B_n^{tpz}$, with K as in Step 1. By (3.16) we thus get

$$(3.18) \quad \sup_{u \in \mathcal{A}_t} J^c(t, p, z; u) = \sup_{u \in \mathcal{A}_t \setminus \bigcup_{\{i \in \mathbb{N}: c(i) \leq c\}} B_i^{tpz}} J^c(t, p, z; u)$$

for each $c \geq \rho^{-2}$. The sequence $\{c(n)\}_n$ is obviously increasing; hence, there exists a function m from $[\rho^{-2}, +\infty[$ to \mathbb{N} such that $\{i \in \mathbb{N} : c(i) \leq c\} = \{1, \dots, m(c)\}$ for each $c \geq \rho^{-2}$. As a consequence, we can rewrite (3.18) as follows:

$$(3.19) \quad \sup_{u \in \mathcal{A}_t} J^c(t, p, z; u) = \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t, p, z; u).$$

Notice that $m(\cdot)$ is increasing and that $m(c) \rightarrow +\infty$.

Let $\varepsilon > 0$. By (3.19), for $c \geq \rho^{-2}$ inequality (3.15) is equivalent to

$$(3.20) \quad -\varepsilon \leq \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t, p, z; u) - \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} J(t, p, z; u) \leq \varepsilon.$$

Therefore, we have to prove that there exists $\delta \geq \rho^{-2}$ such that (3.20) holds for each $c \geq \delta$ and for each $(t, p, z) \in A$. In Step 3 we will prove the right inequality in (3.20), while in Step 4 the left inequality will be proved, thus concluding the proof.

Step 3: Right inequality in (3.20). Let us show that there exists $\delta_1 \geq \rho^{-2}$ independent of $(t, p, z) \in A$ such that

$$(3.21) \quad \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J(t, p, z; u) \leq \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} J(t, p, z; u) + \varepsilon$$

for each $c \geq \delta_1$. Since $J^c \leq J$, by (3.21) we get the right inequality in (3.20).

Let $c \geq \rho^{-2}$, $(t, p, z) \in A$, $u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}$. We set

$$\Pi^u = \{Z^u(T) \notin [m, M]\};$$

notice that $0 \leq \mathbb{P}_{tpz}(\Pi^u) \leq 1/(m(c) + 1)$. Let \tilde{u} be the process defined in the following way: \tilde{u} coincides in Π^u with the process which ensures the reachability of $[m + \rho, M - \rho]$ (see the definition of \mathcal{D}^ρ), and $\tilde{u} \equiv u$ in $\Omega \setminus \Pi^u$. A simple check shows that $\tilde{u} \in \mathcal{A}_t$ and that

$$(3.22) \quad Z^{\tilde{u}}(T) \in [m, M] \quad \mathbb{P}_{tpz}\text{-a.s.}$$

By recalling that $\tilde{u} \equiv u$ in $\Omega \setminus \Pi^u$, by the Hölder inequality (twice), and by estimates as in (3.10), we obtain that

$$(3.23) \quad |J(t, p, z; u) - J(t, p, z; \tilde{u})| \leq C_3(1 + |p|^k + |z|^k) \mathbb{P}_{tpz}(\Pi^u)^{\frac{1}{2}} \leq \frac{C_3(1 + 2R^k)}{(m(c) + 1)^{\frac{1}{2}}}$$

for some constant $C_3 > 0$. Then, by (3.23) and (3.22) it follows that

$$J(t, p, z; u) \leq J(t, p, z; \tilde{u}) + \frac{C_3(1 + 2R^k)}{(m(c) + 1)^{\frac{1}{2}}} \leq \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} J(t, p, z; u) + \frac{C_3(1 + 2R^k)}{(m(c) + 1)^{\frac{1}{2}}}.$$

This inequality holds for each $(t, p, z) \in A$ and $u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}$. Since $m(c) \rightarrow +\infty$, for sufficiently large c (and this choice is independent of (t, p, z) and u), we have that $C_3(1 + 2R^k)/(m(c) + 1)^{\frac{1}{2}} \leq \varepsilon$, thus obtaining (3.21).

Step 4: Left inequality in (3.20). We still have to prove the left inequality in (3.20), i.e.,

$$(3.24) \quad \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} J(t, p, z; u) \leq \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t, p, z; u) + \varepsilon$$

for $c \geq \delta_2$, with $\delta_2 \geq \rho^{-2}$ independent of $(t, p, z) \in A$.

Let $c \geq \rho^{-2}$, $(t, p, z) \in A$, $u \in \mathcal{A}_{tz}^{\text{adm}}$. By Assumption 2, let $\tilde{u} \in \mathcal{A}_t$ be such that

$$(3.25) \quad |J(t, p, z; u) - J(t, p, z; \tilde{u})| \leq \varepsilon$$

and with the property

$$(3.26) \quad Z^{\tilde{u}}(T) \in [m + \eta(\varepsilon), M - \eta(\varepsilon)] \quad \mathbb{P}_{tpz}\text{-a.s.}$$

First of all, notice that

$$(3.27) \quad \tilde{u} \in \mathcal{A}_{tz}^{\text{adm}} \subseteq \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}.$$

By (3.25) we obtain that

$$\begin{aligned} & |J(t, p, z; u) - J^c(t, p, z; \tilde{u})| \\ &= |J(t, p, z; u) - J(t, p, z; \tilde{u}) - \mathbb{E}_{tpz}[e^{-r(T-t)} \Phi^c(Z^{\tilde{u}}(T))]| \leq \varepsilon + \mathbb{E}_{tpz}[|\Phi^c(Z^{\tilde{u}}(T))|]. \end{aligned}$$

Notice that by (3.26) the second term equals zero for $c \geq \eta(\varepsilon)^{-2}$ (in fact, $\Phi^c \equiv 0$ in $[m + c^{-\frac{1}{2}}, M - c^{-\frac{1}{2}}]$); by recalling (3.27), we therefore have that

$$J(t, p, z; u) \leq J^c(t, p, z; \tilde{u}) + \varepsilon \leq \sup_{u \in \mathcal{A}_t \setminus \bigcup_{i=1}^{m(c)} B_i^{tpz}} J^c(t, p, z; u) + \varepsilon$$

for each $c \geq \max\{\eta(\varepsilon)^{-2}, \rho^{-2}\}$. Since this inequality holds for each $(t, p, z) \in A$ and $u \in \mathcal{A}_{tz}^{\text{adm}}$, we get (3.24). ■

Corollary 3.6. *Let the assumptions of section 3.1 hold. Then the functions V^c converge pointwise to V in $\tilde{\mathcal{D}}$, and V is continuous on $\tilde{\mathcal{D}}$.*

Proof. It follows immediately from Theorem 3.5 (recall that $\bigcup_{\rho} \mathcal{D}^{\rho} = \tilde{\mathcal{D}}$). ■

Corollary 3.7. *Let the assumptions of section 3.1 hold. Then the function V is a viscosity solution of (3.13) in $\tilde{\mathcal{D}}$.*

Proof. Due to Theorem 3.4, the functions V^c are viscosity solutions of the same equation, that is,

$$(3.28) \quad -V_t^c(t, p, z) + rV^c(t, p, z) + \min_{v \in U} \left[-f(t, p, v) \cdot D_p V^c(t, p, z) - g(t, v) V_z^c(t, p, z) \right. \\ \left. - \frac{1}{2} \text{tr}(\sigma(t, p, v) \sigma^t(t, p, v) D_p^2 V^c(t, p, z)) - L(t, p, z, v) \right] = 0 \quad \forall (t, p, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}.$$

Moreover, by Theorem 3.5 for each $0 < \rho < (M - m)/2$ the functions V^c locally uniformly converge in \mathcal{D}^{ρ} to the function V . So the conclusion follows by the stability property of viscosity solutions with respect to the uniform convergence and the fact that $\tilde{\mathcal{D}} = \bigcup_{\rho} \mathcal{D}^{\rho}$. ■

Remark 2. We have proved that the value function in the set $\tilde{\mathcal{D}}$ is the locally uniform limit of the functions V^c . In some particular cases, stronger conclusions can be achieved: the value function is characterized in its whole domain \mathcal{D} by the HJB equation. See section 4.

We now face the problem of the regularity of the function V . In control theory, regularity results are usually achieved by passing to the supremum in estimates on quantities such as $|J(t, p', z; u) - J(t, p'', z; u)|$ or $|J(t, p, z'; u) - J(t, p, z''; u)|$, so as to obtain the corresponding inequality for V . In the case of constrained problems, this approach cannot be applied to $V(t, p, \cdot)$. In fact, consider $|J(t, p, z'; u) - J(t, p, z''; u)|$: on the one hand, such a quantity is defined only for $u \in \mathcal{A}_{tz'}^{\text{adm}} \cap \mathcal{A}_{tz''}^{\text{adm}}$; on the other hand, the supremum should be with respect to different sets (precisely, $\mathcal{A}_{tz'}^{\text{adm}}$ and $\mathcal{A}_{tz''}^{\text{adm}}$). Of course, in particular cases some regularity results can also be achieved for $V(t, p, \cdot)$; see section 4. The only case when that approach still works regards estimates on $V(t, \cdot, z)$, given that, t and z fixed, the set of admissible controls does not depend on p . Hence, as for $V(t, \cdot, z)$ we can follow this approach.

Proposition 3.8. *Let the assumptions of section 3.1 hold. Assume that there exists a constant $\bar{C} > 0$ such that*

$$(3.29) \quad |L(t, p, z, v) - L(t, q, z, v)| \leq \bar{C}|p - q|, \quad |\Phi(p, z) - \Phi(q, z)| \leq \bar{C}|p - q|$$

for each $p, q \in \mathbb{R}^n$, $t \in [0, T]$, $v \in U$, $z \in \mathbb{R}$. Then the function $V(t, \cdot, z)$ is Lipschitz continuous, uniformly in (t, z) . Moreover, the gradient $D_p V(t, p, z)$ exists for a.e. $(t, p, z) \in \mathcal{D}$, and we have $|D_p V(t, p, z)| \leq M_1$ for some constant $M_1 > 0$ depending only on U , T , \bar{C} and on the constants in (3.1) and (3.4).

Proof. Let $(t, p, z) \in \mathcal{D}$, $h > 0$, $\xi \in \mathbb{R}^n$, with $|\xi| = 1$, and $u \in \mathcal{A}_t$. In order to avoid ambiguity, we will omit the subscripts in the notation of the mean value (initial data are different, but the probability is obviously the same). By (3.29) and estimates (D.8) in [13,

Appendix D] we have

$$\begin{aligned}
 & |J(t, p, z; u) - J(t, p + h\xi, z; u)| \\
 & \leq \bar{C} \mathbb{E} \left[\int_t^T |P^{t,p;u}(s) - P^{t,p+h\xi;u}(s)| ds + |P^{t,p;u}(T) - P^{t,p+h\xi;u}(T)| \right] \\
 & \leq \bar{C}(T - t + 1) \mathbb{E}[\|P^{t,p;u}(\cdot) - P^{t,p+h\xi;u}(\cdot)\|_\infty] \\
 & \leq C_1 \bar{C}(T + 1) |p - (p + h\xi)| \\
 (3.30) \quad & = C_1 \bar{C}(T + 1) h
 \end{aligned}$$

for some constant $C_1 > 0$. Estimate (3.30) holds for each $u \in \mathcal{A}_t$; thus, it follows that

$$(3.31) \quad |V(t, p, z) - V(t, p + h\xi, z)| \leq M_0 h,$$

where $M_0 := C_1 \bar{C}(T + 1)$. The function $V(t, \cdot, z)$ is therefore Lipschitz continuous, uniformly in (t, z) , and then a.e. differentiable by the Rademacher theorem. By classical results it follows that $D_p V(t, p, z)$ exists for a.e. $(t, p, z) \in \mathcal{D}$. Finally, if the gradient exists and $e_i \in \mathbb{R}^n$ is a vector of the canonical basis ($i = 1, \dots, n$), by (3.31) we get

$$|(D_p V(t, p, z))_i| = \lim_{h \rightarrow 0^+} \frac{|V(t, p, z) - V(t, p + h e_i, z)|}{h} \leq M_0,$$

and then the estimate on the gradient immediately follows. ■

Proposition 3.9. *Let the assumptions of section 3.1 hold. Assume that $\Phi \in C^2(\mathbb{R}^{n+1})$, that the functions $f(t, \cdot, v)$, $\sigma(t, \cdot, v)$, $L(t, \cdot, \cdot, v)$ are of class C^2 for each $(t, v) \in [0, T] \times U$, and that there exist constants $\bar{C} \geq 0$, $j \in \mathbb{N}$ such that*

$$\begin{aligned}
 & |D_p f(t, p, v)| + |D_p^2 f(t, p, v)| + |D_p \sigma(t, p, v)| + |D_p^2 \sigma(t, p, v)| \leq \bar{C}, \\
 & |D_{(p,z)} L(t, p, z, v)| + |D_{(p,z)}^2 L(t, p, z, v)| \leq \bar{C}(1 + |p|^j + |z|^j), \\
 & |D_{(p,z)} \Phi(p, z)| + |D_{(p,z)}^2 \Phi(p, z)| \leq \bar{C}(1 + |p|^j + |z|^j)
 \end{aligned}$$

for each $p \in \mathbb{R}^n$, $t \in [0, T]$, $v \in U$, $z \in \mathbb{R}$. The function $V(t, \cdot, z)$ is then locally semiconvex, uniformly in t , and a.e. twice differentiable.

Proof. Since $\Phi \in C_p^2(\mathbb{R}^{n+1})$, it is possible to rewrite the problem so that $\Phi \equiv 0$ (see [13, Remark IV.6.1]). By arguing as in the proof of [13, Lemma IV.9.1] (with minor modifications: the assumptions are slightly different), we get

$$V(t, p + h\xi, z) + V(t, p - h\xi, z) - 2V(t, p, z) \geq -M_2(1 + |p|^j)h^2$$

for each $(t, p, z) \in \mathcal{D}$, $h > 0$, $\xi \in \mathbb{R}^n$, with $|\xi| = 1$, where $M_2 > 0$ is a constant. The function $V(t, \cdot, z)$ is therefore locally semiconvex, uniformly in (t, z) , and then a.e. twice differentiable by the Alexandrov theorem. ■

3.4. Examples. We now show two wide classes of problems satisfying Assumption 2. We first consider problems where U is a compact interval of \mathbb{R} and $g(s, v) = v$, so that the constraint is $z + \int_t^T u(s) \in [m, M]$.

Proposition 3.10. *Let $a, b \in \mathbb{R}$, with $a < b$. Let the assumptions of section 3.1 hold, with $U = [a, b]$ and $g(s, v) = v$. Moreover, assume that there exist $\Gamma, l > 0$ such that for $\xi = f, \sigma$ the following condition holds:*

$$(3.32) \quad |\xi(s, p, v') - \xi(s, p, v'')| \leq \Gamma(1 + |p|)|v' - v''|^l \quad \forall (s, p) \in [0, T] \times \mathbb{R}^n, \quad \forall v', v'' \in U.$$

Then Assumption 2 is satisfied.

Proof. For the sake of simplicity, in this proof we assume $l = 1$ (for the general case, in the definition of \tilde{u} it suffices to substitute δ^2 by δ^i , where $i > 1/l$).

Let $0 < \rho < (M - m)/2$, let A be a compact subset of \mathcal{D}^ρ , and let $R > 0$ be such that $\overline{\mathbb{B}(0, R)} \supseteq A$, $\varepsilon > 0$. Fix $(t, p, z) \in A$ and $u \in \mathcal{A}_{tz}^{\text{adm}}$.

Let $\gamma > 0$ (it will afterwards be precisely defined). Since the functions L and Φ are continuous, there exists $\delta = \delta(\varepsilon, \gamma) > 0$ such that

$$(3.33) \quad |L(s, p', z', v') - L(s, p'', z'', v'')| \leq \frac{\varepsilon}{4T} \quad \text{and} \quad |\Phi(p', z') - \Phi(p'', z'')| \leq \frac{\varepsilon}{4}$$

for each $s \in [0, T]$, for each $p', p'' \in \overline{\mathbb{B}(0, \gamma)}$, with $|p' - p''| \leq \delta$, for each $z', z'' \in [m, M]$, with $|z' - z''| \leq T\delta$, and for each $v', v'' \in U$, with $|v' - v''| \leq \delta$.

We now define, starting from u , a suitable process \tilde{u} . Let $\Pi^M = \{\omega \in \Omega : Z^u(T) \in]M - \rho/2, M]\}$, and in Π^M let \tilde{u} be defined in the following way:

$$\tilde{u}(s) = \begin{cases} u(s) - \delta^2 & \text{if } s \in E, \\ u(s) & \text{if } s \in [t, T] \setminus E, \end{cases}$$

where $E = E(\omega) = \{s \in [t, T] : u(s) - \delta^2 \in]a, b]\} = \{s \in [t, T] : u(s) > a + \delta^2\}$. Let $\Pi_m = \{\omega \in \Omega : Z^u(T) \in [m, m + \rho/2]\}$, and in Π_m let \tilde{u} be defined in the following way:

$$\tilde{u}(s) = \begin{cases} u(s) + \delta^2 & \text{if } s \in F, \\ u(s) & \text{if } s \in [t, T] \setminus F, \end{cases}$$

where $F = F(\omega) = \{s \in [t, T] : u(s) + \delta^2 \in]a, b]\} = \{s \in [t, T] : u(s) < b - \delta^2\}$. Finally, in $\Omega \setminus (\Pi^M \cup \Pi_m)$ let

$$\tilde{u} \equiv u.$$

We will show that such a process \tilde{u} satisfies the required properties.

Step 1. We prove that

$$(3.34) \quad Z^{t, z; \tilde{u}}(T) \in [m + \eta(\varepsilon), M - \eta(\varepsilon)] \quad \mathbb{P}_{tpz}\text{-a.s.}$$

for a suitable function $0 < \eta \leq (M - m)/2$ depending only on ρ, T, R, a, b .

Consider the case $\omega \in \Pi^M$, i.e.,

$$(3.35) \quad Z^u(T) \in]M - \rho/2, M].$$

Let us first of all notice that

$$(3.36) \quad Z^{\tilde{u}}(T) = z + \int_t^T \tilde{u}(s) ds = z + \int_t^T u(s) ds - \delta^2 \mu(E) = Z^u(T) - \delta^2 \mu(E),$$

where μ denotes the Lebesgue measure in \mathbb{R} . We now look for an estimate for $\mu(E)$. By definition of E , we have

$$\int_t^T u(s)ds = \int_E u(s)ds + \int_{[t,T]\setminus E} u(s)ds \leq b\mu(E) + (a + \delta^2)(T - t - \mu(E))$$

and then

$$(3.37) \quad \mu(E) \in \left[\frac{\int_t^T u(s)ds - (a + \delta^2)(T - t)}{b - a - \delta^2}, T - t \right] \subseteq \left[\frac{\rho/2 - \delta T}{b - a}, T \right],$$

where the inclusion follows by $z + \int_t^T u(s)ds \geq M - \rho/2$ (since $\omega \in \Pi^M$) and $z \leq M - \rho - a(T - t)$ (since $(t, p, z) \in \mathcal{D}^\rho$). By possibly decreasing δ (and the choice depends only on a, b, ρ, T), we can assume that the lower bound in (3.37) is positive. Recall (3.36): by (3.35) and (3.37) we get

$$Z^{\tilde{u}}(T) \in \left] M - \frac{\rho}{2} - \delta^2 T, M - \frac{\delta^2 \rho/2 - \delta^3 T}{b - a} \right] \subseteq \left] m + \frac{\rho}{2}, M - \frac{\delta^2 \rho/2 - \delta^3 T}{b - a} \right],$$

where the inclusion follows by $M - \rho/2 > m + \rho/2$ and by assuming δ sufficiently small. This estimate holds for each $\omega \in \Pi^M$; by arguing in the same way, for each $\omega \in \Pi_m$ we get

$$Z^{\tilde{u}}(T) \in \left[m + \frac{\delta^2 \rho/2 - \delta^3 T}{b - a}, M - \frac{\rho}{2} \right].$$

Finally, in $\Omega \setminus (\Pi^M \cup \Pi_m)$ we have $Z^{\tilde{u}} \in [m + \rho/2, M - \rho/2]$. To summarize, condition (3.34) is verified with

$$\eta(\varepsilon) = \min \left\{ \frac{\rho}{2}, \frac{\delta(\varepsilon, \gamma)^2 \rho/2 - \delta(\varepsilon, \gamma)^3 T}{b - a} \right\}.$$

Step 2. We still have to prove that

$$(3.38) \quad |J(t, p, z; u) - J(t, p, z; \tilde{u})| \leq \varepsilon.$$

Let $\Pi \subseteq \Omega$ be defined by

$$\Pi = \{ \|P^u(\cdot)\| \leq \gamma, \|P^{\tilde{u}}(\cdot)\| \leq \gamma, \|P^u(\cdot) - P^{\tilde{u}}(\cdot)\| \leq \delta \};$$

first of all, we set for brevity

$$\begin{aligned} \Gamma(t, p, z; u, \tilde{u}) = \int_t^T & |L(s, P^u(s), Z^u(s), u(s)) - L(s, P^{\tilde{u}}(s), Z^{\tilde{u}}(s), \tilde{u}(s))| ds \\ & + |\Phi(P^u(T), Z^u(T)) - \Phi(P^{\tilde{u}}(T), Z^{\tilde{u}}(T))|, \end{aligned}$$

and notice that

$$(3.39) \quad |J(t, p, z; u) - J(t, p, z; \tilde{u})| \leq \mathbb{E}_{tpz}[\Gamma(t, p, z; u, \tilde{u})1_\Pi] + \mathbb{E}_{tpz}[\Gamma(t, p, z; u, \tilde{u})1_{\Pi^c}].$$

As for the first term in (3.39), for $s \in [t, T]$ we have

$$|Z^u(s) - Z^{\tilde{u}}(s)| \leq (s - t)\delta^2 \leq T\delta \quad \text{and} \quad |u(s) - \tilde{u}(s)| \leq \delta^2 \leq \delta$$

so that by (3.33) it follows that

$$(3.40) \quad \mathbb{E}_{tpz}[\Gamma(t, p, z; u, \tilde{u})1_{\Pi}] \leq \left(\int_0^T \frac{\varepsilon}{4T} ds + \frac{\varepsilon}{4} \right) \mathbb{P}_{tpz}(\Pi) = \frac{\varepsilon}{2} \mathbb{P}_{tpz}(\Pi) \leq \frac{\varepsilon}{2}.$$

We now consider the second term in (3.39). By (D.8) in [13, Appendix D], (3.32), and (2.9), we get

$$(3.41) \quad \mathbb{E}_{tpz}[\|P^u(\cdot) - P^{\tilde{u}}(\cdot)\|] \leq C_1(1 + |p|)\delta^2$$

for a suitable constant $C_1 > 0$. By the Markov inequality, (3.41), and (2.9), we then get

$$(3.42) \quad \mathbb{P}_{tpz}(\Pi^c) \leq \mathbb{E}_{tpz}[\|P^u(\cdot)\|_{\infty}] \gamma^{-1} + \mathbb{E}_{tpz}[\|P^{\tilde{u}}(\cdot)\|_{\infty}] \gamma^{-1} + \mathbb{E}_{tpz}[\|P^u(\cdot) - P^{\tilde{u}}(\cdot)\|_{\infty}] \delta^{-1} \\ \leq C_2(1 + |p|)(\gamma^{-1} + \delta),$$

where $C_2 > 0$ is a constant. By the Hölder inequality (twice), estimates as in (3.10), and (3.42), we obtain that

$$\begin{aligned} \mathbb{E}_{tpz}[\Gamma(t, p, z; u, \tilde{u})1_{\Pi^c}]^2 &\leq \mathbb{E}_{tpz}[\Gamma(t, p, z; u, \tilde{u})^2] \mathbb{P}_{tpz}(\Pi^c) \\ &\leq C_3(1 + |p|^{2k} + |z|^{2k})(1 + |p|)(\gamma^{-1} + \delta(\varepsilon, \gamma)) \\ &\leq C_4(1 + R^{2k+1})\gamma^{-1} + C_4(1 + R^{2k+1})\delta(\varepsilon, \gamma), \end{aligned}$$

with $C_3, C_4 > 0$. First by choosing a suitable γ and then by possibly taking a lesser δ (and these choices depends only on R and ε), we get

$$(3.43) \quad \mathbb{E}_{tpz}[\Gamma(t, p, z; u, \tilde{u})1_{\Pi^c}] \leq \frac{\varepsilon}{2}.$$

Estimates (3.39), (3.40), and (3.43) imply (3.38), thus ending the proof. \blacksquare

Let us now consider problems where U is a closed ball of \mathbb{R}^l and $g(s, v) = |v|^p$, so that the constraint is $z + \int_t^T |u(s)|^p \in [m, M]$, with $p \geq 1$.

Proposition 3.11. *Let $p \geq 1$ and $b > 0$. Let the assumptions of section 3.1 and (3.32) hold, with $U = \overline{B}(0, b) \subseteq \mathbb{R}^l$ and $g(s, v) = |v|^p$. Assumption 2 is then satisfied.*

Proof. The proof is similar to that of Proposition 3.10, with the following modifications:

- In the definitions of E and F replace $u(s)$ by $|u(s)|$. Process \tilde{u} in E is now

$$\tilde{u}(s) = u(s) - \delta^2 \frac{u(s)}{|u(s)|}.$$

Notice that $|\tilde{u}(s)| = |u(s)| - \delta^2$. The modification is analogous in F .

– It is easy to check that

$$\begin{aligned} (\zeta - \delta^2)^p &\leq \zeta^p - \delta^{2p} && \text{for } \zeta \geq \delta^2, \\ (\zeta + \delta^2)^p &\geq \zeta^p + \delta^{2p} && \text{for } \zeta \geq 0. \end{aligned}$$

By the first estimate, in Π^M we have that

$$\begin{aligned} Z^{\tilde{u}}(T) &= z + \int_t^T |\tilde{u}(s)|^p ds \\ &= z + \int_E (|u(s)| - \delta^2)^p ds + \int_{[t,T] \setminus E} |u(s)|^p ds \\ &\leq z + \int_t^T |u(s)|^p ds - \delta^{2p} \mu(E), \end{aligned}$$

and then we can argue as in the proof of Proposition 3.10. As for Π_m , use the second estimate and the same argument. ■

4. Swing contracts with strict constraints. We now use the results of section 3 to study the problem of optimally exercising swing contracts with strict constraints (see the introduction). In this case we will obtain results stronger than the general ones proved in section 3.3.

4.1. Formulation of the problem. Let $T > 0$, and let $(\Omega, \mathcal{F}_T, \{\mathcal{F}_s\}_{s \in [0,T]}, \mathbb{P})$, U , \mathcal{A}_t , $P^{t,p}$, and $Z^{t,z;u}$ be as in section 2. If $(t, p, z) \in [0, T] \times \mathbb{R}^2$ and $s \in [t, T]$, recall, in particular, that $P^{t,p}(s)$ models the price of energy at time s and that $Z^{t,z;u}(s)$ represents the energy bought up to time s , where $u \in \mathcal{A}_t$ is the usage strategy from time t on.

Given $m, M \geq 0$, with $m < M$, here we ask the following constraint to hold:

$$Z^{t,z;u}(T) \in [m, M] \quad \mathbb{P}_{tpz}\text{-a.s.}$$

The problem of the optimal exercise of this contract (i.e., to find a process u satisfying all the conditions and providing the maximal expected earning) is clearly a constrained stochastic optimal control problem as described in section 3.1 (here P does not depend on u), whose value function is

$$V(t, p, z) = \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} \mathbb{E}_{tpz} \left[\int_t^T e^{-r(s-t)} (P^{t,p}(s) - K) u(s) ds \right].$$

In this problem the sets $\mathcal{D}, \tilde{\mathcal{D}}, \mathcal{D}^\rho$ are formed by all points $(t, p, z) \in [0, T] \times \mathbb{R}^2$ such that (t, z) belongs, respectively, to the marked surfaces in Figure 2.

In more detail, we have

$$\begin{aligned} \mathcal{D} &= \{(t, p, z) \in [0, T] \times \mathbb{R}^2 : m - \bar{u}(T - t) \leq z \leq M\}, \\ \tilde{\mathcal{D}} &= \{(t, p, z) \in [0, T] \times \mathbb{R}^2 : m - \bar{u}(T - t) < z < M\}, \\ \mathcal{D}^\rho &= \{(t, p, z) \in [0, T] \times \mathbb{R}^2 : m + \rho - \bar{u}(T - t) \leq z \leq M - \rho\}. \end{aligned}$$

Notice that these sets include initial data that are inconsistent with the practical problem: in fact, our mathematical formulation admits negative starting values for p and z .

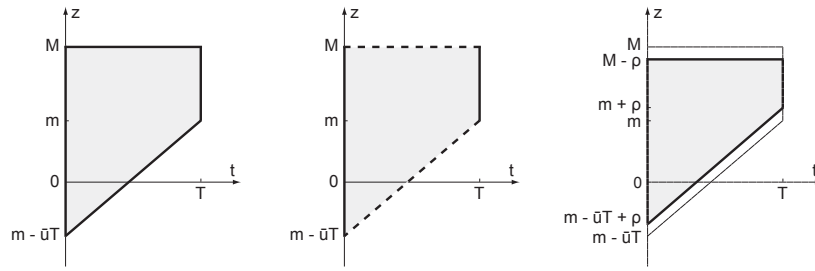


Figure 2. The sets $\mathcal{D}, \tilde{\mathcal{D}}, \mathcal{D}^\rho$.

The functions V^c are here defined by

$$V^c(t, p, z) = \sup_{u \in \mathcal{A}_t} \mathbb{E}_{t, p, z} \left[\int_t^T e^{-r(s-t)} (P^{t,p}(s) - K) u(s) ds + e^{-r(T-t)} \Phi^c(Z^{t,z;u}(T)) \right]$$

for each $c > 0$ and $(t, p, z) \in [0, T] \times \mathbb{R}^2$, where Φ^c is defined in (3.11). This also has a nice economical interpretation: in fact, here we are approximating a swing contract with the strict constraint $Z(T) \in [m, M]$ with a sequence of suitable contracts with increasing penalties for $Z(T) \notin [m + \frac{1}{\sqrt{c}}, M - \frac{1}{\sqrt{c}}]$.

The HJB equation for the function V^c is

$$(4.1) \quad -V_t^c(t, p, z) + rV^c(t, p, z) - f(t, p)V_p^c(t, p, z) - \frac{1}{2}\sigma^2(t, p)V_{pp}^c(t, p, z) \\ + \min_{v \in [0, \bar{u}]} [-v(V_z^c(t, p, z) + p - K)] = 0 \quad \forall (t, p, z) \in [0, T] \times \mathbb{R}^2,$$

with final condition

$$V^c(T, p, z) = \Phi^c(z) \quad \forall (p, z) \in \mathbb{R}^2.$$

4.2. Properties of the value function. The problem described in section 4.1 belongs to the class treated in Proposition 3.10. Therefore, Theorem 3.5 and Corollaries 3.6 and 3.7 hold, but it turns out that in this case we can strengthen such results.

We set for brevity

$$\alpha = \{(t, p, z) \in \mathcal{D} : z = M\}, \quad \beta = \{(t, p, z) \in \mathcal{D} : z + \bar{u}(T-t) = m\}, \quad \gamma = \{T\} \times \mathbb{R} \times [m, M]$$

so that $\mathcal{D} \setminus \tilde{\mathcal{D}} = \alpha \cup \beta$.

Let us first consider Theorem 3.5 and adapt it to our problem, as here something about $\mathcal{D} \setminus \tilde{\mathcal{D}}$ can also be said.

Proposition 4.1. *Let the assumptions of section 4.1 hold. The functions V^c converge to V uniformly on compact subsets of $\tilde{\mathcal{D}}$. Moreover, if $(t, p, z) \in \alpha$, we have $V(t, p, z) = 0$. Finally, if $(t, p, z) \in \beta$, we have*

$$(4.2) \quad V(t, p, z) = \bar{u} \mathbb{E}_{t, p, z} \left[\int_t^T e^{-r(s-t)} (P^{t,p}(s) - K) ds \right] =: \xi(t, p).$$

Proof. As for the first part, notice that each compact subset of $\tilde{\mathcal{D}}$ is contained in some \mathcal{D}^ρ and use Theorem 3.5. For the second and third items, in $\alpha \cup \beta$ there exists a unique admissible control, respectively, $u \equiv 0$ and $u \equiv \bar{u}$. ■

Notice that the boundary condition ξ in (4.2) is continuous and can be computed in many models used in practice (see [3]).

Corollary 3.6 ensures continuity of V on $\tilde{\mathcal{D}}$. We now prove that in this case a stronger result holds, i.e., the value function is continuous on the whole domain \mathcal{D} . For this, we first need a technical lemma (see [13, Appendix D] or [16]), where we give a bound for the mean distance between solutions of (2.1) starting from different data.

Lemma 4.2. *Let the assumptions of section 4.1 hold. Let $t_1, t_2 \in [0, T]$, with $t_1 < t_2$, and $p_1, p_2 \in \mathbb{R}^n$. Then*

$$\mathbb{E}[|P^{t_1, p_1}(s) - P^{t_2, p_2}(s)|] \leq M[|p_2 - p_1| + (t_2 - t_1)^{\frac{1}{2}}(1 + |p_1|)]$$

for each $s \in [t_2, T]$, where \mathbb{E} denotes the mean value with respect to the probability \mathbb{P} and $M > 0$ is a constant depending only on T, U and on the constants in (3.1) and (3.4).

Proposition 4.3. *Let the assumptions of section 4.1 hold. Then V is continuous on \mathcal{D} .*

Proof. As Corollary 3.6 holds, we have to prove that V is continuous on $\mathcal{D} \setminus \tilde{\mathcal{D}} = \alpha \cup \beta$.

Step 1: Continuity on α . Let $(\tilde{t}, \tilde{p}, \tilde{z}) \in \alpha$. Since in this case the only admissible control is $u \equiv 0$, we have to prove that

$$(4.3) \quad \lim_{\substack{(t, p, z) \rightarrow (\tilde{t}, \tilde{p}, \tilde{z}) \\ (t, p, z) \in \mathcal{D}}} V(t, p, z) = V(\tilde{t}, \tilde{p}, \tilde{z}) = 0.$$

Let $(t, p, z) \in \mathcal{D}$ and $u \in \mathcal{A}_{tz}^{\text{adm}}$. Given arbitrary $\gamma > 0$, we first of all observe that

$$(4.4) \quad \begin{aligned} & \mathbb{E}_{tpz} \left[\left(\int_t^T |P^{t,p}(s) - K|u(s)ds \right) 1_{\{\|P(\cdot)\| \leq \gamma\}} \right] \\ & \leq (\gamma + K) \mathbb{E}_{tpz} \left[\int_t^T u(s)ds \right] \leq (\gamma + K)(M - z), \end{aligned}$$

where in the last passage we have used condition $Z^u(T) \leq M$. By the Hölder inequality (twice), estimates as in (3.10), the Markov inequality, and (2.9), we get

$$(4.5) \quad \begin{aligned} & \mathbb{E}_{tpz} \left[\left(\int_t^T |P^{t,p}(s) - K|u(s)ds \right) 1_{\{\|P^{t,p}(\cdot)\| > \gamma\}} \right] \\ & \leq T \mathbb{E}_{tpz} \left[\int_t^T (P^{t,p}(s) - K)^2 u(s)^2 ds \right]^{\frac{1}{2}} \mathbb{P}_{tpz}(\|P^{t,p}(\cdot)\| > \gamma)^{\frac{1}{2}} \leq C_1(1 + |p|)^{\frac{3}{2}} \gamma^{-\frac{1}{2}} \end{aligned}$$

for some constant $C_1 > 0$. By (4.4) and (4.5), it follows that

$$(4.6) \quad \left| \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} \mathbb{E}_{tpz} \left[\int_t^T e^{-r(s-t)} (P^{t,p}(s) - K)u(s)ds \right] \right| \leq (\gamma + K)(M - z) + C_1(1 + |p|)^{\frac{3}{2}} \gamma^{-\frac{1}{2}}.$$

Inequality (4.6) holds for each $\gamma > 0$ and for each $(t, p, z) \in \mathcal{D}$. We get (4.3) by passing to the limit first as $(t, p, z) \rightarrow (\tilde{t}, \tilde{p}, \tilde{z})$ (recall that $\tilde{z} = M$) and then as $\gamma \rightarrow \infty$.

Step 2: Continuity on β . Let $(\tilde{t}, \tilde{p}, \tilde{z}) \in \beta$. Since in β function V is as in (4.2), we have to prove that

$$(4.7) \quad \lim_{\substack{(t,p,z) \rightarrow (\tilde{t}, \tilde{p}, \tilde{z}) \\ (t,p,z) \in \mathcal{D}}} V(t, p, z) = V(\tilde{t}, \tilde{p}, \tilde{z}) = \bar{u} \mathbb{E}_{\tilde{t}\tilde{p}\tilde{z}} \left[\int_{\tilde{t}}^T e^{-r(s-\tilde{t})} (P^{\tilde{t}, \tilde{p}}(s) - K) ds \right].$$

From now on, we will omit the subscripts in the notation of the mean value (the initial data are different, but the probability is clearly the same). Let $(t, p, z) \in \mathcal{D}$ (notice that necessarily $t \leq \tilde{t}$), and fix $u \in \mathcal{A}_{tz}^{\text{adm}}$; for simplicity we will write $P = P^{t,p}$ and $\tilde{P} = P^{\tilde{t}, \tilde{p}}$. Since for $s \in [\tilde{t}, T]$ we have

$$\begin{aligned} e^{-r(s-t)}(P(s) - K)u(s) - e^{-r(s-\tilde{t})}(\tilde{P}(s) - K)\bar{u} \\ = e^{-r(s-t)}(P(s) - \tilde{P}(s))u(s) - e^{-r(s-t)}(\tilde{P}(s) - K)(\bar{u} - u(s)) \\ - (e^{-r(s-\tilde{t})} - e^{-r(s-t)})(\tilde{P}(s) - K)\bar{u}, \end{aligned}$$

let us first of all observe that

$$(4.8) \quad \begin{aligned} & \mathbb{E} \left[\left| \int_t^T e^{-r(s-t)}(P(s) - K)u(s) ds - \bar{u} \int_{\tilde{t}}^T e^{-r(s-\tilde{t})}(\tilde{P}(s) - K) ds \right| \right] \\ & \leq \mathbb{E} \left[\int_t^{\tilde{t}} |P(s) - K|u(s) ds \right] + \mathbb{E} \left[\int_{\tilde{t}}^T |e^{-r(s-t)}(P(s) - K)u(s) - e^{-r(s-\tilde{t})}(\tilde{P}(s) - K)\bar{u}| ds \right] \\ & \leq \mathbb{E} \left[\int_t^{\tilde{t}} |P(s) - K|u(s) ds \right] + \mathbb{E} \left[\int_{\tilde{t}}^T |P(s) - \tilde{P}(s)|u(s) ds \right] \\ & \quad + \mathbb{E} \left[\int_{\tilde{t}}^T |\tilde{P}(s) - K|(\bar{u} - u(s)) ds \right] + \mathbb{E} \left[\int_{\tilde{t}}^T (e^{-r(s-\tilde{t})} - e^{-r(s-t)})|\tilde{P}(s) - K|\bar{u} ds \right]. \end{aligned}$$

Consider the first term in (4.8). By estimates as in (3.10), we have that

$$(4.9) \quad \mathbb{E} \left[\int_t^{\tilde{t}} |P(s) - K|u(s) ds \right] \leq C_2(\tilde{t} - t)(1 + |p|)$$

for some constant $C_2 > 0$. As for the second term in (4.8), by the Fubini–Tonelli theorem and Lemma 4.2, we get

$$(4.10) \quad \mathbb{E} \left[\int_{\tilde{t}}^T |P(s) - \tilde{P}(s)|u(s) ds \right] \leq C_3[|p - \tilde{p}| + (\tilde{t} - t)^{\frac{1}{2}}(1 + |p|)],$$

where $C_3 > 0$ is a constant. Let us now estimate the third term in (4.8). Given arbitrary $\gamma > 0$, we observe that

$$\begin{aligned} & \mathbb{E} \left[\left(\int_{\tilde{t}}^T |\tilde{P}(s) - K|(\bar{u} - u(s)) ds \right) \mathbf{1}_{\{\|\tilde{P}(\cdot)\| \leq \gamma\}} \right] \leq (\gamma + K) \mathbb{E} \left[\int_{\tilde{t}}^T (\bar{u} - u(s)) ds \right] \\ & = (\gamma + K) \mathbb{E} \left[\bar{u}(T - t) - \int_t^T u(s) ds \right] \leq (\gamma + K)(\bar{u}(T - t) - m + z), \end{aligned}$$

where in the last passage we have used condition $Z^u(T) \geq m$. By arguing as in (4.5), we get

$$(4.11) \quad \mathbb{E} \left[\left(\int_{\tilde{t}}^T |\tilde{P}(s) - K|(\bar{u} - u(s))ds \right) 1_{\{\|\tilde{P}(\cdot)\| > \gamma\}} \right] \leq C_4(1 + |\tilde{p}|)^{\frac{3}{2}}\gamma^{-\frac{1}{2}}$$

for some constant $C_4 > 0$. We finally consider the fourth term in (4.8). By local Lipschitzianity of the exponential function and by estimates as in (3.10), we obtain

$$(4.12) \quad \mathbb{E} \left[\int_{\tilde{t}}^T (e^{-r(s-\tilde{t})} - e^{-r(s-t)}) |\tilde{P}(s) - K| \bar{u} ds \right] \leq C_5(\tilde{t} - t)(1 + |\tilde{p}|),$$

where $C_5 > 0$ is constant.

By estimates from (4.9) to (4.12), it follows from (4.8) that

$$(4.13) \quad \left| \sup_{u \in \mathcal{A}_{tz}^{\text{adm}}} \mathbb{E} \left[\int_t^T e^{-r(s-t)} (P(s) - K)u(s)ds \right] - \bar{u} \mathbb{E} \left[\int_{\tilde{t}}^T e^{-r(s-\tilde{t})} (\tilde{P}(s) - K)ds \right] \right| \\ \leq C_2(\tilde{t} - t)(1 + |p|) + C_3[|p - \tilde{p}| + (\tilde{t} - t)^{\frac{1}{2}}(1 + |p|)] + C_5(\tilde{t} - t)(1 + |\tilde{p}|) \\ + (\gamma + K)(\bar{u}(T - t) - m + z) + C_4(1 + |\tilde{p}|)^{\frac{3}{2}}\gamma^{-\frac{1}{2}}.$$

Estimate (4.13) holds for each $\gamma > 0$ and for each $(t, p, z) \in \mathcal{D}$. We get (4.7) by passing to the limit first as $(t, p, z) \rightarrow (\tilde{t}, \tilde{p}, \tilde{z})$ (recall that $\tilde{z} + \bar{u}(T - \tilde{t}) = m$) and then as $\gamma \rightarrow \infty$. ■

Let us now consider the HJB equation and prove a result which is stronger than Corollary 3.7: in this case the value function is, in its whole domain \mathcal{D} , the unique viscosity solution of the HJB equation with polynomial growth and the boundary conditions given below. Thus, we get another characterization of the value function, in addition to that of Proposition 4.1.

Theorem 4.4. *Let the assumptions of section 4.1 hold. Then the function V is the unique continuous viscosity solution of (4.1) in the domain $\mathcal{D} \setminus (\alpha \cup \beta \cup \gamma)$, with boundary conditions*

$$(4.14) \quad \begin{aligned} V(t, p, z) &= 0 \quad \forall (t, p, z) \in \alpha, \\ V(t, p, z) &= \xi(t, z) \quad \forall (t, p, z) \in \beta, \\ V(T, p, z) &= 0 \quad \forall (p, z) \in \mathbb{R} \times [m, M] \end{aligned}$$

such that

$$(4.15) \quad |V(t, p, z)| \leq \check{C}(1 + |p|^2 + |z|^2) \quad \forall (t, p, z) \in \mathcal{D}$$

for some constant $\check{C} > 0$.

Proof. In this problem, $k = 2$ in (3.9). Thus, by (3.9), Corollary 3.7, and Proposition 4.1, the function V is a viscosity solution of problem (4.1)–(4.14)–(4.15). Moreover, it satisfies in the viscosity sense the boundary conditions (see [8]).

We now need a uniqueness result. By the following change of variables,

$$t' = t, \quad p' = p, \quad z' = \frac{z - M}{M - m + \bar{u}(T - t)} + 1,$$

problem (4.1)–(4.14) becomes

$$\begin{aligned}
& -V_{t'}(t', p', z') - \frac{\bar{u}(z' - 1)}{M - m + \bar{u}(T - t')} V_{z'}(t', p', z') \\
& + rV(t', p', z') - f(t', p')V_{p'}(t', p', z') - \frac{1}{2}\sigma^2(t', p')V_{p'p'}(t', p', z') \\
& + \min_{v \in [0, \bar{u}]} \left[-v \left(\frac{1}{M - m + \bar{u}(T - t')} V_{z'}(t', p', z') + p' - K \right) \right] = 0 \\
& \qquad \qquad \qquad \forall (t', p', z') \in [0, T[\times \mathbb{R} \times]0, 1[,
\end{aligned}$$

with boundary condition

$$\begin{aligned}
V(T, p', z') &= 0 \quad \forall (p', z') \in \mathbb{R} \times [0, 1], \\
V(t', p', 1) &= 0 \quad \forall (t', p') \in [0, T] \times \mathbb{R}, \\
V(t', p', 0) &= \xi(t', p') \quad \forall (t', p') \in [0, T] \times \mathbb{R}.
\end{aligned}$$

Moreover, the polynomial growth (4.15) is preserved and the domain is $[0, T[\times \mathbb{R} \times]0, 1[$. We can adapt to the case of bounded domain the comparison principle stated in [9, Theorem 2.1], which is based on the standard argument of doubling the variables in viscosity solution theory. This argument is easily extended to deal with boundary conditions in the viscosity sense. From this we get uniqueness of the solution. ■

The previous result generalizes an analogous result in [3], valid in the case $m = 0$. We now turn to proving some properties of the value function with respect to the variables p and z . As for $V(t, \cdot, z)$, Propositions 3.8 and 3.9 hold.

Proposition 4.5. *Let the assumptions of section 4.1 hold. Let $(t, z) \in [0, T] \times \mathbb{R}$ be such that $(t, p, z) \in \mathcal{D}$ for each $p \in \mathbb{R}$. Then the following hold:*

- *The function $V(t, \cdot, z)$ is Lipschitz continuous, uniformly in (t, z) . Moreover, the derivative $V_p(t, p, z)$ exists for a.e. $(t, p, z) \in \mathcal{D}$, and we have $|V_p(t, p, z)| \leq M_1$ for some constant $M_1 > 0$ depending only on T, \bar{u} and on the constants in (3.1), (3.4), and (3.29).*
- *If $f(s, \cdot), \sigma(s, \cdot) \in C_b^2(\mathbb{R})$, uniformly in $s \in [0, T]$, the function $V(t, \cdot, z)$ is locally semiconvex, uniformly in t , and a.e. twice differentiable.*

Proof. The first part follows from Proposition 3.8 (notice that function $p \mapsto (p - K)v$ is Lipschitz continuous). As for the second item, it suffices to rewrite Proposition 3.9 (notice that the function $p \mapsto (p - K)v$ is of class $C^\infty(\mathbb{R})$, with bounded derivatives). ■

Let us now consider the function $V(t, p, \cdot)$. Recall that its domain is $[m - \bar{u}(T - t), M]$.

Proposition 4.6. *Let the assumptions of section 4.1 hold. For each $(t, p) \in [0, T] \times \mathbb{R}$ the function $V(t, p, \cdot)$ has the following characteristics:*

- *It is concave, Lipschitz continuous, and a.e. twice differentiable.*
- *It is nondecreasing in $[m - (T - t)\bar{u}, M - (T - t)\bar{u}]$ and nonincreasing in $[m, M]$. In particular, if $M - (T - t)\bar{u} \geq m$, then the function $V(t, p, \cdot)$ is constant in $[m, M - (T - t)\bar{u}]$ (they all are maximum points).*

Proof. Item 1 (this is an adaptation of [3, Proposition 3.4], which takes into account only an upper bound on $Z^{t, z; u}(T)$). Let $(t, p) \in [0, T] \times \mathbb{R}$, $z_1, z_2 \in [m - \bar{u}(T - t), M]$, $u_1 \in \mathcal{A}_{t, z_1}^{\text{adm}}$,

and $u_2 \in \mathcal{A}_{tz_2}^{\text{adm}}$. By (2.10), the process $(u_1 + u_2)/2$ belongs to the set of admissible controls for initial point $(t, (z_1 + z_2)/2)$. By the linearity of the function $v \mapsto (P^{t,p}(s) - K)v$, we have

$$(4.16) \quad \frac{J(t, p, z_1; u_1) + J(t, p, z_2; u_2)}{2} = J\left(t, p, \frac{z_1 + z_2}{2}; \frac{u_1 + u_2}{2}\right) \leq V\left(t, p, \frac{z_1 + z_2}{2}\right).$$

Since (4.16) holds for each $u_1 \in \mathcal{A}_{tz_1}^{\text{adm}}$ and $u_2 \in \mathcal{A}_{tz_2}^{\text{adm}}$, it follows that

$$\frac{V(t, p, z_1) + V(t, p, z_2)}{2} \leq V\left(t, p, \frac{z_1 + z_2}{2}\right),$$

which implies the concavity of the function $V(t, p, \cdot)$. Local Lipschitzianity is a well-known property of concave functions (and here the domain is a compact set), while the a.e. existence of the second derivative follows from the Alexandrov theorem.

Item 2. Let $(t, p) \in [0, T] \times \mathbb{R}$. If $m \leq z_1 \leq z_2 \leq M$, it is easy to check that $\mathcal{A}_{tz_2}^{\text{adm}} \subseteq \mathcal{A}_{tz_1}^{\text{adm}}$ so that $V(t, p, z_1) \geq V(t, p, z_2)$. Similarly, if $m - (T - t)\bar{u} \leq z_1 \leq z_2 \leq M - (T - t)\bar{u}$, we have $\mathcal{A}_{tz_1}^{\text{adm}} \subseteq \mathcal{A}_{tz_2}^{\text{adm}}$, and then $V(t, p, z_2) \leq V(t, p, z_1)$. The second part immediately follows, since $[m - (T - t)\bar{u}, M - (T - t)\bar{u}] \cap [m, M] = [m, M - (T - t)\bar{u}]$. ■

The monotonicity result in Proposition 4.6 is described in Figure 3.

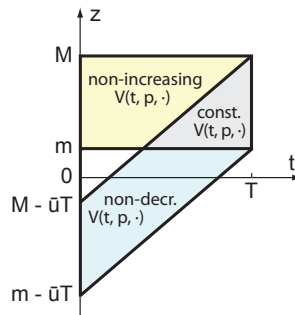


Figure 3. Monotonicity of $V(t, p, \cdot)$.

As in section 2, it was foreseeable that the function $V(t, p, \cdot)$ would be constant in an interval: if $M - (T - t)\bar{u} \geq m$ and $z \in [m, M - (T - t)\bar{u}]$, then $\mathcal{A}_{tz}^{\text{adm}} = \mathcal{A}_t$ (i.e., all controls satisfy the constraint), which implies that the initial value z does not influence the value function. This generalizes an intuitive result in [3, Lemma 3.2]: for (t, z) such that the volume constraint is de facto absent, the value function V does not depend on z .

Finally, also in this case Remark 1 holds: by Proposition 4.6, the candidate in (2.13)–(2.14) is well defined.

5. Conclusions. We characterize the value of swing contracts in continuous time as the unique viscosity solution of an HJB equation with suitable boundary conditions. In more detail, swings can be divided into two broad contract classes: those with penalties on the cumulated quantity of energy $Z(T)$ at the end T of the contract, and those with strict constraints on the same quantity. Usually these constraints and penalties are meant to make $Z(T)$ belong to an interval $[m, M]$, with $m > 0$ (in real contracts usually $m > 0.8M$; see [17]).

In section 2 we treat the case of contracts with penalties, which results in a straightforward application of classical optimal control theory, and in that case only a terminal condition is needed. For swing contracts with penalties, we prove that their value is the unique viscosity solution of the HJB equation (2.5), and that is Lipschitz both in p (spot price of energy) and in z (current cumulated quantity), with first weak derivatives with sublinear growth. We also prove that the value function is also concave with respect to z , nonincreasing for $z \leq M - (T - t)\bar{u}$, where t is the current time and \bar{u} is the maximum marginal energy that can be purchased, and nondecreasing for $z \geq m$. In this respect, we extend and generalize previous results of [3], which were proved only for swing contracts with strict penalties. These results make the candidate optimal exercise policy in (2.13)–(2.14) well defined.

Conversely, the case of contracts with strict constraints gives rise to a stochastic control problem with a nonstandard state constraint in $Z(T)$. In section 3 we approach a suitable generalization of this problem by a penalty method: we consider a general constrained problem and approximate the value function with a sequence of value functions of appropriate unconstrained problems with a penalization term in the objective functional, showing that they converge uniformly on compact sets to the value function of the constrained problem.

In section 4 we come back to the case of swing contracts with strict constraints: in this case the penalty functions used in section 3 turn out to be penalties of suitable swing contracts, so that we also have the economic interpretation that a swing contract with strict constraints can be approximated by swing contracts with suitable penalties. In this context we succeed in strengthening the results of section 3, by characterizing the value function as the unique viscosity solution with polynomial growth of the HJB equation (2.5) subject to the boundary conditions in (4.14). As for the smoothness of the value function with respect to p and z , we find exactly the same results as in section 2, extending previous results of [3] to the case $m > 0$. These results make the candidate optimal exercise policy in (2.13)–(2.14), i.e., the same as in section 2, again well defined.

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