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A Note on Open Loop Nash Equilibrium in Linear-State Differential Games

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Abstract

In this paper we focus on non-cooperative two-player linear-state differential games. In the standard definition this family is introduced assuming that there is no multiplicative interaction among state and control variables. In this paper we show that a multiplicative interaction between the state and the control of one player does not destroy the analytical features of the linear-state differential games if it appears in the objective functional of the other player. We prove that this slightly new definition preserves not only the solvability of the differential game, but also the subgame perfectness of an Open Loop Nash Equilibrium.

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1 Introduction

In non-cooperative differential games the definition of an Open Loop Nash Equilibrium (OLNE for short) presents two drawbacks. First, it is not simple to find an OLNE in an explicit form; second, an OLNE is not a stable equilibrium. The analytical issue is connected with the necessary conditions: in order to characterize an OLNE in closed form we have to solve a couple of interdependent optimal control problems and it is not an easy task. The stability issue is connected with the idea of subgame perfectness; roughly speaking, if the players use an OLNE and the state of the system deviates from the optimal trajectory (for example because of a mistake in the estimation in some parameters of the motion equation), then they cannot adjust their strategies and they get a non-optimal output. Therefore, in applications, special structures for the differential games are frequently used in order to find explicit and stable equilibria. A class of differential games which solves both the previous drawbacks is the linear-state one. A complete approach about the analytically tractable differential games can be found in [4]; while a very clear and beautiful paper about the importance of the linear-state assumption for the subgame perfectness is [8]. Moreover, the reader may be interested in some recent weaker definitions of subgame perfectness which can be found in [2].

In this paper we consider the class of linear-state differential games as defined in [5, Chapter 7, page 188], in [7] or in [6, Chapter 7, page 262]. The authors in [5, Chaper 7, page 188, line 27] describe this set of problems in the following way: "Condition (7.36) implies that there is no multiplicative interaction at all between state and control variables in the game." In this paper we show that a multiplicative interaction between the state and the control of the first player is allowed if it appears in the objective functional of the second player. This new formulation does not destroy the features of the linear-state differential games and it slightly extends this family of analytically tractable problems. This approach is not new in literature (see [4]), however in the following we take into account not only the solvability of the differential game, but also the propriety of the subgame perfectness of the OLNE.

The paper is organized as follows: in Section 2 the definition of two-player non-cooperative differential games, the definition of OLNE, and the analytical form of the necessary conditions for an OLNE are revised. In Section 3 we introduce a slightly new form for the linear-state differential games, we compare this definition with the previous ones, and we prove the main results of this paper: a particular multiplicative interaction between the state and the control does not destroy the attractive features of this family of differential games. Finally, in Section 4 we perform all the computations in a simple practical example.

2 Differential games

In the rest of the paper we deal with a two-player, non-cooperative differential game which is described by the definition:

Definition 2.1 *A two-player, non-cooperative differential game is defined* *as follows*

$$\begin{aligned}\max_{u_i(\cdot)} J_i\left[u_i\left(\cdot\right), u_j\left(\cdot\right)\right] = \max_{u_i(\cdot)} \textstyle\int_0^T & L_i\left(t, x\left(t\right), u_i\left(t\right), u_j\left(t\right)\right) dt + \ell_i\left(x\left(T\right)\right) \\ & \dot{x}\left(t\right) = f\left(t, x\left(t\right), u_1\left(t\right), u_2\left(t\right)\right) \\ & x\left(0\right) = x_0 \in \mathbb{R}\end{aligned} \tag{1}$$

with $i, j \in \{1, 2\}$ *and* $i \neq j$.

We assume that all the functions L_i, ℓ_i, f with $i \in \{1, 2\}$ are regular enough to guarantee the existence and the uniqueness of the solution of the motion equation for all choices of the feasible control functions $u_i\left(\cdot\right)$. Moreover, we require that for all the choices of the control functions the objective integrals converge (for a very complete and detailed book about the analytical conditions to correctly define an optimal control problem and a differential game we suggest [3]). A standard equilibrium concept in differential games is the OLNE (see [5, Chapter 4, page 86]).

Definition 2.2 *A couple of feasible controls* $\left(u_1^N\left(\cdot\right), u_2^N\left(\cdot\right)\right)$ *is an OLNE if and only if for all feasible controls* $u_i\left(\cdot\right)$

$$J_i\left[u_i^N\left(\cdot\right), u_j^N\left(\cdot\right)\right] \geq J_i\left[u_i\left(\cdot\right), u_j^N\left(\cdot\right)\right] \tag{2}$$

holds with $i, j \in \{1, 2\}$ *and* $i \neq j$.

If we want to characterize an OLNE, we have to solve a couple of interdependent optimal control problems. Given the player i's Hamiltonian function

$$H_i\left(t, x, u_i, \lambda_i; u_j\right) = L_i\left(t, x, u_i, u_j\right) + \lambda_i\ f\left(t, x, u_1, u_2\right) \tag{3}$$

we have to carry out the following four steps. In the rest of the paper we will refer to this method as the "solution scheme." Even if it is a standard way to solve an optimal control and a differential game, in my very personal opinion this approach is masterly described in [1].

Solution scheme

Step 1 for all $i, j \in \{1, 2\}$

$$u_i^{\#}\left(t, x, \lambda_i; u_j\right) = \arg\max_{u_i} H_i\left(t, x, u_i, \lambda_i; u_j\right) \tag{4}$$

and this maximization introduces a well-defined function $u_i^{\#}$

Step 2 the system

$$\begin{cases} u_i = u_i^{\#}(t, x, \lambda_i; u_j) \\ u_j = u_j^{\#}(t, x, \lambda_j; u_i) \end{cases} \tag{5}$$

in the two unknowns u_i, u_j has a unique solution

$$\left(u_i^{OL}(t, x, \lambda_i, \lambda_j), u_j^{OL}(t, x, \lambda_i, \lambda_j)\right) \tag{6}$$

Step 3 the two-point boundary value problem (with $i, j \in \{1, 2\}$ and $i \neq j$)

$$\begin{cases} \dot{x} = f\left(t, x, u_1^{OL}(t, x, \lambda_i, \lambda_j), u_2^{OL}(t, x, \lambda_i, \lambda_j)\right) \\ x(0) = x_0 \\ \dot{\lambda}_i = -\partial_x H_i\left(t, x, u_i^{OL}(t, x, \lambda_i, \lambda_j), \lambda_i; u_j^{OL}(t, x, \lambda_i, \lambda_j)\right) \\ \lambda_i(T) = \partial_x \ell_i(x(T)) \end{cases} \tag{7}$$

has a unique solution $(x^*(t), \lambda_1^*(t), \lambda_2^*(t))$ (to simplify the notation in the previous ODEs we have neglected the time dependence in the functions: x, λ_1, λ_2)

Step 4 the function $H_i\left(t, x, u_i, \lambda_i^*(t); u_j^{OL}\left(t, \lambda_i^*(t), \lambda_j^*(t)\right)\right)$ is concave in (x, u_i) for all $t \in [0, T]$ and for all $i, j \in \{1, 2\}$ with $i \neq j$

If all the previous steps can be performed, then

$$u_i^N(t) = u_i^{OL}\left(t, x^*(t), \lambda_i^*(t), \lambda_j^*(t)\right) \tag{8}$$

with $i, j \in \{1, 2\}$ and $i \neq j$ is an OLNE for (1).

Step $1, 2, 3$ represent the necessary conditions for an OLNE, while Step 4 represents the sufficient conditions [5, Chapter 4, page 93]. We notice that it is very difficult to carry out all these steps, hence some further assumptions are generally introduced in order to obtain a closed form solution for an OLNE. In the next Section we will see what this solution scheme will be like when we introduce the linear-state assumption.

In the definition of an OLNE the information structure of the differential game is implicitly taken into account. Using this definition we assume that both players know all the data of problem (1). Hence, they know x_0 (the value of the state at the initial time $t = 0$), but they cannot observe the state after this time. The drawback of an OLNE is the instability of this kind of equilibrium: if the state of the system deviates from the optimal trajectory, then the players cannot adjust their strategies. This difficulty becomes clear when we introduce the definition of subgame perfectness.

Definition 2.3 *We denote as $\Gamma(0, x_0)$ the two-player non-cooperative differential game (1). An OLNE $\left(u_1^N, u_2^N\right)$ is called subgame perfect if and only if for all $\tau \in (0, T)$ and for any reachable state $x_\tau \in \mathbb{R}$ at the time τ, the couple of feasible controls $\left(u_1^N\big|_{[\tau,T]}, u_2^N\big|_{[\tau,T]}\right)$ is an OLNE for $\Gamma(\tau, x_\tau)$.*

Roughly speaking, an OLNE is subgame perfect if its restriction on the subinterval $[\tau, T]$ remains an OLNE even if the original game is played in $[\tau, T]$ choosing any reachable state x_τ as initial condition. In the following example we show that an OLNE is generally not subgame perfect.

Example 2.4 *Given the differential game*

$$\begin{aligned} \max_{u_i(\cdot)} J_i\left[u_i\left(\cdot\right), u_j\left(\cdot\right)\right] = \max_{u_i(\cdot)} \int_0^1 & x\left(t\right) u_i\left(t\right) - u_i^2\left(t\right)/2dt \\ & \dot{x}\left(t\right) = u_1\left(t\right) + u_2\left(t\right) \\ & x\left(0\right) = 0 \end{aligned} \tag{9}$$

the Hamiltonian function of player i is

$$H_i\left(t, x, u_i, \lambda_i; u_j\right) = xu_i - u_i^2/2 + \lambda_i\ \left(u_1 + u_2\right) \tag{10}$$

and then

$$u_i^{\#}\left(t, x, \lambda_i; u_j\right) = \left(x + \lambda_i\right). \tag{11}$$

In this example $u_i^{\#}$ does not depend on u_j, hence

$$u_i^{OL}\left(t, x, \lambda_i, \lambda_j\right) = \left(x + \lambda_i\right). \tag{12}$$

The two-point boundary value problem becomes

$$\begin{cases} \dot{x} = 2x + \lambda_1 + \lambda_2, & x\left(0\right) = 0 \\ \dot{\lambda}_i = -x - \lambda_i, & \lambda_i\left(1\right) = 0 \end{cases} \tag{13}$$

and its solution is

$$\begin{cases} x^*\left(t\right) = 0 \\ \lambda_i^*\left(t\right) = 0. \end{cases} \tag{14}$$

We notice that $H_i\left(t, x, u_i, \lambda_i^*\left(t\right); u_j^{OL}\left(t, \lambda_i^*\left(t\right), \lambda_j^*\left(t\right)\right)\right)$ is concave in (x, u_i) because its Hessian is constant and equal to

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Therefore

$$u_i^N\left(t\right) = 0 \tag{16}$$

is really an OLNE for (9). Now, let us consider the subgame starting at the time $\tau = \ln(2)$ from the state $x_\tau = 1$. Steps $1, 2,$ and 4 of the previous solution scheme do not change, whereas the two-point boundary value problem (13) in step 3 becomes

$$\begin{cases} \dot{x} = 2x + \lambda_1 + \lambda_2, & x\left(\ln\left(2\right)\right) = 1 \\ \dot{\lambda}_i = -x - \lambda_i, & \lambda_i\left(1\right) = 0. \end{cases} \tag{17}$$

The new solution is

$$\begin{cases} x^*(t) = (2e^t - e)/(4-e) \\ \lambda_i^*(t) = -(e^t - e)/(4-e) \end{cases} \tag{18}$$

therefore

$$u_i^N(t) = e^t/(4-e) \tag{19}$$

is the new OLNE which is different from the previous one (16).

3 Linear-state differential games

In the example of Section 2 we have shown a well-known result: an OLNE is generally not subgame perfect [2]. However, for the linear-state differential games everything works well: we can find an OLNE in a closed form and it turns out to be subgame perfect.

Definition 3.1 *A two-player, non-cooperative differential game is defined as linear-state if and only if it has the following analytical form*

$$\begin{array}{l} \max\limits_{u_i(\cdot)} J_i[u_i,u_j] = \max\limits_{u_i(\cdot)} \int_0^T \alpha_i(t,u_j(t))\,x(t) + \beta_i(t,u_i(t),u_j(t))\,dt + \gamma_i x(T) \\ \qquad\qquad \dot{x}(t) = \delta(t)\,x(t) + \varepsilon(t,u_i(t),u_j(t)) \\ \qquad\qquad x(0) = x_0 \end{array} \tag{20}$$

with $i,j \in \{1,2\}$ and $i \neq j$.

We notice that this definition is quite different from the other ones (see e.g. the book [5, Chaper 7, page 187], the book [6, Chaper 7, page 262], or the paper [7]): in the objective functional of player i a multiplicative interaction between the state and the control player j is allowed. We assume that all the functions $\alpha_i, \beta_i, \delta, \varepsilon$ are regular enough to guarantee the existence and the uniqueness of the solution of the motion equation for all choices of the control functions $u_i(\cdot), u_j(\cdot)$. Moreover, we require that for all the choices of the feasible control functions the objective integrals converge.

In the following theorem we will prove that in a linear-state differential game an OLNE is subgame perfect. In the proof it will become clear that linear-state differential games are "analytically tractable" as already shown in [4].

Theorem 3.2 *Let us assume that, using the solution scheme described in Section 2, it is possible to characterize an OLNE for (20). Therefore this OLNE is subgame perfect.*

Proof. The Hamiltonian function of player i is

$$H_i(t, x, u_i, \lambda_i; u_j) = \alpha_i(t, u_j)\, x + \beta_i(t, u_i, u_j) + \lambda_i(\delta(t)\, x + \varepsilon(t, u_i, u_j)). \quad (21)$$

For all $i, j \in \{1, 2\}$

$$\begin{aligned} u_i^{\#}(t, \lambda_i; u_j) &= \arg\max_{u_i} H_i(t, x, u_i, \lambda_i; u_j) \\ &= \arg\max_{u_i} \{\beta_i(t, u_i, u_j) + \lambda_i \varepsilon(t, u_i, u_j)\} \end{aligned}$$

is a well-defined function. We notice that $u_i^{\#}$ does not depend on the state x because of the analytical form of a linear-state differential game. We are also assuming that the system

$$\begin{cases} u_i = u_i^{\#}(t, \lambda_i; u_j) \\ u_j = u_j^{\#}(t, \lambda_j; u_i) \end{cases} \quad (22)$$

in the two unknowns u_i, u_j has a unique solution

$$\left(u_i^{OL}(t, \lambda_i, \lambda_j), u_j^{OL}(t, \lambda_i, \lambda_j)\right) \quad (23)$$

which depends on the adjoint variables only. Moreover, we are assuming that the two-point boundary value problem (with $i, j \in \{1, 2\}$ and $i \neq j$)

$$\begin{cases} \dot{x} = \delta(t)\, x + \varepsilon\left(t, u_i^{OL}(t, \lambda_i, \lambda_j), u_j^{OL}(t, \lambda_i, \lambda_j)\right) \\ x(0) = x_0 \\ \dot{\lambda}_i = -\alpha_i\left(t, u_j^{OL}(t, \lambda_i, \lambda_j)\right) - \lambda_i \delta(t) \\ \lambda_i(T) = \gamma_i \end{cases} \quad (24)$$

has a unique solution $(x^*(t), \lambda_1^*(t), \lambda_2^*(t))$. We notice that the two adjoint differential equations are coupled together, but they are decoupled by the motion equation. Finally, we are assuming that the function

$$H_i\left(t, x, u_i, \lambda_i^*(t); u_j^{OL}\left(t, \lambda_i^*(t), \lambda_j^*(t)\right)\right)$$

is concave in (x, u_i) for all $i, j \in \{1, 2\}$ and $i \neq j$. Therefore, we have that

$$u_i^N(t) = u_i^{OL}\left(t, \lambda_i^*(t), \lambda_j^*(t)\right) \qquad i, j \in \{1, 2\}, \quad i \neq j; \quad (25)$$

is an OLNE for (20).

Now, let us consider the subgame starting at the time $\tau \in (0, T)$ from any feasible state $x_\tau \in \mathbb{R}$. The first two steps of the solution scheme for this subgame are identical to the previous ones. On the other hand, the new two-point boundary value problem for the subgame becomes

$$\begin{cases} \dot{x} = \delta(t)\, x + \varepsilon\left(t, u_i^{OL}(t, \lambda_i, \lambda_j), u_j^{OL}(t, \lambda_i, \lambda_j)\right) \\ x(\tau) = x_\tau \\ \dot{\lambda}_i = -\alpha_i\left(t, u_j^{OL}(t, \lambda_i, \lambda_j)\right) - \lambda_i \delta(t) \\ \lambda_i(T) = \gamma_i. \end{cases} \quad (26)$$

The adjoint functions are solved backward and they are decoupled from the motion equation; hence, the previous functions $\lambda_i^*(t), \lambda_j^*(t)$ still solve the adjoint equations. Using $\lambda_i^*(t), \lambda_j^*(t)$ we can characterize $u_i^N(t) = u_i^{OL}\left(t, \lambda_i^*(t), \lambda_j^*(t)\right)$ which is exactly the previous OLNE restricted to the subinterval $[\tau, T]$ (we are using again the fact that $u_i^{\#}$ does not depend on x). Finally, we can solve the motion equation with the new initial condition $x(\tau) = x_\tau$ and we obtain a new optimal motion function which we denote as $x_\tau^*(t)$.
To complete the proof we have to show that the sufficient conditions can be applied also to this new optimal state function $x_\tau^*(t)$. The Hessian matrix of function $H_i\left(t, x, u_i, \lambda_i^*; u_j^{OL}\left(t, \lambda_i^*, \lambda_j^*\right)\right)$ with respect to (x, u_i) is

$$\begin{pmatrix} 0 & 0 \\ 0 & \partial_{u_i u_i}^2 \beta_i\left(t, u_i, u_j^{OL}\left(t, \lambda_i^*, \lambda_j^*\right)\right) + \lambda_i^* \partial_{u_i u_i}^2 \varepsilon\left(t, u_i, u_j^{OL}\left(t, \lambda_i^*, \lambda_j^*\right)\right) \end{pmatrix} \tag{27}$$

where we have neglected the time dependence in the functions λ_i^*, λ_j^*. We can observe that $x_\tau^*(t)$ does not enter this matrix; therefore, if we can apply the sufficient conditions to (20), then we can apply the same sufficient conditions also to the subgame starting at the time $\tau \in (0, T)$ from any feasible state $x_\tau \in \mathbb{R}$.

We notice that using the original formulation of linear-state differential games as presented in [5, Chaper 7, page 187], in [6, Chaper 7, page 262], or in [7] we obtain that $\alpha_i(t, u_j)$ only depends on t; hence, not only the adjoint equations are decoupled from the motion equation, but they are also decoupled from each other. This is useful because it simplifies the computations, but it is not necessary to obtain the stability of the equilibrium.

4 Practical example

We close the paper with an example where we can perform all the computations and we can obtain a subgame perfect OLNE even if there exists a particular multiplicative interaction between the state and the control. The example is very close to the previous one, but a different multiplicative interaction gives a new light to the problem.

Example 4.1 *Let us consider the differential game*

$$\begin{aligned} \max_{u_i} J_i[u_i, u_j] = \max_{u_i} & \int_0^1 x(t)\, u_j(t) - u_i^2(t)/2 dt \\ & \dot{x}(t) = u_1(t) + u_2(t) \\ & x(0) = 0. \end{aligned} \tag{28}$$

The Hamiltonian function of player i is

$$H_i(t, x, u_i, \lambda_i; u_j) = x u_j - u_i^2/2 + \lambda_i \ (u_1 + u_2) \tag{29}$$

then

$$u_i^{\#}(t, \lambda_i; u_j) = \lambda_i. \tag{30}$$

Again $u_i^{\#}$ *does not depend on* u_j*, hence*

$$u_i^{OL}(t, \lambda_i, \lambda_j) = \lambda_i. \tag{31}$$

The two-point boundary value problem is

$$\begin{cases} \dot{x} = \lambda_1 + \lambda_2, & x(0) = 0 \\ \dot{\lambda}_i = -\lambda_j, & \lambda_i(1) = 0 \end{cases} \tag{32}$$

and the solution is straightforward

$$\begin{cases} x^*(t) = 0 \\ \lambda_i^*(t) = 0 \end{cases} \tag{33}$$

therefore

$$u_i^N(t) = 0. \tag{34}$$

We notice that $H_i\left(t, x, u_i, \lambda_i^*(t); u_j^{OL}\left(t, \lambda_i^*(t), \lambda_j^*(t)\right)\right)$ *is concave in* (x, u_i) *because its Hessian with respect to these two variables is constant and equal to*

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \tag{35}$$

Hence, (34) is really an OLNE. Now, we already know by Section 3 that this OLNE is subgame perfect, however it may be useful to see how and why this occurs. As in the previous example, let us consider the subgame starting at the time $\tau = \ln(2)$ *from the state* $x_\tau = 1$*. The two-point boundary value problem (32) in step* 3 *becomes*

$$\begin{cases} \dot{x} = \lambda_1 + \lambda_2, & x(\ln(2)) = 1 \\ \dot{\lambda}_i = -\lambda_j, & \lambda_i(1) = 0 \end{cases} \tag{36}$$

and we can notice that the adjoint functions do not change. Even if the adjoint functions are coupled with each other, they are decoupled from the motion equation and this is sufficient to guarantee that the OLNE (34) is subgame perfect.

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