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Length functions, multiplicities and algebraic entropy

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Abstract. We consider algebraic entropy defined using a general discrete length function L and will consider the resulting entropy in the setting of $R[X]$-modules. Then entropy will be viewed as a function h_L on modules over the polynomial ring $R[X]$ extending L. In this framework we obtain the main results of this paper, namely that under some mild conditions the induced entropy is additive, thus entropy becomes an operator from the length functions on R-modules to length functions on $R[X]$-modules. Furthermore, if one requires that the induced length function h_L satisfies two very natural conditions, then this process is uniquely determined. When R is Noetherian, we will deduce that, in this setting, entropy coincides with the multiplicity symbol as conjectured by the second named author. As an application we show that if R is also commutative, the L-entropy of the right Bernoulli shift on the infinite direct product of a module of finite positive length has value ∞, generalizing a result proved for Abelian groups by A. Giordano Bruno.

Keywords. Length functions, endomorphisms of modules, algebraic entropy, multiplicity.

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1 Introduction

In module theory one encounters a host of functions that measure some sort of finiteness, the most useful ones have values in the non-negative reals and infinity and are additive in some sense. The first axiomatic approach to these functions was given by D. G. Northcott and M. Reufel in their 1965 paper [14] where they called them *length functions*. An interesting feature in this paper is the appearance of a non-discrete length function induced by a non-Archimedean real valuation on a valuation domain. Recently, further results on length functions for valuation domains have been obtained by Zanardo in [24]. A more systematic study of these length functions were undertaken by the second author in [18] and [19]. In particular, all the upper continuous additive functions on a category with Krull dimension (in the sense of Gabriel–Rentschler) were classified in [19]. In this situation,

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which includes the case of all Noetherian rings, the length functions arising are essentially discrete in a sense to be defined below.

In 1965, a new invariant from a topological setting appeared in algebra when Adler, Konheim and McAndrew defined the (algebraic) entropy of an endomorphism of an Abelian group in [1]. In 1974, Weiss [23] studied some basic properties of this entropy. After Weiss's paper no further study of the algebraic entropy has been performed until 2009, when the first named author jointly with Dikranjan, Goldsmith and Zanardo in [6] and [15] investigated the main properties of the algebraic entropy for abelian groups proving, in particular, the additivity property and a uniqueness theorem for the entropy function.

The algebraic entropy defined in [1] has the intrinsic limitation to be trivial on the torsion-free groups. To overcome this problem, in [16] the entropy based on the (torsion-free) rank was defined and investigated in detail. Importantly, in the same paper, a general notion of algebraic entropy related to invariants satisfying the subadditive property (which is weaker than the property of being a length function) was given. The advantage of this definition is that it applies to endomorphisms of modules over arbitrary rings. In this paper we introduce new tools to investigate entropy based on a general length function, inspired by the techniques used in both [6] and [16].

In what follows $\mathrm{Mod}(R)$ will denote the category of right R-modules and $\subset$ (resp. $\supset$) will denote strict inclusion. We will often say R-module to mean right R-module. Let us start with the following

Definition 1.1 ([16]). Given a ring R, an *invariant* of the category $\mathrm{Mod}(R)$ is a map $i : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that $i(0) = 0$ and $i(M) = i(M')$ whenever $M \cong M'$.

To be able to deal with the invariants of $\mathrm{Mod}(R)$, without any assumption on the structure of the ring R, we will impose three strong hypotheses considered in [14], [18] and [16], on an invariant i of $\mathrm{Mod}(R)$. More precisely we will always suppose that:

(i) for every exact sequence $0 \to A \to B \to C \to 0$, the equality $i(B) = i(A) + i(C)$ holds true. In such a case i is said to be *additive*;

(ii) for every $M \in \mathrm{Mod}(R)$, $i(M) = \sup_{F \in \mathcal{F}(M)} i(F)$, where $\mathcal{F}(M)$ denotes the set of the finitely generated submodules of M. In such a case i is said to be *upper continuous*;

(iii) the set of finite values of i is order-isomorphic to $\mathbb{N}$. In such a case i is said to be *discrete*.

Definition 1.2 ([14]). An additive upper continuous invariant is said to be a *length function*.

In the sequel, we will denote by the symbol L a length function on $\mathrm{Mod}(R)$; furthermore, we will use the fact that, if A and B are two submodules of an R-module, then $L(A+B)+L(A\cap B)=L(A)+L(B)$, which immediately follows from the additivity of L considering the exact sequence $0\to A\cap B\to A\oplus B\to A+B\to 0$.

The most ubiquitous examples of discrete length functions are the 'classical' composition length $l(M)$ of an R-module M, for R an arbitrary ring; the (torsion-free) rank $\mathrm{rk}(M)$ of an arbitrary R-module M over an integral domain R; and $\log|M|$ when M is an Abelian group. It is understood that the value of these length functions is infinity whenever it is not finite.

We now define the L-entropy following [16]. Let L be a length function on the category $\mathrm{Mod}(R)$, $M\in\mathrm{Mod}(R)$ and $\phi\in\mathrm{End}_R(M)$. Let us consider the class $\mathrm{Fin}_L=\{F\in\mathrm{Mod}(R)\mid L(F)<\infty\}$ of the *L-finite modules* and set

$$\mathrm{Fin}_L(M)=\{N\subseteq M\mid L(N)<\infty\}.$$

Since L is additive, Fin_L is a Serre class. For every submodule $F\in\mathrm{Fin}_L(M)$ and every $n\in\mathbb{N}_+$ ($\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_+$ the set of positive integers),

$$T_n(\phi,F)=F+\cdots+\phi^{n-1}F$$

is called the *n-th partial ϕ-trajectory of F*, and

$$T(\phi,F)=\bigcup_{n\in\mathbb{N}_+}T_n(\phi,F)=\sum_{n\in\mathbb{N}}\phi^nF \tag{1.1}$$

is called the *ϕ-trajectory of F*. Since Fin_L is a Serre class, $T_n(\phi,F)\in\mathrm{Fin}_L(M)$ for every $n\in\mathbb{N}_+$. We can now give the following

Definition 1.3. The *L-entropy of ϕ with respect to $F\in\mathrm{Fin}_L(M)$* is

$$\mathrm{ent}_L(\phi,F)=\lim_{n\to\infty}\frac{L(T_n(\phi,F))}{n}$$

and the *L-entropy of ϕ* is $\mathrm{ent}_L(\phi)=\sup\{\mathrm{ent}_L(\phi,F)\mid F\in\mathrm{Fin}_L(M)\}$.

We now put entropy into a different setting where many of the notions relative to an endomorphism will become 'absolute' as proposed in [20]. More importantly, this is the setting where entropy will become just another length function and will meet multiplicity. Let R be a ring. We can define a category whose objects are the pairs (ϕ, M) with $M \in \mathrm{Mod}(R)$ and $\phi \in \mathrm{End}_R(M)$. In this category a morphism $\alpha : (\phi, M) \to (\psi, N)$ is a commutative square of the form

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ N & \xrightarrow{\psi} & N \end{array} \tag{1.2}$$

where α is an R-homomorphism from M to N. But this category, in turn, is just equivalent to the category $\mathrm{Mod}(R[X])$ of modules over the polynomial ring over R; the equivalence functor is given by $(\phi, M) \mapsto M_\phi \in \mathrm{Mod}(R[X])$, where M_ϕ, as an R-module, is just M and X acts on M_ϕ via ϕ. Also the homomorphism α in (1.2) becomes an $R[X]$-homomorphism, as it commutes with X. In this way, a ϕ-invariant submodule of M is just an $R[X]$-submodule of M_ϕ; furthermore M_ϕ and N_ψ are isomorphic as $R[X]$-modules if and only if there exists an R-isomorphism $\alpha : M \to N$ such that $\psi = \alpha \circ \phi \circ \alpha^{-1}$, that is, ϕ and ψ are conjugated (see [11, Chapter 12] and [22, 659–674]). Every $R[X]$-module can be viewed as an R-module M with the multiplication by X acting as an R-endomorphism. So, in the following, when dealing with $R[X]$-modules, we will always consider objects written in the form M_ϕ; we will sometimes abuse notation, denoting a ϕ-invariant submodule N of M with the structure of $R[X]$-module induced by the restriction of ϕ to N simply by N_ϕ. Notice also that the ϕ-trajectory $T(\phi, F)$ of F in (1.1) is nothing but the $R[X]$-submodule of M_ϕ generated by F. Furthermore, any finitely generated $R[X]$-submodule of M_ϕ is of the form $T(\phi, F)$ with $F \in \mathcal{F}(M)$.

Now we can modify our point of view on the L-entropy as indicated above. Instead of looking at ent_L as a function from the endomorphism ring of an arbitrary R-module M, we consider it as a function $\mathrm{ent}_L : \mathrm{Mod}(R[X]) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ sending the $R[X]$-module M_ϕ to $\mathrm{ent}_L(\phi)$. Thanks to [16, Proposition 1.8 and Property (1)] (see Section 2.2 for the precise statements) we have that the function $\mathrm{ent}_L : \mathrm{Mod}(R[X]) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is an invariant, and [16, Proposition 1.10] implies that ent_L is discrete. For example, when $R = \mathbb{Z}$ and $L = \mathrm{rk}$, from Definition 1.3 we obtain the definition of rank-entropy $\mathrm{ent}_{\mathrm{rk}}$ which was studied in [16]. For example, in the above setting it follows from Theorem 3.10 (c) of that paper that

$$\mathrm{ent}_{\mathrm{rk}}(\phi) = \mathrm{rk}_{\mathbb{Z}[X]}(M_\phi) \tag{1.3}$$

for any endomorphism of Abelian groups $\phi : M \to M$. It is not difficult to see that the above equality holds not only for Abelian groups, but also for modules over arbitrary domains.

The goal of this paper is to show how equality (1.3) can be transferred to the more general setting of modules over an arbitrary ring R, replacing the rank of Abelian groups by an arbitrary discrete length function L of $\mathrm{Mod}(R)$, the rank-entropy by the L-entropy, and $\mathrm{rk}_{\mathbb{Z}[X]}$ by the induced length function of $\mathrm{Mod}(R[X])$. The main results are collected in two theorems, named Addition Theorem and Uniqueness Theorem, respectively.

We describe now in more detail the two main theorems. Let R be a ring and L be a discrete length function of $\mathrm{Mod}(R)$. Let us consider the subclass lFin_L consisting of the locally L-finite modules (i.e., their cyclic submodules are L-finite, see Definition 2.2). Obviously Fin_L is contained in lFin_L. The unavoidable restriction to lFin_L, needed in the forthcoming Addition Theorem, allows us to make use of the full force of the upper continuity of L, that replaces other assumptions on the invariants made in [16], namely smallness and liftability. Section 2 will be entirely devoted to developing the techniques needed to prove the following very general

Addition Theorem. *Let R be a ring and $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a discrete length function. Let M be an R-module and $\phi : M \to M$ be an endomorphism. If M is locally L-finite and N is a ϕ-invariant submodule of M, then*

$$\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi\restriction_N) + \mathrm{ent}_L(\bar{\phi})$$

where $\bar{\phi} : M/N \to M/N$ denotes the map induced by ϕ on the quotient module M/N.

We emphasize that our proof of the Addition Theorem strongly depends on the discreteness of L since it makes use of inductive arguments on the values of L.

Next Corollary 2.16 will show that ent_L is also upper continuous. Now denote by $\mathrm{lFin}_L[X]$ the class of the $R[X]$-modules that are locally L-finite when considered as R-modules. The Addition Theorem shows that ent_L is also additive once restricted to the class $\mathrm{lFin}_L[X]$. Thus we can say that ent_L is a discrete length function on $\mathrm{lFin}_L[X]$ (this restriction is unavoidable because, outside of $\mathrm{lFin}_L[X]$, ent_L is not even monotone under taking quotients, as [6, Example 1.11] shows).

Therefore, one can look at the L-entropy essentially as a tool that allows one to 'extend' a discrete length function L on $\mathrm{Mod}(R)$ to a length function on a the subclass of $\mathrm{Mod}(R[X])$ consisting of the locally L-finite R-modules. We recall that the idea of extending a length function from $\mathrm{Mod}(R)$ to $\mathrm{Mod}(R[X])$ was already considered by Irite in [10], but in the much simpler context of commutative Noetherian rings, using the Euler characteristic of Koszul complexes as in [7, p. 408].

Section 3 will be devoted to the proof of the uniqueness of the L-entropy function, provided two natural conditions are satisfied. In such a way we will obtain an axiomatic characterization of the L-entropy similar to that given by Stojanov in [17] of the topological entropy for compact groups and to that of the algebraic entropy and of the rank entropy given respectively in [6] and [16].

Before stating the Uniqueness Theorem, we need to define the *Bernoulli functor*

$$\beta : \mathrm{Mod}(R) \to \mathrm{Mod}(R[X]), \quad \beta(M) = (M^{(\mathbb{N})})_{\beta_M}$$

where β_M is the Bernoulli shift on the direct sum $M^{(\mathbb{N})}$ defined in Example 2.14. If $\phi : M \to N$ is an R-morphism, then $\beta(\phi)(x_n)_{n\geq 0} = (\phi(x_n))_{n\geq 0}$ for every $(x_n)_{n\geq 0} \in M^{(\mathbb{N})}$. It is not difficult to see that the Bernoulli functor β is isomorphic to the tensor product by $R[X]$, that is, for each R-module M, $\beta(M)$ is naturally isomorphic as $R[X]$-module to $M \otimes_R R[X]$ (this fact has the remarkable consequence that β has as right adjoint the forgetful functor).

We can now give the precise statement of the following:

Uniqueness Theorem. *Let R be a ring, $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a discrete length function and $\beta : \mathrm{Mod}(R) \to \mathrm{Mod}(R[X])$ be the Bernoulli functor. Then there exists a unique length function $h_L : \mathrm{lFin}_L[X] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for every L-finite module $M \in \mathrm{Fin}_L$:*

(i) $h_L(M_\phi) = 0$ *for every* $\phi \in \mathrm{End}_R(M)$;

(ii) $h_L(\beta(M)) = L(M)$.

Moreover, h_L is discrete and $h_L(M_\phi) = \mathrm{ent}_L(\phi)$ for every $M_\phi \in \mathrm{lFin}_L[X]$.

Note that condition (ii) in the Uniqueness Theorem, written in the equivalent form $h_L(M \otimes_R R[X]) = L(M)$, is exactly the condition required in Theorem 1.2 of [10].

In Section 4 we apply the results described above to modules over a Noetherian ring R. First we compare the multiplicity (as defined in [19]) associated to a length function L_X of $\mathrm{Mod}(R[X])$, derived by a length function L of $\mathrm{Mod}(R)$ 'forgetting' the action of X, with the L-entropy. Then, assuming R to be also commutative, we compute the L-entropy of the right Bernoulli shift on the direct product $M^{\mathbb{N}}$, where M is an R-module of finite positive length, thus extending a result proved recently by Giordano Bruno in the setting of Abelian groups in [8].

2 The Addition Theorem

This section is devoted to the proof of the Addition Theorem. Notice that analogous results were proved in [6] and [16] respectively for the invariant $\log|-|$ and rk of $\mathrm{Mod}(\mathbb{Z})$. The techniques introduced in Section 2.1 and Section 2.4, namely the

L-purifications and the L-minimal elements, generalize to modules over arbitrary rings techniques belonging to Abelian group theory. This will allow us to adapt proofs of [6] and [16] to our, more general, setting.

Throughout this section R will denote a fixed ring and $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ a discrete length function of $\mathrm{Mod}(R)$.

2.1 L-Purifications

The proof of the following lemma is an immediate consequence of the upper continuity and of the discreteness of L; so it is left to the reader.

Lemma 2.1. *If $M \in \mathrm{Fin}_L$, then there exists $F \in \mathcal{F}(M)$ such that $L(F) = L(M)$.*

Definition 2.2. Let L be a length function on $\mathrm{Mod}(R)$ and M be an R-module. For any element $x \in M$ we will use the notation $L(x)$ for $L(xR)$. Then define

(i) $z_L(M) = \{x \in M \mid L(x) = 0\}$ the *L-singular submodule* of M;

(ii) $f_L(M) = \{x \in M \mid L(x) < \infty\}$ the *locally L-finite submodule* of M.

If $z_L(M) = M$ (resp. $f_L(M) = M$), we will say that M is *L-singular* (resp. *locally L-finite*).

It is not difficult to prove that both $f_L(M)$ and $z_L(M)$ are fully invariant R-submodules of M. In the next example we will consider the most often used and well-known length functions.

Example 2.3. (a) Let R be any ring and $L = l$ be the 'classical' composition length. An R-module M is locally l-finite if all its cyclic submodules have finite length, whereas no non-trivial R-module is l-singular.

(b) Let R be a domain and $L = \mathrm{rk}$ be the rank. Every R-module M is locally rk-finite, whereas M is rk-singular if and only if it is a torsion module.

(c) Let $R = \mathbb{Z}$ and $L = \log|-|$. If M is an Abelian group, then M is locally $\log|-|$-finite if and only if it is torsion, whereas no non-trivial Abelian group is $\log|-|$-singular.

The following examples are more particular.

Example 2.4 ([14, Example 2]). Given a commutative ring R and a non-zero idempotent ideal $I = I^2$, for every R-module M we set

$$L(M) = 0 \text{ if } IM = 0, \quad L(M) = \infty \text{ otherwise.}$$

Then $z_L(M) = f_L(M) = \{x \in M \mid \mathrm{Ann}_R(x) \supseteq I\}$.

Example 2.5. Let K be a field and V be an infinite dimensional K-vector space. Let $R = K(+)V$ be the idealization of V (see for example [12] and [9]). Further, let $J = \{0\}(+)V$ be the unique maximal ideal of R, which satisfies $J^2 = 0$. Let L be the classical composition length. Thus L is a discrete length function on $\mathrm{Mod}(R)$ with $L(R) = \infty$. It is easily seen that $L(R/J) = 1$ and, if $I \subset J$, then $I = \{0\}(+)W$ where $W \subset V$ is a K-subspace, and

$$L(R/I) = 1 + \mathrm{rk}_K(V/W) = \mathrm{rk}_K(R/J) + \mathrm{rk}_K(J/I).$$

Furthermore, given an R-module M, $z_L(M) = 0$ and $f_L(M) = M[J]$. Consequently $f_L(R) = J$ and $f_L(R/J) = R/J$.

The previous example shows that f_L is not in general a radical, while it is easy to see that f_L is a radical provided R is Noetherian.

The next lemma is an easy observation. Recall that L denotes a length function.

Lemma 2.6. *Let M be an R-module. Then $L(M) = 0$ if and only if $L(x) = 0$ for every $x \in M$. In particular, $L(z_L(N)) = 0$ for every $N \in \mathrm{Mod}(R)$.*

In the next definition we introduce three notions that are crucial for the rest of the section.

Definition 2.7. Let L be a length function on the category $\mathrm{Mod}(R)$, $N \subseteq M$ and $\pi : M \to M/N$ be the natural projection. Define

$$N_{L*} = \pi^{-1}(z_L(M/N))$$

to be the *L-purification* of N in M. A submodule $N \subseteq M$ is said to be *L-pure* in M if $N_{L*} = N$ and N is said to be *L-dense in M* if $L(M/N) = 0$, that is, if $N_{L*} = M$.

The reason for calling N_{L*} the L-purification of N is that, if $R = \mathbb{Z}$, $L = \mathrm{rk}$ and M is a torsion-free abelian group, then N_{L*} is the classical purification of N in M and N is L-dense in M exactly if M/N is torsion. Note that, if $H \subseteq G$ are Abelian groups and $L = \log|-|$, then H is always L-pure in G and H is L-dense in G if and only if $H = G$.

The following proposition summarizes some basic properties of the L-purifications. Notice that the class of modules of zero length is in fact a torsion class and $z_L(-)$ the associated 'torsion' part, cf. part (ii) in Proposition 2.8 below. The proofs of the following propositions could be deduced from this observation and are straightforward, but we include them for the sake of completeness.

Proposition 2.8. *Let* $N \subseteq M \in \mathrm{Mod}(R)$. *Then*

(i) $L(N_{L*}/N) = 0$;

(ii) $z_L(M/N_{L*}) = 0$;

(iii) $N \subseteq N'_{L*}$ *whenever* N' *is an* L*-dense submodule of* N;

(iv) $(N_{L*})_{L*} = N_{L*}$;

(v) $((N + N')/N)_{L*} = (N + N')_{L*}/N$ *with* $N' \subseteq M$.

Proof. (i) Follows from Lemma 2.6 since $N_{L*}/N = z_L(M/N)$.

(ii) Let $x \in z_L(M/N_{L*})$ and consider the following exact sequence:

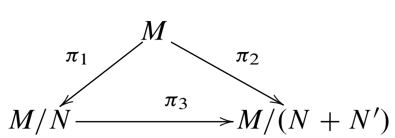
$$0 \to \frac{(xR + N) \cap N_{L*}}{N} \to \frac{xR + N}{N} \to \frac{xR + N_{L*}}{N_{L*}} \to 0.$$

By item (i), $L(((xR+N) \cap N_{L*})/N) = 0$ and so $L(x+N) = L(x+N_{L*}) = 0$. Since $x + N \in z_L(M/N)$, the definition of L-purification gives $x \in N_{L*}$. This means that $x + N_{L*} = 0$.

(iii) Since $L(N/N') = 0$, we have that $N/N' \subseteq z_L(M/N')$. Thus (iii) follows immediately.

(iv) It is not difficult to see that $N \subseteq N_{L*} \subseteq (N_{L*})_{L*}$. Applying twice item (i), we get that N is L-dense in $(N_{L*})_{L*}$. Now we can apply item (iii) to obtain that $(N_{L*})_{L*} \subseteq N_{L*}$.

(v) Consider the following commutative diagram:



where π_1, π_2 and π_3 denote the natural projections. Let $x \in M$; then $x + N \in (N + N'/N)_{L*}$ if and only if $L(\pi_3(x + N)) = L(\pi_3(\pi_1(x))) = L(\pi_2(x)) = 0$. This happens if and only if $x \in (N + N')_{L*}$. □

In the following proposition we collect all the results we will need on the L-purifications related to the action of an endomorphism.

Proposition 2.9. *Let* M *be an* R*-module,* $N \subseteq M$ *and* $\phi : M \to M$ *be an endomorphism. Then*

(i) $\phi(N_{L*}) \subseteq (\phi N)_{L*}$;

(ii) $L((\phi N)_{L*}/\phi(N_{L*})) = 0$;

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(iii) *whenever* $N' \subseteq N$ *is* L*-dense,* $\phi N'$ *is* L*-dense in* ϕN*;*

(iv) $N' \subseteq N$*,* $L(N) = L(\phi N) < \infty$ *imply* $L(N') = L(\phi N')$.

Proof. (i) Consider the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M/N & \xrightarrow{\bar{\phi}} & M/\phi N \end{array}$$

where π_1 and π_2 denote the natural projections. Let $x \in N_{L*}$, that is to say $L(\pi_1(x)) = 0$. Then $L(\bar{\phi}(\pi_1(x))) = L(\pi_2(\phi(x))) = 0$ and so $\phi(x) \in (\phi N)_{L*}$.

(ii) By item (i), we get $\phi N \subseteq \phi(N_{L*}) \subseteq (\phi N)_{L*}$. By Proposition 2.8 (i), ϕN is L-dense in $(\phi N)_{L*}$ and so the claim follows easily.

(iii) By Proposition 2.8 (iii), $N \subseteq N'_{L*}$ and this gives $\phi N' \subseteq \phi N \subseteq \phi N'_{L*} \subseteq (\phi N')_{L*}$ (where the last inclusion is given by item (i)). By Proposition 2.8 (i), we get $L((\phi N')_{L*}/\phi N') = 0$ so our claim follows.

(iv) By the additivity of L, we obtain $L(\mathrm{Ker}(\phi) \cap N) = L(N) - L(\phi N) = 0$. As $\mathrm{Ker}(\phi) \cap N \geq \mathrm{Ker}(\phi) \cap N'$, we get $0 = L(\mathrm{Ker}(\phi) \cap N) \geq L(\mathrm{Ker}(\phi) \cap N') = L(N') - L(\phi N')$. □

From part (i) in the previous proposition it follows immediately that if N is a ϕ-invariant submodule of M, then N_{L*} is also ϕ-invariant.

In the following proposition we study the behaviour of the L-purifications with respect to the sum of two submodules.

Proposition 2.10. *Let* M *be an* R*-module and* $A, B \subseteq M$*. Then*

(i) $A_{L*} + B_{L*} \subseteq (A + B)_{L*}$*;*

(ii) $A + B$ *is* L*-dense in* $A_{L*} + B_{L*}$*;*

(iii) $A' + B'$ *is* L*-dense in* $A + B$ *whenever* $A' \subseteq A$*,* $B' \subseteq B$ *are* L*-dense in* A *and* B *respectively;*

(iv) $(A_{L*} + B)_{L*} = (A + B)_{L*}$.

Proof. (i) Let $a \in A_{L*}$ and $b \in B_{L*}$. Then we have $L((a + b) + (A + B)) \leq L(a + (A + B)) + L(b + (A + B)) \leq L(a + A) + L(b + B) = 0$.

(ii) By part (i), $A + B \subseteq A_{L*} + B_{L*} \subseteq (A + B)_{L*}$. Now statement (ii) follows from Proposition 2.8 (i).

(iii) We have $A \subseteq A'_{L*}$ and $B \subseteq B'_{L*}$ by Proposition 2.8 (iii) whence $A' + B' \subseteq A + B \subseteq A'_{L*} + B'_{L*}$ shows that $A' + B'$ is L-dense in $A + B$ by part (ii) above.

(iv) The inclusion $(A_{L*} + B)_{L*} \supseteq (A + B)_{L*}$ is clear. Next, $L((A_{L*} + B)/(A + B)) \leq L(A_{L*}/A) = 0$. Making use of Proposition 2.8 (iii), we obtain $A_{L*} + B \subseteq (A + B)_{L*}$, from which the result follows. □

2.2 Background on L-entropy

In this paragraph we collect some useful facts from [16], adapted to our context in which L is a discrete length function. Notice that most of these facts hold also for invariants satisfying weaker hypotheses.

Let M be an R-module and $\phi : M \to M$ be an R-endomorphism.

(i) [16, Proposition 1.8] If $L(M)$ is finite, then $\mathrm{ent}_L(\phi) = 0$.

(ii) [16, Proposition 1.10] If $F \in \mathrm{Fin}_L(M)$, then $\mathrm{ent}_L(\phi, F) = a \in \mathbb{R}$ where $a = L(T_{n+1}(\phi, F)) - L(T_n(\phi, F))$ for any sufficiently large n. In particular, $\mathrm{ent}_L(\phi, F) = 0$ if and only if $L(T(\phi, F)) = L(T_n(\phi, F))$ for some $n \in \mathbb{N}_+$.

A consequence of this fact and the equality $T_n(\phi, T_m(\phi, F)) = T_{n+m-1}(\phi, F)$ $(n, m \geq 1)$ is that $\mathrm{ent}_L(\phi, F) = \mathrm{ent}_L(\phi, T_n(\phi, F))$ for every $n \in \mathbb{N}_+$ and every $F \in \mathrm{Fin}_L(M)$.

(iii) [16, Lemma 1.13] If $N \subseteq M$ is ϕ-invariant and $F \in \mathrm{Fin}_L(M)$, then

$$\mathrm{ent}_L(\phi, F) \geq \mathrm{ent}_L(\bar{\phi}, (F + N)/N) + \mathrm{ent}_L(\phi \restriction_N, F \cap N)$$

where $\bar{\phi} : M/N \to M/N$ denotes the map induced by ϕ on the quotient.

(iv) [16, Property (1)] If $\alpha : M \to M'$ is an isomorphism, then

$$\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\alpha\phi\alpha^{-1}).$$

In particular, this fact, together with Fact (i), shows that ent_L is an invariant of $\mathrm{Mod}(R[X])$.

(v) [16, Property (3)] $\mathrm{ent}_L(\phi) \geq \mathrm{ent}_L(\phi \restriction_N)$ for every ϕ-invariant submodule N of M.

(vi) [16, Property (5)] Let $\phi_j : M_j \to M_j$ be endomorphisms $(j = 1, 2)$; then $\mathrm{ent}_L(\phi_1 \oplus \phi_2) = \mathrm{ent}_L(\phi_1) + \mathrm{ent}_L(\phi_2)$.

2.3 First properties

In this section we fix an R-module M and an R-endomorphism $\phi : M \to M$.

Lemma 2.11. *Let $N \in \mathrm{Fin}_L(M)$ and $F \subseteq N$ be an L-dense submodule. Then we have* $\mathrm{ent}_L(\phi, N) = \mathrm{ent}_L(\phi, F)$.

Proof. Using Proposition 2.9 (iii), we obtain that $L(\phi^n N/\phi^n F) = 0$, for every $n \in \mathbb{N}$. Now we can use Proposition 2.10 (iii) to obtain that, for every $n \in \mathbb{N}_+$, $L(T_n(\phi, N)/T_n(\phi, F)) = 0$, therefore $L(T_n(\phi, N)) = L(T_n(\phi, F))$. This easily gives $\mathrm{ent}_L(\phi, N) = \mathrm{ent}_L(\phi, F)$. □

The next proposition is the first step that will enable us to say that ent_L is an upper continuous invariant (see Corollary 2.16). This is the main reason to suppose L to be upper continuous.

Proposition 2.12. *If M is locally L-finite, $\phi \in \mathrm{End}(M)$, then*

$$\mathrm{ent}_L(\phi) = \sup_{F \in \mathcal{F}(M)} \mathrm{ent}_L(\phi, F).$$

Proof. Since $M = f_L(M)$, for every $F \in \mathcal{F}(M)$ we have that $L(F) < \infty$. So we have that $\sup_{F \in \mathcal{F}(M)} \mathrm{ent}_L(\phi, F) \leq \mathrm{ent}_L(\phi)$ by definition. On the other hand, consider $N \in \mathrm{Fin}_L(M)$. By Lemma 2.1, there exists $F \in \mathcal{F}(N)$ which is L-dense in N and thanks to Lemma 2.11 $\mathrm{ent}_L(\phi, F) = \mathrm{ent}_L(\phi, N)$. Hence we obtain that $\mathrm{ent}_L(\phi) = \sup_{N \in \mathrm{Fin}_L(M)} \mathrm{ent}_L(\phi, N) \leq \sup_{F \in \mathcal{F}(M)} \mathrm{ent}_L(\phi, F)$. □

The following corollary deals with the case when M_ϕ is the $R[X]$-module generated by an R-submodule $N \in \mathrm{Fin}_L(M)$.

Corollary 2.13. *If $M = T(\phi, N)$ for some submodule N with $L(N) < \infty$, then* $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi, N)$.

Proof. We first prove that $M = f_L(M)$. Pick $x \in M = \bigcup_{n \in \mathbb{N}_+} T_n(\phi, N)$. Then there exists $n \in \mathbb{N}_+$ such that $x \in T_n(\phi, N)$. In particular, we have that $L(x) \leq L(T_n(\phi, N)) < \infty$. Consider now $F \in \mathcal{F}(M)$; there exists $n \in \mathbb{N}_+$ such that $F \subseteq T_n(\phi, N)$. This shows that $\mathrm{ent}_L(\phi, F) \leq \mathrm{ent}_L(\phi, T_n(\phi, N))$. The consequence of Fact (ii) in Section 2.2 and Proposition 2.12 imply that $\mathrm{ent}_L(\phi) \leq \mathrm{ent}_L(\phi, N)$. The converse inequality trivially holds, so we are done. □

Example 2.14. Let $M \in \mathrm{Fin}_L$ and $N = \bigoplus_{n \in \mathbb{N}} M_n$ where $M_n \cong M$ for every $n \in \mathbb{N}$. The *right Bernoulli shift* on N is the endomorphism $\beta_M : M^{(\mathbb{N})} \to M^{(\mathbb{N})}$, defined by

$$\beta_M(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, \ldots).$$

Clearly, $N = T(\beta_M, M_0)$, so, by Corollary 2.13, we obtain that $\mathrm{ent}_L(\beta_M) = \mathrm{ent}_L(\beta_M, M_0)$. Since $T_{n+1}(\beta_M, M_0)/T_n(\beta_M, M_0) \cong M$ for every $n \in \mathbb{N}_+$, we can apply Fact (ii) in Section 2.2 to obtain $\mathrm{ent}_L(\beta_M) = L(M)$.

It is worthwhile remarking that a result similar to that of Example 2.14 was proved in [16, Property (7), Section 2] under the assumption that the considered invariant was 'small'. In Example 2.14 we could avoid this assumption since the upper continuity of L allows us to reach the same conclusion.

The next two lemmata show that, to evaluate the L-entropy of an endomorphism ϕ, we can always assume $M = f_L(M)$ and $z_L(M) = 0$.

Lemma 2.15. *For all M and $\phi \in \mathrm{End}(M)$, $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi\restriction_{f_L(M)})$.*

Proof. Since $f_L(M)$ is a fully invariant submodule of M, the inequality $\mathrm{ent}_L(\phi) \geq \mathrm{ent}_L(\phi\restriction_{f_L(M)})$ derives from Fact (v) in Section 2.2. For the converse inequality, consider $N \in \mathrm{Fin}_L(M)$. Then obviously $N \subseteq f_L(M)$ holds, therefore $\mathrm{Fin}_L(M) = \mathrm{Fin}_L(f_L(M))$. Thus $\mathrm{ent}_L(\phi, N) = \mathrm{ent}_L(\phi\restriction_{f_L(M)}, N) \leq \mathrm{ent}_L(\phi\restriction_{f_L(M)})$, and we are done. □

An immediate consequence of Lemma 2.15 is the announced result that ent_L is an upper continuous invariant. Recall that the finitely generated $R[X]$-submodules of M_ϕ are of the form $T(\phi, F)$ for some $F \in \mathcal{F}(M)$.

Corollary 2.16. $\mathrm{ent}_L(\phi) = \sup_{F\in\mathcal{F}(M)} \mathrm{ent}_L(\phi\restriction_{T(\phi,F)})$.

Proof. Since $T(\phi, F)$ is a ϕ-invariant submodule of M for every $F \in \mathcal{F}(M)$, from Fact (v) in Section 2.2 we get $\mathrm{ent}_L(\phi) \geq \sup_{F\in\mathcal{F}(M)} \mathrm{ent}_L(\phi\restriction_{T(\phi,F)})$. For the converse inequality, from Lemma 2.15 we get $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi\restriction_{f_L(M)})$ and, since $f_L(M)$ is a locally L-finite module, we can apply to it Proposition 2.12, deriving that $\mathrm{ent}_L(\phi) = \sup_{F\in\mathcal{F}(f_L(M))} \mathrm{ent}_L(\phi\restriction_{f_L(M)}, F)$. If $F \in \mathcal{F}(f_L(M))$, then $L(F) < \infty$, therefore from Corollary 2.13 we deduce $\mathrm{ent}_L(\phi\restriction_{f_L(M)}, F) = \mathrm{ent}_L(\phi\restriction_{T(\phi,F)})$. It follows that $\mathrm{ent}_L(\phi) = \sup_{F\in\mathcal{F}(f_L(M))} \mathrm{ent}_L(\phi\restriction_{T(\phi,F)}) \leq \sup_{F\in\mathcal{F}(M)} \mathrm{ent}_L(\phi\restriction_{T(\phi,F)})$, as desired. □

Lemma 2.17. *For all M and $\phi \in \mathrm{End}(M)$ the following hold:*

(i) $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\bar\phi)$ *where $\bar\phi : M/z_L(M) \to M/z_L(M)$ is the map induced by ϕ on the quotient;*

(ii) $\mathrm{ent}_L(\phi) = 0$ *provided $M = z_L(M)$;*

(iii) $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi\restriction_N)$, *provided N is an L-dense ϕ-invariant submodule of M.*

Proof. (i) Looking at the exact sequence for any $H \subseteq M$,

$$0 \to z_L(M) \cap H \to H \to (H + z_L(M))/z_L(M) \to 0,$$

we see that $L(H) = L((H + z_L(M))/z_L(M))$, by Lemma 2.6. This shows that $N \in \mathrm{Fin}_L(M)$ if and only if $(N + z_L(M))/z_L(M) \in \mathrm{Fin}_L(M/z_L(M))$ and that, for every $n \in \mathbb{N}_+$,

$$L(T_n(\phi, N)) = L(T_n(\bar{\phi}, (N + z_L(M))/z_L(M))).$$

Dividing by n and passing to the limit, we obtain that

$$\mathrm{ent}_L(\phi, N) = \mathrm{ent}_L(\bar{\phi}, (N + z_L(M))/z_L(M)),$$

from which the conclusion follows.

(ii) Follows from (i).

(iii) Let $F \in \mathrm{Fin}_L(M)$ and consider the following exact sequence:

$$0 \to F \cap N \to F \to (F + N)/F \to 0.$$

Since $(F + N)/N \subseteq M/N$, $L((F + N)/N) = 0$ and then $F \cap N$ is L-dense in F. Using Lemma 2.11, we obtain $\mathrm{ent}_L(\phi, F) = \mathrm{ent}_L(\phi\restriction_N, F \cap N)$. This proves the inequality $\mathrm{ent}_L(\phi\restriction_N) \geq \mathrm{ent}_L(\phi)$; the converse inequality follows from Fact (v) in Section 2.2. □

As a consequence of the last lemma, we can see how the entropy works with respect to L-purifications.

Corollary 2.18. *Let N be a ϕ-invariant submodule of M. Denote by*

$$\bar{\phi} : M/N \to M/N \quad \text{and} \quad \bar{\phi}_* : M/N_{L*} \to M/N_{L*}$$

the maps induced by ϕ on the quotients. Then

(i) $\mathrm{ent}_L(\phi\restriction_{N_{L*}}) = \mathrm{ent}_L(\phi\restriction_N)$;

(ii) $\mathrm{ent}_L(\bar{\phi}) = \mathrm{ent}_L(\bar{\phi}_*)$.

Proof. (i) Follows from Lemma 2.17 (iii) since $L(N_{L*}/N) = 0$.

(ii) Follows from Lemma 2.17 (i) since $M/N_{L*} \cong (M/N)/z_L(M/N)$. □

We are now ready to prove one part of the Addition Theorem. A similar result was proved in [16, Proposition 2.1] with the assumption of upper continuity of L replaced by the assumption that L is 'liftable'.

Proposition 2.19. *Suppose that $M = f_L(M)$ and $z_L(M) = 0$. Let $N \subseteq M$ be a ϕ-invariant submodule and denote by $\bar{\phi} : M/N \to M/N$ the map induced on the quotient. Then* $\mathrm{ent}_L(\phi) \geq \mathrm{ent}_L(\phi\restriction_N) + \mathrm{ent}_L(\bar{\phi})$.

Proof. First observe that the inequality $\mathrm{ent}_L(\bar{\phi}) \leq \mathrm{ent}_L(\phi)$ holds. In fact, fix an arbitrary $F/N \in \mathcal{F}(M/N)$ and let $F' \in \mathcal{F}(M)$ be such that $(F' + N)/N = F/N$. Then clearly $\mathrm{ent}_L(\phi, F') \geq \mathrm{ent}_L(\bar{\phi}, F/N)$. The inequality follows from Proposition 2.12. Now, if $\mathrm{ent}_L(\phi\restriction_N) = \infty$ or $\mathrm{ent}_L(\bar{\phi}) = \infty$ the inequality follows respectively by Fact (v) in Section 2.2 and the preceding observation. Thus suppose both $\mathrm{ent}_L(\phi\restriction_N)$ and $\mathrm{ent}_L(\bar{\phi})$ to be finite. Since L is discrete, we can find submodules $F_1 \in \mathcal{F}(N)$ and $F_2 \in \mathcal{F}(M)$ such that $\mathrm{ent}_L(\phi\restriction_N) = \mathrm{ent}_L(\phi, F_1)$ and $\mathrm{ent}_L(\bar{\phi}) = \mathrm{ent}_L(\bar{\phi}, (F_2 + N)/N)$. Let $F = F_1 + F_2$; then, according to Fact (iii) in Section 2.2,

$$\mathrm{ent}_L(\phi, F) \geq \mathrm{ent}_L(\phi\restriction_N, F \cap N) + \mathrm{ent}_L(\bar{\phi}, (F + N)/N).$$

The result follows from the fact that $(F + N)/N = (F_2 + N)/N$ and

$$\mathrm{ent}_L(\phi\restriction_N, F_1) \geq \mathrm{ent}_L(\phi, F \cap N) \geq \mathrm{ent}_L(\phi\restriction_N, F_1)$$

where the first inequality holds since $\mathrm{ent}_L(\phi\restriction_N) = \mathrm{ent}_L(\phi\restriction_N, F_1)$, while the second one follows by the inclusion $F_1 \subseteq F \cap N$. □

The next proposition is the starting step of the inductive proof of the Addition Theorem.

Proposition 2.20. *Suppose that $M = f_L(M)$ and $z_L(M) = 0$. Let $N \subseteq M$ be a ϕ-invariant submodule and denote by $\bar{\phi} : M/N \to M/N$ the morphism induced on the quotient.*

(i) *If* $\mathrm{ent}_L(\bar{\phi}) = 0$*, then* $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi\restriction_N)$*;*

(ii) *if $L(N) < \infty$, then we have* $\mathrm{ent}_L(\phi, F) = \mathrm{ent}_L(\bar{\phi}, (F + N)/N)$ *for every* $F \in \mathrm{Fin}_L(M)$*. In particular,* $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\bar{\phi})$.

Proof. (i) By Fact (v) in Section 2.2 and Proposition 2.12, we have to prove that, for any $F \in \mathcal{F}(M)$, there is $F' \in \mathcal{F}(N)$ such that $\mathrm{ent}_L(\phi\restriction_N, F') \geq \mathrm{ent}_L(\phi, F)$. Fix $F \in \mathcal{F}(M)$; we will write $\bar{F}$ for $(F + N)/N$. Since $\mathrm{ent}_L(\bar{\phi}) = 0$, we get $\mathrm{ent}_L(\bar{\phi}, \bar{F}) = 0$. Hence there is $n \in \mathbb{N}_+$ such that $L(T(\bar{\phi}, \bar{F})) = L(T_n(\bar{\phi}, \bar{F}))$. In particular, $L(T_n(\bar{\phi}, \bar{F})) = L(T_{n+1}(\bar{\phi}, \bar{F}))$. By Proposition 2.8 (iii), we get

$$\bar{\phi}^n(\bar{F}) \subseteq T_{n+1}(\bar{\phi}, \bar{F}) \subseteq (T_n(\bar{\phi}, \bar{F}))_{L*}.$$

By Proposition 2.8 (v), $((T_n(\phi, F) + N)/N)_{L*} = (T_n(\phi, F) + N)_{L*}/N$ and then $\phi^n F \subseteq (T_n(\phi, F) + N)_{L*}$. Since F is finitely generated, so is $\phi^n F$ and then there exists $F' \in \mathcal{F}(N)$ with $\phi^n F \subseteq (T_n(\phi, F) + F')_{L*}$. For every $k \in \mathbb{N}_+$,

$$\phi^{k+n} F \subseteq \phi^k((T_n(\phi, F) + F')_{L*}) \subseteq (\phi^k T_n(\phi, F) + \phi^k F')_{L*} \tag{2.1}$$

where the last inclusion follows by Proposition 2.9 (i). We will prove, using induction on k, that $T_{n+k}(\phi, F) \subseteq (T_n(\phi, F) + T_k(\phi, F'))_{L*}$ for every $k \in \mathbb{N}_+$. We already have the case $k = 1$. Suppose now $k > 1$; then

$$\begin{aligned} T_{n+k}(\phi, F) &= T_{n+k-1}(\phi, F) + \phi^{n+k-1}F \\ &\subseteq (T_n(\phi, F) + T_{k-1}(\phi, F'))_{L*} + (\phi^{k-1}T_n(\phi, F) + \phi^{k-1}F')_{L*} \\ &\subseteq (T_n(\phi, F) + T_{k-1}(\phi, F') + \phi^{k-1}T_n(\phi, F) + \phi^{k-1}F')_{L*} \\ &= (T_n(\phi, F) + T_k(\phi, F'))_{L*} \end{aligned}$$

where the first inclusion follows by the inductive hypothesis and equation (2.1), the second inclusion follows by Proposition 2.10 (i), and the last equality holds since $\phi^{k-1}T_n(\phi, F) \subseteq T_{n+k-1}(\phi, F) \subseteq (T_n(\phi, F) + T_{k-1}(\phi, F'))_{L*}$ again by the inductive hypothesis. So we have proved that

$$\begin{aligned} \mathrm{ent}_L(\phi, F) &\leq \lim_{k\to\infty} \frac{L(T_n(\phi, F) + T_k(\phi, F'))}{n+k} \\ &\leq \lim_{k\to\infty} \frac{L(T_n(\phi, F))}{n+k} + \lim_{k\to\infty} \frac{L(T_k(\phi, F'))}{n+k} = \mathrm{ent}_L(\phi\restriction_N, F'). \end{aligned}$$

(ii) Fix $F \in \mathcal{F}(M)$; then

$$\begin{aligned} L(T_n(\bar{\phi}, (F+N)/N)) &= L(T_n(\phi, F)) - L(T_n(\phi, F) \cap N) \\ &\geq L(T_n(\phi, F)) - L(N). \end{aligned}$$

Using Proposition 2.12, we get $\mathrm{ent}_L(\phi, F) \leq \mathrm{ent}_L(\bar{\phi}, (F+N)/N)$ after dividing by n and passing to the limit. Since the other inequality is trivial, we get the desired equality. □

2.4 *L*-minimal elements

All along this paragraph M will denote a fixed R-module such that $M = f_L(M)$ and $z_L(M) = 0$; $\phi : M \to M$ will denote a fixed endomorphism. The following definition is a key-point in the proof of the Addition Theorem.

Definition 2.21. Let M be an R-module such that $M = f_L(M)$ and $z_L(M) = 0$. An element $0 \neq x \in M$ is said to be L-*minimal* if $L(x) = L(xr)$ for every $r \in R$ such that $xr \neq 0$.

Notice that, if $R = \mathbb{Z}$ and $L = \log|-|$, then an element of a torsion group is L-minimal if and only if its order is a prime number. In particular this shows that, if x is L-minimal, the value $L(x)$ is not necessarily minimal in the set $\mathrm{Im}(L) \setminus \{0\}$. In fact, for the invariant $\log|-|$ we can find minimal elements of arbitrarily large order. On the other hand, if $R = \mathbb{Z}$ and M is a torsion-free Abelian group, then every non-zero element is L-minimal. Furthermore, if R is an arbitrary ring and L is the 'classical' length, then a non-zero element x of an R-module M is L-minimal if and only if xR is simple.

The reader should be warned that the notion of L-minimality given above can be developed, albeit in a more complicated way, to prove the results in this subsection without supposing that M is locally L-finite and $z_L(M) = 0$ (see [21]).

The following lemma gives another characterization of the L-minimal elements.

Lemma 2.22. *Let M be an R-module such that $M = f_L(M)$ and $z_L(M) = 0$. An element $x \in M$ is L-minimal if and only if, for every non-zero submodule H of xR, $L(H) = L(xR)$.*

Proof. The fact that $0 \subset H$ means that there exists $0 \neq y \in H$. Hence $L(x) \geq L(H) \geq L(y) = L(x)$ where the last equality holds by the L-minimality of x. □

Clearly, if $x \in M$ is L-minimal, then all the generators of xR are also L-minimal.

Proposition 2.23. *Let M be a locally L-finite R-module such that $z_L(M) = 0$. If $x \in M$ is L-minimal and $\phi(x) \neq 0$, then $xR \cong \phi(xR)$, so $L(\phi x) = L(x)$ and $\phi(x)$ is L-minimal.*

Proof. Let $\phi' : xR \to \phi(xR)$ be the restriction of ϕ. Then ϕ' is clearly surjective. Furthermore, $\mathrm{Ker}(\phi') \subseteq xR$ and so either $\mathrm{Ker}(\phi') = 0$ or it is L-dense in xR, by Lemma 2.22. From $z_L(M) = 0$ and $\phi(x) \neq 0$ we get that $L(\phi(x)) > 0$, so we derive that $\mathrm{Ker}(\phi')$ cannot be L-dense in xR and thus $\mathrm{Ker}(\phi') = 0$. This proves that ϕ' is an isomorphism between xR and $\phi(xR)$. □

From now on, to simplify the notation, we will write $T(\phi, x)$ (resp. $T_n(\phi, x)$ and $\mathrm{ent}_L(\phi, x)$) for the ϕ-trajectory (resp. n-th partial ϕ-trajectory and L-entropy) of the cyclic module generated by x.

Proposition 2.24. *Let M be a locally L-finite R-module such that $z_L(M) = 0$. Let $x \in M$ be an L-minimal element. If $\mathrm{ent}_L(\phi, x) > 0$, then $\sum_{n\in\mathbb{N}} \phi^n xR = \bigoplus_{n\in\mathbb{N}} \phi^n xR$ and $\mathrm{ent}_L(\phi, x) = L(x)$.*

Proof. We prove that $\phi^n xR \cap \sum_{j=0}^{n-1}(\phi^j xR) = 0$ for all $n \in \mathbb{N}_+$. Since, by assumption, $\mathrm{ent}_L(\phi, x) > 0$, it follows that $\phi^n x \neq 0$ for every $n \in \mathbb{N}$ and so $\phi^n x$ is L-minimal. Hence

$$\begin{aligned} L(T_{n+1}(\phi, x)) &= L(T_n(\phi, x)) + L(\phi^n x) - L(T_n(\phi, x) \cap \phi^n xR) \\ &= L(T_n(\phi, x)) + L(x) - L(T_n(\phi, x) \cap \phi^n xR) \end{aligned}$$

and $L(T_n(\phi, x) \cap \phi^n xR)$ equals $L(x)$ or 0. If $L(T_n(\phi, x) \cap \phi^n xR) = L(x)$, then $L(T_n(\phi, x)) = L(T_{n+1}(\phi, x))$, but this contradicts the fact that $\mathrm{ent}_L(\phi, x) > 0$. So $L(T_n(\phi, x) \cap \phi^n xR) = 0$ and this is to say that $T_n(\phi, x) \cap \phi^n xR = 0$ since $z_L(M) = 0$. To prove the last assertion just look at Example 2.14. □

Suppose that $R = \mathbb{Z}$ (in particular M is an Abelian group) and $L = \log|-|$. In [6] it is proved that $\mathrm{ent}_L(\phi) > 0$ if and only if there exists an element $x \in M[p]$, for some prime p, such that $\mathrm{ent}_L(\phi, x) > 0$. We do not know whether it is true that $\mathrm{ent}_L(\phi) > 0$ implies the existence of an L-minimal element x in M such that $\mathrm{ent}_L(\phi, x) > 0$. Even if we cannot answer the above question, the next lemma shows that we can find such elements in a suitable quotient of M.

Lemma 2.25. *Let M be an R-module such that $M = f_L(M)$ and $z_L(M) = 0$. Let N be an L-pure ϕ-invariant submodule of M and denote by $\bar{\phi} : M/N \to M/N$ the map induced by ϕ on the quotient. If* $\mathrm{ent}_L(\bar{\phi}) > 0$, *then there exists an L-pure ϕ-invariant submodule N' of M containing N and an element $x \in M$ such that*

(i) $L(N'/N) < \infty$;

(ii) $x + N'$ *is L-minimal in M/N';*

(iii) $\mathrm{ent}_L(\bar{\phi}', x + N') > 0$ *where $\bar{\phi}' : M/N' \to M/N'$ is the map induced by ϕ.*

Proof. Since $\mathrm{ent}_L(\bar{\phi}) > 0$, there exists an element $\bar{y} = y + N \in M/N$ such that $\mathrm{ent}_L(\bar{\phi}, \bar{y}R) > 0$ by the additivity of L and Proposition 2.12.

If there exists $r \in R$ such that $yr + N$ is L-minimal and $\mathrm{ent}_L(\bar{\phi}, yr + N) > 0$, then we can conclude setting $x = yr$ and $N' = N$.

Otherwise, choose any $r_1 \in R$ such that $yr_1 + N$ is L-minimal; for such an r_1 we have that $L(T(\bar{\phi}, yr_1 + N) < \infty$. Call $y_1 = yr_1$, $N_1 = (N + T(\phi, y_1))_{L*}$ and $\bar{\phi}_1 : M/N_1 \to M/N_1$ the map induced by ϕ. By Proposition 2.20 (ii), we have that

$$\mathrm{ent}_L(\bar{\phi}, y + N) = \mathrm{ent}_L(\bar{\phi}_1, y + N_1);$$

in fact, since $L(T(\bar{\phi}, yr_1 + N)) < \infty$, hence $L(N_1/N) < \infty$. Consider now the following exact sequence,

$$0 \to \frac{N_1 \cap (yR + N)}{N} \to \frac{yR + N}{N} \to \frac{yR + N_1}{N_1} \to 0,$$

that shows (since $0 \neq y_1 + N \in (N_1 \cap (yR + N))/N$) that $L(y + N) > L(y + N_1)$.

If there is an $r \in R$ such that $yr + N_1$ is L-minimal and $\mathrm{ent}_L(\bar{\phi}_1, yr + N_1) > 0$, then we can conclude setting $x = yr$ and $N' = N_1$.

Otherwise, arguing as above, we can find an L-pure ϕ-invariant submodule $N_2 \supset N_1$ such that

$$\mathrm{ent}_L(\bar{\phi}_1, y + N_1) = \mathrm{ent}_L(\bar{\phi}_2, y + N_2)$$

and $L(y + N) > L(y + N_1) > L(y + N_2)$.

Going on in this way, we will find the desired $x = y_n$ and $N' = N_n$ for some n in a finite number of steps. In fact, otherwise we will obtain an infinite sequence of submodules of M

$$N \subset N_1 \subset \cdots \subset N_k \subset \cdots$$

such that $0 < \mathrm{ent}_L(\bar{\phi}, y + N) = \cdots = \mathrm{ent}_L(\bar{\phi}_k, y + N_k) = \cdots$ and

$$L(y + N) > \cdots > L(y + N_k) > \cdots .$$

This is impossible, because $L(y + N)$ is finite, and L is discrete. □

2.5 Proof of the Addition Theorem

The next proposition is a particular case of the Addition Theorem and a crucial step in proving it. The proof is technical, but the idea is simple. The key point is to prove that $\mathrm{ent}_L(\phi \restriction_{T(\phi,x)\cap N}) = \mathrm{ent}_L(\phi, xR \cap N)$ (even if $T(\phi, xR \cap N) \subset T(\phi, x) \cap N$ in general). Steps 1, 2 and 3 are devoted to proving this equality.

Proposition 2.26. *Let M be a locally L-finite R-module such that $z_L(M) = 0$. Furthermore, let $\phi \in \mathrm{End}_R(M)$, $N \subseteq M$ be a ϕ-invariant L-pure submodule and $x + N$ be an L-minimal element of M/N. If $\bar{\phi} : M/N \to M/N$ is the induced map and* $\mathrm{ent}_L(\bar{\phi}, x + N) > 0$, *then*

$$\mathrm{ent}_L\big(\phi \restriction_{T(\phi,x)}\big) = \mathrm{ent}_L\big(\phi \restriction_{T(\phi,x)\cap N}\big) + \mathrm{ent}_L\big(\bar{\phi} \restriction_{T(\bar{\phi},x+N)}\big).$$

Proof. Since L is discrete, $\mathrm{ent}_L(\phi, x) > 0$ and $L(x) < \infty$, it follows that the sequence $\{L(\phi^n x)\}_{n\in\mathbb{N}}$ stabilizes, i.e., there exists $\tilde{n} \in \mathbb{N}$ such that, for every $m \geq \tilde{n}$, $L(\phi^m x) = L(\phi^{\tilde{n}} x)$. Without loss of generality, we can suppose $\tilde{n} = 0$. In the remaining part of the proof we will write T_n for $T_n(\phi, x)$ and $\overline{T}_n$ for $T_n(\bar{\phi}, x + N)$.

Step 1. Let $A_n = (xR \cap N) + \cdots + (\phi^{n-1} xR \cap N)$; then $A_n = T_n \cap N$ for every $n \in \mathbb{N}_+$.

The inclusion $A_n \subseteq T_n \cap N$ is obvious. For the converse, let us assume that $y = \sum_{i=0}^{n-1} \phi^i x r_i \in T_n(\phi, x) \cap N$. Passing modulo N, we get $\bar{y} = \sum_{i=0}^{n-1} \bar{\phi}^i \bar{x} r_i = \bar{0}$ which implies, in view of Proposition 2.24, $\bar{\phi}^i \bar{x} r_i = \bar{0}$ for every i. This clearly implies that $y \in A_n$.

Step 2. $L(T_n \cap N) = L(T_n(\phi, xR \cap N))$ for every $n \in \mathbb{N}_+$.

Since, for every $n \in \mathbb{N}$, $L(\phi^n x) = L(x)$ and $L(\bar{\phi}^n(x + N)) = L(x + N)$, it follows that $L(\phi^n xR \cap N) = L(x) - L(x + N)$. Furthermore, using Proposition 2.9 (iv), we get $L(\phi^n(xR \cap N)) = L(xR \cap N) = L(x) - L(x + N)$. This shows that $L(\phi^n xR \cap N) = L(\phi^n(xR \cap N))$. Using now Proposition 2.10 (iii), we get $L(T_n(\phi, xR \cap N)) = L(A_n)$. The result now follows from Step 1.

Step 3. $\mathrm{ent}_L(\phi \restriction_{T(\phi,x)\cap N}) = \mathrm{ent}_L(\phi, xR \cap N)$.

From Step 2 we easily deduce the equality $\mathrm{ent}_L(\phi, T_n \cap N) = \mathrm{ent}_L(\phi, xR \cap N)$. Let $F \in \mathcal{F}(T(\phi, x) \cap N)$. Since $T(\phi, x) \cap N = \bigcup_{n\in\mathbb{N}_+}(T_n \cap N)$, there exists $n \in \mathbb{N}_+$ such that $F \in T_n \cap N$. Then

$$\mathrm{ent}_L(\phi, F) \leq \mathrm{ent}_L(\phi, T_n \cap N) = \mathrm{ent}_L(\phi, xR \cap N).$$

Step 4. Consider the following exact sequence,

$$0 \to T_n \cap N \to T_n \to \overline{T}_n \to 0,$$

that gives $L(T_n) = L(T_n \cap N) + L(\overline{T}_n)$; but $L(\overline{T}_n) = n \cdot L(x + N)$ in view of Proposition 2.24, so we obtain $L(T_n) = L(T_n(\phi, xR \cap N)) + n \cdot L(x + N)$, using Step 2. Hence, dividing by n and passing to the limit, we get with $T = T(\phi, x)$:

$$\mathrm{ent}_L(\phi \restriction_T) = \mathrm{ent}_L(\phi, xR \cap N) + L(x + N) = \mathrm{ent}_L(\phi \restriction_{T\cap N}) + L(x + N).$$

Note that $\mathrm{ent}_L(\phi \restriction_T) = \mathrm{ent}_L(\phi, x)$ holds. This is indeed the desired equality since $L(x + N)$ is nothing but $\mathrm{ent}_L(\bar{\phi} \restriction_{T(\bar{\phi},x+N)})$ in view of Proposition 2.24. □

The next lemma was used for the proof of the Addition Theorem in [6]. The proof holds verbatim also in our general case; it makes substantial use of Fact (vi) in Section 2.2.

Lemma 2.27 ([6, Lemma 3.7]). *Suppose that M is a locally L-finite R-module with $z_L(M) = 0$ and $\phi \in \mathrm{End}_R(M)$. If $M = H + K$ where H, K are ϕ-invariant submodules of M, then*

$$\mathrm{ent}_L(\phi) + \mathrm{ent}_L(\phi \restriction_{H\cap K}) \leq \mathrm{ent}_L(\phi \restriction_H) + \mathrm{ent}_L(\phi \restriction_K).$$

We can now give the

Proof of the Addition Theorem. By Lemma 2.17, we can assume $z_L(M) = 0$. In view of Proposition 2.19, we can assume both $\mathrm{ent}_L(\phi \restriction_N)$ and $\mathrm{ent}_L(\bar{\phi})$ to be finite. Furthermore, Corollary 2.18 shows that we may suppose that N is L-pure in M.

Since L is discrete, the set $\mathrm{Im}(L) \setminus \{\infty\}$ is order-isomorphic to $\mathbb{N}$ and so we can write $\mathrm{Im}(L) \setminus \{\infty\} = \{k_n \mid n \in \mathbb{N}\}$ with $k_0 = 0$ and $k_{n+1} > k_n$. We will prove our result using induction on $n \in \mathbb{N}$ such that $\mathrm{ent}_L(\bar{\phi}) = k_n$. The case $n = 0$ is Proposition 2.20 (i). So we can suppose $n > 0$. In this case we can apply Lemma 2.25 to find an L-pure and ϕ-invariant submodule $N' \geq N$, and $x \in M$ such that $x + N'$ is L-minimal, $\mathrm{ent}_L(\bar{\phi}, x + N') > 0$ and $L(N'/N) < \infty$. Since clearly $\mathrm{ent}_L(\phi \restriction_{N'}) = \mathrm{ent}_L(\phi \restriction_N)$ and $\mathrm{ent}_L(\bar{\phi}) = \mathrm{ent}_L(\bar{\phi}')$ where $\bar{\phi}' : M/N' \to M/N'$ is the map induced by ϕ, we can suppose $N = N'$. Consider the map induced by ϕ

$$\bar{\bar{\phi}} : \frac{M}{T(\phi, x) + N} \to \frac{M}{T(\phi, x) + N}.$$

By Proposition 2.19, $\mathrm{ent}_L(\bar{\phi}) \geq \mathrm{ent}_L(\bar{\phi} \restriction_{T(\bar{\phi}, x+N)}) + \mathrm{ent}_L(\bar{\bar{\phi}})$. Furthermore, we have $\mathrm{ent}_L(\bar{\bar{\phi}}) < \mathrm{ent}_L(\bar{\phi})$ since $\mathrm{ent}_L(\bar{\phi} \restriction_{T(\bar{\phi}, x+N)}) = L(x + N) > 0$ being $x + N$ L-minimal. Hence we can use the inductive hypothesis to get

$$\mathrm{ent}_L(\bar{\phi}) = \mathrm{ent}_L(\bar{\phi} \restriction_{T(\bar{\phi}, x+N)}) + \mathrm{ent}_L(\bar{\bar{\phi}}), \tag{2.2}$$

$$\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi \restriction_{T(\phi, x)+N}) + \mathrm{ent}_L(\bar{\bar{\phi}}). \tag{2.3}$$

If $\mathrm{ent}_L(\bar{\phi}) > \mathrm{ent}_L(\bar{\phi} \restriction_{T(\bar{\phi}, x+N)})$, we can use the inductive hypothesis to obtain

$$\mathrm{ent}_L(\phi \restriction_{N+T(\phi, x)}) = \mathrm{ent}_L(\phi \restriction_N) + \mathrm{ent}_L(\bar{\phi} \restriction_{T(\bar{\phi}, x+N)}).$$

Then, using (2.2) and (2.3) completes the proof. So it remains to consider only the case when $\mathrm{ent}_L(\bar{\phi}) = \mathrm{ent}_L(\bar{\phi} \restriction_{T(\bar{\phi}, x+N)})$, so $\mathrm{ent}_L(\bar{\bar{\phi}}) = 0$ by (2.2). Thanks to Proposition 2.20 (i), $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi \restriction_{N+T(\phi, x)})$. Now we can use Lemma 2.27 to get (again writing $T = T(\phi, x)$):

$$\begin{aligned} \mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi \restriction_T) &\leq \mathrm{ent}_L(\phi \restriction_N) + \mathrm{ent}_L(\phi \restriction_T) - \mathrm{ent}_L(\phi \restriction_{N \cap T}) \\ &= \mathrm{ent}_L(\phi \restriction_N) + \mathrm{ent}_L(\bar{\phi}) \end{aligned}$$

where the last equality is given by Proposition 2.26. We have proved the inequality $\mathrm{ent}_L(\phi) \leq \mathrm{ent}_L(\phi \restriction_N) + \mathrm{ent}_L(\bar{\phi})$. Since the converse inequality holds by Proposition 2.19, we reached the desired conclusion. □

An immediate consequence of the Addition Theorem is the following improvement of Lemma 2.27:

Corollary 2.28. *Suppose that M is a locally L-finite R-module and $\phi \in \mathrm{End}_R(M)$. If $M = H + K$ where H, K are ϕ-invariant submodules, then*

$$\mathrm{ent}_L(\phi) + \mathrm{ent}_L(\phi \restriction_{H \cap K}) = \mathrm{ent}_L(\phi \restriction_H) + \mathrm{ent}_L(\phi \restriction_K).$$

Proof. If $\mathrm{ent}_L(\phi\restriction_H) = \infty$ or $\mathrm{ent}_L(\phi\restriction_{K\cap H}) = \infty$, then the claim follows from the Addition Theorem. So suppose these entropies to be finite. Consider the following exact sequence:

$$0 \to H \to M \to \frac{K}{K \cap H} \to 0.$$

Using the Addition Theorem, we get that $\mathrm{ent}_L(\phi) = \mathrm{ent}_L(\phi\restriction_H) + \mathrm{ent}_L(\bar{\phi})$ where $\bar{\phi} : K/(K \cap H) \to K/(K \cap H)$ is the morphism induced by $\phi\restriction_K$ on the quotient. The claim follows from the fact that $\mathrm{ent}_L(\bar{\phi}) = \mathrm{ent}_L(\phi\restriction_K) - \mathrm{ent}_L(\phi\restriction_{H\cap K})$. □

3 Uniqueness of the entropy function

This section is devoted to the proof of the Uniqueness Theorem. Our strategy will be to prove that, given a length function $h_L : \mathrm{lFin}_L[X] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying the hypotheses of the theorem, h_L coincides with ent_L restricted to $\mathrm{lFin}_L[X]$.

As we noted in the introduction, what we have proved up to now shows that ent_L is a discrete length function on $\mathrm{lFin}_L[X]$. So the results proved in Section 2 for the discrete length function L on $\mathrm{Mod}(R)$ hold true for the discrete length function ent_L, but on the subcategory $\mathrm{lFin}_L[X]$ of $\mathrm{Mod}(R[X])$. In particular, given a locally L-finite module M and an endomorphism $\phi \in \mathrm{End}_R(M)$, denoting by $Z_L(M_\phi)$ the ent_L-singular $R[X]$-submodule of M_ϕ (in symbols: $Z_L(M_\phi) = z_{\mathrm{ent}_L}(M_\phi)$), $\mathrm{ent}_L(\phi) = 0$ if and only if the restriction of ϕ to any cyclic ϕ-trajectory $T(\phi, x)$ has zero L-entropy (see Lemma 2.6). This submodule $Z_L(M_\phi)$ is called the 'Pinsker submodule' and is investigated in [3, 4].

The $R[X]$-module $Z_L(M_\phi)$ is an ent_L-pure submodule of M_ϕ, but it is also an L-pure R-submodule of M, as is easy to check. It coincides also with the maximal $R[X]$-submodule $N \subseteq M_\phi$ such that $\mathrm{ent}_L(\phi\restriction_N) = 0$.

The next proposition is the key point in the proof of the Uniqueness Theorem and it gives a characterization of the $R[X]$-modules having finite L-entropy.

Proposition 3.1. *Let $M_\phi \in \mathrm{lFin}_L[X]$. Then $\mathrm{ent}_L(\phi) < \infty$ if and only if there exist a finite chain of L-pure ϕ-invariant submodules of M,*

$$Z_L(M_\phi) = N_0 \subset N_1 \subset \cdots \subset N_n = M,$$

and a sequence of L-minimal elements $x_1 + N_0, \ldots, x_n + N_{n-1}$ ($x_k \in N_k$ for $1 \leq k \leq n$) such that

$$\mathrm{ent}_L(\phi) = \sum_{k=1}^{n} L(x_k + N_{k-1}).$$

Proof. The sufficiency is trivial. For the necessity, we construct the N_k inductively. To this end, let $N_0 = Z_L(M_\phi)$. For every $k \in \mathbb{N}_+$, let x_k be an element such that $x_k \notin N_{k-1}$ and $x_k + N_{k-1}$ is L-minimal (if it exists; otherwise set $N_k = N_{k-1}$). Set $A_k = N_{k-1} + T(\phi, x_k)$ and let N_k be the ent_L-purification of $(A_k)_\phi$ in M_ϕ, that is, $N_k/A_k = Z_L((M/A_k)_\phi)$. Let us denote for every $k \in \mathbb{N}_+$ by $\phi_k : M/N_{k-1} \to M/N_{k-1}$ the map induced by ϕ. We have to check two things:

(i) there exists $n \in \mathbb{N}$ such that $N_n = M$;

(ii) $\mathrm{ent}_L(\phi) = \sum_{k=1}^n L(x_k + N_{k-1})$.

(i) Assume, looking for a contradiction, that $M/N_k \neq 0$ for any $k \in \mathbb{N}$. Choose, for each $k \in \mathbb{N}$, an L-minimal element $x'_k + N_k$ in M/N_k; as $Z_L((M/N_k)_{\phi_k}) = 0$, it follows that $\mathrm{ent}_L(\phi_k \restriction_{T(\phi_k, x'_k + N_k)}) > 0$. Since L is discrete, we can use the Addition Theorem to get $\mathrm{ent}_L(\phi) = \infty$, and this is a contradiction.

(ii) By the Addition Theorem, $\mathrm{ent}_L(\phi) = \sum_{k=1}^n \mathrm{ent}_L(\phi_k \restriction_{N_k/N_{k-1}})$. Since N_k is the ent_L-purification of A_k in M, to evaluate the L-entropy of $\phi_k \restriction_{N_k/N_{k-1}}$ we can consider the restriction of ϕ_k to $A_k/N_{k-1} = T(\phi_k, x_k + N_{k-1})$; but this is the ϕ_k-trajectory of the L-minimal element $x_k + N_{k-1}$. So the conclusion follows from Proposition 2.24. □

The second preparatory result in order to prove the Uniqueness Theorem is the following proposition, which proves the Uniqueness Theorem when the value of the L-entropy is 0.

Proposition 3.2. *Let* $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *be a discrete length function and* $h_L : \mathrm{lFin}_L[X] \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *be a length function satisfying part* (i) *in the statement of the Uniqueness Theorem, namely,* $h_L(M_\phi) = 0$ *for every* $M \in \mathrm{Fin}_L$ *and* $\phi \in \mathrm{End}_R(M)$. *Then for any* $M_\phi \in \mathrm{lFin}_L[X]$, $\mathrm{ent}_L(\phi) = 0$ *implies* $h_L(M_\phi) = 0$.

Proof. Let $N_0 = 0$; for any ordinal σ define $N_{\sigma+1} = N_\sigma + T(\phi, x_{\sigma+1})$ where $x_{\sigma+1} + N_\sigma \in M/N_\sigma$ is a non-zero element (if such an element exists). For a limit ordinal τ define $N_\tau = \bigcup_{\sigma<\tau} N_\sigma$. Note that all the submodules N_σ of M are actually $R[X]$-submodules of M_ϕ. There exists an ordinal λ such that

$$0 = (N_0)_\phi \subset (N_1)_\phi \subset \cdots \subset (N_\sigma)_\phi \subset \cdots \subset (N_\lambda)_\phi = M_\phi.$$

Then $h_L(M_\phi) = \sup_\sigma h_L((N_\sigma)_\phi)$ since $h_L(M_\phi) = \sup_{F \in \mathcal{F}(M_\phi)} h_L(F)$ and every $F \in \mathcal{F}(M_\phi)$ is contained in N_σ for some σ. As noted at the beginning of the section, $L(T(\phi, x)) < \infty$ for every $x \in M$. This shows that $L(N_{\sigma+1}/N_\sigma) < \infty$ for every σ. By hypothesis (i) in the statement of the Uniqueness Theorem, we have that $h_L((N_{\sigma+1}/N_\sigma)_\phi) = 0$. Thanks to the additivity and the upper continuity of h_L, this gives $h_L(M_\phi) = 0$ as desired. □

Proof of the Uniqueness Theorem. First we deal with the case when $\text{ent}_L(\phi) < \infty$. If $\text{ent}_L(\phi) = 0$, just apply Proposition 3.2. If $\text{ent}_L(\phi) > 0$, we can apply Proposition 3.1 to construct a finite chain of submodules of M_ϕ:

$$Z_L(M_\phi) = (N_0)_\phi \subset \cdots \subset (N_n)_\phi = M_\phi$$

where $\phi_k : N_k/N_{k-1} \to N_k/N_{k-1}$ $(k = 1, \ldots, n)$ acts as a Bernoulli shift. We can now finish the proof using additivity and the fact that both the L-entropy and h_L take the same values on the Bernoulli shifts by hypothesis (i) and Example 2.14. Finally, suppose that $\text{ent}_L(\phi) = \infty$. Let $F \subseteq M$ be a finitely generated submodule; then we obtain that $\text{ent}_L(\phi\restriction_F) = \text{ent}_L(\phi, F) < \infty$, by Fact (ii) in Section 2.2 and Proposition 2.12. So, using the first part of the proof, we get $h_L(T(\phi, F)) = \text{ent}_L(\phi\restriction_{T(\phi,F)})$. Hence we have that $\infty = \text{ent}_L(\phi) = \sup_{F\in\mathcal{F}(M)} \text{ent}_L(\phi, F) = \sup_{F\in\mathcal{F}(M)} h_L(T(\phi, F)) \le h_L(M)$. □

4 Two applications

The rings R considered in this section are always assumed to be Noetherian.

In order to introduce the first application, we explain the connection between multiplicity and L-entropy. Multiplicity was firstly defined in terms of the 'classical' length function. For further information about this multiplicity see [13, Chapter 7]. The ideas of [13] on multiplicities were extended by the second named author in his PhD Thesis [19] to obtain a multiplicity function from an arbitrary length function $L_X : \text{Mod}(R[X]) \to \mathbb{R}_{\ge 0} \cup \{\infty\}$ (actually, he considered an arbitrary ring R' with a distinguished central element γ instead of $R[X]$ with the indeterminate X commuting with R; R' was not necessarily Noetherian, but L_X was 'continuous on Noetherian modules', a stronger notion than upper continuity and equivalent to it when R' is Noetherian – see [19] for details).

We now give this definition following Vámos [19].

Definition 4.1. Let $L_X : \text{Mod}(R[X]) \to \mathbb{R}_{\ge 0} \cup \{\infty\}$ be a length function and N be a finitely generated $R[X]$-module. If $L_X(N/N \cdot X) < \infty$, set

$$e[X]L_X(N) = L_X(N/N \cdot X) - L_X(\text{Ann}_N(X));$$

otherwise set $e[X]L_X(N) = \infty$. For any $M \in \text{Mod}(R[X])$ define

$$e[X]L_X(M) = \sup e[X]L_X(N)$$

where N ranges over all the finitely generated $R[X]$-submodules of M, and call it the *multiplicity of M associated with L_X*.

We recall the following known results on multiplicity:

(i) [19, Chapter 5, Corollary 1] If $L_X(N/N\cdot X)<\infty$, the quantity $e[X]L_X(N)$ is non-negative;

(ii) [19, Chapter 5, Proposition 3] $e[X]L_X$ is additive.

As $e[X]L_X$ is upper continuous by definition, we can say that $e[X]L_X$ is a length function on $\mathrm{Mod}(R[X])$.

If we start now with a length function $L:\mathrm{Mod}(R)\to\mathbb{R}_{\geq 0}\cup\{\infty\}$, we can define a length function $\bar{L}_X$ on $\mathrm{Mod}(R[X])$ by 'forgetting' the action of X. To this end, we set

$$\bar{L}_X:\mathrm{Mod}(R[X])\to\mathbb{R}_{\geq 0}\cup\{\infty\},\quad \bar{L}_X(M_\phi)=L(M)$$

for every $M_\phi\in\mathrm{Mod}(R[X])$. It is obvious that $\bar{L}_X$ is additive, and its upper continuity depends on the fact that a finitely generated $R[X]$-submodule of M_ϕ is the ϕ-trajectory of a finitely generated R-submodule of M.

With this notation we will prove the following result, that was conjectured by the second named author. Notice that the restriction to the class $\mathrm{lFin}_L[X]$ is necessary since, as we said, ent_L is not even a length function outside of that class.

Multiplicity Theorem. *Let R be a Noetherian ring and $L:\mathrm{Mod}(R)\to\mathbb{R}_{\geq 0}\cup\{\infty\}$ be a discrete length function. Then*

$$\mathrm{ent}_L(\phi)=e[X]\bar{L}_X(M_\phi)\quad \textit{for every}\ M_\phi\in\mathrm{lFin}_L[X].$$

Before proving the Multiplicity Theorem, we need two preparatory lemmata.

Lemma 4.2. *Let $M_\phi\in\mathrm{Mod}(R[X])$ and $\bar{L}_X(M_\phi)=L(M)<\infty$. Then*

$$e[X]\bar{L}_X(M_\phi)=0.$$

Proof. Consider, for any finitely generated $R[X]$-submodule N of M_ϕ, the following exact sequences

$$0\to\mathrm{Ann}_N(X)\to N\to N\cdot X\to 0,\quad 0\to N\cdot X\to N\to N/(N\cdot X)\to 0.$$

Using additivity we get that $\bar{L}_X(\mathrm{Ann}_N(X))=\bar{L}_X(N/(N\cdot X))$. This shows that $e[X]\bar{L}_X(M_\phi)=0$. □

It only remains to evaluate the multiplicity when X acts as a Bernoulli shift.

Lemma 4.3. *Let M be a locally L-finite R-module such that $z_L(M)=0$. Further, let $\phi:M\to M$ be an R-endomorphism, $x\in M$ be an L-minimal element, and $T=T(\phi,x)$. If $L(T)=\infty$, then $e[X]\bar{L}_X(T_\phi)=L(x)$.*

Proof. Since T_ϕ is a finitely generated $R[X]$-module, we have that $e[X]\bar{L}_X(T_\phi)=L(T/T\cdot X)-L(\mathrm{Ann}_{T_\phi}(X))$. But $L(T)=\infty$ implies that $\phi|_T$ is injective, by Proposition 2.24. Thus $\mathrm{Ann}_{T_\phi}(X) = 0$ and consequently $e[X]\bar{L}_X(T_\phi) = L(x)$ since, again by Proposition 2.24, $T/T \cdot X = T/\phi(T) \cong xR$. □

We can now give the

Proof of the Multiplicity Theorem. It is a direct consequence of the Uniqueness Theorem that states that all the length functions on $\mathrm{lFin}_L[X]$ satisfying (i) and (ii) (in its statement) coincide with ent_L. Looking at the proof of the Uniqueness Theorem, we can see that condition (ii) can be weakened in the following form:

(ii') $h_L(T(\phi,x)_\phi) = L(x)$ whenever x is an L-minimal element in an R-module M, $\phi : M \to M$ is an endomorphism and $L(T(\phi,x)) = \infty$.

Now $e[X]\bar{L}_X$ satisfies (i) in view of Lemma 4.2 and (ii') in view of Lemma 4.3. This concludes the proof. □

We now pass to the second application. In [5], the first named author together with Dikranjan and Giordano Bruno defined the adjoint entropy $\mathrm{ent}^\star$, that is an invariant of the endomorphisms of Abelian groups dual, under the action of the Pontryagin duality, to the algebraic entropy ent with respect to the length function $\log|-|$. One of the main results about $\mathrm{ent}^\star$ is the proof of a dichotomy, namely $\mathrm{ent}^\star$ takes values in $\{0,\infty\}$. The proof of the dichotomy is based on the fact that the algebraic entropy of the right shift on the direct product $\mathbb{Z}(p)^{\mathbb{N}}$ is infinite. This result was proved, with some pages of technical combinatorial computations, by Giordano Bruno in [8]. We give here an alternative proof of that result in the more general setting of an arbitrary length function on the category of modules over a commutative Noetherian ring (recall that these length functions have been characterized in [18]). Recall that the *right Bernoulli shift* $\widehat{\beta} : M^{\mathbb{N}} \to M^{\mathbb{N}}$ of the direct product of $\mathbb{N}$ copies of a given module M is the endomorphism defined by $\widehat{\beta}(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots)$. So the right Bernoulli shift on the direct sum $M^{(\mathbb{N})}$ considered in Example 2.14 is just the restriction of $\widehat{\beta}$.

Theorem 4.4. *Let R be a commutative Noetherian ring and let $L : \mathrm{Mod}(R) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be a discrete length function. Let M be a locally L-finite R-module such that $L(M) > 0$. If $\widehat{\beta} : M^{\mathbb{N}} \to M^{\mathbb{N}}$ denotes the right Bernoulli shift, then* $\mathrm{ent}_L(\widehat{\beta}) = \infty$.

Proof. Without loss of generality, we can assume M to be finitely generated and such that $z_L(M) = 0$. By a well-known result on modules over Noetherian rings (see Theorem 1 of Section 1 in [2]), there exists a chain of submodules

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = M$$

such that $A_k/A_{k-1} \cong R/P_k$ for some prime ideal P_k, with $k \in \{1, \dots, n\}$. Since we are supposing $z_L(M) = 0$, we get that $L(A_1) = L(R/P_1) > 0$. Let us set $P = P_1$; it is enough to prove now that the right shift $\widehat{\beta}$ on $(R/P)^{\mathbb{N}}$ has infinite L-entropy since $(R/P)^{\mathbb{N}}$ is isomorphic to a $\widehat{\beta}$-invariant submodule of $M^{\mathbb{N}}$ and because of Fact (v) in Section 2.2. The canonical isomorphism of R-modules

$$((R/P)^{\mathbb{N}})_{\widehat{\beta}} \cong (R/P)[[X]]$$

is also an isomorphism of modules over the integral domain $S = (R/P)[X]$. Now L induces a discrete length function $\bar{L} : \mathrm{Mod}(R/P) \to \mathbb{R}_{\geq 0} \cup \infty$. But $\mathrm{lFin}_{\bar{L}} = \mathrm{Mod}(R/P)$ because $\bar{L}(R/P) < \infty$; therefore $\mathrm{ent}_{\bar{L}} : \mathrm{Mod}(S) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a discrete length function and, in view of [14, Theorem 2], $\mathrm{ent}_{\bar{L}} = \mathrm{ent}_{\bar{L}}(S) \cdot \mathrm{rk}_S$ where $\mathrm{ent}_{\bar{L}}(S) = \bar{L}(R/P)$ (see Example 2.14). Therefore

$$\mathrm{ent}_{\bar{L}}(\widehat{\beta}) = \bar{L}(R/P) \cdot \mathrm{rk}_S((R/P)[[X]]).$$

It is well known and easily seen that $\mathrm{rk}_S((R/P)[[X]])$ is infinite. The conclusion now follows from the remark that on (R/P)-modules ent_L and $\mathrm{ent}_{\bar{L}}$ coincide. □

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