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AN INVERSE IMAGE THEOREM FOR SHEAVES WITH APPLICATIONS TO THE CAUCHY PROBLEM

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CONTENTS

0. Introduction. There is a wide literature on the so-called Cauchy problem. Let us recall in particular the following papers.

(i) In 1976, Hamada, Leray, and Wagschal solved the initial value problem for a linear partial differential equation when the data are ramified along the characteristic hypersurfaces. Their proof of this result relies essentially on the precised Cauchy-Kowalevski theorem of Leray. (See [HLW].)

(ii) In 1978, Kashiwara and Schapira proposed a new proof and an extension of the previous work to general systems, when the data are of logarithmic type. This time microdifferential operators and complex contact transformations were involved. (See [KS1].)

(iii) In 1988, Schiltz showed how the holomorphic solution for the Cauchy problem can be expressed as a sum of functions which are holomorphic in domains whose boundary is given by the real characteristic hypersurfaces issued from the boundary of a strictly pseudoconvex domain where the data are defined. (See [Sc].)

The aim of this paper is to propose a new approach to the Cauchy problem based on sheaf theory, or more precisely, on its microlocal version. By this method we shall, in particular, recover the above results and even extend (i) and (iii) to general systems of partial differential equations (i.e., to $\mathscr{D}$-modules).

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Let us describe our work with some details.

Throughout this paper we shall systematically use the language of derived categories and sheaves. (E.g., see [KS4, Ch. I, II].)

Let $f\colon Y \to X$ be a morphism of complex analytic manifolds. Let $\mathcal{M}$ be a left coherent module over the ring $\mathcal{D}_X$ of holomorphic differential operators and assume f is noncharacteristic for $\mathcal{M}$. Our starting point is the Cauchy-Kowalevski theorem as stated by Kashiwara (see [K1], [K2, Th. 2.5.16]):

$$(0.1)\qquad \begin{cases} \text{the natural morphism:} \\ f^{-1}\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \to \mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \\ \text{is an isomorphism.} \end{cases}$$

Here, $\mathcal{M}_Y$ denotes the induced system on Y, and $\mathcal{O}_X$ is the sheaf of holomorphic functions on X.

Let K be an object of $\mathrm{D}^b(X)$, the derived category of sheaves on X. By an appropriate choice of K to be explained in §3, the complex $\mathrm{R}\mathcal{H}om(K, \mathcal{O}_X)$ is a complex of $\mathcal{D}_X$-modules representing a sheaf of "ramified" holomorphic functions.

Next, let L be an object of $\mathrm{D}^b(Y)$, let $\psi\colon L \to f^{-1}K$ be a morphism, and set $F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Due to (0.1), theorems like those given in (i), (ii), or (iii), may be stated as follows.

$$(0.2)\qquad \begin{cases} \text{the morphism induced by } \psi\text{:} \\ f^{-1}\mathrm{R}\mathcal{H}om(K, F)|_Z \to \mathrm{R}\mathcal{H}om(L, f^{-1}F)|_Z \\ \text{is an isomorphism on some subset } Z \text{ of } Y. \end{cases}$$

Hence, we are reduced to the following sheaf theoretical problem.

PROBLEM 0.1. *Give conditions on f, F, K, L, and ψ as above so that* (0.2) *holds.*

Our answer (Theorem 2.1.1 below) relies on the microlocal theory of sheaves as developed in [KS4]. We shall apply it to the Cauchy problem using (0.1) and the inclusion (see [KS4, Th. 11.3.3]):

$$(0.3)\qquad \mathrm{SS}(\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \subset \mathrm{char}(\mathcal{M})$$

where $\mathrm{SS}(F)$ denotes the microsupport of the object F of $\mathrm{D}^b(X)$ (see [KS4, Def. 5.1.2]) and $\mathrm{char}(\mathcal{M})$ denotes the characteristic variety of $\mathcal{M}$.

We now summarize the plan of this paper.

In §1 we make a short review on the theory of sheaves, mainly to fix the notations. In particular we recall the notion of microsupport, the functor μhom, the category

$D^b(X; p_X)$ that is the localization of the derived category of sheaves on X at $p_X \in T^*X$.

In §2 we state and prove our main theorem, namely Theorem 2.1.1, on the well-posedness for the Cauchy problem, in a sheaf theoretical frame.

In §3, as an application of Theorem 2.1.1, we show how to recover the results obtained in the papers cited at the beginning of this introduction.

Let us put the emphasis on the fact that all of our proofs rely only on the following tools: the Cauchy-Kowalevski theorem in the precised version by Leray [L] from which the inclusion (0.3) is deduced, the isomorphism (0.1) which is deduced by Kashiwara [K] from the classical Cauchy-Kowalevski theorem by purely algebraic tools, and, of course, the techniques of [KS3], [KS4]. We never use pseudo-differential operators and quantized contact transformations nor any estimates.

Of course, our purely sheaf theoretical methods do not allow us to treat holomorphic functions with growth conditions (e.g., meromorphic solutions). For these questions we refer to [H], [MF], [La].

The results of this paper were announced in our note [D'A-S].

Let us mention that further developments of Theorem 2.1.1 allow us, in particular, to recover a result of [KS2] on the hyperbolic Cauchy problem in the frame of hyperfunctions. (See [D'A].)

1. Review on sheaves. For what follows refer to [KS4].

Until §3 all manifolds and morphisms of manifolds will be real and of class C^∞.

1.1. *Geometry.* Let X be a manifold, τ_X: $TX \to X$ its tangent bundle, and π_X: $T^*X \to X$ its cotangent bundle. We will identify X with T^*_XX, the zero section of T^*X. Set $\dot{T}^*X = T^*X \setminus T^*_XX$ and denote by $\dot{\pi}_X$ the projection $\dot{T}^*X \to X$.

Let M be a closed submanifold of X. One denotes by T^*_MX the conormal bundle to M in X. If A is a subset of X, one denotes by $C_M(A)$ the normal cone of A along M. This is a closed conic subset of T_MX, the normal bundle to M in X. More generally, if B is another subset of X, one denotes by $C(A, B)$ the normal cone of A along B, a closed conic subset of TX. (See [KS4, Def. 4.1.1].)

If γ is a conic subset of TX, one notes $\gamma^a = -\gamma$. One says that γ is proper if its fibers contain no lines. One denotes by γ° the polar cone to γ, a convex conic subset of T^*X. Recall that its fibers are given by $\gamma^\circ_x = \{\theta \in T^*_xX; \langle v, \theta\rangle \geqslant 0\ \forall v \in \gamma_x\}$.

Let f: $Y \to X$ be a morphism of manifolds. One denotes by ${}^tf'$ and f_π the natural mappings associated to f:

$$T^*Y \xleftarrow{{}^tf'} Y \times_X T^*X \xrightarrow{f_\pi} T^*X.$$

One sets

$$T^*_YX = {}^tf'^{-1}(T^*_YY).$$

Let N (resp. M) be a closed submanifold of Y (resp. X) with $f(N) \subset M$. One denotes by ${}^t f'_N$ and $f_{N\pi}$ the natural mappings associated to f:

$$T_N^* Y \xleftarrow{{}^t f'_N} N \times_M T_M^* X \xrightarrow{f_{N\pi}} T_M^* X.$$

Let A be a closed conic subset of T^*X. One says that f is *noncharacteristic* for A if $T_Y^*X \cap f_\pi^{-1}(A) \subset Y \times_X T_X^*X$. Let V be a subset of T^*Y. We refer to [KS4, Def. 6.2.7] for the intrinsic definition of f being noncharacteristic for A on V. Let (x) (resp. (y)) be a system of local coordinates on X (resp. Y) and let $(x; \xi)$ (resp. $(y; \eta)$) be the associated coordinates on T^*X (resp. T^*Y). We recall that f is noncharacteristic for A on V if and only if for every $(y; \eta) \in V$ there are no sequences $\{(y_n, (x_n; \xi_n))\}$ in $Y \times A$ such that

$$\begin{cases} y_n \xrightarrow[n]{} y, \\ x_n \xrightarrow[n]{} f(y), \\ {}^t f'(y_n) \cdot \xi_n \xrightarrow[n]{} \eta, \\ |x_n - f(y_n)| \, |\xi_n| \xrightarrow[n]{} 0, \\ |\xi_n| \xrightarrow[n]{} \infty. \end{cases}$$

1.2. *The category* $\mathrm{D}^b(X)$. We fix a commutative ring A with finite global dimension (e.g., $A = \mathbb{Z}$).

Let X be a manifold. One denotes by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of A-modules on X.

To an object F of $\mathrm{D}^b(X)$ one associates the *microsupport* $\mathrm{SS}(F)$ of F, a closed conic involutive subset of T^*X. (See [KS4].)

Let M be a closed submanifold of X. One associates to F the Sato's microlocalization of F along M, denoted by $\mu_M(F)$. (See [KS4, Def. 4.3.1].) This is an object of $\mathrm{D}^b(T_M^*X)$. More generally, if G is another object of $\mathrm{D}^b(X)$, the *microlocalization of* F *along* G, denoted by $\mu\mathrm{hom}(G, F)$, is defined in [KS4] as

$$\mu\mathrm{hom}(G, F) = \mu_\Delta \mathrm{R}\mathcal{H}om(q_2^{-1} G, q_1^! F)$$

where Δ is the diagonal of $X \times X$ and where q_1, q_2 denote the projections from $X \times X$ to X. By identifying $T_\Delta^*(X \times X)$ to T^*X by the first projection $T^*X \times T^*X \to T^*X$, one has that $\mu\mathrm{hom}(G, F)$ belongs to $\mathrm{D}^b(T^*X)$. Concerning this functor, one proves that

$$\mathrm{R}\pi_{X*}\mu\mathrm{hom}(G, F) \cong \mathrm{R}\mathcal{H}om(G, F), \tag{1.2.1}$$

$$\mu\mathrm{hom}(A_M, F) \cong \mu_M(F), \tag{1.2.2}$$

$$\mathrm{supp}(\mu\mathrm{hom}(G, F)) \subset \mathrm{SS}(G) \cap \mathrm{SS}(F). \tag{1.2.3}$$

Here, for Z a locally closed subset of X, one denotes by A_Z the sheaf which is 0 on $X\setminus Z$ and is the constant sheaf with stalk A on Z.

Let $f\colon Y \to X$ be a morphism of manifolds. One defines the *relative dualizing complex* $\omega_{Y/X}$ by $\omega_{Y/X} = f^! A_X$. One sets $\omega_X = a_X^! A$, where $a_X\colon X \to \{pt\}$. Recall that one has an isomorphism $\omega_{Y/X} \otimes f^{-1}\omega_X \cong \omega_Y$. Recall, moreover, that $\omega_X \cong or_X[\dim X]$ where or_X is the orientation sheaf and $\dim X$ is the dimension of X. If there is no risk of confusion, when working on T^*X we shall write ω_X, or_X, $\omega_{X/Z}$, etc. instead of $\pi_X^{-1}\omega_X$, $\pi_X^{-1} or_X$, $\pi_X^{-1}\omega_{X/Z}$, etc.

Let F be an object of $\mathrm{D}^b(X)$ and let V be a subset of T^*Y. One says that f is noncharacteristic for F on V if f is noncharacteristic for $\mathrm{SS}(F)$ on V.

1.3. *The category* $\mathrm{D}^b(X; p_X)$. Let X be a manifold and let Ω be a subset of T^*X. One denotes by $\mathrm{D}^b(X;\Omega)$ the localized category $\mathrm{D}^b(X)/\mathrm{D}^b_\Omega(X)$, where $\mathrm{D}^b_\Omega(X)$ is the null system: $\mathrm{D}^b_\Omega(X) = \{F \in \mathrm{Ob}(\mathrm{D}^b(X));\ \mathrm{SS}(F)\cap\Omega = \varnothing\}$. Recall that the objects of $\mathrm{D}^b(X;\Omega)$ are the same as those of $\mathrm{D}^b(X)$ and that a morphism $u\colon F \to G$ in $\mathrm{D}^b(X)$ becomes an isomorphism in $\mathrm{D}^b(X;\Omega)$ if $\Omega \cap \mathrm{SS}(H) = \varnothing$, H being the third term of a distinguished triangle: $F \xrightarrow{u} G \to H \xrightarrow{+1}$. Such a u is called an isomorphism on Ω. If $p_X \in T^*X$, one writes $\mathrm{D}^b(X; p_X)$ instead of $\mathrm{D}^b(X; \{p_X\})$.

Let $f\colon Y \to X$ be a morphism of manifolds. Take a point $p \in Y \times_X T^*X$ and set $p_X = f_\pi(p)$, $p_Y = {}^t f'(p)$. The functors $\mathrm{R}f_*$, $\mathrm{R}f_!$ (resp. f^{-1}, $f^!$) are not microlocal, i.e., are not well defined as functors from $\mathrm{D}^b(Y; p_Y)$ (resp. $\mathrm{D}^b(X; p_X)$) to $\mathrm{D}^b(X; p_X)$ (resp. $\mathrm{D}^b(Y; p_Y)$). To give a microlocal meaning to these functors one may enlarge the category $\mathrm{D}^b(X; p_X)$ and work with ind-objects and pro-objects. This is what has been done in [KS4, Ch. 6], but there is another way to attack this problem. In this section we will give the definition of microlocal images for complexes with some prescribed conditions on the microsupport, and to this end we need the following result on the cutting of the microsupport.

Let X be a vector space and let $x_0 \in X$. In what follows we will often identify X with $T_{x_0}X$.

Let γ be a (not necessarily proper) closed convex cone of $T_{x_0}X$. Let ω be an open neighborhood of x_0 in X with smooth boundary. We shall denote by q_1 and q_2 the projections from $X \times X$ to X and by s the map $s(x_1, x_2) = x_1 - x_2$. The following definition is a slight modification of that of [KS4, Prop. 6.1.4, 6.1.8].

Definition 1.3.1. Let γ and ω be as above and let F be an object of $\mathrm{D}^b(X)$. We set

$$\begin{cases}\Phi_X(\gamma, \omega; F) = \mathrm{R}q_{2*}(s^{-1}A_\gamma \overset{\mathrm{L}}{\otimes} q_1^{-1}F_\omega),\\ \Psi_X(\gamma, \omega; F) = \mathrm{R}q_{2!}\mathrm{R}\Gamma_{s^{-1}(\gamma^a)}(q_1^! \mathrm{R}\Gamma_\omega(F)).\end{cases}$$

Notice that for $\gamma' \subset \gamma$, $\omega' \supset \omega$, one has natural morphisms in $\mathrm{D}^b(X)$

$$\begin{cases}\Phi_X(\gamma, \omega; F) \to \Phi_X(\gamma', \omega'; F),\\ \Psi_X(\gamma', \omega'; F) \to \Psi_X(\gamma, \omega; F).\end{cases} \tag{1.3.1}$$

In particular, recalling the isomorphisms $\mathrm{R}q_{2*}(s^{-1}A_{\{0\}} \overset{\mathrm{L}}{\otimes} q_1^{-1}F) \cong F$, $F \cong$

$Rq_{2!}R\Gamma_{s^{-1}(0)}(q_1^! F)$, we get natural morphisms

$$\begin{cases} \Phi_X(\gamma, \omega; F) \to F, \\ F \to \Psi_X(\gamma, \omega; F). \end{cases} \tag{1.3.2}$$

After [KS4, Prop. 6.1.4] we give the following definition. (Here, $x_0 = 0$.)

Definition 1.3.2. Take $\xi_0 \in \dot{T}_0^* X$, $\omega \subset X$, such that
(i) γ is a closed proper convex cone,
(ii) $\partial\gamma \setminus \{0\}$ is C^1,
(iii) $\xi_0 \in \operatorname{Int} \gamma^{0a}$,
(iv) ω is an open neighborhood of 0,
(v) $\partial\omega$ is C^1,
(vi) $\omega \subset \{x; |x| < \varepsilon\}$ for some $\varepsilon > 0$, and
(vii) $\partial\omega$ and $\partial\gamma$ are tangent at their intersection.
We will call a pair (γ, ω) satisfying (i)–(vii) a *refined cutting pair* on X at $(0; \xi_0)$.

Remark that, if $g(x) < 0$ (resp. $h(x) < 0$) is a local equation for ω (resp. γ) at $x \in \partial\omega \cap \partial\gamma$, condition (vii) means that $-\mathrm{d}g(x) \in \mathbb{R}^+ \mathrm{d}h(x)$.
One has the following sharp result.

PROPOSITION 1.3.3. (See [KS4, Prop. 6.1.4, 6.1.8].) *Let K be a proper closed convex cone of $T_0^* X$ and let $U \subset K$ be an open cone. Let $F \in \mathrm{Ob}(\mathrm{D}^b(X))$ and let W be a conic neighborhood of $K \cap (\mathrm{SS}(F) \setminus \{0\})$. Then*
(a) (refined microlocal cutoff lemma) *there exists $F' \in \mathrm{Ob}(\mathrm{D}^b(X))$ and a morphism u: $F' \to F$ satisfying*
(1) *u is an isomorphism on U and*
(2) $\pi_X^{-1}(0) \cap \mathrm{SS}(F') \subset W \cup \{0\}$,
(b) (dual refined microlocal cutoff lemma) *the same as* (a) *with u: $F \to F'$.*

Sketch of the proof. Keep the same notations as in Definition 1.3.1, 1.3.2. It is not restrictive to assume $\overline{U} \subset \{0\} \cup \operatorname{Int} K$. Take $\xi_0 \in U$ and choose a refined cutting pair (γ, ω) on X at $(0; \xi_0)$ with $K^{0a} \subset \gamma \subset U^{0a}$. Then it is possible to show that $\Phi_X(\gamma, \omega; F)$ (resp. $\Psi_X(\gamma, \omega; F)$) satisfies the requirements. Q.E.D.

We can now define microlocal inverse images.
Let $F \in \mathrm{Ob}(\mathrm{D}^b(X; p_X))$ and consider the condition

$$ {}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset \{p\} \text{ in a neighborhood of } p. \tag{1.3.3}$$

LEMMA 1.3.4. (See [KS4, Prop. 6.1.9].) *Let $F \in \mathrm{Ob}(\mathrm{D}^b(X; p_X))$ satisfy* (1.3.3); *then*
(a) *there exist $F' \in \mathrm{Ob}(\mathrm{D}^b(X))$ with*

$$ {}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F')) \subset \{p\}, \ f \text{ is noncharacteristic for } F' \tag{1.3.4}$$

and a morphism $F' \to F$ (resp. $F \to F'$) in $\mathrm{D}^b(X)$ which is an isomorphism at p_X,

(b) *for* $F \in \mathrm{Ob}(\mathrm{D}^b(X; p_X))$ *satisfying* (1.3.3), *the object* $f^{-1}F'$ (*resp.* $f^!F'$) *of* $\mathrm{D}^b(Y; p_Y)$ *does not depend* (*up to isomorphism*) *on the choice of* F'.

Definition 1.3.5. Let $F \in \mathrm{Ob}(\mathrm{D}^b(X; p_X))$ satisfy (1.3.3). We define the microlocal inverse image (resp. extraordinary inverse image) of F by $f_{\mu,p}^{-1}F := f^{-1}F'$ (resp. $f_{\mu,p}^!F := f^!F'$), where F' is the complex constructed in Lemma 1.3.4.

$f_{\mu,p}^{-1}$ (resp. $f_{\mu,p}^!$) is a functor from the full subcategory of $\mathrm{D}^b(X; p_X)$ whose objects verify (1.3.3) to $\mathrm{D}^b(Y; p_Y)$.

Note that the definition of $f_{\mu,p}^{-1}F$, $f_{\mu,p}^!F$ is coherent with the definition given in [KS4, §6.1].

Proof of Lemma 1.3.4. Since the proofs for f^{-1} and $f^!$ are similar, we will treat only the statement concerning f^{-1}. If $p_X \in T_X^*X$, then f is noncharacteristic for F, and we may take $F' = F$. Assume then that $p_X \in \dot{T}^*X$. Following the proof of Proposition 1.3.3, we may choose a refined cutting pair (γ, ω) on X at p_X so that $F' = \Phi_X(\gamma, \omega; F)$ satisfies (1.3.4). This proves (a). As for (b), we must show that, if $F' \cong F''$ at p_X, F'' satisfying (1.3.4), then $f^{-1}F' \cong f^{-1}F''$ at p_Y. It is not restrictive to assume the isomorphism $F' \cong F''$ being induced by a morphism $u\colon F' \to F''$ in $\mathrm{D}^b(X)$. Embedding u in a distinguished triangle $F' \xrightarrow{u} F'' \to F_0 \xrightarrow{+1}$, we see that f is noncharacteristic for F_0 and that ${}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F_0)) = \emptyset$. Then $f^{-1}F' \cong f^{-1}F''$ at p_Y. Q.E.D.

2. An inverse image theorem for sheaves

2.1. *The main theorem.* In this paragraph we will give an answer to Problem 0.1.

Let X be a real analytic manifold. We say that $K \in \mathrm{Ob}(\mathrm{D}^b(X))$ is *weakly cohomologically constructible* (w.c.c. for short) if

(i) for any $x \in X$, $\text{“}\varinjlim_{U \ni x}\text{”}\, \mathrm{R}\Gamma(U; K)$ is represented by K_x, and

(ii) for any $x \in X$, $\text{“}\varprojlim_{U \ni x}\text{”}\, \mathrm{R}\Gamma_c(U; K)$ is represented by $\mathrm{R}\Gamma_{\{x\}}K$.

Here, U ranges over an open neighborhood system of x, and we make use of the notion of “$\varinjlim$” and “$\varprojlim$”. (E.g., see [KS4, §1.11].)

Note that weakly $\mathbb{R}$-constructible complexes on a real analytic manifold are w.c.c. (See [KS4, §8.4].)

Let $f\colon Y \to X$ be a morphism of manifolds. Let Z be a subset of Y (e.g., $Z = \{y\}$ for $y \in Y$).

THEOREM 2.1.1. *Let F and K be objects of* $\mathrm{D}^b(X)$ *and let L be an object of* $\mathrm{D}^b(Y)$. *Assume to be given a morphism $\psi\colon L \to f^{-1}K$. Let V be an open neighborhood of* $\dot{\pi}_Y^{-1}(Z)$. *Assume that*

(i) *f is noncharacteristic for F on V and*

(ii) *f_π is noncharacteristic for* $C(\mathrm{SS}(F), \mathrm{SS}(K))$ *on* ${}^t f'^{-1}(V)$.

Assume that for every $p_Y \in \dot{\pi}_Y^{-1}(Z)$ *there exist* $p_1, \ldots, p_r$ *in* ${}^t f'^{-1}(p_Y)$ *with*

(iii) ${}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset \{p_1, \ldots, p_r\}$,

(iv) f *noncharacteristic for* K *on* V *and* $p_1, \ldots, p_r$ *isolated in* ${}^t f'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(K))$, *and*

(v) *the morphism induced by* ψ, $L \to f_{\mu,p_j}^{-1} K$, *an isomorphism in* $\mathrm{D}^b(Y; p_Y)$ *for* $j = 1, \ldots, r$.

Finally, assume

(vi) K *and* L *are w.c.c. and*

(vii) *the morphism induced by* ψ, $\mathrm{R}\Gamma_{\{y\}}(L \otimes \omega_Y) \to \mathrm{R}\Gamma_{\{x\}}(K \otimes \omega_X)$, *is an isomorphism for every* $y \in Z$, $x = f(y)$.

Then the natural morphism induced by ψ:

$$f^{-1}\mathrm{R}\mathscr{H}om(K, F)|_Z \to \mathrm{R}\mathscr{H}om(L, f^{-1}F)|_Z, \tag{2.1.1}$$

is an isomorphism.

Recall that $f_{\mu,p_j}^{-1} K$ in (v) has been defined in Definition 1.3.5.

Before going into the proof, let us explain how the morphism in (vii) is constructed. From ψ and the natural morphism $f^{-1}K \otimes \omega_{Y/X} \to f^! K$, we obtain the arrow $L \otimes \omega_{Y/X} \to f^! K$ and hence the arrow $L \otimes \omega_Y \to f^! K \otimes f^{-1}\omega_X \cong f^!(K \otimes \omega_X)$. Applying the functor $\mathrm{R}\Gamma_{\{y\}}$, we obtain

$$\begin{aligned} \mathrm{R}\Gamma_{\{y\}}(L \otimes \omega_Y) &\to \mathrm{R}\Gamma_{\{y\}} f^!(K \otimes \omega_X) \\ &\cong \mathrm{R}\Gamma_{\{x\}}(K \otimes \omega_X). \end{aligned}$$

Proof. We shall adapt to our situation the proof of [KS1, Theorem 3.3]. Recall that one has a distinguished triangle in $\mathrm{D}^b_{\mathbb{R}^+}(T^*X)$:

$$\mathrm{R}\pi_{Y!}(\cdot) \to \mathrm{R}\pi_{Y*}(\cdot) \to \mathrm{R}\dot{\pi}_{Y*}(\cdot) \xrightarrow{+1}.$$

(Here, $\mathrm{D}^b_{\mathbb{R}^+}(T^*X)$ denotes the full subcategory of $\mathrm{D}^b(X)$ whose objects have locally constant cohomology along the orbits of the action of $\mathbb{R}^+$ on T^*X.) If we apply this triangle to the morphism

$$\mathrm{R}{}^t f'_! f_\pi^{-1} \mu\mathrm{hom}(K, F) \to \mu\mathrm{hom}(L, f^{-1}F),$$

we get the morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathrm{R}\pi_{Y!}A & \longrightarrow & \mathrm{R}\pi_{Y*}A & \longrightarrow & \mathrm{R}\dot{\pi}_{Y*}A & \xrightarrow{+1} & \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{R}\pi_{Y!}B & \longrightarrow & \mathrm{R}\pi_{Y*}B & \longrightarrow & \mathrm{R}\dot{\pi}_{Y*}B & \xrightarrow{+1} & \end{array} \tag{2.1.2}$$

where $A = \mathrm{R}{}^t f'_! f_\pi^{-1} \mu\mathrm{hom}(K, F)$ and $B = \mu\mathrm{hom}(L, f^{-1}F)$.

Recall that $R\pi_{Y*}\mu\mathrm{hom}(L, f^{-1}F) \cong R\mathcal{H}om(L, f^{-1}F)$. On the other hand, consider the diagram

$$\begin{array}{ccccc} T^*Y & \xleftarrow{{}^tf'} & Y \times_X T^*X & \xrightarrow{f_\pi} & T^*X \\ \downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_X} \\ Y & \xleftarrow{\sim} & Y & \xrightarrow{f} & X. \end{array}$$

Since f is noncharacteristic for F, ${}^tf'$ is proper on $f_\pi^{-1}(\mathrm{SS}(F))$ and hence on $f_\pi^{-1}(\mathrm{supp}(\mu\mathrm{hom}(K, F)))$. Then one has the isomorphisms

$$\begin{aligned} R\pi_{Y*}R{}^tf'_!f_\pi^{-1}\mu\mathrm{hom}(K, F) &\cong R\pi_{Y*}R{}^tf'_*f_\pi^{-1}\mu\mathrm{hom}(K, F) \\ &\cong R\pi_*f_\pi^{-1}\mu\mathrm{hom}(K, F) \\ &\cong f^{-1}R\pi_{X*}\mu\mathrm{hom}(K, F) \\ &\cong f^{-1}R\mathcal{H}om(K, F) \end{aligned}$$

where the third isomorphism follows from the fact that, $\mu\mathrm{hom}(K, F)$ being in $D^b_{\mathbb{R}^+}(T^*X)$, $R\pi_{X*}\mu\mathrm{hom}(K, F) \cong i^{-1}\mu\mathrm{hom}(K, F)$, i denoting the immersion of the zero-section X in T^*X.

Thus, the second vertical arrow in (2.1.2) is nothing but the morphism (2.1.1), and it is enough to prove that the first and third vertical arrows in (2.1.2) are isomorphisms at every $y \in Z$.

A. *First vertical arrow.* Consider the following diagram, where δ_Y, δ_X denote the diagonal embeddings and δ the graph embedding:

$$\begin{array}{ccccc} Y \times Y & \xrightarrow{f_2} & Y \times X & \xrightarrow{f_1} & X \times X \\ \uparrow{\scriptstyle \delta_Y} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta_X} \\ Y & \xrightarrow{\sim} & Y & \xrightarrow{f} & X. \end{array}$$

We shall denote by $\tilde{f}$ the composite $\tilde{f} = f_1 \circ f_2$ and by q_1, q_2 the first and second projections defined on a product of two spaces.

Recall (see [KS4, §4.3]) that

$$R\pi_{Y!}\mu\mathrm{hom}(L, f^{-1}F) \cong \delta_Y^{-1}R\mathcal{H}om(q_2^{-1}L, q_1^!f^{-1}F) \otimes \omega_Y^{\otimes -1}.$$

On the other hand, one has the chain of isomorphisms

$$\begin{aligned} R\pi_{Y!}R^t f_!' f_\pi^{-1}\mu\hom(K,F) &\cong R\pi_! f_\pi^{-1}\mu\hom(K,F) \\ &\cong f^{-1}R\pi_{X!}\mu\hom(K,F) \\ &\cong f^{-1}\delta_X^{-1}R\mathscr{H}om(q_2^{-1}K, q_1^! F)\otimes\omega_X^{\otimes -1} \\ &\cong \delta_Y^{-1}\tilde{f}^{-1}R\mathscr{H}om(q_2^{-1}K, q_1^! F)\otimes\omega_X^{\otimes -1}. \end{aligned}$$

The map f being noncharacteristic for F, we get

$$\begin{aligned} \tilde{f}^{-1}R\mathscr{H}om(q_2^{-1}K, q_1^! F)\otimes\omega_X^{\otimes -1} &\cong f_2^{-1}f_1^{-1}R\mathscr{H}om(q_2^{-1}(K\otimes\omega_X), q_1^! F) \\ &\cong f_2^{-1}R\mathscr{H}om(q_2^{-1}(K\otimes\omega_X), q_1^! f^{-1}F). \end{aligned}$$

It is enough then to prove the isomorphism for every $y \in Z$, $x = f(y)$:

$$R\mathscr{H}om(q_2^{-1}(K\otimes\omega_X), q_1^! f^{-1}F)_{(y,x)} \xrightarrow{\sim} R\mathscr{H}om(q_2^{-1}(L\otimes\omega_Y), q_1^! f^{-1}F)_{(y,y)}. \tag{2.1.3}$$

For an open neighborhood $V \subset Y$ of y and for $U \subset Y$ ranging over an open neighborhood system of y, by (vi) we have

$$\begin{aligned} &\text{“}\varinjlim_{U\ni y}\text{”}\, R\Gamma(V\times U; R\mathscr{H}om(q_2^{-1}(L\otimes\omega_Y), q_1^! f^{-1}F)) \\ &\quad\cong \text{“}\varinjlim_{U\ni y}\text{”}\, R\mathrm{Hom}(R\Gamma_c(U; L\otimes\omega_Y), R\Gamma(V; f^{-1}F)) \\ &\quad\cong R\mathrm{Hom}\left(\text{“}\varprojlim_{U\ni y}\text{”}\, R\Gamma_c(U; L\otimes\omega_Y), R\Gamma(V; f^{-1}F)\right) \\ &\quad\cong R\mathrm{Hom}(R\Gamma_{\{y\}}(Y; L\otimes\omega_Y), R\Gamma(V; f^{-1}F)). \end{aligned} \tag{2.1.4}$$

(Recall that the first equality follows from the Poincaré-Verdier duality). Similarly to (2.1.4) we get

$$\begin{aligned} &\text{“}\varprojlim_{W\ni x}\text{”}\, R\Gamma(V\times W; R\mathscr{H}om(q_2^{-1}(K\otimes\omega_X), q_1^! f^{-1}F)) \\ &\quad\cong R\mathrm{Hom}(R\Gamma_{\{x\}}(X; K\otimes\omega_X), R\Gamma(V; f^{-1}F)) \end{aligned}$$

where now $V \subset Y$ is an open neighborhood of y and $W \subset X$ ranges over an open neighborhood system of x. Recalling the hypothesis (vii), (2.1.3) follows.

B. *Third vertical arrow.* We have to prove that the natural morphism

$$\text{(2.1.5)} \qquad \mathrm{R}^t f'_! f_\pi^{-1} \mu hom(K, F)_{p_Y} \to \mu hom(L, f^{-1}F)_{p_Y}$$

is an isomorphism for every $p_Y \in \dot{\pi}_Y(Z)$.

This will follow from the following Proposition 2.1.2. Q.E.D.

PROPOSITION 2.1.2. *Let F and K be objects of $\mathrm{D}^b(X)$ and let L be an object of $\mathrm{D}^b(Y)$. Assume that a morphism $\psi\colon L \to f^{-1}K$ is given. Let V be an open neighborhood of p_Y and assume that*

(i) *f is noncharacteristic for F and for K on V and*

(ii) *f_π is noncharacteristic for $C(\mathrm{SS}(F), \mathrm{SS}(K))$ on ${}^tf'^{-1}(V)$.*

Assume, moreover, that there exist $p_1, \ldots, p_r$ in ${}^tf'^{-1}(p_Y)$ with

(iii) ${}^tf'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset \{p_1, \ldots, p_r\}$,

(iv) *$p_1, \ldots, p_r$ isolated in ${}^tf'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(K))$, and*

(v) *the morphism induced by ψ, $L \to f_{\mu,p_j}^{-1}K$, an isomorphism in $\mathrm{D}^b(Y; p_Y)$ for $j = 1, \ldots, r$.*

Then the natural morphism (2.1.5) induced by ψ is an isomorphism.

For the proof of this proposition we will need the following lemma which is an immediate consequence of [KS4, Prop. 6.7.1].

LEMMA 2.1.3. *Let F, K be objects of $\mathrm{D}^b(X)$. Let V be an open neighborhood of p_Y and assume*

(i) *f is noncharacteristic for F and for K on V and*

(ii) *f_π is noncharacteristic for $C(\mathrm{SS}(F), \mathrm{SS}(K))$ on ${}^tf'^{-1}(V)$.*

Assume that there exists $p \in {}^tf'^{-1}(p_Y)$ with

(iii) ${}^tf'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset \{p\}$ *and*

(iv) ${}^tf'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(K)) \subset \{p\}$.

Then the natural morphism $\mathrm{R}^t f'_! f_\pi^{-1} \mu hom(K, F)_{p_Y} \to \mu hom(f^{-1}K, f^{-1}F)_{p_Y}$ is an isomorphism.

Proof of Proposition 2.1.2. Following the proof of Proposition 1.3.3 and due to assumptions (iii) and (iv), we can find refined cutting pairs (γ_i, ω_i) $(i = 1, \ldots, r)$ on X at p_X such that, setting $F_i = \Phi_X(\gamma_i, \omega_i; F)$, $K_i = \Phi_X(\gamma_i, \omega_i; K)$, and for K_0 being the third term of a distinguished triangle $\bigoplus_{i=1}^r K_i \to K \to K_0 \xrightarrow{+1}$, we have

— $f_\pi^{-1}\mathrm{SS}(F_i) \cap {}^tf'^{-1}(p_Y) \subset \{p_i\}$,

— $f_\pi^{-1}\mathrm{SS}(K_i) \cap {}^tf'^{-1}(p_Y) \subset \{p_i\}$ and $p_i \notin \mathrm{SS}(K_0)$ for $i = 1, \ldots, r$,

— $F \cong \bigoplus_{i=1}^r F_i$ in $\mathrm{D}^b(X; f_\pi {}^tf'^{-1}(p_Y))$.

Moreover, since f is noncharacteristic for the F_i's, we have

— $f^{-1}F \cong \bigoplus_{i=1}^r f^{-1}F_i$ in $\mathrm{D}^b(Y; p_Y)$.

We get the chain of isomorphisms

$$\begin{aligned}
\mathrm{R}^t f'_! f_\pi^{-1} \mu hom(K, F)_{p_Y} &\cong (\mathrm{R}^t f'_! f_\pi^{-1} \textstyle\bigoplus_{i=1}^r \mu hom(K, F_i))_{p_Y} \\
&\cong (\mathrm{R}^t f'_! f_\pi^{-1} \textstyle\bigoplus_{i=1}^r \mu hom(K_i, F_i))_{p_Y}
\end{aligned}$$

$$\cong \bigoplus_{i=1}^{r} \mu\mathrm{hom}(f^{-1}K_i, f^{-1}F_i)_{p_Y}$$

$$\cong \bigoplus_{i=1}^{r} \mu\mathrm{hom}(f_{\mu,p_i}^{-1}K, f^{-1}F_i)_{p_Y}$$

$$\cong \bigoplus_{i=1}^{r} \mu\mathrm{hom}(L, f^{-1}F_i)_{p_Y}$$

$$\cong \mu\mathrm{hom}(L, f^{-1}F)_{p_Y}.$$

Here, the second isomorphism follows from the fact that μhom is a microlocal functor, the third one follows from Lemma 2.1.3, and the fourth from assumption (iv). Q.E.D.

2.2. *A particular case.* Let $f: Y \to X$ be a morphism of manifolds. Let Z be a closed subset of Y. Let F be an object of $\mathrm{D}^b(X)$ such that

(2.2.1) $$f \text{ is noncharacteristic for } F.$$

Assume that for every $p_Y \in \dot{\pi}_Y^{-1}(Z)$ there exist $p_1, \ldots, p_r$ in ${}^tf'^{-1}(p_Y)$ with

(2.2.2) $${}^tf'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(F)) \subset \{p_1, \ldots, p_r\}.$$

Let K_i $(i = 1, \ldots, r)$ be objects of $\mathrm{D}^b(X)$ such that

(2.2.3) $$K_i \text{ is w.c.c.},$$

(2.2.4) $$f \text{ is noncharacteristic for } K_i, \text{ and}$$

(2.2.5) $$\text{a morphism } \tau_i: K_i \to A_X \text{ is given}.$$

Assume that for every $p_Y \in \dot{\pi}_Y^{-1}(Z)$ and for $p_1, \ldots, p_r$ as in (2.2.2)

(2.2.6) $${}^tf'^{-1}(p_Y) \cap f_\pi^{-1}(\mathrm{SS}(K_i)) \subset \{p_i\}.$$

Let L be an object of $\mathrm{D}^b(Y)$ such that

(2.2.7) $$\text{an isomorphism } \psi_i: L \to f^{-1}K_i \text{ is given}$$

and

(2.2.8) $$f^{-1}(\tau_i) \circ \psi_i \text{ induces an isomorphism } \mathrm{R}\Gamma_Z L \to \mathrm{R}\Gamma_Z A_Y.$$

Assume, moreover, that for an open neighborhood V of $\dot{\pi}_Y^{-1}(Z)$

(2.2.9) $$f_\pi \text{ is noncharacteristic for } C(\mathrm{SS}(F), \mathrm{SS}(K_i)) \text{ on } {}^tf'^{-1}(V).$$

We shall give here a way to build up a complex $K \in \mathrm{Ob}(\mathrm{D}^b(X))$ in order to satisfy the hypotheses of Theorems 2.1.1.

Let h be the composite of the map $\bigoplus_{i=1}^r \tau_j\colon \bigoplus_{i=1}^r K_i \to \bigoplus_{i=1}^r A_X$ and the map $\bigoplus_{i=1}^r A_X \to \bigoplus_{i=1}^{r-1} A_X$, given by $(a_1, \ldots, a_r) \mapsto (a_2 - a_1, \ldots, a_r - a_{r-1})$.

Let K be defined by embedding h in a distinguished triangle

$$K \to \bigoplus_{i=1}^r K_i \xrightarrow{h} \bigoplus_{i=1}^{r-1} A_X \xrightarrow{+1}. \tag{2.2.10}$$

It is now straightforward to verify the following lemma.

LEMMA 2.2.1. *K is w.c.c. in $\mathrm{D}^b(X)$ and the following estimate holds*:

$$\mathrm{SS}(K) = \bigcup_i \mathrm{SS}(K_i) \cup T_X^* X.$$

By completing the morphism of distinguished triangles (where the vertical arrows are the natural ones)

$$\begin{array}{ccccccc} f^{-1}K & \longrightarrow & \bigoplus_{i=1}^r f^{-1}K_j & \longrightarrow & \bigoplus_{i=1}^{r-1} f^{-1}A_X & \xrightarrow{+1} \\ & & \uparrow & & \uparrow & \\ L & \longrightarrow & \bigoplus_{i=1}^r L & \longrightarrow & \bigoplus_{i=1}^{r-1} L & \xrightarrow{+1}, \end{array} \tag{2.2.11}$$

we get a morphism

$$\psi\colon L \to f^{-1}K. \tag{2.2.12}$$

LEMMA 2.2.2. *The morphism ψ induces an isomorphism*

$$\mathrm{R}\Gamma_Z L \xrightarrow{\sim} \mathrm{R}\Gamma_Z f^{-1}K.$$

Proof. One has the morphism of distinguished triangles

$$\begin{array}{ccccccc} \mathrm{R}\Gamma_Z f^{-1}K & \longrightarrow & \bigoplus_{i=1}^r \mathrm{R}\Gamma_Z f^{-1}K_j & \longrightarrow & \bigoplus_{i=1}^{r-1} \mathrm{R}\Gamma_Z f^{-1}A_X & \xrightarrow{+1} \\ \downarrow & & \downarrow & & \downarrow & \\ \mathrm{R}\Gamma_Z f^{-1}A_X & \longrightarrow & \bigoplus_{i=1}^r \mathrm{R}\Gamma_Z f^{-1}A_X & \longrightarrow & \bigoplus_{i=1}^{r-1} \mathrm{R}\Gamma_Z f^{-1}A_X & \xrightarrow{+1}. \end{array}$$

The third vertical arrow is the identity, and the second one is an isomorphism by (2.2.7) and (2.2.8). Hence, $\mathrm{R}\Gamma_Z f^{-1}K \cong \mathrm{R}\Gamma_Z A_Y \cong \mathrm{R}\Gamma_Z L$. Q.E.D.

We can then state the following proposition.

PROPOSITION 2.2.3. *Let f, F, K_i, and L be given satisfying the hypotheses* (2.2.1)–(2.2.9). *For K and ψ constructed in* (2.2.10) *and* (2.2.12), *the hypotheses of Theorem* 2.1.1 *are satisfied.*

The proof is immediate if one just notices that, due to (2.2.4), the hypothesis (vii) of Theorem 2.2.1 reads as

$$\text{(vii)}' \qquad \mathrm{R}\Gamma_{\{y\}}L \xrightarrow{\sim} \mathrm{R}\Gamma_{\{y\}}(f^{-1}K).$$

3. Applications to the Cauchy problem. In this section, as an application of Theorem 2.1.1, we will show how to recover the results obtained in the papers cited at the beginning of the introduction.

For our purpose, the base ring A introduced at the beginning §1.2 is now the field $\mathbb{C}$. In order to avoid confusion with the complex line, we still denote it by A.

3.1. *Ramified solutions.* (See [HLW].) Let Y be a complex analytic manifold and let Z be a smooth complex hypersurface of Y. First, we shall construct a sheaf L on Y such that, for $G \in \mathrm{Ob}(\mathrm{D}^b(Y))$, $\mathrm{R}\mathcal{H}om(L, G)$ describes in some sense the "ramified sections of G along Z". (See [G] and [D].)

Let $p\colon \tilde{\mathbb{C}}^* \to \mathbb{C}$ be the universal covering of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Recall that one can choose a coordinate $t \in \mathbb{C} \cong \tilde{\mathbb{C}}^*$ so that $p(t) = \exp(2\pi i t)$. Choose a complex analytic function $g\colon Y \to \mathbb{C}$ such that $Z = g^{-1}(0)$. (This is possible locally.) Set $\tilde{Y}^* = \tilde{\mathbb{C}}^* \times_{\mathbb{C}} Y$ and consider the Cartesian diagram

$$\text{(3.1.1)} \qquad \begin{array}{ccc} \tilde{Y}^* & \longrightarrow & \tilde{\mathbb{C}}^* \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\;g\;} & \mathbb{C}. \end{array}$$

We will call the complex $\mathrm{R}p_{Y*}p_Y^{-1}G$ the complex of ramified sections of G along Z. Notice that $p_{Y!}$ is an exact functor and that $p_Y^! = p_Y^{-1}$. By the Poincaré-Verdier duality one gets

$$\mathrm{R}p_{Y*}p_Y^{-1}G = \mathrm{R}\mathcal{H}om(p_{Y!}A_{\tilde{Y}^*}, G)$$

$$= \mathrm{R}\mathcal{H}om(g^{-1}p_!A_{\tilde{\mathbb{C}}^*}, G).$$

Set $L = g^{-1}p_!A_{\tilde{\mathbb{C}}^*}$ and notice that, by adjunction, there is a natural morphism

$$\text{(3.1.2)} \qquad \tau\colon L \to A_Y.$$

Let us now describe what will be our geometrical frame.

(3.1.3) X is a complex analytic manifold, Y is a smooth hypersurface of X, Z is a smooth hypersurface of Y, Z_i ($i = 1, \dots, r$) are smooth hypersurfaces of X pairwise transversal, transversal to Y, and such that $Z_i \cap Y = Z$ for every i. Let $f\colon Y \to X$ be the embedding. Assume given complex analytic functions $g\colon Y \to \mathbb{C}$, $g_i\colon X \to \mathbb{C}$ with $dg \neq 0$, $dg_i \neq 0$, such that $g_i \circ f = g$ and $Z = g^{-1}(0)$, $Z_i = g_i^{-1}(0)$.

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module such that for a neighborhood V of $\dot{T}_Z^* Y$

$$(3.1.4)\qquad \begin{cases} \text{(i) } f_\pi \text{ is noncharacteristic for } C(\mathrm{char}(\mathcal{M}), \dot{T}_{Z_i}^* X) \text{ on } {}^t f'^{-1}(V), \\ \text{(ii) } \mathrm{char}(\mathcal{M}) \cap {}^t f'^{-1}(T_Z^* Y) \subset \bigcup_i T_{Z_i}^* X \cup T_X^* X \end{cases}$$

where char($\mathcal{M}$) denotes the characteristic variety of $\mathcal{M}$.

Note that (3.1.4), (ii), implies that f is noncharacteristic for $\mathcal{M}$. (I.e., f is noncharacteristic for char($\mathcal{M}$).)

Set $K_i = g_i^{-1} p_! A_{\tilde{\mathbb{C}}^*}$ $(i = 1, \ldots, r)$ and $F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Of course, $\mathrm{SS}(K_i) \subset T_{Z_i}^* X \cup T_X^* X$. One has $L = f^{-1} K_i$, and the natural morphisms $\tau_i\colon K_i \to A_X$ may be constructed as in (3.1.2).

LEMMA 3.1.1. *The morphism τ induces an isomorphism*

$$\mathrm{R}\Gamma_Z L \xrightarrow{\sim} \mathrm{R}\Gamma_Z A_Y .$$

Proof. Since g is a smooth map, one has

$$\mathrm{R}\Gamma_Z L \simeq \mathrm{R}\Gamma_Z g^{-1} p_! A_{\tilde{\mathbb{C}}^*} \simeq g^{-1} \mathrm{R}\Gamma_{\{0\}} p_! A_{\tilde{\mathbb{C}}^*},$$

and, similarly,

$$\mathrm{R}\Gamma_Z A_Y \simeq \mathrm{R}\Gamma_Z g^{-1} A_{\mathbb{C}} \simeq g^{-1} \mathrm{R}\Gamma_{\{0\}} A_{\mathbb{C}} .$$

It is therefore enough to prove the isomorphism

$$\mathrm{R}\Gamma_{\{0\}} p_! A_{\tilde{\mathbb{C}}^*} \xrightarrow{\sim} \mathrm{R}\Gamma_{\{0\}} A_{\mathbb{C}} .$$

Since $p_! A_{\tilde{\mathbb{C}}^*}$ is $\mathbb{R}^+$-conic for the action of $\mathbb{R}^+$ on $\mathbb{C}$, one has

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{C}, \mathrm{R}\Gamma_{\{0\}} p_! A_{\tilde{\mathbb{C}}^*}) &\cong \mathrm{R}\Gamma_c(\mathbb{C}, p_! A_{\tilde{\mathbb{C}}^*}) \\ &= \mathrm{R}\Gamma_c(\tilde{\mathbb{C}}^*, A_{\tilde{\mathbb{C}}^*}) \\ &\cong A[2], \end{aligned}$$

and hence $\mathrm{R}\Gamma_{\{0\}} p_! A_{\tilde{\mathbb{C}}^*} \cong A_{\{0\}}[2] \cong \mathrm{R}\Gamma_{\{0\}} A_{\mathbb{C}}$. Q.E.D.

The complex $\mathcal{O}_{Z|Y}^{\mathrm{ram}} := \mathrm{R}\mathcal{H}om(L, \mathcal{O}_Y)$ is concentrated in degree 0, and its sections are the holomorphic functions ramified along the hypersurface Z and similarly for the complex $\mathcal{O}_{Z_i|X}^{\mathrm{ram}} = \mathrm{R}\mathcal{H}om(K_i, \mathcal{O}_X)$. For K constructed from the K_i's as in (2.2.10), set $\sum_i \mathcal{O}_{Z_i|X}^{\mathrm{ram}} = \mathrm{R}\mathcal{H}om(K, \mathcal{O}_X)$. Applying Theorem 2.1.1 (see Proposition 2.2.3) we get the following theorem.

THEOREM 3.1.2. *Keeping the same notation as above, the natural morphism*

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \sum_i \mathcal{O}^{\mathrm{ram}}_{Z_i|X}\right)\Big|_Z \to \mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}^{\mathrm{ram}}_{Z|Y})|_Z \tag{3.1.5}$$

is an isomorphism.

Let us now explain how Theorem 3.1.2 gives an extension to $\mathcal{D}_X$-modules of [HLW]'s result.

Let X be an open subset of $\mathbb{C}^n$ with $0 \in X$. Take a local system $z = (z_1, z') = (z_1, \ldots, z_n)$ of complex coordinates on X. Let $(z; \zeta)$ be the associated coordinates in T^*X. Set $D = \partial/\partial z$. Let $P = P(z, D)$ be a linear partial differential operator of order m on X with holomorphic coefficients for which the hyperplane $Y = \{z \in X; z_1 = 0\}$ is noncharacteristic. Denote by $f\colon Y \to X$ the embedding and set $Z = \{z \in X; z_1 = z_2 = 0\}$. Suppose that P has characteristics with constant multiplicities transversal to $Y \times_X T^*X$ at ${}^tf'^{-1}(T^*_Z Y) \cap \mathrm{char}(P)$.

PROPOSITION 3.1.3. *Under the previous hypotheses there exist smooth hypersurfaces $Z_1, \ldots, Z_r$ of X as in* (3.1.3) *and satisfying* (3.1.4) *for $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$.*

This result is classical. For the reader's convenience we recall its proof [S, Prop. 2.2.2].

Proof. Let $p(z; \zeta)$ be the principal symbol of $P(z, D)$. Decompose it as a product of irreductible polynomials $p(z; \zeta) = \prod_{i=1}^r p_i(z; \zeta)^{d_i}$, and set $q(z; \zeta) = \prod_{i=1}^r p_i(z; \zeta)$. One has $\mathrm{char}(P) = q^{-1}(0)$. The fact that P has characteristics with constant multiplicities transversal to $Y \times_X T^*X$ at ${}^tf'^{-1}(T^*_Z Y) \cap \mathrm{char}(P)$ means that the Poisson bracket $\{z_1, q\}$ does not vanish at any point of ${}^tf'^{-1}(T^*_Z Y) \cap \mathrm{char}(P)$. Since $\partial q(z; \zeta)/\partial \zeta_1 = -\{z_1, q\}$ and $q(z; 1, 0, \ldots, 0) \neq 0$ for $z \in Y$, the equation $q(z; \tau, 1, 0, \ldots, 0) = 0$ has r distinct roots for $z \in Z$. Hence, $\Lambda' = {}^tf'^{-1}(T^*_Z Y) \cap \mathrm{char}(P)$ is the disjoint union of r isotropic smooth manifolds $\Lambda'_1, \ldots, \Lambda'_r$ of T^*X. The Hamiltonian vector field H_q is transversal to $Y \times_X T^*X$ at Λ'_i, and, therefore, the union of the integral curves of H_q issuing from Λ'_i is a Lagrangian manifold Λ_i of T^*X contained in $\mathrm{char}(P)$. Let $Z_i = \pi_X(\Lambda_i)$. Since the projection on X of the vector field $(H_q)|_{\Lambda_i}$ is a vector field transversal to Y, Z_i is a smooth hypersurface of X transversal to Y, and, Λ_i being Lagrangian, $\Lambda_i = \dot{T}^*_{Z_i} X$. Finally, f_π is noncharacteristic for $C(\mathrm{char}(P), \dot{T}^*_{Z_i} X)$ on ${}^tf'^{-1}(T^*_Z Y)$ since $\dot{T}^*_{Z_i} X \subset \mathrm{char}(P)$. Q.E.D.

Consider the Cauchy problem

$$\begin{cases} P(z, D)u(z) = v(z), \\ D_1^h u(z)|_Y = w_h(z'), \quad 0 \leqslant h < m. \end{cases} \tag{3.1.6}$$

In [HLW], Hamada, Leray, and Wagschal proved that the Cauchy problem (3.1.6) has a unique solution $u(z) = \sum \varphi_i(z)$ for any $v(z) = \sum \psi_i(z)$, where w_h is a ramified holomorphic function on Y along Z and where φ_i (resp. ψ_i) is ramified on X along Z_i.

We can then apply Theorem 3.1.2 to $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$.

One has $\mathcal{M}_Y \cong (\mathcal{D}_Y)^m$, and hence $\mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}^{\mathrm{ram}}_{Z|Y}) \cong (\mathcal{O}^{\mathrm{ram}}_{Z|Y})^m$ represents the sheaf of ramified Cauchy data. The complex $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y$ is concentrated in degree zero. From (2.2.10) we get a distinguished triangle

(3.1.7)
$$\bigoplus_{i=1}^{r-1} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \to \bigoplus_{i=1}^{r} \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathrm{ram}}_{Z_i|X}) \to \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{M}, \sum_i \mathcal{O}^{\mathrm{ram}}_{Z_i|X}\right) \xrightarrow{+1}.$$

By (3.1.5) the complex $\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sum_i \mathcal{O}^{\mathrm{ram}}_{Z_i|X})|_Z$ is concentrated in degree zero, and by (3.1.7) its sections may be expressed as a sum $\sum_i \varphi_i$, where the φ_i's are holomorphic functions ramified along Z_i, satisfying the equation $P\varphi_i = 0$. That is to say, by taking the zeroth cohomology group in (3.1.5), Theorem 3.1.2 ensures the existence and the uniqueness for the solution of the Cauchy problem (3.1.6) with $v = 0$, the solution being a sum $\sum_i \varphi_i$, where φ_i is a holomorphic function on X ramified along Z_i.

Moreover, the vanishing of $H^1(\mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathfrak{X}}}(\mathcal{M}, \sum_i \mathcal{O}^{\mathrm{ram}}_{Z_i|X}))|_Z$, which follows from (3.1.5), ensures the solvability of the equation

(3.1.8)
$$P(z, D)u(z) = v(z)$$

for u and v of the form $\sum_i \varphi_i$ as above.

Remark 3.1.4. In [Le] Leichtnam solves the Cauchy problem $Pf = g$, where f and g are ramified on $X \setminus \cup Z_i$ (with the above notations). Theorem 2.1.1 does not allow us to treat Leichtnam's result, but, nevertheless, we hope to recover it by similar geometrical methods in the future.

Remark 3.1.5. Other interesting results concerning this kind of problems are announced in [NSS].

3.2. *Ramified solutions of logarithmic type.* In [KS1] a theorem on the existence and uniqueness for the solution of (3.1.6) is given when the ramifications involved are of logarithmic type. We will give here a new proof of their theorem.

Let $z \in \mathbb{C}$ be a coordinate and set $D = \partial/\partial z$. Consider the left coherent $\mathcal{D}_{\mathbb{C}}$-module $\mathcal{N} = \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}} DzD$.

For $L^1_{\{0\}|\mathbb{C}} := \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{N}, \mathcal{O}_{\mathbb{C}})$ set $\mathcal{O}^1_{\{0\}|\mathbb{C}} := \mathrm{R}\mathcal{H}om(L^1_{\{0\}|\mathbb{C}}, \mathcal{O}_{\mathbb{C}}) = \mathcal{N}^{\infty}$. This represents a complex of holomorphic functions with logarithmic ramifications.

Take $\mathcal{M}_{\mathbb{C}}$ to be a left coherent $\mathcal{D}_{\mathbb{C}}$-module.

If one makes the choice $F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{M}_{\mathbb{C}}, \mathcal{O}_{\mathbb{C}})$, then

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{M}_{\mathbb{C}}, \mathcal{O}^1_{\{0\}|\mathbb{C}}) \cong \mathrm{R}\mathcal{H}om(L^1_{\{0\}|\mathbb{C}}, F).$$

Moreover, one has the following lemma.

LEMMA 3.2.1. (a) *There is a natural map*

$$\tau\colon L^1_{\{0\}|\mathbb{C}} \to A_{\mathbb{C}}.$$

(b) *The morphism induced by* τ

$$\mathrm{R}\Gamma_{\{0\}} L^1_{\{0\}|\mathbb{C}} \to \mathrm{R}\Gamma_{\{0\}} A_{\mathbb{C}}$$

is an isomorphism.

Proof. To prove (a) notice that $L^1_{\{0\}|\mathbb{C}} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{N}, \mathcal{O}_{\mathbb{C}})$ is concentrated in degree 0 and is represented by the complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}} \xrightarrow{DzD} \mathcal{O}_{\mathbb{C}} \longrightarrow 0.$$

A section of $L^1_{\{0\}|\mathbb{C}}$ is then represented by a function $\varphi \in \mathcal{O}_{\mathbb{C}}$ with $DzD\varphi = 0$. We define τ by $\tau(\varphi) = zD\varphi$.

As for (b), consider the exact sequence of $\mathcal{D}_{\mathbb{C}}$-modules

$$0 \to \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}D \to \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}DzD \to \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}zD \to 0.$$

Here, for $P \in \mathcal{D}_{\mathbb{C}}$ the second arrow is given by $[P] \mapsto [PzD]$ and the third by $[P] \mapsto [P]$, $[P]$ denoting the class of P modulo the corresponding ideal.

Applying $\mathrm{R}\Gamma_{\{0\}}\, \mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\cdot, \mathcal{O}_{\mathbb{C}})$ to this exact sequence, we get the distinguished triangle

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_{\mathbb{C}}}(\mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}}zD, \mathrm{R}\Gamma_{\{0\}}\mathcal{O}_{\mathbb{C}}) \to \mathrm{R}\Gamma_{\{0\}} L^1_{\{0\}|\mathbb{C}} \to \mathrm{R}\Gamma_{\{0\}} A_{\mathbb{C}} \xrightarrow{+1}.$$

Recall that $\mathrm{R}\Gamma_{\{0\}}\mathcal{O}_{\mathbb{C}}$ is concentrated in degree 1. One concludes by observing that the first term of the triangle is 0 due to the fact that zD is an automorphism of $\mathcal{H}^1_{\{0\}}(\mathcal{O}_{\mathbb{C}})$. Q.E.D.

Let us adopt the notations (3.1.3) and take $\mathcal{M}$ as in (3.1.4). Set $L = g^{-1} L^1_{\{0\}|\mathbb{C}}$ and $K_i = g_i^{-1} L^1_{\{0\}|\mathbb{C}}$. Let K be the complex defined by (2.2.10) with this choice of the K_i's. Setting $\mathcal{O}^1_{Z|Y} = \mathrm{R}\mathcal{H}om(L, \mathcal{O}_Y)$, $\sum_i \mathcal{O}^1_{Z_i|X} = \mathrm{R}\mathcal{H}om(K, \mathcal{O}_X)$, we recover the results of [KS1, Prop. 4.2] by simply applying Theorem 2.1.1.

THEOREM 3.2.2. *With the same notation as above, the natural morphism*

$$\mathrm{R}\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \sum_i \mathcal{O}^1_{Z_i|X}\Big)\Big|_Z \to \mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}^1_{Z|Y})|_Z$$

is an isomorphism.

3.3. *Decomposition at the boundary.* In [Sc] Schiltz shows how the solution of the noncharacteristic Cauchy problem may be expressed as a sum of functions which are holomorphic in domains determined by the characteristic real hypersurfaces issuing from the boundary of the domain where the data are defined.

Here, we are going to recover his result and extend it to general systems. Let us describe our geometrical frame.

Let Y be a smooth complex analytic hypersurface of a complex analytic manifold X and let $f\colon Y \to X$ be the embedding. In what follows we will consider T^*X endowed with the underlying real symplectic structure. Recall that, if σ is the symplectic two-form on T^*X, the corresponding real two-form is given by $\sigma^{\mathbb{R}} = 2\operatorname{Re}\sigma$. We consider the following situation.

(3.3.1) ω is an open subset of Y with smooth boundary Z, and Ω_i $(i = 1, \dots, r)$ are open subsets of X with smooth boundaries Z_i such that $\Omega_i \cap Y = \omega$ and the Z_i's are pairwise transversal and transversal to Y.

Set $T^*_\omega Y = \mathrm{SS}(A_\omega) \subset T^*Y$. Recall that $\mathrm{SS}(A_\omega)$ is identified to the subset of $T^*_Z Y$ consisting of the conormals pointing outside ω.

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module such that for a neighborhood V of $\dot{T}^*_\omega Y$

$$(3.3.2)\qquad \begin{cases} \text{(i) } f_\pi \text{ is noncharacteristic for } C(\operatorname{char}(\mathcal{M}), \dot{T}^*_{\Omega_i}X) \text{ on } {}^t f'^{-1}(V) \text{ and} \\ \text{(ii) } \operatorname{char}(\mathcal{M}) \cap {}^t f'^{-1}(T^*_\omega Y) \subset \bigcup_i T^*_{\Omega_i}X \cup T^*_X X. \end{cases}$$

Note that (3.3.2), (ii), implies that f is non-characteristic for $\mathcal{M}$.

Let $F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, set $K_i = A_{\Omega_i}$, an object of $\mathrm{D}^b(X)$, and set $L = A_\omega$, an object of $\mathrm{D}^b(Y)$. Of course, $f^{-1}K_i = L$, and, moreover, one has canonical morphisms $\tau_i\colon K_i \to A_X$. We can then define K by (2.2.10) with this choice of the K_i's. Set $\sum_i \mathrm{R}\Gamma_{\Omega_i}\mathcal{O}_X = \mathrm{R}\mathcal{H}om(K, \mathcal{O}_X)$.

THEOREM 3.3.6. *With the same notations as above, the canonical morphism*

$$(3.3.3)\qquad \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}\Big(\mathcal{M}, \sum_i \mathrm{R}\Gamma_{\Omega_i}\mathcal{O}_X\Big)\Big|_Y \to \mathrm{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathrm{R}\Gamma_\omega\mathcal{O}_Y)$$

is an isomorphism.

Proof. It is enough to prove the isomorphism at each point $y \in Y$.

If $y \notin \bar{\omega}$, both complexes are zero.

If $y \in \omega$, one has

$$\sum_i \mathrm{R}\Gamma_{\Omega_i}\mathcal{O}_X|_\omega \cong \mathcal{O}_X|_\omega,$$

$$\mathrm{R}\Gamma_\omega\mathcal{O}_Y|_\omega \cong \mathcal{O}_Y|_\omega,$$

and hence the isomorphism (3.3.3) is nothing but the Cauchy-Kowaleski theorem (0.1).

If $y \in \partial\omega$, we apply Theorem 2.1.1 (see Proposition 2.2.3) for $Z = \{y\}$. The only nontrivial property to check is

$$\mathrm{R}\Gamma_{\{y\}}(A_\omega) \cong \mathrm{R}\Gamma_{\{y\}}(A_Y).$$

Since ω has a smooth boundary, $A_\omega \cong \mathrm{R}\mathscr{H}om(A_{\bar{\omega}}, A_Y)$, and hence

$$\begin{aligned}\mathrm{R}\Gamma_{\{y\}}(A_\omega) &\cong \mathrm{R}\mathscr{H}om(A_{\{y\}}, \mathrm{R}\mathscr{H}om(A_{\bar{\omega}}, A_Y)) \\ &\cong \mathrm{R}\mathscr{H}om(A_{\{y\}} \otimes A_{\bar{\omega}}, A_Y) \\ &\cong \mathrm{R}\Gamma_{\{y\}}(A_Y).\end{aligned}$$

Q.E.D.

Let us explain why Theorem 3.3.6 contains the result of [Sc].

Let X be an open subset of $\mathbb{C}^n$ with $0 \in X$. Let $z = (z_1, \ldots, z_n)$ denote the complex coordinates in $\mathbb{C}^n$ and let $(z; \zeta)$ be the associated coordinates in T^*X. Let ω be an open subset of $Y = \{z \in X; z_1 = 0\}$ with smooth boundary Z. Denote by $f\colon Y \to X$ the embedding. Consider a linear partial differential operator with holomorphic coefficients $P = P(z, D)$ on X for which Y is noncharacteristic. Assume that P has characteristics with constant multiplicities transversal to $Y \times_X T^*X$ at ${}^t f'^{-1}(T^*_Z Y) \cap \mathrm{char}(P)$. Let $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X P$.

PROPOSITION 3.3.7. *Under the previous hypotheses there exist open subsets Ω_i of X as in* (3.3.1) *and satisfying* (3.3.2).

Proof. Take $q(z; \zeta)$ as in Proposition 3.1.3. Set $\Lambda' = {}^t f'^{-1}(T^*_Z Y) \cap \mathrm{char}(\mathscr{M})$, a disjoint union of r isotropic smooth manifolds $\Lambda'_1, \ldots, \Lambda'_r$ of $(T^*X)^{\mathbb{R}}$, the cotangent bundle endowed with the underlying real analytic symplectic structure. From the fact that $\partial q(z; \zeta)/\partial\bar{\zeta}_1 = 0$ and that $\{z_1, q\} \neq 0$ on ${}^t f'^{-1}(T^*_Z Y) \cap \mathrm{char}(\mathscr{M})$, one easily sees that the integral leaves of $H^{\mathbb{R}}_{\mathrm{Re}\, q}$, $H^{\mathbb{R}}_{\mathrm{Im}\, q}$, are transversal to $Y \times_X T^*X$ at Λ'_i. Therefore, the union of these integral leaves issuing from Λ'_i is a Lagrangian manifold Λ_i of $(T^*X)^{\mathbb{R}}$, contained in $\mathrm{char}(\mathscr{M})$. One constructs the hypersurfaces Z_i of $X^{\mathbb{R}}$ similarly as in Proposition 3.1.3 and then chooses Ω_i as the half-spaces delimited by Z_i and containing ω. Q.E.D.

Consider the distinguished triangle

$$\bigoplus_{i=1}^{r-1} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \to \bigoplus_{i=1}^{r} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\Gamma_{\Omega_i}\mathscr{O}_X) \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}\Big(\mathscr{M}, \sum_i \mathrm{R}\Gamma_{\Omega_i}\mathscr{O}_X\Big) \overset{+1}{\to}$$

and apply the functor $\mathrm{R}\Gamma(Y; \cdot|_Y)$. By the Cauchy-Kowalevski theorem (0.1) we find that, if Y is Stein, the sequence

$$0 \to \bigoplus_{i=1}^{r-1} \Gamma(Y; \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y) \to \bigoplus_{i=1}^{r} \Gamma(Y; \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_i}\mathscr{O}_X)|_Y)$$

$$\to \Gamma(Y; \mathscr{H}om_{\mathscr{D}_X}\left(\mathscr{M}, \sum_i \Gamma_{\Omega_i}\mathscr{O}_X\right)\Bigg|_Y) \to 0$$

is exact. Hence, by (3.3.3) we get that the holomorphic solution of the Cauchy problem

$$\begin{cases} P(z, D)u(z) = 0, \\ D_1^h u(z)|_Y \in \Gamma(\omega; \mathscr{O}_Y), \quad 0 \leqslant h < m, \end{cases}$$

may be written as a sum $u = \sum_{i=1}^r u_i$, where $u_i \in \Gamma(\Omega_i \cap V; \mathscr{O}_X)$ satisfies the equation $Pu_i = 0$, V being an open neighborhood of Y in X.

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