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CAUCHY PROBLEM FOR HYPERBOLIC $\mathcal{D}$-MODULES WITH REGULAR SINGULARITIES

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Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module (e.g., an overdetermined system of partial differential equations) on the complexification of a real analytic manifold M. Assume that the characteristic variety of $\mathcal{M}$ is hyperbolic with respect to a submanifold $N \subset M$. Then, it is well-known that the Cauchy problem for $\mathcal{M}$ with data on N is well posed in the space of hyperfunctions.

In this paper, under the additional assumption that $\mathcal{M}$ has regular singularities along a regular involutive submanifold of real type, we prove that the Cauchy problem is well posed in the space of distributions.

When $\mathcal{M}$ is induced by a single differential operator (or by a normal square system) with characteristics of constant multiplicities, our hypotheses correspond to Levi conditions, and we recover a classical result.

1. Statement of the Result.

It is a classical result that the hyperbolic Cauchy problem for a linear partial differential operator with analytic coefficients is well posed in the framework of distributions, assuming that the operator has real simple characteristics, or better that it has real constant multiplicities and it satisfies the Levi conditions. The aim of this paper is to extend this last result to general systems of linear partial differential equations, that is, to $\mathcal{D}$-modules. The assumption of having real constant multiplicities is replaced by the requirement that the characteristic variety of the $\mathcal{D}$-module is contained in the complexification of a regular involutive submanifold of the conormal bundle to the real ambient space (under the hypothesis of hyperbolicity), and the Levi conditions are replaced by Kashiwara-Oshima's notion of regular singularities.

In order to make this statement precise, we begin by reviewing some basic notions in the theory of $\mathcal{D}$-modules. The reader not acquainted with $\mathcal{D}$-module theory may have a glimpse at Section 6: There, we make a comparison with more classical statements.

1.1. Let $f\colon Y \to X$ be a morphism of complex analytic manifolds, denote by $\pi\colon T^*X \to X$ the cotangent bundle to X, and consider the microlocal correspondence associated to f:

$$T^*Y \underset{{}^t f'}{\longleftarrow} Y \times_X T^*X \underset{f_\pi}{\longrightarrow} T^*X.$$

One says that f is *non-characteristic* for a closed conic subset $V \subset T^*X$ if

$${}^t f'^{-1}(T^*_Y Y) \cap f_\pi{}^{-1}(V) \subset Y \times_X T^*_X X,$$

where $T^*_X X$ denotes the zero-section of T^*X.

Let $\mathcal{O}_X$ be the structural sheaf of X, and $\mathcal{D}_X$ the sheaf of rings of holomorphic linear differential operators on X. We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_X)$ the abelian category of (left) coherent $\mathcal{D}_X$-modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ the full triangulated subcategory of the derived category of $\mathcal{D}_X$-modules, whose objects have bounded amplitude and coherent cohomology groups. To $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ one associates its characteristic variety $\mathrm{char}(\mathcal{M})$, a closed $\mathbb{C}^\times$-conic involutive subvariety of T^*X. One says that f is non-characteristic for $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ if f is non-characteristic for $\mathrm{char}(\mathcal{M})$.

A natural operation of inverse image is defined in the derived category of $\mathcal{D}$-modules by:

$$\underline{f}^{-1}\mathcal{M} = \mathcal{D}_{Y\to X} \otimes^L_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M},$$

where f^{-1} denotes the sheaf-theoretical inverse image, and $\mathcal{D}_{Y\to X}$ the transfer bi-module. If f is non-characteristic for $\mathcal{M}$, $\underline{f}^{-1}\mathcal{M}$ belongs to $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_Y)$. Moreover, as shown by Kashiwara [10, Theorems 2.2.1, 2.3.4], the Cauchy-Kovalevskaya theorem for general systems (i.e., for $\mathcal{D}$-modules) is expressed in terms of derived categories as follows:

Theorem 1.2. *Let $f\colon Y \to X$ be a morphism of complex manifolds. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. Assume that f is non-characteristic for $\mathcal{M}$. Then, the natural morphism:*

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \to R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, \mathcal{O}_Y), \tag{1.1}$$

is an isomorphism.

More generally, an isomorphism like (1.1), with $\mathcal{O}$ replaced by another function space $\mathcal{F}$, asserts the well-poseness of the Cauchy problem for $\mathcal{M}$ in $\mathcal{F}$.

1.3. Let now $f_N\colon N\to M$ be a morphism of real analytic manifolds, and let $f\colon Y\to X$ be a complexification of f_N. Denote by $\pi_M\colon T^*_MX\to M$ the conormal bundle to M in X. Recall that the normal bundle $T_{T^*_MX}T^*X$ is naturally identified to $T^*T^*_MX$ via the Hamiltonian isomorphism $H\colon TT^*X\to T^*T^*X$. One denotes by $C_{T^*_MX}(V)$ the Whitney normal cone along T^*_MX to a conic subset $V\subset T^*X$. This is a closed conic subset of $T_{T^*_MX}T^*X\simeq T^*T^*_MX$, which describes the set of normal directions to T^*_MX lying in V. Consider the microlocal correspondence associated to f_N:

$$T^*_NY \xleftarrow[{}^tf_N{}']{} N\times_M T^*_MX \xrightarrow[f_{N\pi}]{} T^*_MX.$$

Definition 1.4. One says that f_N is *hyperbolic* for V if $f_{N\pi}$ is non-characteristic for $C_{T^*_MX}(V)$. One says that f_N is hyperbolic for $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ if f_N is hyperbolic for $\mathrm{char}(\mathcal{M})$.

We refer to Section 2 for a more detailed discussion on the notion of hyperbolicity, and to Section 6 for the relation between this notion and the classical notion of (weakly) hyperbolic differential operators.

As it was proved by Bony-Schapira [4] for a single differential operator, and by Kashiwara-Schapira [13] for general systems, the hyperbolic Cauchy problem is well posed in the sheaf $\mathcal{B}$ of Sato's hyperfunctions:

Theorem 1.5. *Let $f_N\colon N\to M$ be a morphism of real analytic manifolds, and let $f\colon Y\to X$ be a complexification of f_N. Let $\mathcal{M}\in\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. Assume that f_N is hyperbolic for $\mathcal{M}$. Then, there is a natural isomorphism:*

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{B}_M)\xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M},\mathcal{B}_N). \tag{1.2}$$

In the framework of the microlocal theory of sheaves, Kashiwara-Schapira [14, Chapter 10] gave a proof of (1.2) which is almost purely geometrical, in that it borrows from the theory of PDE the only Cauchy-Kovalevskaya theorem (in its sharp form due to Leray).

1.6. As it is well-known, when switching from hyperfunctions to distributions the geometric condition of hyperbolicity is no longer sufficient for the Cauchy problem to be well posed. Additional analytic conditions should be imposed on the system. For example, take $M=\mathbb{R}^2\ni(x_1,x_2)$, $N=\{x_1=0\}$, f_N the natural embedding. Then, the Cauchy problem for the operator $D_1^2-D_2$ is not well-posed for distributions.

When $\mathcal{M}$ is the $\mathcal{D}$-module associated to a single differential operator or to a normal square system whose characteristics have constant multiplicities, these additional conditions are the so-called Levi conditions (see e.g., Hörmander [8], Matsumoto [17], Vaillant [22]). In the case of general $\mathcal{D}$-modules, a related notion is that of system with regular singularities in the sense of Kashiwara-Oshima [12] (see Example 1.1 of loc. cit.). We postpone to Section 3 a short review on the definition and the microlocal model for $\mathcal{D}$-modules with regular singularities. We refer to Section 6 for a discussion on the relations between Levi conditions and systems with regular singularities.

For $V \subset T^*X$, we set $\dot{V} = V \cap \dot{T}^*X$, where $\dot{\pi}_X \colon \dot{T}^*X \to X$ denotes the cotangent bundle with the zero-section removed. The canonical one-form α_X endows T^*X with a structure of homogeneous symplectic manifold. A conic involutive submanifold V of a homogeneous symplectic manifold is called *regular* if the restriction to V of the canonical one-form is everywhere different from zero. The one-form $2\operatorname{Im}\alpha_X$ induces a structure of real homogeneous symplectic manifold on T^*_MX, of which T^*X is a natural complexification.

We can now state our main result. We denote by $\mathcal{D}b$ the sheaf of Schwartz's distributions.

Theorem 1.7. *Let $f_N \colon N \to M$ be a morphism of real analytic manifolds, let $f \colon Y \to X$ be a complexification of f_N, and let $V \subset T^*X$ be a closed $\mathbb{C}^\times$-conic subvariety. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. Assume that:*

(o) *$\dot{V} \cap T^*_MX$ is a closed smooth regular involutive submanifold of $\dot{T}^*_MX$, of which $\dot{V}$ is a complexification,*

(i) *f_N is hyperbolic for V,*

(ii) *$\mathcal{M}$ has regular singularities along $\dot{V}$.*

Then, there is a natural isomorphism:

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_Y}(\underline{f}^{-1}\mathcal{M}, \mathcal{D}b_N). \tag{1.3}$$

Under hypothesis (o), the condition of f_N being hyperbolic for V can be reformulated in terms of other properties, easier to be checked. We refer to Section 2 (and in particular to Proposition 2.6) for details on this point.

1.8. In general, it is not possible to restrict a distribution to a submanifold, unless a transversality condition is imposed on its wavefront set. In other words, the morphism underlying the isomorphism (1.3) is not defined for an arbitrary $\mathcal{D}_X$-module $\mathcal{M}$. It exists only because f is non-characteristic for $\mathcal{M}$, due to Lemma 2.4. A way to construct this morphism is by the use of microlocal analysis, i.e., by replacing the sheaf of distributions by the sheaf of tempered microfunctions. We refer to [13] or [14] for such a construction in the framework of hyperfunctions. Here, we will choose another approach. The proof of Theorem 1.7 that we will give in Section 5 is divided in several steps. Let us describe the main lines of our arguments.

Decomposing f by its graph, we first reduce to the case of a closed embedding. Using the classical division theorem, we further reduce to prove a result of propagation for the distribution solutions to $\mathcal{M}$. Expressed in terms of the tempered microlocalization functor, we see that such a problem is of a microlocal nature. By performing a contact transformation, we then reduce to a simple geometric model. Moreover, the theory of microdifferential operators, and an argument due to [10], allow us to replace $\mathcal{M}$ with a partial de Rham system. The propagation result becomes evident.

1.9. As we will recall in Lemma 2.7, if V is a hypersurface hypothesis (o) in Theorem 1.7 is partly implied by hypothesis (i). We thus get the following statement:

Let $f_N\colon N \to M$ be the embedding of a smooth hypersurface, let $f\colon Y \to X$ be a complexification of f_N, and let $V \subset T^*X$ be a closed $\mathbb{C}^\times$-conic hypersurface. Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$. Assume hypotheses (i) *and* (ii) *of Theorem* 1.7, *and:*
(o)′ *$\dot{V}$ is a smooth regular hypersurface of $\dot{T}^*X$.*
Then, the isomorphism (1.3) *holds true.*

The case of $\dot{V}$ being a smooth hypersurface occurs for example when $V = \mathrm{char}(\mathcal{M})$, $\mathcal{M}$ being associated to a single operator or to a normal square system whose characteristics have constant multiplicities.

1.10. The plan of the rest of this paper goes as follows. In Section 2 we discuss the geometric notion of hyperbolicity. In Section 3 we review the notion of $\mathcal{D}$-modules with regular singularities, and we recall a theorem of [12] which gives the normal form of a microdifferential system with regular singularities along a smooth regular involutive submanifold. We describe in Section 4 the functor $T\mathcal{H}om$ of temperate cohomology introduced in [9], and the functor $T\mu hom$ of tempered microlocalization introduced in [3]. Having all the necessary tools at hand, we give in Section 5 a proof of our main theorem. We discuss in Section 6 the relations between our results and some more classical statements (namely, statements that do not make use of derived categories, nor of $\mathcal{D}$-module theory). Finally, we comment on some possible developments in Section 7.

2. Geometric Hyperbolicity.

In this section, mostly following Chapter 10 and the Appendix of [14], we review the geometric notion of hyperbolicity, and we recall some constructions in microlocal geometry. By the standard argument of decomposing a morphism via its graph, we will see in §5.1 that the only interesting case to consider here is that of a closed embedding.

2.1. Let M be a real analytic manifold, X be a complexification of M, and let $V \subset T^*X$ be a conic subset. Let us recall how the Whitney normal cone

$$C_{T^*_MX}(V) \subset T_{T^*_MX}T^*X \simeq T^*T^*_MX$$

of V along $T^*_M X$ is defined (see e.g. [14, Definition 4.1.1]). Let $(z) = (x+iy)$ be a system of local coordinates on X, and denote by $(z;\zeta) = (x+iy;\xi+i\eta)$ the associated system of symplectic coordinates on T^*X. Then, $(x;\eta)$ and $(x,\eta;\xi,y)$ are natural systems of coordinates on $T^*_M X$ and $T^*T^*_M X$, respectively. With these notations, $(x_\circ,\eta_\circ;\xi_\circ,y_\circ) \in C_{T^*_M X}(V)$ if and only if there exist sequences:

$$\begin{cases} (z_n;\zeta_n) \in V, \quad c_n \in \mathbb{R}_{>0}, \\ (x_n,y_n;\xi_n,\eta_n) \to (x_\circ,0;0,\eta_\circ), \quad c_n(y_n,\xi_n) \to (y_\circ,\xi_\circ). \end{cases}$$

Let $f_N\colon N \to M$ be a closed embedding of real analytic manifolds, and let $f\colon Y \to X$ be a complexification of f_N. Consider the identification

$$T^*M \hookrightarrow T^*_M X \times_M T^*M \underset{{}^t\pi_M{}'}{\hookrightarrow} T^*T^*_M X, \tag{2.1}$$

where the first map abuts to the zero section of $T^*_M X$. Using (2.1), one has the alternative descriptions below of the notion of hyperbolicity recalled in Definition 1.4 (we thank Pierre Schapira for pointing out to us the implication (iii) $\implies$ (ii)).

Lemma 2.2. *Let f_N be a closed embedding. Then, the following conditions are equivalent:*

(i) *N (i.e., f_N) is hyperbolic for V,*

(ii) *$\dot{T}^*_N M \cap C_{T^*_M X}(V) = \varnothing$,*

(iii) *$T^*_N X \cap V \subset T^*_M X$ and $\dot{T}^*_N M \cap C_{\dot{T}^*_M X}(\dot{V}) = \varnothing$.*

Proof. The equivalence of (i) and (ii) is clear. To prove the equivalence of (ii) and (iii) one may work in a local coordinate system as above. Let N be given by $x' = 0$ in M, for $(x) = (x',x'')$. Then, (ii) is satisfied if and only if, given $(x''_\circ,\xi'_\circ) \in T^*_N M$ with $\xi'_\circ \neq 0$, there are no sequences

$$\begin{cases} (z_n;\zeta_n) \in V, \quad c_n \in \mathbb{R}_{>0}, \\ (x'_n,x''_n,y_n;\xi_n,\eta_n) \to (0,x''_\circ,0;0,0), \quad c_n(y_n,\xi'_n,\xi''_n) \to (0,\xi'_\circ,0). \end{cases} \tag{2.2}$$

To prove that (ii) implies (iii), assume by absurd that $T^*_N X \cap V \not\subset T^*_M X$. In other words, assume that there exists a point $(z'_\circ,z''_\circ;\zeta'_\circ,\zeta''_\circ) = (0,x''_\circ;\zeta'_\circ,i\eta''_\circ) \in V$ with $\xi'_\circ \neq 0$. Then, the sequences $c_n = n$ (or any diverging sequence, for that matters) and $(z'_n,z''_n;\zeta'_n,\zeta''_n) = (0,x''_\circ;\zeta'_\circ/c_n,i\eta''_\circ/c_n)$ satisfy (2.2), thus contradicting the hypothesis. The proof that (iii) implies (ii) is similar. ∎

2.3. Lemma 2.2 is a statement that does not take into account the fact that X is a complexification of M. On the contrary, the following results depend on the complex structure.

Lemma 2.4. *Let $N \subset M$ be real analytic manifolds, let $Y \subset X$ be their respective complexifications, and let $V \subset T^*X$ be a closed $\mathbb{C}^\times$-conic subset. Assume that $T^*_N X \cap V \subset T^*_M X$. Then, Y is non-characteristic for V (i.e., $T^*_Y X \cap V \subset T^*_X X$).*

Proof. Since V is closed, after possibly shrinking X around M, it is enough to check that $(N \times_Y T^*_Y X) \cap V \subset T^*_X X$. This is clear since, on one hand $(N \times_Y T^*_Y X) \cap V$ is $\mathbb{C}^\times$-conic, and on the other hand $(N \times_Y T^*_Y X) \cap V \subset T^*_M X$ by hypothesis. □

Definition 2.5. A closed conic subset $W \subset T^*X$ is called *non-glancing* with respect to a conic involutive submanifold $V \subset T^*X$ if, for any $p \in W \cap V$ and any germ at p of regular holomorphic function φ vanishing on W, the Hamiltonian vector $H_\varphi(p)$ is not tangent to V.

Proposition 2.6. *Let $N \subset M$ be real analytic manifolds, let $Y \subset X$ be their respective complexifications, and let $V \subset T^*X$ be a closed $\mathbb{C}^\times$-conic subset. Assume hypothesis* (o) *of Theorem 1.7. Then, N is hyperbolic for V if and only if:*
(i)′ *$T^*_N X \cap V \subset T^*_M X$ and $Y \times_X T^*X$ is non-glancing with respect to $\dot{V}$.*

Proof. Since $\dot{V}$ is a complexification of $\dot{T}^*_M X \cap V$, $Y \times_X T^*X$ is non-glancing with respect to $\dot{V}$ if and only if for any local equation φ of N in M, $H_{\varphi^{\mathbb{C}}} \notin T\dot{V}$, where $\varphi^{\mathbb{C}}$ denotes a complexification of φ. By Lemma 2.2, we thus have to prove that this is equivalent to $\dot{T}^*_N M \cap C_{\dot{T}^*_M X}(\dot{V}) = \varnothing$.

Consider the maps:

$$T^*M \underset{j}{\longrightarrow} T_{\dot{T}^*_M X} T^*X \underset{p}{\longleftarrow} \dot{T}^*_M X \times_{\dot{T}^*X} T\dot{T}^*X,$$

where j is obtained by composing (2.1) with the Hamiltonian $-H$, and p is the natural projection. Since $\dot{V}$ is smooth, $C_{\dot{T}^*_M X}(\dot{V}) = p(\dot{T}^*_M X \times_{\dot{T}^*X} T\dot{V})$. Moreover, one checks that $j(d\varphi) = p(H_{\varphi^{\mathbb{C}}})$. The equivalence is then clear. □

Lemma 2.7. *Let $N \subset M$ be real analytic manifolds, let $Y \subset X$ be their respective complexifications, and let $V \subset T^*X$ be a closed $\mathbb{C}^\times$-conic subset. Assume that N is hyperbolic for V, and that $\dot{V}$ is a hypersurface of T^*X. Then, $\dot{V}$ is a complexification of $T^*_M X \cap \dot{V}$.*

Proof. Locally on X, V is given as the zero-set of a single homogeneous polynomial $g(z;\zeta) = \sum_{|\alpha|\leq m} g_\alpha(z)\zeta^\alpha$, with holomorphic coefficients in the base variables. It is then classical to prove that, up to a non-zero multiplicative constant, the coefficients of $g(z;\zeta)$ are real valued for $(z;\zeta) \in T^*_M X$ (see e.g. [8]). In fact, these are given by the product of the leading term by the elementary symmetric functions in the roots of $g(x;i\eta)$, and these roots are real due to the hyperbolicity hypothesis. □

Remark 2.8. The simple example $M \simeq \mathbb{R}^n \ni (x_1, \dots x_n)$, $X \simeq \mathbb{C}^n \ni (z_1, \dots z_n)$, $T^*X \ni (z;\zeta)$, $N = \{x_1 = 0\} \subset M$, $V = \{\zeta_1 = \zeta_2 - i\zeta_3 = 0\} \subset T^*X$, shows that the claim of the above lemma is false if one removes the hypothesis of V being a hypersurface.

2.9. Let us recall a result of "geometric optics" asserting that, locally, a pair of non-glancing involutive submanifolds can be simultaneously straight-en-out (see [7] or [21]).

Lemma 2.10. *Let V be a conic regular involutive submanifold of $\dot{T}^*X$ of codimension c_V. Assume that $Y \times_X T^*X$ is non-glancing with respect to V. Then, at every point $p \in V \cap (Y \times_X T^*X)$ there exists a germ of contact transformation*

$$\chi_p \colon T^*X \to T^*(X' \times X''),$$

such that $\chi_p(V) = T^*_{X'}X' \times U$, $\chi_p(Y \times_X T^*X) = \{(z;\zeta)\colon z' = 0\} \times T^*X''$. Here, X' and X'' are manifolds of dimension c_V and $\dim X - c_V$, respectively, U is an open subset of $\dot{T}^*X''$, $(z) = (z', z'')$ is a local coordinate system on X', and $(z;\zeta)$ is the associated system of symplectic coordinates on T^*X'.

If moreover V is a complexification of $T^*_M X \cap V$, then we may assume that at every $p \in V \cap (N \times_M T^*_M X)$ the contact transformation χ_p is the complexification of a contact transformation on $T^*_M X$.

3. Modules with Regular Singularities.

We review in this section some notions and results from Kashiwara-Oshima [12].

3.1. Let $\mathcal{E}_X$ be the ring of microdifferential operators of finite order. We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X)$ the abelian category of (left) coherent $\mathcal{E}_X$-modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X)$ the full triangulated subcategory of the derived category of $\mathcal{E}_X$-modules, whose objects have bounded amplitude and coherent cohomology groups.

The ring $\mathcal{E}_X$ is naturally endowed with a $\mathbb{Z}$-filtration by the degree, and we denote by $\mathcal{E}_X(k)$ the sheaf of operators of degree at most k. Denote by $\mathcal{O}_{T^*X}(k)$ the sheaf of holomorphic functions on T^*X, homogeneous of degree k in the fiber variables.

Let $V \subset T^*X$ be a conic regular involutive submanifold. Denote by $\mathcal{I}_V(k)$ the sheaf ideal of sections of $\mathcal{O}_{T^*X}(k)$ vanishing on V. Let $\mathcal{E}_V$ be the subalgebra of $\mathcal{E}_X$ generated over $\mathcal{E}_X(0)$ by the sections P of $\mathcal{E}_X(1)$ such that $\sigma_1(P)$ belongs to $\mathcal{I}_V(1)$ (here $\sigma_1(\cdot)$ denotes the symbol of order 1). For example, if $X = X' \times X''$, and $V = T^*_{X'}X' \times U$ for an open subset $U \subset \dot{T}^*X''$, then $\mathcal{E}_V$ is the subalgebra of $\mathcal{E}_X$ generated over $\mathcal{E}_X(0)$ by the differential operators of X'.

Definition 3.2. Let $\mathcal{P}$ be a coherent $\mathcal{E}_X$-module. One says that $\mathcal{P}$ has *regular singularities along* V if locally there exists a coherent sub-$\mathcal{E}_X(0)$-module $\mathcal{P}_0$ of $\mathcal{P}$ which generates it over $\mathcal{E}_X$, and such that $\mathcal{E}_V\mathcal{P}_0 \subset \mathcal{P}_0$. One says that $\mathcal{P}$ is *simple along* V if locally there exists an $\mathcal{E}_X(0)$-module $\mathcal{P}_0$ as above such that $\mathcal{P}_0/\mathcal{E}_X(-1)\mathcal{P}_0$ is a locally free $\mathcal{O}_V(0)$-module of rank one.

We will denote by $\mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$ the thick abelian subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{E}_X)$ whose objects have regular singularities along V. We denote by $\mathbf{D}^{\mathrm{b}}_{RS(V)}(\mathcal{E}_X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{E}_X)$ whose objects have cohomology groups with regular singularities along V. To $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ we associate its microlocalization

$$\mathcal{E}\mathcal{M} = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1}\mathcal{M}.$$

We say that $\mathcal{M}$ has regular singularities along V if $\mathcal{E}\mathcal{M}$ has regular singularities along V.

Remark 3.3. Let us denote by $\mathrm{char}^1_V(\mathcal{P})$ the 1-micro-characteristic variety of a coherent $\mathcal{E}_X$-module, introduced by Laurent [16] and Monteiro Fernandes [18] (see also [21]). This is a subvariety of the normal bundle $\tau\colon T_VT^*X \to T^*X$. Then one can show that $\mathcal{P}$ has regular singularities along V if and only if $\mathrm{char}^1_V(\mathcal{P}) \subset V$, the zero section of T_VT^*X.

3.4. One of the main results of Sato-Kawai-Kashiwara [20] asserts that contact transformations can be quantized to give an equivalence of categories at the level of $\mathcal{E}$-modules. Note that the notion of module with regular singularities is invariant by quantized contact transformations, and that a module with regular singularities along V is supported by V (see [11, Lemma 1.13]).

Theorem 3.5 ([12, Theorem 1.9])**.** *Let $V = T^*_{X'}X' \times U \subset T^*(X' \times X'')$, where $U \subset \dot{T}^*X''$ is an open subset. If $\mathcal{P}$ is simple along V, then it is* *isomorphic to the partial de Rham system* $\mathcal{O}_{X'} \boxtimes \mathcal{E}_{X''}$. *If* $\mathcal{P} \in \mathrm{Mod}_{RS(V)}(\mathcal{E}_X)$, *then it is a quotient of a multiple partial de Rham system:*

$$(\mathcal{O}_{X'} \boxtimes \mathcal{E}_{X''})^N \to \mathcal{P} \to 0.$$

Here, $\boxtimes$ *denotes the exterior tensor product for* $\mathcal{E}$*-modules.*

According to Lemma 2.10, $T^*_{X'}X' \times U$ is the microlocal model of any regular involutive submanifold. The above characterization of systems with regular singularities will be a key point in the proof of our main result.

4. Functors of Temperate Cohomology.

Here, we review some definitions and results from [14], [9], [15], and [3].

4.1. Let M be a real analytic manifold. We denote by $\mathrm{Mod}(\mathbb{C}_M)$ the category of sheaves of $\mathbb{C}$-vector spaces on M, by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ its bounded derived category, and we consider the "six operations" of sheaf theory $R\mathcal{H}om(\cdot,\cdot)$, $\cdot \otimes \cdot$, $Rf_!$, Rf_*, f^{-1}, $f^!$. If $A \subset M$ is a locally closed subset, we denote by $\mathbb{C}_A$ the sheaf on M which is the constant sheaf on A with stalk $\mathbb{C}$, and zero on $M \setminus A$. For $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$, we set $D'F = R\mathcal{H}om(F, \mathbb{C}_M)$. Let $f\colon N \to M$ be a morphism of manifolds. One defines the relative dualizing complex by $\omega_{N/M} = f^!\mathbb{C}_M$. Recall that one has an isomorphism $\omega_{N/M} \cong or_{N/M}[\dim N - \dim M]$, where $or_{N/M}$ denotes the relative orientation sheaf.

4.2. One denotes by $SS(F)$ the micro-support of $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$, a closed conic involutive subset of T^*M which describes the directions of non-propagation for the cohomology of F. Let X be a complex manifold, and let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. By using the Cauchy-Kovalevskaya theorem (in its sharp form due to Leray), the following estimate is obtained in [**14**, Theorem 11.3.3]:

$$SS(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) \subset \mathrm{char}(\mathcal{M}). \tag{4.1}$$

4.3. For $p \in T^*X$, one considers the category $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; p)$, which is the localization of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$ by the null system $\mathcal{N} = \{F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X) \colon p \notin SS(F)\}$. Let $\chi_p\colon T^*X \to T^*X$ be a germ of contact transformation. By [**14**, Theorem 7.2.1], to χ_p one may associate a kernel $K \in \mathbf{D}^{\mathrm{b}}(X \times X; (p, -\chi_p(p)))$, whose associated integral transform:

$$\Phi_K \colon \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; p) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; \chi_p(p))$$

is an equivalence of categories. Moreover, [14, Theorem 7.5.6] asserts that simple sheaves along a Lagrangian manifold Λ are interchanged by Φ_K with simple sheaves along $\chi_p(\Lambda)$. In particular, if $\chi_p(T^*_S X) = T^*_Z X$ for (real) submanifolds $S, Z \subset X$, then

$$\Phi_K(\mathbb{C}_S) \simeq \mathbb{C}_Z[d], \tag{4.2}$$

for a shift d determined by means of Maslov's inertia index.

4.4. Denote by $\mathrm{Mod}_{\mathbb{R}-c}(\mathbb{C}_M)$ the abelian category of $\mathbb{R}$-constructible sheaves on a real analytic manifold M, and by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ whose objects have cohomology groups belonging to $\mathrm{Mod}_{\mathbb{R}-c}(\mathbb{C}_M)$. The functor of temperate cohomology

$$T\mathcal{H}om(\cdot, \mathcal{D}b_M)\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_M),$$

was introduced in [9]. It is characterized in [15] by being exact, and by the requirement that if Z is a closed subanalytic subset of M, then

$$T\mathcal{H}om(\mathbb{C}_Z, \mathcal{D}b_M) = \Gamma_Z \mathcal{D}b_M.$$

Let X be a complex manifold, and denote by $\overline{X}$ the associated anti-holomorphic manifold, by $X^{\mathbb{R}}$ the underlying real analytic manifold, and identify $X^{\mathbb{R}}$ to the diagonal of $X \times \overline{X}$. For $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$, one sets:

$$T\mathcal{H}om(F, \mathcal{O}_X) = R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, T\mathcal{H}om(F, \mathcal{D}b_{X^{\mathbb{R}}})).$$

In other words, one defines $T\mathcal{H}om(F, \mathcal{O}_X)$ as the Dolbeault complex with coefficients in $T\mathcal{H}om(F, \mathcal{D}b_{X^{\mathbb{R}}})$.

4.5. Let M be a real analytic manifold, let X be a complexification of M, and denote by $j\colon M \hookrightarrow X$ the natural embedding. For $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)$, there is a natural isomorphism (see [3]):

$$T\mathcal{H}om(Rj_*F, \mathcal{O}_X) \simeq Rj_*T\mathcal{H}om(F, \mathcal{D}b_M) \otimes \omega_{M/X}. \tag{4.3}$$

For example, if $N \subset M$ is a submanifold, one has:

$$\begin{cases} T\mathcal{H}om(D'\mathbb{C}_M, \mathcal{O}_X) = \mathcal{D}b_M, \\ T\mathcal{H}om(D'\mathbb{C}_N, \mathcal{O}_X) = \Gamma_N \mathcal{D}b_M \otimes \omega_{N/M}^{\otimes -1}. \end{cases}$$

4.6. A tempered microlocalization of the functor $T\mathcal{H}om$ has been introduced in [3]. As it was the case for $T\mathcal{H}om$, this functor is constructed in two steps. First, one defines on a real analytic manifold M the functor:

$$T\mu hom(\cdot, \mathcal{D}b_M)\colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\pi_M^{-1}\mathcal{D}_M).$$

Then, on a complex analytic manifold X one sets:

$$T\mu hom(F, \mathcal{O}_X) = R\mathcal{H}om_{\pi^{-1}_{X^{\mathbb{R}}}\mathcal{D}_{\overline{X}}}(\pi^{-1}_{X^{\mathbb{R}}}\mathcal{O}_{\overline{X}}, T\mu hom(F, \mathcal{D}b_{X^{\mathbb{R}}})).$$

For example, if X is a complexification of M, one recovers the sheaf of tempered microfunctions on $T^*_M X$ by:

$$\mathcal{C}^f_M \simeq T\mu hom(D'\mathbb{C}_M, \mathcal{O}_X).$$

Recall that, for $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$, the tempered analogous of Sato's distinguished triangle reads:

$$D'F \otimes \mathcal{O}_X \to T\mathcal{H}om(F, \mathcal{O}_X) \to R\dot{\pi}_{X*} T\mu hom(F, \mathcal{O}_X) \underset{+1}{\longrightarrow} . \tag{4.4}$$

By the estimate:

$$\mathrm{supp}(T\mu hom(F, \mathcal{O}_X)) \subset SS(F),$$

it follows that, for $p \in T^*X$, the functor $T\mu hom(\cdot, \mathcal{O}_X)$ induces a functor:

$$T\mu hom(\cdot, \mathcal{O}_X) \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X; p)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{T^*X}; p).$$

Using the notations of §4.3, for a suitable choice of the kernel K, there is an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_{T^*X}; \chi_p(p))$:

$$\chi_{p*} T\mu hom(F, \mathcal{O}_X) \simeq T\mu hom(\Phi_K(F), \mathcal{O}_X). \tag{4.5}$$

5. Proof of the Main Result.

We give here a proof of Theorem 1.7.

5.1. Recall that we are given a morphism of real analytic manifolds $f_N \colon N \to M$, of which $f \colon Y \to X$ is a complexification. Let us decompose f by its graph:

$$Y \underset{\delta_f}{\longrightarrow} Y \times X \underset{q}{\longrightarrow} X,$$

where δ_f is the closed embedding $\delta_f(y) = (y, f(y))$, and q is the second projection. Accordingly, we may factorize the isomorphism (1.3) as:

$$\begin{aligned} \delta_f^{-1} q^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M) &\to \delta_f^{-1} R\mathcal{H}om_{\mathcal{D}_{Y\times X}}(\underline{q}^{-1}\mathcal{M}, \mathcal{D}b_{N\times M}) \\ &\to R\mathcal{H}om_{\mathcal{D}_Y}(\underline{\delta_f}^{-1}(\underline{q}^{-1}\mathcal{M}), \mathcal{D}b_N). \end{aligned} \tag{5.1}$$

The first isomorphism is induced by the natural morphism:

$$q^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M) \to R\mathcal{H}om_{\mathcal{D}_{Y\times X}}(\underline{q}^{-1}\mathcal{M}, \mathcal{D}b_{N\times M}).$$

By "dévissage", in order to prove that it is an isomorphism, it is enough to consider the case $\mathcal{M} = \mathcal{D}_X$. Thus, we have to prove the isomorphism:

$$q^{-1}\mathcal{D}b_M \simeq R\mathcal{H}om_{\mathcal{D}_{Y\times X}}(\mathcal{O}_Y \underline{\boxtimes} \mathcal{D}_X, \mathcal{D}b_{N\times M}).$$

This is obvious, since the left hand side is the space of distributions on $N \times M$ which are constant along the fibers of q, and the right hand side is the solution complex of the q-relative de Rham system with values in $\mathcal{D}b_{N\times M}$.

It is easy to see that hypotheses (o), (i) and (ii) of Theorem 1.7 imply the analogous hypotheses when replacing the morphism f_N by $\delta_{f_N}\colon N\to N\times M$, the module $\mathcal{M}$ by $\underline{q}^{-1}\mathcal{M}\simeq\mathcal{O}_Y\boxtimes\mathcal{D}_X$, and V by $T^*_YY\times V$. Hence, in order to prove that the second morphism in (5.1) is an isomorphism, we are reduced to treat the case of f_N being a closed embedding.

5.2. Let $f_N\colon N\to M$ be a closed embedding of real analytic manifolds, and let $f\colon Y\to X$ be a complexification of f_N. In this case, the morphism underlying the isomorphism (1.3) may be constructed as the composition of the morphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{D}b_M)|_N\to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_N\mathcal{D}b_M)\otimes\omega_{N/M}^{\otimes -1},\tag{5.2}$$

with the inverse of the "division" isomorphism (where we use the hypothesis of f being non-characteristic for $\mathcal{M}$):

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_N\mathcal{D}b_M)\otimes\omega_{N/M}^{\otimes -1}\xleftarrow{\sim}R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{D}b_N).\tag{5.3}$$

These results are classical. In the language of temperate cohomology, the morphism in (5.2) is obtained by applying the functor

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},T\mathcal{H}om(D'(\cdot),\mathcal{O}_X))|_Y\tag{5.4}$$

to the natural morphism $\mathbb{C}_M\to\mathbb{C}_N$, and (5.3) is a simple application of [15, Theorem 7.2] (in their notations, take $G=\omega_{N/Y}$).

Our statement will be proved if we show that (5.2) is also an isomorphism.

5.3. Let $H\subset M$ be a submanifold such that $N\subset H$. Since N is hyperbolic for $\mathcal{M}$, a fortiori H is hyperbolic for $\mathcal{M}$. Thus, the morphism (5.2) is an isomorphism if and only if for any such H the morphism:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_H\mathcal{D}b_M)|_N\otimes\omega_{H/M}^{\otimes -1}\to R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\Gamma_N\mathcal{D}b_M)\otimes\omega_{N/M}^{\otimes -1}\tag{5.5}$$

is an isomorphism. By considering a chain of submanifolds:

$$N=H_0\subset H_1\subset\cdots\subset H_{\operatorname{codim}_H N}=H,$$

we easily reduce to the case $\operatorname{codim}_H N=1$. Let $H^\pm$ be the two open connected components of $H\setminus N$ (locally at N). Applying the functor (5.4) to the short exact sequence

$$0\to\mathbb{C}_{H^+}\oplus\mathbb{C}_{H^-}\to\mathbb{C}_H\to\mathbb{C}_N\to 0,\tag{5.6}$$

we get a distinguished triangle attached to the morphism (5.5), whose third term is given by the complex $C_+ \oplus C_-$, where we set:

$$C_\pm = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, T\mathcal{H}om(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))|_N. \tag{5.7}$$

Thus, (5.5) is an isomorphism if and only if the complexes $C_\pm$ vanish. Following a classical argument, by considering the distinguished triangle associated to the truncation functors:

$$\tau^{\leq n-1}(\cdot) \to \tau^{\leq n}(\cdot) \to H^n(\cdot)[-n] \xrightarrow[+1]{}, \tag{5.8}$$

and reasoning by induction on the amplitude of $\mathcal{M}$, we may further assume that $\mathcal{M}$ is concentrated in degree zero.

5.4. Consider the complexes

$$D_\pm = R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}\mathcal{M}, T\mu hom(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))|_N. \tag{5.9}$$

(This makes sense, since $T\mu hom(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X)$ is concentrated in degree zero, and hence well defined in the derived category of $\mathcal{E}_X$-modules.) Using Sato's distinguished triangle (4.4) for $F = D'\mathbb{C}_{H^\pm}$, we get the distinguished triangle

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathbb{C}_{H^\pm} \otimes \mathcal{O}_X)|_N \to C_\pm \to R\dot{\pi}_*(D_\pm) \xrightarrow[+1]{}. \tag{5.10}$$

Since $\mathcal{M}$ is coherent and $H^\pm \cap N = \varnothing$, the first term in (5.10) vanishes. We are thus left to prove the vanishing of the complexes $D_\pm$ outside of the zero-section. Note that one has the estimates:

$$\begin{aligned} \pi^{-1}(N) \cap \operatorname{supp}(D_\pm) &\subset \pi^{-1}(N) \cap \operatorname{char}(\mathcal{M}) \cap SS(D'\mathbb{C}_{H^\pm}) \\ &\subset V \cap T^*_N X \\ &\subset V \cap (N \times_M T^*_M X), \end{aligned}$$

where the second inclusion follows the fact that $\mathcal{M}$ has regular singularities along V, and hence $\operatorname{supp}(\mathcal{E}\mathcal{M}) = \operatorname{char}(\mathcal{M}) \subset V$, and the last inclusion follows from Lemma 2.2. Setting $\mathcal{P} = \mathcal{E}\mathcal{M}$, and using Proposition 2.6, we are thus reduced to prove the following statement:

5.5. *Assume that $\dot{V} \cap T^*_M X$ is a smooth regular involutive submanifold of $T^*_M X$, of which $\dot{V}$ is a complexification, assume that $Y \times_X T^*X$ is non-glancing with respect to $\dot{V}$, and let $\mathcal{P} \in \mathrm{Mod}_{RS(\dot{V})}(\mathcal{E}_X)$. Then:*

(5.11)
$$R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{P}, T\mu hom(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))_p = 0, \quad \text{for } p \in \dot{V} \cap (N \times_M T^*_M X).$$

5.6. Denote by c_V the codimension of V in T^*X, let M' and M'' be real analytic manifolds of dimension c_V and $\dim X - c_V$, respectively, and let $X' \supset M'$, $X'' \supset M''$ be complexifications. Let $(z) = (z_1, z', z'')$ be a local coordinate system on X', $z = x + iy$. For any $p \in \dot{V} \cap (N \times_M T^*_M X)$, by performing a contact transformation χ_p as in Lemma 2.10 (which is the complexification of a contact transformation on $T^*_M X$), we may assume:

$$\begin{cases} M = M' \times M'', \\ X = X' \times X'', \\ N = \{x_1 = x' = 0\} \times M'' \subset M, \\ H = \{x' = 0\} \times M'' \subset M, \\ V = T^*_{X'} X' \times U \subset T^*X, \quad U \text{ open in } \dot{T}^*X''. \end{cases}$$

In other words, χ_p straightens out V and preserves $N \times_M T^*_M X$ and $H \times_M T^*_M X$. Note that the Lagrangian manifolds $T^*_N X$ and $T^*_H X$ are also preserved by χ_p, since they can be realized as the union of the complexified bicharacteristic leaves of $N \times_M T^*_M X$ and $H \times_M T^*_M X$, respectively.

5.7. Let us now investigate the behavior of a quantization of χ_p on some locally constant sheaves.

Lemma 5.8. *In the above notations, associating to χ_p a quantization Φ_K, we have:*

$$\Phi_K(D'\mathbb{C}_{H^\pm}) \simeq D'\mathbb{C}_{H^\pm} \qquad \text{in } \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; \chi_p(p)).$$

Proof. Since N and H are completely real submanifolds of X, the shift d appearing in (4.2) is zero for $S = N, M$. Hence:

$$\Phi_K(\mathbb{C}_N) \simeq \mathbb{C}_N, \quad \Phi_K(\mathbb{C}_H) \simeq \mathbb{C}_H \qquad \text{in } \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; \chi_p(p)).$$

Since locally (i.e., neglecting the relative orientation) $D'\mathbb{C}_N \simeq \mathbb{C}_N[-\operatorname{codim}_X N]$, we also have:

$$\Phi_K(D'\mathbb{C}_N) \simeq D'\mathbb{C}_N, \quad \Phi_K(D'\mathbb{C}_H) \simeq D'\mathbb{C}_H \qquad \text{in } \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; \chi_p(p)). \tag{5.12}$$

Set $\Lambda^\pm = SS(D'\mathbb{C}_{H^\pm})$. There are isomorphisms in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; q)$:

$$D'\mathbb{C}_{H^\pm} \simeq \begin{cases} D'\mathbb{C}_H, & \text{for } q \in \Lambda^\pm \setminus T^*_N X, \\ D'\mathbb{C}_N[1], & \text{for } q \in \Lambda^\pm \setminus T^*_H X. \end{cases}$$

By (5.12), a similar description holds for $\Phi_K(D'\mathbb{C}_{H^\pm})$, possibly interchanging H^+ and H^-. Since $\Lambda^\pm$ is a Lagrangian manifold of class $\mathcal{C}^1$, in a neighborhood of p there is essentially (i.e., up to isomorphism) one simple sheaf of a given shift along $\Lambda^\pm$. Since $\Phi_K(D'\mathbb{C}_{H^\pm})$ and $D'\mathbb{C}_{H^\pm}$ are isomorphic at every $q \in \Lambda^\pm \setminus (T^*_N X \cap T^*_H X)$, it follows that they are isomorphic in $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_X; \chi_p(p))$. □

Recall that χ_p can also be quantized to induce an equivalence of categories at the level of $\mathcal{E}$-modules. By the above lemma, and using (4.5), we may thus assume that the claim in §5.5 is expressed in the geometrical setting of §5.6.

5.9. We use here an argument similar to [10, page 32]. By Theorem 3.5, $\mathcal{P}$ is a quotient of a multiple de Rham system. Hence, we have an exact sequence

$$0 \to \mathcal{Q} \to (\mathcal{O}_{X'} \boxtimes \mathcal{E}_{X''})^N \underset{\alpha}{\to} \mathcal{P} \to 0, \tag{5.13}$$

where $\mathcal{Q} = \ker \alpha$. Since $\mathrm{Mod}_{RS(\dot{V})}(\mathcal{E}_X)$ is an abelian category, $\mathcal{Q}$ has regular singularities along V. In particular, the hypothesis of the claim in §5.5 are satisfied if one replaces $\mathcal{P}$ by any of the three modules appearing in the above short exact sequence.

Let us suppose that we have proved (5.11) for the system $(\mathcal{O}_{X'} \boxtimes \mathcal{E}_{X''})^N$. Considering the long exact cohomology sequence obtained by applying the functor

$$R\mathcal{H}om_{\mathcal{E}_X}(\cdot, T\mu hom(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))_p$$

to (5.13), we would have isomorphisms:

$$\mathcal{E}xt^{j+1}_{\mathcal{E}_X}(\mathcal{P}, T\mu hom(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))_p \simeq \mathcal{E}xt^{j}_{\mathcal{E}_X}(\mathcal{Q}, T\mu hom(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))_p.$$

Starting from $j = -1$, we would thus get (5.11) for the system $\mathcal{P}$ by induction on j. We are then reduced to prove

$$R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{O}_{X'} \boxtimes \mathcal{E}_{X''}, T\mu hom(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))_p = 0, \quad p \in \dot{V} \cap (N \times_M T^*_M X).$$

5.10. Applying Sato's distinguished triangle as in §5.4, we further reduce to prove:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''}, T\mathcal{H}om(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X))|_N = 0.$$

Since $D'\mathbb{C}_{H^\pm} \simeq \mathbb{C}_{\overline{H}^\pm} \otimes \omega_{H/X}$, it follows from (4.3) that $T\mathcal{H}om(D'\mathbb{C}_{H^\pm}, \mathcal{O}_X) = \Gamma_{\overline{H}^\pm}\mathcal{D}b_M \otimes \omega_{H/M}^{\otimes -1}$. The above vanishing statement is then equivalent to:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''}, \Gamma_{\overline{H}^\pm}\mathcal{D}b_M)|_N = 0. \tag{5.14}$$

Denote by $j\colon L \to X$ a complexification of the embedding $H \hookrightarrow M$. We have:

$$\begin{aligned}&R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''}, \Gamma_{\overline{H}^\pm}\mathcal{D}b_M)|_N \\ &\simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''}, \underline{j}_*\Gamma_{\overline{H}^\pm}\mathcal{D}b_H)|_N \\ &\simeq R\mathcal{H}om_{\mathcal{D}_L}(\underline{j}^{-1}(\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''}), \Gamma_{\overline{H}^\pm}\mathcal{D}b_H)|_N.\end{aligned}$$

Here, $\underline{j}_*$ denotes the functor of direct image for $\mathcal{D}$-modules, and to get the second isomorphism we used the fact that j is non-characteristic for $\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''}$. Since $\underline{j}^{-1}(\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''})$ is again a partial de Rham system, we may assume that $H = M$, so that $\operatorname{codim}_M N = 1$ and $\overline{H}^\pm = \overline{M}^\pm$ are closed half-spaces of M.

5.11. Denoting by $D_{X'}$ the (commutative) ring of differential operators with constant coefficients on $X' \ni (z) = (z_1, \dots, z_{c_V})$, we have:

$$\mathcal{O}_{X'} = \mathcal{D}_{X'}/(\partial_{z_1}, \dots, \partial_{z_{c_V}}) = \mathcal{D}_{X'} \otimes_{D_{X'}} [D_{X'}/(\partial_{z_1}, \dots, \partial_{z_{c_V}})].$$

Since

$$D_{X'}/(\partial_{z_1}, \dots, \partial_{z_{c_V}}) = [D_{X'}/(\partial_{z_2}, \dots, \partial_{z_{c_V}})] \otimes_{D_{X'}} [D_{X'}/(\partial_{z_1})],$$

for any $\mathcal{D}_X$-module $\mathcal{N}$ we have

$$\begin{aligned}&R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_{X'} \boxtimes \mathcal{D}_{X''}, \mathcal{N}) \\ &\simeq R\mathcal{H}om_{D_X}(D_X/(\partial_{z_1}, \dots, \partial_{z_{c_V}}), \mathcal{N}) \\ &\simeq R\mathcal{H}om_{D_X}(D_X/(\partial_{z_2}, \dots, \partial_{z_{c_V}}), R\mathcal{H}om_{D_X}(D_X/(\partial_{z_1}), \mathcal{N})).\end{aligned}$$

Hence, (5.14) is implied by:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/(\partial_{z_1}), \Gamma_{\overline{M}^\pm}\mathcal{D}b_M)|_N = 0.$$

Recall that, in our geometrical setting, $M^\pm$ are the open half-spaces:

$$M^\pm = \{\pm x_1 > 0\} \times M'' \subset M.$$

It is then obvious that ∂_{x_1} acts bijectively on $\Gamma_{\overline{M}^\pm}\mathcal{D}b_M$, an inverse being given by the convolution with the Heaviside function.

The proof of Theorem 1.7 is thus complete.

6. Relations with Classical Statements.

In this section, for a $\mathcal{D}$-module $\mathcal{M}$ associated to a single differential operator or to a normal square system, we compare the statements of Section 1 with more classical statements of the Cauchy problem (i.e., statements that do not make use of derived categories, nor of $\mathcal{D}$-module theory). Nothing here is new. We just collect some remarks from [10], [12], [1], [2], for the convenience of a reader not acquainted with $\mathcal{D}$-module theory.

6.1. Let $X = \mathbb{C}^n$, and denote by $(z) = (z_1, z')$ the canonical coordinates. Let A be a system of linear partial differential operators with holomorphic coefficients in X, consisting of N_1 equations with N_0 unknown functions. To A, one associates the coherent $\mathcal{D}_X$-module $\mathcal{M}_A$ defined by the exact sequence:

$$(\mathcal{D}_X)^{N_1} \underset{\cdot A}{\longrightarrow} (\mathcal{D}_X)^{N_0} \to \mathcal{M}_A \to 0.$$

(In fact, by definition any coherent $\mathcal{D}_X$-module locally admits such a presentation.) In particular, this shows that for a $\mathcal{D}_X$-module $\mathcal{F}$,

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A, \mathcal{F}) \simeq \ker(\mathcal{F}^{N_0} \underset{A\cdot}{\longrightarrow} \mathcal{F}^{N_1})$$

is the space of homogeneous solutions $Au = 0$ in $\mathcal{F}$. More generally, using Hilbert's syzygy theorem one checks that the higher cohomology groups of the complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A, \mathcal{F})$ describe the compatibility conditions for solving non-homogeneous equations $Au = v$ (see the Introduction of [10]).

6.2. Denote by $(z;\zeta)$ the system of symplectic coordinates in T^*X. Assume that $A = P$ is a single differential operator of order m, $P(z,\partial) = \sum_{|\alpha|\leq m} a_\alpha(z)\partial^\alpha$, where $a_\alpha(z)$ are holomorphic functions on X, and $\partial^\alpha = \partial_{z_1}^{\alpha_1} \cdots \partial_{z_n}^{\alpha_n}$. Denote by $\sigma(P)(z;\zeta) = \sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha$ the principal symbol of P. In this case, one checks that the characteristic variety of $\mathcal{M}_P = \mathcal{D}_X/\mathcal{D}_X P$ is given by $\mathrm{char}(\mathcal{M}_P) = \sigma(P)^{-1}(0)$. More generally, Sato-Kashiwara [19] associated to a square N by N system A a determinant $\det A(z;\zeta)$, which is a homogeneous polynomial in ζ with holomorphic coefficients in z. It coincides with the principal symbol if A is given by a single differential operator, and it satisfies $\mathrm{char}(\mathcal{M}_A) = \det A^{-1}(0)$. If $\det A$ is not zero, the morphism $\cdot A$ is injective, and we thus have a short exact sequence:

$$0 \to (\mathcal{D}_X)^N \underset{\cdot A}{\longrightarrow} (\mathcal{D}_X)^N \to \mathcal{M}_A \to 0.$$

In particular, A has no compatibility conditions, and the only non-vanishing cohomology groups of the solution complex $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A, \mathcal{F})$ are given by:

$$\begin{cases} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}_A, \mathcal{F}) \simeq \ker(\mathcal{F}^N \underset{A\cdot}{\longrightarrow} \mathcal{F}^N), \\ \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{M}_A, \mathcal{F}) \simeq \mathrm{coker}(\mathcal{F}^N \underset{A\cdot}{\longrightarrow} \mathcal{F}^N). \end{cases} \tag{6.1}$$

Note that a result of [1] asserts that a coherent $\mathcal{D}$-module $\mathcal{M}$ admits a representation by a square matrix with non-vanishing determinant if and only if it is purely one-codimensional.

6.3. Let $f\colon Y \to X$ be the embedding of the hyperplane Y of equation $z_1 = 0$. Since $\mathrm{char}(\mathcal{M}_A) = \det A^{-1}(0)$, it follows that f is non-characteristic for $\mathcal{M}_A$ if and only if

$$\det A(0, z'; \zeta_1, 0) \neq 0 \qquad \forall\ (0, z'; \zeta_1, 0) \in T^*_Y X, \quad \zeta_1 \neq 0, \tag{6.2}$$

which is the classical notion of Y being non-characteristic for A. In this case (see [1]), the induced system $\underline{f}^{-1}\mathcal{M}_A$ is locally projective, and its rank equals the homogeneity degree, say m, of the polynomial $\det A$.

Recall that the square system A is classically called normal if there are integers $m_i, n_j \in \mathbb{Z}$ (called Leray-Volevich weights), such that $\mathrm{ord}(A_{ij}) \leq m_i - n_j$, and $\det(\sigma_{m_i - n_j}(A_{ij})) \not\equiv 0$, where $\sigma_{m_i - n_j}$ denotes the symbol of order $m_i - n_j$. If A is normal, then $\det A$ coincides with $\det(\sigma_{m_i - n_j}(A_{ij}))$, which is the notion of determinant usually encountered in the literature. In [2], it is proved that in this case the induced system is free, i.e., $\underline{f}^{-1}\mathcal{M}_A \simeq (\mathcal{D}_Y)^m$. In particular, recalling (6.1), we see that Theorem 1.2 for $\mathcal{M} = \mathcal{M}_A$ asserts indeed the well poseness of the holomorphic Cauchy problem:

$$\begin{cases} Au = v, \\ \partial^j_{z_1} u_i\big|_Y = w_{ij} \quad \text{for} \ \ j = 0, \dots, k_i \quad i = 1, \dots, N \end{cases}$$

where $v \in (\mathcal{O}_X|_Y)^N$, $u = (u_1, \dots, u_N)$, $w_{ij} \in \mathcal{O}_Y$, and the k_i's are suitable non-negative integers satisfying $k_1 + \cdots + k_N = m$ (see [**23**]).

6.4. Let $M = \mathbb{R}^n$, and let $f_N\colon N \to M$ be the embedding of the hyperplane N given by $x_1 = 0$ in the canonical system of coordinates $(x) = (x_1, x')$. Writing $(z) = (x + iy)$ we regard $f\colon Y \to X$ as complexifications of f_N.

Let A be a square system on X. One then easily checks that N is hyperbolic for $\mathcal{M}_A$ in the sense of Lemma 2.2 if and only if:

$$\det A(0, x'; t + i\eta_1, i\eta_2, \dots, i\eta_n) \neq 0, \qquad \text{for } t \in \mathbb{R} \setminus \{0\},\ \eta \in \mathbb{R}^n,$$

which is the classical notion of A being (weakly) hyperbolic with respect to N. By the same reasoning as above, it follows that Theorem 1.5 for $\mathcal{M} = \mathcal{M}_A$ indeed asserts the well poseness of the Cauchy problem for A in the space of hyperfunctions.

6.5. One says that A has constant multiplicities along Y if every root in ζ_1 of the polynomial $\det A(z; \zeta_1, \zeta')$ has a constant multiplicity. This implies in particular that the hypersurface $V = \mathrm{char}(\mathcal{M}_A)$ of T^*X is smooth outside of the zero-section. The classical notion of Levi conditions for a single differential operator has been extended to square systems with constant multiplicities (see [17], [22]). It is possible to prove (see [6]) that this notion is equivalent the fact that $\mathcal{M}_A$ has regular singularities along $\dot{V}$.

We thus see that for these choices of $\mathcal{M} = \mathcal{M}_A$ and $V = \mathrm{char}(\mathcal{M}_A)$, Theorem 1.7 is equivalent to the well poseness of the hyperbolic Cauchy problem for distributions, and that our hypotheses are equivalent to say that A has constant multiplicities and satisfies the Levi conditions.

7. Comments and Remarks.

7.1. A functor of formal cohomology, dual to $T\mathcal{H}om$, has been constructed in [15]. With the same notations as in §4.4, this is a functor:

$$\cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_X \colon \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X),$$

which satisfies, for example:

$$\mathbb{C}_M \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = \mathcal{C}^\infty_M, \qquad \mathbb{C}_N \overset{\mathrm{w}}{\otimes} \mathcal{O}_X = \widehat{\mathcal{C}^\infty_M|}_N,$$

where $N \subset M$ is a submanifold, and $\widehat{\mathcal{C}^\infty_M|}_N$ denotes the formal completion of $\mathcal{C}^\infty_M$ along N.

The statement in Theorem 1.7 should still hold when replacing $\mathcal{D}b$ by the sheaf of $\mathcal{C}^\infty$-functions. In fact, the proof should proceed along the same lines, using $\overset{\mathrm{w}}{\otimes}$ instead of $T\mathcal{H}om$, until §5.4. There, a microlocalization of $\overset{\mathrm{w}}{\otimes}$ playing the same role as $T\mu hom$ plays for $T\mathcal{H}om$ should be needed. This should be the Fourier transform of the Whitney specialization functor recently introduced in Colin [5]. Unfortunately, at present such a theory is not available.

In the case of normal square systems with constant multiplicities, Levi conditions are proved to be necessary and sufficient for the well-poseness of the Cauchy problem in the framework of $\mathcal{C}^\infty$-functions (see [17], [22]). The above mentioned result would recover the sufficiency part of this statement, and generalize it to $\mathcal{D}$-modules with regular singularities.

7.2. Let $f_N \colon N \to M$ be a morphism of real analytic manifolds, and let $f \colon Y \to X$ be a complexification of f_N. If $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ is regular holonomic, it follows from Corollary 8.3 of [9] that:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}b_M) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M).$$

Assume that f_N is hyperbolic for $\mathcal{M}$. In particular, f is non-characteristic for $\mathcal{M}$, and hence a similar isomorphism holds for $\underline{f}^{-1}\mathcal{M}$. Then, (1.3) is a consequence of Theorem 1.5.

A natural question is to ask whether this still holds for general modules, i.e., whether Theorem 1.7 remains true when suppressing hypothesis (o).

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