JOURNAL OF FUNCTIONAL ANALYSIS **153**, 343–356 (1998)
ARTICLE NO. FU973201

The Radon–Penrose Correspondence II: Line Bundles and Simple $\mathcal{D}$-Modules

Andrea D'Agnolo and Pierre Schapira

Institut de Mathématiques, Analyse Algébrique, Université Pierre et Marie Curie, Case 82, 4 place Jussieu, F-75252 Paris Cedex 05, France
E-mail: dagnolo@math.jussieu.fr, schapira@math.jussieu.fr

Received March 5, 1997; revised September 15, 1997; accepted September 15, 1997

On a complex manifold X of dimension $\geqslant 3$, we show that coherent $\mathcal{D}_X$-modules which are "simple" all over P^*X are classified by Pic(X), the family of holomorphic line bundles on X. As a corollary, using the Penrose transform, we obtain that on the complex Minkowski space $\mathbb{M}$, simple $\mathcal{D}_{\mathbb{M}}$-modules along the characteristic variety of the wave equation are classified by (half) integers, the so-called helicity.

AMS Mathematics Subject Classification Numbers: 30E20, 32L25, 58G37.

INTRODUCTION

Consider a complex manifold X of dimension $\geqslant 3$ and a regular involutive submanifold V of P^*X, the projective cotangent bundle to X. A natural problem is to classify all systems of linear PDEs (i.e., coherent $\mathcal{D}_X$-modules) with simple characteristics along V. In the extreme case where $V = P^*X$, we solve this problem by showing that such modules are classified—modulo flat connections—by Pic(X), the family of holomorphic line bundles on X. Starting from this result, the theory of integral transformations for $\mathcal{D}$-modules allows us to treat the case of other involutive manifolds, such as the characteristic variety of the wave equation in the conformally compactified Minkowski space or, more generally, the case where X is a Grassmannian manifold $G_p(\mathbb{C}^n)$, and V is identified by the natural projection with the conormal bundle to the incidence relation in $G_1(\mathbb{C}^n) \times G_p(\mathbb{C}^n)$. We show in particular that on such a space, simple modules along V are determined, up to flat connections, by an (half) integer corresponding to the so-called helicity.

In [5] it was shown that the Penrose transform allows one to obtain the whole family of massless field equations. By the results of this paper, one gets that there are no other simple modules than those of this family.

0022-1236/98 $25.00

1. SIMPLE $\mathscr{D}$-MODULES

We shall use the classical notations concerning $\mathscr{D}$-modules. References are made to [6], [10].

Let X be a complex manifold. We denote by d_X its dimension, by $\pi : T^*X \to X$ its cotangent bundle, and by $\mathscr{O}_X$ its structural sheaf. The sheaf $\mathscr{D}_X$ of linear holomorphic partial differential operators on X is naturally endowed with a structure of filtered ring, the filtration being given by the subsheaves $\mathscr{D}_X(k)$ of operators of degree at most k. The associated graded ring $G\mathscr{D}_X$ is identified to $\bigoplus_{k \in \mathbb{N}} \pi_* \mathscr{O}_{T^*X}(k)$, where $\mathscr{O}_{T^*X}(k)$ denotes the subsheaf of $\mathscr{O}_{T^*X}$ whose sections are homogeneous of degree k in the fiber variables.

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. A filtration on $\mathscr{M}$ is an increasing sequence $\{\mathscr{M}_k\}_{k \in \mathbb{Z}}$ of $\mathscr{O}_X$-submodules of $\mathscr{M}$, such that $\mathscr{M} = \bigcup_k \mathscr{M}_k$, and $\mathscr{D}_X(l)\mathscr{M}_k \subset \mathscr{M}_{k+l}$. A filtration $\{\mathscr{M}_k\}_{k \in \mathbb{Z}}$ is called good if the $\mathscr{M}_k$'s are $\mathscr{O}_X$-coherent and, locally on X, $\mathscr{M}_k = 0$ for $k \ll 0$, and $\mathscr{D}_X(l)\, \mathscr{M}_k = \mathscr{M}_{k+l}$ for any $l \geqslant 0$ and for $k \gg 0$. If $\mathscr{M}$ is endowed with a good filtration, we set

$$\bar{G}\mathscr{M} = \mathscr{O}_{T^*X} \otimes_{\pi^{-1}G\mathscr{D}_X} \pi^{-1}G\mathscr{M},$$

where $G\mathscr{M} = \bigoplus_k \mathscr{M}_k/\mathscr{M}_{k-1}$ is the associated graded module. This is a coherent $\mathscr{O}_{T^*X}$-module. Recall that any coherent $\mathscr{D}_X$-module locally admits a good filtration, that $\mathrm{supp}(\bar{G}\mathscr{M})$ does not depend on the choice of the good filtration, and that $\mathrm{char}(\mathscr{M}) \subset T^*X$, the characteristic variety of $\mathscr{M}$, is defined as the support of $\bar{G}\mathscr{M}$.

Let us denote by $\mathrm{Mod}(\mathscr{D}_X)$ the abelian category of $\mathscr{D}_X$-modules, and by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ its full abelian subcategory consisting of coherent $\mathscr{D}_X$-modules. We denote by $\mathbf{D}^{\mathrm{b}}(\mathscr{D}_X)$ the bounded derived category of $\mathrm{Mod}(\mathscr{D}_X)$, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathbf{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have coherent cohomology groups.

DEFINITION 1.1. Let $\varphi : \mathscr{M} \to \mathscr{N}$ be a morphism of coherent $\mathscr{D}_X$-modules. We shall say that φ is an isomorphism modulo-flat-connections (an m-f-c isomorphism for short) if $\ker \varphi$ and $\mathrm{coker}\, \varphi$ are flat connections (i.e., coherent $\mathscr{D}_X$-modules whose characteristic varieties are contained in the zero-section).

We denote by $T^*_X X$ the zero-section of T^*X, by $\dot{\pi} : \dot{T}^*X \to X$ the cotangent bundle with the zero-section removed, by $\tau : P^*X \to X$ the projective cotangent bundle, and by $\gamma : \dot{T}^*X \to P^*X$ the natural projection. For $V \subset T^*X$, we set $\dot{V} = V \cap \dot{T}^*X$.

Denote by $\hat{\mathscr{E}}_X$ the sheaf of formal microdifferential operators on P^*X of [10] (see [11] for an exposition). If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, we set:

$$\hat{\mathscr{M}} = \hat{\mathscr{E}}_X \otimes_{\tau^{-1}\mathscr{D}_X} \tau^{-1}\mathscr{M}. \tag{1.1}$$

If $\varphi : \mathscr{M} \to \mathscr{N}$ is a morphism of coherent $\mathscr{D}_X$-modules, we denote by $\hat{\varphi} : \hat{\mathscr{M}} \to \hat{\mathscr{N}}$ the associated morphism of $\hat{\mathscr{E}}_X$-modules.

LEMMA 1.2. (i) *A morphism φ: $\mathscr{M} \to \mathscr{N}$ is an m-f-c isomorphism if and only if $\hat{\varphi}$ is an isomorphism.*

(ii) *The set of m-f-c isomorphisms is a multiplicative system (as defined, e.g., in [7, Definition 1.6.1]) in* $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$.

Proof. Set $\mathscr{K} = \ker \varphi$, $\mathscr{H} = \operatorname{coker} \varphi$. Since $\hat{\mathscr{E}}_X$ is flat over $\tau^{-1}\mathscr{D}_X$, $\hat{\mathscr{K}} = \ker \hat{\varphi}$, $\hat{\mathscr{H}} = \operatorname{coker} \hat{\varphi}$. Recall that $\dot{T}^*X \cap \mathrm{char}(\mathscr{M}) = \gamma^{-1} \operatorname{supp}(\hat{\mathscr{M}})$, and that $\mathscr{M}$ is a flat connection if and only if $\mathrm{char}(\mathscr{M}) \subset T^*_X X$. It is then clear that $\mathscr{K}$ and $\mathscr{H}$ are flat connections if and only if $\hat{\varphi}$ is an isomorphism. This proves (i).

To prove (ii), we have to check that properties (S1)–(S4) of [7, Definition 1.6.1]) are satisfied. (S1), asserting that the identity morphisms are m-f-c isomorphisms, is obvious. (S2) requires that a composition of two m-f-c isomorphisms be again an m-f-c isomorphism, and follows from (i). A simple proof of (S3) and (S4) is obtained by working in the derived category $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, along the lines of Proposition 1.6.7 of loc. cit. Let us prove for example that any diagram in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$

$$\begin{array}{ccc} & & \mathscr{M} \\ & & \downarrow{\scriptstyle f} \\ \mathscr{P} & \xrightarrow{\varphi} & \mathscr{N} \end{array}$$

where φ is an m-f-c isomorphism, can be completed into a commutative diagram

$$\begin{array}{ccc} \mathscr{Q} & \xrightarrow{\psi} & \mathscr{M} \\ \downarrow & & \downarrow{\scriptstyle f} \\ \mathscr{P} & \xrightarrow{\varphi} & \mathscr{N}, \end{array} \tag{1.2}$$

with ψ an m-f-c isomorphism. (We shall leave the other verifications to the reader.) Embed φ into a distinguished triangle $\mathscr{P} \xrightarrow[\varphi]{} \mathscr{N} \xrightarrow[g]{} \mathscr{R} \xrightarrow[+1]{}$, and embed $h := g \circ f$ into a distinguished triangle $\tilde{\mathscr{Q}} \xrightarrow[\tilde{\psi}]{} \mathscr{M} \xrightarrow[h]{} \mathscr{R} \xrightarrow[+1]{}$. The diagram

$$\begin{array}{ccccccc}
\widetilde{\mathscr{Q}} & \xrightarrow{\tilde{\psi}} & \mathscr{M} & \xrightarrow{h} & \mathscr{R} & \xrightarrow[+1]{} & \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \\
\mathscr{P} & \xrightarrow{\varphi} & \mathscr{N} & \xrightarrow{g} & \mathscr{R} & \xrightarrow[+1]{} & ,
\end{array}$$

may be completed as a morphism of distinguished triangles. Since $\mathrm{char}(\mathscr{R}) \subset T^*_X X$, $\psi = H^0(\tilde{\psi})$ is an m-f-c isomorphism. Setting $\mathscr{Q} = H^0(\widetilde{\mathscr{Q}})$, we get (1.2). ∎

We denote by $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X; \mathscr{O}_X)$ the quotient category of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ by the multiplicative system of m-f-c isomorphisms. Note that an m-f-c isomorphism of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ becomes an isomorphism in $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X; \mathscr{O}_X)$.

Recall that a conic involutive submanifold V of T^*X is called regular if the restriction to V of the canonical 1-form never vanishes.

DEFINITION 1.3. (i) Let V be a closed conic regular involutive submanifold of $\dot{T}^*X$, and let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. We say that $\mathscr{M}$ is simple along V if $\mathscr{M}$ can be endowed with a good filtration $\{\mathscr{M}_k\}$ such that $\overline{G}\mathscr{M}|_{\dot{T}^*X}$ is locally isomorphic to $\mathscr{O}_V$ as an $\mathscr{O}_{\dot{T}^*X}$-module.

We denote by $\mathrm{Simp}(V; \mathscr{O}_X)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X; \mathscr{O}_X)$ whose objects are simple along V.

(ii) If $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module, we set:

$$\mathscr{D}\mathscr{F} = \mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{F}.$$

We denote by $\mathrm{Line}(\mathscr{D}_X)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ whose objects are of the type $\mathscr{D}\mathscr{L}$ for some line bundle $\mathscr{L}$.

The $\mathscr{D}_X$-module $\mathscr{D}\mathscr{L}$ has a natural good filtration $\mathscr{D}\mathscr{L}_k = \mathscr{D}_X(k) \otimes_{\mathscr{O}_X} \mathscr{L}$, and is thus an example of simple module along $\dot{T}^*X$. This gives a natural functor:

$$F\colon \mathrm{Line}(\mathscr{D}_X) \to \mathrm{Simp}(\dot{T}^*X; \mathscr{O}_X).$$

Recall the definition of $\hat{\mathscr{M}}$ given in (1.1), and consider the functor:

$$\begin{aligned} G\colon \mathrm{Mod}(\mathscr{D}_X) &\to \mathrm{Mod}(\mathscr{D}_X) \\ \mathscr{M} &\mapsto \tau_* \hat{\mathscr{M}}. \end{aligned}$$

If $\varphi : \mathscr{M} \to \mathscr{N}$ is an m-f-c isomorphism, $\hat{\varphi}$ is an isomorphism, and hence G factorizes through a functor that we will also denote by G:

$$G\colon \mathrm{Simp}(\dot{T}^*X; \mathscr{O}_X) \to \mathrm{Mod}(\mathscr{D}_X).$$

THEOREM 1.4. *With the above notations, assume* $d_X \geqslant 3$.

(i) *If* $\mathcal{M} \in \mathrm{Simp}(\dot{T}^*X; \mathcal{O}_X)$, *then* $G(\mathcal{M}) \in \mathrm{Line}(\mathcal{D}_X)$.

(ii) *The functors*:

$$\mathrm{Line}(\mathcal{D}_X) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Simp}(\dot{T}^*X; \mathcal{O}_X)$$

are quasi-inverse to each other, and thus establish an equivalence of categories.

We get that if $\mathcal{M}$ is simple along $\dot{T}^*X$ there exists a unique (up to $\mathcal{O}_X$-linear isomorphism) line bundle $\mathcal{L}$ on X and an m-f-c isomorphism $\mathcal{M} \to \mathcal{D}\mathcal{L}$. In other words, simple $\mathcal{D}_X$-module along $\dot{T}^*X$ are classified, up to flat connections, by $\mathrm{Pic}(X)$, the family of holomorphic line bundles on X.

Proof. (i) If $\{\mathcal{M}_k\}$ is a good filtration of $\mathcal{M}$, $\hat{\mathcal{M}}$ has a natural filtration given by

$$\hat{\mathcal{M}}_k = \sum_{l \in \mathbb{Z}} \hat{\mathcal{E}}_X(k-l)\, \tau^{-1}\mathcal{M}_l,$$

where $\hat{\mathcal{E}}_X(k)$ is the subsheaf of $\hat{\mathcal{E}}_X$ of operators of degree at most k. Since $\{\mathcal{M}_k\}$ is a good filtration, the above sum is locally finite, and hence the $\hat{\mathcal{M}}_k$'s are $\hat{\mathcal{E}}_X(0)$-coherent. Moreover,

$$\hat{\mathcal{M}}_k = \hat{\mathcal{E}}_X(k)\, \hat{\mathcal{M}}_0$$

for all $k \in \mathbb{Z}$. Since $\mathcal{M}$ is simple along $\dot{T}^*X$, $\hat{\mathcal{M}}_0/\hat{\mathcal{M}}_{-1}$ is a line bundle on P^*X, and by Lemma 1.5 below, there exists a line bundle $\mathcal{F}$ on X and $m \in \mathbb{Z}$, such that

$$\hat{\mathcal{M}}_0/\hat{\mathcal{M}}_{-1} \simeq \tau^{-1}\mathcal{F} \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_{P^*X}(m),$$

where $\mathcal{O}_{P^*X}(m) = \gamma_*(\mathcal{O}_{\dot{T}^*X}(m))$. By shifting the filtration, we may assume $m = 0$. Let us cover X by open polydiscs (in some local chart). Let U be such a polydisc. Then, $\hat{\mathcal{M}}_0/\hat{\mathcal{M}}_{-1}|_{P^*U} \simeq \mathcal{O}_{P^*U}(0)$. In particular, this implies:

$$\hat{\mathcal{M}}_k/\hat{\mathcal{M}}_{k-1}|_{P^*U} \simeq \mathcal{O}_{P^*U}(k). \tag{1.3}$$

Since U is affine, $P^*U \simeq U \times \mathbb{P}$, where $\mathbb{P}$ is a $(d_X - 1)$-dimensional complex projective space. Since $d_X - 1 > 1$ and U is Stein, $H^1(P^*U; \mathcal{O}_{P^*U}(k)) \simeq \Gamma(U; \mathcal{O}_U) \otimes H^1(\mathbb{P}; \mathcal{O}_{\mathbb{P}}(k)) = 0$ for $k < 0$. Apply the functor $R\Gamma(P^*U; \cdot)$ to the exact sequence:

$$0 \to \hat{\mathcal{M}}_k/\hat{\mathcal{M}}_{k-1} \to \hat{\mathcal{M}}_0/\hat{\mathcal{M}}_{k-1} \to \hat{\mathcal{M}}_0/\hat{\mathcal{M}}_k \to 0.$$

We get, for $k<0$, the surjectivity of the morphism:

$$\Gamma(P^*U;\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_{k-1})\to\Gamma(P^*U;\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_k).$$

Let $\bar{s}_U$ be a free generator of $\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_{-1}$ on P^*U. By induction on k, using the above surjection, we get a section

$$s_U\in\varprojlim_k\Gamma(P^*U;\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_k)\simeq\Gamma(P^*U;\varprojlim_k\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_k)$$
$$\simeq\Gamma(P^*U;\hat{\mathscr{M}}_0)$$

whose class modulo $\hat{\mathscr{M}}_{-1}$ is $\bar{s}_U$ (here, the last isomorphism follows from [10, Proposition II, 3.2.5]). Consider the morphism of $\hat{\mathscr{E}}_U(0)$-modules

$$\hat{\mathscr{E}}_U(0)\xrightarrow[\varphi_U]{}\hat{\mathscr{M}}_0|_{P^*U},$$

given by $\varphi_U(P)=Ps_U$, and set $\mathscr{K}=ker\,\varphi_U$, $\mathscr{H}=\operatorname{coker}\varphi_U$. By construction, φ_U induces an isomorphism $\hat{\mathscr{E}}_U(0)/\hat{\mathscr{E}}_U(-1)\xrightarrow{\sim}\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_{-1}|_{P^*U}$. It follows that $\mathscr{K}/\hat{\mathscr{E}}_U(-1)\mathscr{K}\simeq\mathscr{H}/\hat{\mathscr{E}}_U(-1)\mathscr{H}\simeq 0$, and hence, again by loc. cit., $\mathscr{K}=\mathscr{H}=0$.

Summarizing, we have shown that $\hat{\mathscr{M}}_0|_{P^*U}$ is a free $\hat{\mathscr{E}}_U(0)$-module of rank one. This implies that $\hat{\mathscr{M}}|_{P^*U}$ is a free $\hat{\mathscr{E}}_U$-module of rank one. Hence, $\tau_*\hat{\mathscr{M}}$ is a locally free $\mathscr{D}_X$-module of rank one. Since the only invertible differential operators are of degree zero, $\tau_*\hat{\mathscr{M}}\simeq\mathscr{D}\mathscr{L}$ for a line bundle $\mathscr{L}$ on X.

(ii) For $\mathscr{M}\in\operatorname{Simp}(\dot{T}^*X;\mathscr{O}_X)$, the natural adjunction morphism $\varphi:\mathscr{M}\to\tau_*\hat{\mathscr{M}}$ gives a morphism of functors $Id\to F\circ G$, and we have to check that it is an m-f-c isomorphism. By Lemma 1.2(i), we have to check that $\hat{\varphi}$ is an isomorphism. By the arguments and with the notations in (i), $\hat{\mathscr{M}}$ is free of rank one on P^*U, and we are reduced to prove that the natural morphism:

$$\hat{\mathscr{E}}_U\to\hat{\mathscr{E}}_U\otimes_{\tau^{-1}\mathscr{D}_X}\tau^{-1}\tau_*\hat{\mathscr{E}}_U$$

is an isomorphism, which is obvious since $\tau_*\hat{\mathscr{E}}_U\simeq\mathscr{D}_U$.

For $\mathscr{D}\mathscr{L}\in\operatorname{Line}(\mathscr{D}_X)$, the natural adjunction morphism $\varphi:\mathscr{D}\mathscr{L}\to\tau_*\widehat{\mathscr{D}\mathscr{L}}$ gives a morphism of functors $Id\to G\circ F$, and we have to check that it is an isomorphism. This is a local problem on X, and we may assume $\mathscr{D}\mathscr{L}\simeq\mathscr{D}_X$. Then, as above, the statement follows from the isomorphism $\tau_*\hat{\mathscr{E}}_X\simeq\mathscr{D}_X$. ∎

LEMMA 1.5. *If $d_X \geqslant 2$, there is a natural isomorphism*:

$$\begin{aligned} \mathrm{Pic}(X) \times \mathbb{Z} &\xrightarrow{\sim} \mathrm{Pic}(P^*X) \\ (\mathscr{F}, m) &\mapsto \tau^{-1}\mathscr{F} \otimes_{\tau^{-1}\mathscr{O}_X} \mathscr{O}_{P^*X}(m). \end{aligned} \tag{1.4}$$

Proof. We may assume X is connected.

(o) If $x \in X$ and $\mathscr{L}$ is a line bundle on P^*X, denote by $\mathscr{L}_x$ its restriction to $P^*_x X$ as an $\mathscr{O}$-module. If U is an open ball in $\mathbb{C}^n$, then $\mathrm{Pic}(P^*U) \simeq \mathbb{Z}$. Hence, the Chern class of $\mathscr{L}_x$ is locally constant w.r.t. x.

(i) The map is injective. In fact, if $\tau^{-1}\mathscr{F} \otimes_{\tau^{-1}\mathscr{O}_X} \mathscr{O}_{P^*X}(m)$ is trivial, by restriction to $P^*_x X$ we find that $m = 0$. Next, by taking the direct image on X, we find that $\mathscr{F}$ is trivial.

(ii) The map is surjective. In fact, if $\mathscr{L}$ is a line bundle on P^*X, then $\mathscr{L}_x \simeq \mathscr{O}_{P^*_x X}(m)$ for some m, and m does not depend on $x \in X$ by (o). Let $\mathscr{L}' = \mathscr{L} \otimes_{\mathscr{O}_{P^*X}} \mathscr{O}_{P^*X}(-m)$. Then $\mathscr{L}'_x$ is trivial for all $x \in X$, and hence the natural morphism $\mathscr{L}' \to \tau^*\tau_*\mathscr{L}'$ is an isomorphism. One concludes, since $\mathscr{L}$ is the image of $(\tau_*\mathscr{L}', m)$ by (1.4). ∎

2. INTEGRAL TRANSFORMS

Let X and Y be complex analytic manifolds of dimension d_X and d_Y, respectively. Let $\Lambda \subset \dot{T}^*(X \times Y)$ be a closed smooth Lagrangian submanifold and consider the natural projections

$$X \xleftarrow{q_1} X \times Y \xrightarrow{q_2} Y, \qquad \dot{T}^*X \xleftarrow{p_1} \Lambda \xrightarrow{p_2^a} \dot{T}^*Y, \tag{2.1}$$

where we denote by p_2^a the composition of p_2 with the antipodal map. Here, we will make the following assumptions:

$$\begin{cases} \text{(i)} & \Lambda \cap (\dot{T}^*X \times T^*_Y Y) = \Lambda \cap (T^*_X X \times \dot{T}^*Y) = \varnothing, \\ \text{(ii)} & p_1 \text{ is smooth and surjective on } \dot{T}^*X, \text{ and has} \\ & \text{connected and simply connected fibers,} \\ \text{(iii)} & p_2^a \text{ is a closed embedding identifying } \Lambda \text{ to a} \\ & \text{closed regular involutive submanifold } V \text{ of } \dot{T}^*Y. \end{cases} \tag{2.2}$$

If $f: S \to X$ is a morphism, we denote by $\underline{f}_!$ and $\underline{f}^{-1}$ the proper direct image and inverse image for $\mathscr{D}$-modules, and we denote by $\boxtimes$ the exterior tensor product. To $\mathscr{M} \in \mathbf{D}^{\mathrm{b}}(\mathscr{D}_X)$ we associate its dual

$$\underline{D}'\mathscr{M} = R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}),$$

where Ω_X is the sheaf of holomorphic forms of maximal degree. We also set $\underline{D}\mathcal{M} = \underline{D}'\mathcal{M}[d_X]$. Thus, $\underline{D}'\mathcal{M}$ and $\underline{D}\mathcal{M}$ belong to $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$.

Let $\mathcal{K}$ be a simple $\mathcal{D}_{X\times Y}$-module along Λ. In particular, $\mathcal{K}$ is regular holonomic, and hence $\underline{D}\mathcal{K}$ is concentrated in degree zero. For $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, we set:

$$\underline{\Phi}_{\mathcal{K}}\mathcal{M} = \underline{q}_{2!}(\mathcal{K} \otimes^L_{\mathcal{O}_{X\times Y}} \underline{q}_1^{-1}\mathcal{M}), \qquad \underline{\Phi}^j_{\mathcal{K}}\mathcal{M} = H^j\underline{\Phi}_{\mathcal{K}}\mathcal{M},$$

$$\underline{\Psi}_{\mathcal{K}}\mathcal{N} = \underline{q}_{1!}(\underline{D}\mathcal{K} \otimes^L_{\mathcal{O}_{X\times Y}} \underline{q}_2^{-1}\mathcal{N})[d_X - d_Y], \qquad \underline{\Psi}^j_{\mathcal{K}}\mathcal{N} = H^j\underline{\Psi}_{\mathcal{K}}\mathcal{N}.$$

THEOREM 2.1. *Assume that q_1 and q_2 are proper on* supp($\mathcal{K}$), *and assume* (2.2). *Let $\mathcal{M}$ be a simple $\mathcal{D}_X$-module along $\dot{T}^*X$, and let $\mathcal{N}$ be a simple $\mathcal{D}_Y$-module along V. Then*:

(o) *$\underline{\Phi}^0_{\mathcal{K}}$ and $\underline{\Psi}^0_{\mathcal{K}}$ send m-f-c isomorphisms to m-f-c isomorphisms.*

(i) *$\underline{\Phi}^0_{\mathcal{K}}\mathcal{M}$ is simple along V and $\underline{\Psi}^0_{\mathcal{K}}\mathcal{N}$ is simple along $\dot{T}^*X$. Moreover, $\underline{\Phi}^j_{\mathcal{K}}M$ and $\underline{\Psi}^j_{\mathcal{K}}\mathcal{N}$ are flat connections for $j \neq 0$.*

(ii) *The natural adjunction morphisms $\mathcal{M} \to \underline{\Psi}^0_{\mathcal{K}}\underline{\Phi}^0_{\mathcal{K}}\mathcal{M}$ and $\underline{\Phi}^0_{\mathcal{K}}\underline{\Psi}^0_{\mathcal{K}}\mathcal{N} \to \mathcal{N}$ are m-f-c isomorphisms.*

In particular, the functors

$$\mathrm{Simp}(\dot{T}^*X; \mathcal{O}_X) \underset{\underline{\Psi}^0_{\mathcal{K}}}{\overset{\underline{\Phi}^0_{\mathcal{K}}}{\rightleftarrows}} \mathrm{Simp}(V; \mathcal{O}_Y)$$

are quasi-inverse to each other, and thus establish an equivalence of categories.

Proof. The above theorem has been proved in [3] in the case where $\Lambda = \dot{T}^*_S(X\times Y)$, for a smooth submanifold $S \subset X\times Y$ of codimension d, and $\mathcal{K} = \mathcal{B}_S := H^d_{[S]}(\mathcal{O}_{X\times Y})$, the algebraic cohomology of $\mathcal{O}_{X\times Y}$ supported by S. There, we also show that this statement is of a microlocal nature, i.e., local on Λ. The general case follows since, in a neighborhood of any point of Λ, one may find a quantized contact transformation interchanging the pair $(\Lambda, \mathcal{K})$ with the pair $(\dot{T}^*_S(X\times Y), \mathcal{B}_S)$. ∎

THEOREM 2.2. *With the same hypotheses as in Theorem* 2.1, *assume also* $d_X \geqslant 3$. *Then, with the notations of Theorem* 1.4, *there is an equivalence of categories*:

$$\mathrm{Line}(\mathcal{D}_X) \underset{G\circ\underline{\Psi}^0_{\mathcal{K}}}{\overset{\underline{\Phi}^0_{\mathcal{K}}\circ F}{\rightleftarrows}} \mathrm{Simp}(V; \mathcal{O}_Y)$$

In particular, if $\mathcal{N}$ *is a simple* $\mathcal{D}_Y$*-module along* V *there exists a unique* (*up to* $\mathcal{O}_X$*-linear isomorphisms*) *line bundle* $\mathcal{L}$ *on* X *such that* $\mathcal{N} \simeq \underline{\Phi}^0_{\mathcal{K}} \mathcal{D}\mathcal{L}$ *in* $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Y; \mathcal{O}_Y)$.

In other words, the above theorem says that simple $\mathcal{D}_Y$-modules along V are classified, up to flat connections, by $\mathrm{Pic}(X)$.

Proof. This is an obvious corollary of Theorems 1.4 and 2.1. Let us just describe how the isomorphism $\mathcal{N} \simeq \underline{\Phi}^0_{\mathcal{K}} \mathcal{D}\mathcal{L}$ is obtained. By Theorem 2.1(i), $\underline{\Psi}^0_{\mathcal{K}}(\mathcal{N})$ is simple along $\dot{T}^*X$. Hence, by Theorem 1.4, there exists a line bundle $\mathcal{L}$ on X and an m-f-c isomorphism $\underline{\Psi}^0_{\mathcal{K}}(\mathcal{N}) \to \mathcal{D}\mathcal{L}$. By Theorem 2.1(o), we get an m-f-c isomorphism $\underline{\Phi}^0_{\mathcal{K}}(\underline{\Psi}^0_{\mathcal{K}}(\mathcal{N})) \to \underline{\Phi}^0_{\mathcal{K}} \mathcal{D}\mathcal{L}$. Theorem 2.1(ii) gives an m-f-c isomorphism $\underline{\Phi}^0_{\mathcal{K}}(\underline{\Psi}^0_{\mathcal{K}}(\mathcal{N})) \to \mathcal{N}$. Summarizing, we have obtained m-f-c isomorphisms:

$$\underline{\Phi}^0_{\mathcal{K}} \mathcal{D}\mathcal{L} \leftarrow \underline{\Phi}^0_{\mathcal{K}}(\underline{\Psi}^0_{\mathcal{K}}(\mathcal{N})) \to \mathcal{N}. \quad \blacksquare$$

Theorem 2.1 gives an equivalence of categories between simple $\mathcal{D}_X$-modules on $\dot{T}^*X$, and simple $\mathcal{D}_Y$-modules on V, modulo flat connections. However, if one is interested in calculating explicitly the image of a $\mathcal{D}_X$-module associated to a line bundle, one way to do it consists in "quantizing" this equivalence. This is the purpose of the next result.

With the same notations as in Theorem 2.1, let $\mathcal{M}$ be a simple $\mathcal{D}_X$-module along $\dot{T}^*X$, and let $\mathcal{N}$ be a simple $\mathcal{D}_Y$-module along V. Then $\underline{D}'\mathcal{M} \boxtimes \mathcal{N}$ is a simple $\mathcal{D}_{X\times Y}$-module along $\dot{T}^*X \times V$.

DEFINITION 2.3. (i) Let $p \in \dot{T}^*X \times V$, and let u be a generator at p of $(\underline{D}'\mathcal{M} \boxtimes \mathcal{N})^\wedge$, the $\hat{\mathcal{E}}_{X\times Y}$-module associated to $\underline{D}'\mathcal{M} \boxtimes \mathcal{N}$. Denote by $\mathcal{I}$ the annihilating ideal of u in $\hat{\mathcal{E}}_{X\times Y}$. We say that u is simple if its symbol ideal $\bar{\mathcal{I}}$ is reduced, and hence coincides with the defining ideal $\mathcal{I}_{\dot{T}^*X\times V}$ of $\dot{T}^*X \times V$.

(ii) Let $p \in \Lambda$, we say that a section

$$s \in \mathrm{Hom}_{\mathcal{D}_{X\times Y}}(\underline{D}'\mathcal{M} \boxtimes \mathcal{N}, \mathcal{K}) \tag{2.3}$$

is nondegenerate at p if for a simple generator u of $\underline{D}'\mathcal{M} \boxtimes \mathcal{N}$ at p, $s(u)$ is a nondegenerate section of $\mathcal{K}$ at p in the sense of [10]. (Note that locally simple modules admit simple generators, and one checks immediately that this definition does not depend on the choice of such generators.)

(iii) We say that s is nondegenerate on Λ if s is nondegenerate at any $p \in \Lambda$.

There is a natural isomorphism (see [4, Lemma 3.1]):

$$\alpha : \mathrm{Hom}_{\mathscr{D}_{X\times Y}}(\underline{D}'\mathscr{M} \underline{\boxtimes} \mathscr{N},\ \mathscr{K}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{N}, \underline{\Phi}_{\mathscr{K}}(\mathscr{M})).$$

Hence, a section s as in (2.3) defines a $\mathscr{D}_Y$-linear morphism $\alpha(s) : \mathscr{N} \to \underline{\Phi}_{\mathscr{K}}(\mathscr{M})$.

THEOREM 2.4. *With the above notations, if s is non degenerate on Λ, then $\alpha(s)$ is an m-f-c isomorphism.*

Note that when Λ is the graph of a contact transformation (and hence V is open in T^*Y), the above result reduces to the so-called "quantized contact transformations" of [10].

Proof. Let $\hat{\alpha}(s) : \hat{\mathscr{N}} \to \underline{\Phi}_{\mathscr{K}}(\mathscr{M})^\wedge$ denote the associated $\hat{\mathscr{E}}_{X\times Y}$-linear morphism. It is enough to check that $\hat{\alpha}(s)$ is an isomorphism at each $p \in V$. This can be done as in [4, Theorem 3.3], using [3, Lemma 4.7]. ∎

3. APPLICATION 1: PROJECTIVE DUALITY

By the methods above, we will recall here some results of [4], [8] on the complex projective Radon transform. Let us begin by recalling some well-known facts and introduce some notations.

Let X be a complex manifold, S a closed smooth hypersurface, and set $U = X \backslash S$. Consider the $\mathscr{D}_X$-modules $\mathscr{B}_U = \mathscr{O}_X(*S)$, $\mathscr{B}_S = H^1_{[S]}(\mathscr{O}_X)$, $\mathscr{B}^*_U = \underline{D}\mathscr{B}_U$, where $\mathscr{O}_X(*S)$ denotes the sheaf of meromorphic functions with poles in S. There are natural short exact sequences of $\mathscr{D}_X$-modules:

$$0 \to \mathscr{O}_X \xrightarrow[\alpha]{} \mathscr{B}_U \to \mathscr{B}_S \to 0, \qquad 0 \to \mathscr{B}_S \xrightarrow[\beta]{} \mathscr{B}^*_U \to \mathscr{O}_X \to 0. \tag{3.1}$$

Assume X is an open disc in $\mathbb{C}$ with holomorphic coordinate t, $S = \{t = 0\}$, and $U = \{t \neq 0\}$. Then, $\mathscr{B}_U = \mathscr{D}_X/(\mathscr{D}_X \cdot \partial_t t)$, $\mathscr{B}_S = \mathscr{D}_X/(\mathscr{D}_X \cdot t)$, $\mathscr{B}^*_U = (\mathscr{D}_X/\mathscr{D}_X \cdot t\partial_t)$. One denotes by $1/t$, $\delta(t)$ and $Y(t)$ the canonical generators of $\mathscr{B}_U$, $\mathscr{B}_S$ and $\mathscr{B}^*_U$, respectively. In this case, $\alpha(P \cdot 1) = Pt \cdot 1/t$, $\beta(P \cdot \delta(t)) = P\partial_t \cdot Y(t)$.

Let now $\mathbb{P}$ be a complex n-dimensional projective space, $\mathbb{P}^*$ the dual projective space, $\mathbb{A} = \{(z, \zeta);\ \langle z, \zeta\rangle = 0\}$ the incidence relation, and set $\Omega = (\mathbb{P} \times \mathbb{P}^*) \backslash \mathbb{A}$. Denote, as above, by $\mathscr{O}_{\mathbb{P}}(k)$ the $-k$th tensor power of the tautological line bundle on $\mathbb{P}$, and set $\mathscr{D}_{\mathbb{P}}(k) = \mathscr{D}_{\mathbb{P}} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}}(k)$. For $\mathscr{K}$ a $\mathscr{D}_{\mathbb{P}\times\mathbb{P}^*}$-module, set:

$$\mathscr{K}^{(n,0)}(k, l) = q_1^{-1}[\Omega_{\mathbb{P}} \otimes_{\mathscr{O}_{\mathbb{P}}} \mathscr{O}_{\mathbb{P}}(k)] \otimes_{q_1^{-1}\mathscr{O}_{\mathbb{P}}} \mathscr{K} \otimes_{q_2^{-1}\mathscr{O}_{\mathbb{P}^*}} q_2^{-1}\mathscr{O}_{\mathbb{P}^*}(l).$$

For $k \in \mathbb{Z}$, we set

$$k^* = -n - 1 - k,$$

and we consider the Leray form $\omega(z) = \sum_{j=0}^{n} (-1)^j z_j \, dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n$, which is a section of $\Omega_{\mathbb{P}}(n+1) \simeq \mathcal{O}_{\mathbb{P}}$.

Since $\langle z, \zeta \rangle$ is a section of $\mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}(1,1)$, the analogue of the sections $(1/t)^k$, $\delta^{(k)}(t)$, $t^k Y(t)$, obtained by considering $\langle z, \zeta \rangle$ instead of t, are then "twisted". More precisely, by tensoring them with the Leray form we get sections:

$$s_k(z,\zeta) = \langle z, \zeta \rangle^{k^*} \omega(z) \in \Gamma(\mathbb{P}\times\mathbb{P}^*; \mathscr{B}_{\Omega}^{(n,0)}(-k, k^*)) \quad \text{for } k^* < 0,$$

$$\bar{s}_k(z,\zeta) = \delta^{(-k^*-1)}(\langle z, \zeta \rangle)\, \omega(z) \in \Gamma(\mathbb{P}\times\mathbb{P}^*; \mathscr{B}_{\mathbb{A}}^{(n,0)}(-k, k^*)) \quad \text{for } k^* < 0,$$

$$s_k^*(z,\zeta) = \begin{cases} \delta^{(-k^*-1)}(\langle z, \zeta \rangle)\, \omega(z) & \text{for } k^* < 0 \\ \langle z, \zeta \rangle^{k^*} Y(\langle z, \zeta \rangle)\, \omega(z) & \text{for } k^* \geqslant 0 \end{cases}$$
$$\in \Gamma(\mathbb{P}\times\mathbb{P}^*; \mathscr{B}_{\Omega}^{*,(n,0)}(-k, k^*)).$$

Note that $\Lambda = \dot{T}^*_{\mathbb{A}}(\mathbb{P}\times\mathbb{P}^*)$ is the graph of a globally defined contact transformation (the Legendre transform):

$$\dot{T}^*\mathbb{P} \underset{p_1}{\overset{\sim}{\longleftarrow}} \Lambda \underset{p_2^a}{\overset{\sim}{\longrightarrow}} \dot{T}^*\mathbb{P}^*,$$

and it is thus immediate to check that the above sections are non-degenerate on Λ. For $\mathscr{K}$ equals to $\mathscr{B}_{\Omega}$, $\mathscr{B}_{\mathbb{A}}$ or $\mathscr{B}_{\Omega}^*$, we have:

$$\Gamma(\mathbb{P}\times\mathbb{P}^*; \mathscr{K}^{(n,0)}(-k, k^*)) \simeq \mathrm{Hom}_{\mathscr{D}_{X\times Y}}(\mathscr{D}_{\mathbb{P}}(k) \underline{\boxtimes} \mathscr{D}_{\mathbb{P}^*}(-k^*), \mathscr{K}).$$

Hence, applying Theorem 2.4, we get that

$$H^0\alpha(s_k)\colon \mathscr{D}_{\mathbb{P}^*}(-k^*) \to \Phi^0_{\mathscr{B}_{\Omega}} \mathscr{D}_{\mathbb{P}}(-k) \qquad \text{for} \quad k^* < 0 \text{ (i.e., } k > -n-1\text{)},$$
$$H^0\alpha(\bar{s}_k)\colon \mathscr{D}_{\mathbb{P}^*}(-k^*) \to \underline{\Phi}^0_{\mathscr{B}_{\mathbb{A}}} \mathscr{D}_{\mathbb{P}}(-k) \qquad \text{for} \quad k^* < 0 \text{ (i.e., } k > -n-1\text{)},$$
$$H^0\alpha(s_k^*)\colon \mathscr{D}_{\mathbb{P}^*}(-k^*) \to \underline{\Phi}^0_{\mathscr{B}_{\Omega}^*} \mathscr{D}_{\mathbb{P}}(-k) \qquad \text{for} \quad k^* \in \mathbb{Z} \text{ (i.e., } k \in \mathbb{Z}\text{)},$$

are m-f-c isomorphisms. In fact, a more precise result holds.

THEOREM 3.1 *With the above notations, the morphisms*:

(i) $\alpha(s_k)\colon \mathscr{D}_{\mathbb{P}^*}(-k^*) \to \underline{\Phi}_{\mathscr{B}_{\Omega}} \mathscr{D}_{\mathbb{P}}(-k)$, *for* $k > -n-1$,

(ii) $\alpha(\bar{s}_k)\colon \mathscr{D}_{\mathbb{P}^*}(-k^*) \to \underline{\Phi}_{\mathscr{B}_{\mathbb{A}}} \mathscr{D}_{\mathbb{P}}(-k)$, *for* $-n-1 < k < 0$,

(iii) $\alpha(s_k^*)\colon \mathscr{D}_{\mathbb{P}^*}(-k^*) \to \underline{\Phi}_{\mathscr{B}_{\Omega}^*} \mathscr{D}_{\mathbb{P}}(-k)$, *for* $k < 0$,

are isomorphisms.

Sketch of Proof. (i) First notice that the complex $\Phi_{\mathscr{B}_\Omega}\mathscr{D}_{\mathbb{P}}(-k)$ is concentrated in degree zero (either by a direct calculation along the lines of [4, Proposition 2.8], or, as pointed out by Kashiwara, by using GAGA and by noticing that the fibers of $q_2|_\Omega$ are affine). Then the result follows as in the proof of [4, Theorem 3.3].

(iii) The fact that $\Phi_{\mathscr{B}^*_\Omega}\mathscr{D}_{\mathbb{P}}(-k)$ is concentrated in degree zero for $k<0$ follows by duality from (i), since $\underline{D}\Phi_{\mathscr{B}_\Omega}\mathscr{D}_{\mathbb{P}}(-k^*)\simeq\Phi_{\mathscr{B}^*_\Omega}\mathscr{D}_{\mathbb{P}}(-k)$. One then concludes as above.

(ii) follows from (i) and (iii), using the exact sequences (3.1). ∎

Remark 3.2. (a) Theorem 3.1(ii) was obtained in [4].

(b) The kernels $\mathscr{B}_\Omega$ and $\mathscr{B}^*_\Omega$ were first considered in [8]. Then Kashiwara pointed out the fact that these kernels allow one to treat all values of k.

(c) A generalization of the above result to the case of Grassmannian duality (where the analogue of $\mathbb{A}$ is no longer smooth) is treated in [9].

Recall that any holomorphic line bundle on $\mathbb{P}$ is isomorphic to $\mathscr{O}_{\mathbb{P}}(k)$ for some k, so that $Pic(\mathbb{P})\simeq\mathbb{Z}$. Then, the results of Section 1 imply that simple $\mathscr{D}_{\mathbb{P}}$-modules along $\dot{T}^*\mathbb{P}$ are classified, up to flat connections, by an integer. If $\mathscr{M}$ is simple along $\dot{T}^*\mathbb{P}$, we denote by $ch(\mathscr{M})$ the Chern class of the line bundle $\mathscr{L}$ such that $\mathscr{M}$ is m-f-c isomorphic to $\mathscr{D}\mathscr{L}$.

COROLLARY 3.3. *Let $\mathscr{M}$ be simple along $\dot{T}^*\mathbb{P}$. Then*

$$ch(\underline{\Phi}^0_{\mathscr{B}_{\mathbb{A}}}(\mathscr{M}))=-n-1-ch(\mathscr{M}).$$

4. APPLICATION 2: TWISTOR CORRESPONDENCE

For $1\leqslant q\leqslant p\leqslant n$, denote by $F(q,\ p)$ the flag manifold of type $(q,\ p)$ in an $(n+1)$ dimensional complex vector space V, and denote by $G(p)=F(p,\ p)$ the Grassmannian manifold of p-planes in V. Let $\mathbb{P}=G(1)$, a complex n-dimensional projective space, $\mathbb{M}=G(p)$, and denote by $\mathbb{F}=F(1,\ p)$ the incidence relation. To $\Lambda=\dot{T}^*_{\mathbb{F}}(\mathbb{P}\times\mathbb{M})$ is associated a diagram:

$$\dot{T}^*\mathbb{P}\xleftarrow[p_1]{}\Lambda\xrightarrow[p_2^a]{}V\subset\dot{T}^*\mathbb{M},$$

which satisfies hypotheses (2.2). Consider the kernel $\mathscr{K}=\mathscr{B}_{\mathbb{F}}$. Theorem 2.2 thus implies:

THEOREM 4.1. *Simple $\mathscr{D}_{\mathbb{M}}$-modules along $V\subset\dot{T}^*\mathbb{M}$ are classified, up to flat connections, by $\mathbb{Z}$.*

Let $\mathcal{N}$ be a simple $\mathcal{D}_{\mathbb{M}}$-modules along V, and let k be the unique integer such that $\mathcal{N}$ is isomorphic to $\Phi^0_{\mathcal{B}_{\mathbb{A}}}(\mathcal{D}_{\mathbb{P}}(-k))$ (up to flat connections). Following Penrose and the physics literature, one sets

$$h(\mathcal{N}) = -(1 + k/2),$$

and calls it the "helicity" of $\mathcal{N}$.

Remark 4.2. Consider the case $n = 4$, $p = 2$, as in [5]. Theorem 4.1 shows in particular that there are no other simple $\mathcal{D}_{\mathbb{M}}$-modules along V than those corresponding to the massless field equations in the conformally compactified Minkowski space $\mathbb{M}$. (See [5] and [1] for related results. Even if the language of $\mathcal{D}$-modules is not used in these papers, their statements could be translated in terms of quasi-equivariant $\mathcal{D}$-modules.)

5. COMMENTS

The classification of locally free $\mathcal{D}$-modules (see [2]) or, more generally, $\hat{\mathcal{E}}$-modules globally defined on an involutive manifold V and of constant multiplicities with regular singularities, would be, in our opinion, a very interesting task. This paper should be considered as a very first step in this direction.

ACKNOWLEDGMENTS

We wish to thank Louis Boutet de Monvel, Masaki Kashiwara, and Jean-Pierre Schneiders for useful discussions during the preparation of this paper.

REFERENCES

1. R. J. Baston and M. G. Eastwood, Invariant operators, *in* "Twistors in Mathematics in Physics," London Math. Soc. Lecture Note Ser., Vol. 156, pp. 265–529, Cambridge Univ. Press, Cambridge, UK, 1990.
2. S. C. Coutinho and M. P. Holland, Locally free $\mathcal{D}(\mathrm{P}^n)$-modules, *Math. Proc. Cambridge Philos. Soc.* **112** (1992), 233–245.
3. A. D'Agnolo and P. Schapira, The Radon-Penrose transform for $\mathcal{D}$-modules, *J. Funct. Anal* **138** (1996), 349–382.
4. A. D'Agnolo and P. Schapira, Leray's quantization of projective duality, *Duke Math. J.* **84** (1996), 453–496.
5. M. G. Eastwood, R. Penrose, and R. O. Wells, Jr., Cohomology and massless fields, *Comm. Math. Phys.* **78** (1981), 305–351.
6. M. Kashiwara, Algebraic study of systems of partial differential equations, *Mém. Soc. Math. France* **123** (1995), 63.

7. M. Kashiwara and P. Shapira, "Sheaves on Manifolds," Grundlehren der matematischen Wissenschaften, Vol. 292, Springer-Verlag, Berlin/New York, 1990.
8. M. Kashiwara and T. Tanisaki, Kazhdan–Lusztig conjecture for affine Lie algebras with negative level. II. Non-integral case, *Duke Math. J.* (1996).
9. C. Marastoni, La dualité de Grassmann pour les $\mathscr{D}$-modules, *C. R. Acad. Sci. Paris Sér. I Math.* **322** (1996), 929–933; Quantification de la dualité de Grassmann, Ibid. **324** (1997), 1343–1348; and in *Amer. Sci. École Norm. Sup.*, paper to appear.
10. M. Sato, T. Kawai, and M. Kashiwara, Microfunctions and pseudodifferential equations, *in* "Hyperfunctions and Pseudodifferential Equations" (Komatsu, Ed.), Proceedings Katata 1971, Lecture Notes in Mathematics, Vol. 287, pp. 265–529, Springer-Verlag, Berlin/New York, 1973.
11. P. Schapira, "Microdifferential Systems in the Complex Domain," Grundlehren der Math. Wiss., Vol. 269, Springer-Verlag, Berlin/New York, 1985.