FURTHER RESULTS ON CONTROLLABILITY PROPERTIES OF DISCRETE-TIME NONLINEAR SYSTEMS*

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ABSTRACT

Controllability questions for discrete-time nonlinear systems are addressed in this paper. In particular, we continue the search for conditions under which the group-like notion of transitivity implies the stronger and semigroup-like property of forward accessibility. We show that this implication holds, pointwise, for states which have a weak Poisson stability property, and globally, if there exists a global "attractor" for the system.

1 Introduction

This paper continues the study, initiated in [4] and then developed in [1], of some controllability properties of discrete-time nonlinear systems, of the type:

$$x(t+1) = f(x(t), u(t)), \qquad t = 0, 1, 2, \ldots, \tag{1}$$

where $x(t) \in \mathcal{X}$ and $u(t) \in U$. We will deal, as in the above-mentioned papers, with the class of *invertible* systems, that is with those systems for which the function $f(\cdot, u)$ is a diffeomorphism. Such systems arise, for instance, when dealing with continuous-time models under digital controls via *sampling*. For futher motivations of the study of this class we refer to [4].

If Σ is a system of type (1), and x_0 is any state, then we can define the reachable set from x_0, $R(x_0)$, and the orbit from x_0, $O(x_0)$. We will see later (section 2.1) the precise definition of these objects; intuitively, $R(x_0)$ is the set of all those states that can be reached from x_0 using arbitrary controls, and $O(x_0)$ consists of all those states to which we can steer x_0 using both the motions of Σ and negative-time motions. The concept of reachable sets is certainly more natural than the concept of orbits, since negative-time motions are not allowed. However, orbits are usually easier to study —they arise from group actions— and they have nicer properties. For instance, it is known that each orbit has a natural structure of submanifold of $\mathcal{X}$.

This paper studies some relations between these two concepts. In particular, we will focus our attention on the relation between the notion of forward accessibility (i.e. $\operatorname{int} R(x_0) \neq \emptyset$) and the weaker notion of transitivity (i.e. $\operatorname{int} O(x_0) \neq \emptyset$). We would like to see when transitivity implies forward accessibility. It is a classical result, in the continuous-time framework, that this implications holds always for analytic systems, and under some appropriate Lie-rank conditions in the C^∞ case (this fact is often called the "positive form of Chow's Lemma"). For discrete-time systems it is known that, in general, this implication fails. However, for analytic discrete-time systems, there are some cases in which it has been already established that transitivity

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implies forward accessibility. For instance, this is known when f is a rational map (see [5]), or when x_0 is a positively Poisson stable point for a some fixed diffeomorphism $f(\cdot, u_0)$ (see [1]). In this paper we will strengthen considerably this last result, by proving the implication when x_0 has a weak type of Poisson stability. Moreover it is shown that transitivity implies forward accessibility when there exists a transitive state x_0 which is also a global "attractor" for Σ .

The paper is organized as follows. In section 2 we introduce some basic definitions and notations. In section 3 we associate to a discrete-time system Σ some families of vector fields whose orbits will be correlated to the geometry of reachable sets and orbits. These families of vector fields are an extension of those considered in the previous work [1, 2, 3, 4]; we consider their introduction one of the main contributions of this paper. Section 4 presents some of the connections between the sets of vector fields so introduced and reachability. In section 5 we prove partial results for smooth systems. Finally, in section 6, which deals with the analytic case, we give our main results.

2 Basic Definitions

In this paper we study discrete-time nonlinear systems Σ of the type (1), where the state space $\mathcal{X}$ and the control space U satisfy the following properties:

- $\mathcal{X}$ is a connected, second countable, Hausdorff, differentiable manifold of dimension n,
- U is a subset of $\mathbb{R}^m$ such that $U \subseteq \operatorname{clos} \operatorname{int} U$, and any two points in the same connected component of U can be joined by a smooth curve lying entirely in $\operatorname{int} U$ (except possibly for endpoints).

Notice that when $U \subseteq \mathbb{R}$ then the second assumption on U is automatically satisfied.

The system is *of class* C^k, with $k = \infty$ or ω, if the manifold $\mathcal{X}$ is of class C^k and the function $f: \mathcal{X} \times U \to \mathcal{X}$ is of class C^k (i.e., there exists a C^k extension of f to an open neighborhood of $\mathcal{X} \times U$ in $\mathcal{X} \times \mathbb{R}^m$). We call systems of class C^∞ *smooth systems* and those of class C^ω *analytic systems*.

Definition 2.1 A system Σ is said to be *invertible* if for all $u \in U$, the function $f_u : \mathcal{X} \to \mathcal{X}$ with $f_u(x) = f(x, u)$ is a diffeomorphism.

We will be dealing with the class of invertible systems. For each $u \in U$, we will denote by f_u^{-1} the inverse function of f_u.

From now on, and unless otherwise stated, we assume that a fixed smooth invertible system Σ is given.

2.1 Some Notations

For any fixed state x and any nonnegative integer k define:

$$\psi_{k,x}(\mathbf{u}) \;:=\; f_{u_k,\ldots,u_1}(x) \tag{2}$$

where $\mathbf{u} = (u_k, \ldots, u_1) \in U^k$, and where $f_{u_k,\ldots,u_1}$ denotes $f_{u_k} \circ \ldots \circ f_{u_1}$. If there exists an integer $k \geq 0$ and a $\mathbf{u} = (u_k, \ldots, u_1) \in U^k$ such that $\psi_{k,x}(\mathbf{u}) = z$, we will write:

$$x \underset{k}{\rightsquigarrow} z \; .$$

For each $\mathbf{u}$, let $\rho_{k,x}(\mathbf{u})$ be the rank of $\frac{\partial}{\partial \mathbf{u}}\psi_{k,x}[\mathbf{u}]$. For each $x \in \mathcal{X}$, let:

$$\bar{\rho}_x := \max_{k\geq 0} \max_{\mathbf{u}\in U^k} \rho_{k,x}(\mathbf{u}). \tag{3}$$

Let:

$$\begin{cases} R^0(x) & := & \{x\} \\ R^k(x) & := & \{\ z \mid x \underset{k}{\leadsto} z\ \}, \quad k = 1, 2, \dots, \end{cases}$$

$R^k(x)$ is the set of states *reachable from x in* (exactly) *k steps*. The following sets will also be very helpful:

$$\tilde{R}^k(x) := \ \{\psi_{k,x}(\mathbf{u}) \mid \mathbf{u} \in U^k, \ \ \rho_{k,x}(\mathbf{u}) = \bar{\rho}_x\},$$

which represents the set of states that are *maximal-rank reachable* from x in (exactly) k steps, and

$$\bar{R}^k(x) := \ \{\psi_{k,x}(\mathbf{u}) \mid \mathbf{u} \in U^k, \ \ \rho_{k,x}(\mathbf{u}) = n\}$$

which represents the set of states that are *nonsingularly reachable from x in k steps*. We let:

$$R(x) \ := \bigcup_{k\geq 0} R^k(x)$$

and analogously for $\tilde{R}(x)$ and $\bar{R}(x)$. $R(x)$ is the set of states reachable from x. The set $\tilde{R}(x)$ is always nonempty, but the set $\bar{R}(x)$ may be empty.

Recall that Σ is said to be *forward accessible from x* if and only if $\operatorname{int} R(x) \neq \emptyset$. It can be proved that, for analytic systems Σ ,

$$\boxed{U \ \text{ connected } \ \Rightarrow \ \forall\, x \in \mathcal{X}, \ \ \tilde{R}(x) \ \text{ is dense in } \ R(x).} \tag{4}$$

For the proof of (4) we refer to Proposition 3.3 of [1]; the basic idea is to combine the analyticity assumption, which guarantees that the set of control sequences giving maximal rank is open and dense, with the assumption that $U \subseteq \operatorname{clos}\operatorname{int} U$, which gives that for each $u \in U$ there exists a sequence $u_n \in \operatorname{int} U$ converging to u.

We also define the *controllable set to x*, and the *orbit from x*, as follows. Let:

$$\begin{cases} C^0(x) & := & \{x\} \\ C^k(x) & := & \{\ z \mid z \underset{k}{\leadsto} x\ \}, \quad k = 1, 2, \dots, \end{cases}$$

then the controllable set to x is:

$$C(x) := \bigcup_{k\geq 0} C^k(x).$$

Let:

$$\begin{cases} O^0(x) & := & \{x\} \\ O^k(x) & := & \{\ z \mid \exists z_1 \in O^{k-1} \ \text{ and } \ (z_1 \underset{1}{\leadsto} z \ \text{ or } \ z \underset{1}{\leadsto} z_1)\ \}, \quad k = 1, 2, \dots, \end{cases}$$

then the orbit from x is:

$$O(x) := \bigcup_{k\geq 0} O^k(x).$$

The system Σ is said to be *backward accessible from x* if and only if $\operatorname{int} C(x) \neq \emptyset$; Σ is said to be *transitive from x* if and only if $\operatorname{int} O(x) \neq \emptyset$.

The following Lemma states a well known criterion for forward accessibility (for the proof see Proposition 3.2 in [1]).

Lemma 2.2 Let Σ be a smooth invertible system. For each $x \in \mathcal{X}$, the following are equivalent:

1. $\operatorname{int} R(x) \neq \emptyset$,
2. $\operatorname{int} \bar{R}(x) \neq \emptyset$.

There is an analogous result for the transitivity property.

Lemma 2.3 Let Σ be a smooth invertible system. Then, for each $x \in \mathcal{X}$, and for any positive integer k the following properties hold:

1. $\operatorname{int} O^k(x) \neq \emptyset$ if and only if there exist a sequence of control values $\mathbf{u} = (u_1, \cdots, u_k)$, and a sequence $\epsilon = (\epsilon_1, \cdots, \epsilon_k)$, with $\epsilon_i = \pm 1$, such that, the following map:

$$\Psi_\epsilon : U^k \to \mathcal{X} : \quad (v_1, \cdots, v_k) \mapsto f_{v_k}^{\epsilon_k} \circ \cdots \circ f_{v_1}^{\epsilon_1}(x)$$

   has full rank at $\mathbf{u}$.

2. $\operatorname{int} O(x) \neq \emptyset$ if and only if there exists a positive integer k such that the previous conditions are satisfied for this k.

Proof. We prove 1. The sufficient part follows easily from the Implicit Function Theorem, so we need to see the converse. Notice that $O^k(x)$ is given by the union, over all the different sequences of length k of ± 1's, of the images of maps of the type Ψ_ϵ. Thus, arguing by contradiction, since this union is countable, the neccessary part follows by Sard's Theorem and the fact that a countable union of set of measure zero has again measure zero. The second claim follows immediately from the first, by a similar argument. ∎

3 A New Class of Vector Fields Associated to Systems

Some Lie algebras of vector fields $L,\ L^-,\ L^+,\ \Gamma,\ \Gamma^-,\ \Gamma^+$ were introduced in [4] (see also [2] and [3] for previous work) to study the controllability properties of invertible systems. Here, using the same vector fields, we will define slightly different Lie algebras, which allow us to derive stronger results.

Let Σ be a given smooth invertible system. First, for each $u \in U$, and each $i = 1, \ldots, m$, we let $X^+_{u,i}$, and $X^-_{u,i}$ be the following vector fields:

$$X^+_{u,i}(x) = \left.\frac{\partial}{\partial v_i}\right|_{v=0} f_u^{-1} \circ f_{u+v}(x), \tag{5}$$

$$X^-_{u,i}(x) = \left.\frac{\partial}{\partial v_i}\right|_{v=0} f_u \circ f_{u+v}^{-1}(x). \tag{6}$$

Given a vector field Y and a control value $u \in U$, we can define another vector field from Y by applying the change of coordinates given by the diffeomorphism f_u,

$$(\mathrm{Ad}_u Y)(x) = (\mathrm{d}f_u(x))^{-1} Y(f_u(x)). \tag{7}$$

Here $\mathrm{d}f_u$ stands for the differential of f_u with respect to x. This is sometimes called the "pull-back of Y under the diffeomorphism f_u". In the same way, but now using the diffeomorphism f_u^{-1}, we also define Ad_u^{-1}. We let:

$$\mathrm{Ad}_{u_k\cdots u_1}^{\epsilon_k\cdots\epsilon_1}\, Y = \mathrm{Ad}_{u_1}^{\epsilon_1}\cdots \mathrm{Ad}_{u_k}^{\epsilon_k}\, Y. \tag{8}$$

We define now:

$$\begin{aligned}
\Gamma^+ &= \{\mathrm{Ad}_{u_k\cdots u_1} X_{u_0,i}^+ | k \geq 0,\ 1 \leq i \leq m,\ u_0,\ldots,u_k \in U\},\\
\Gamma^- &= \{\mathrm{Ad}_{u_k\cdots u_1}^{-1\cdots -1} X_{u_0,i}^- | k \geq 0,\ 1 \leq i \leq m,\ u_0,\ldots,u_k \in U\},\\
\Gamma &= \{\mathrm{Ad}_{u_k\cdots u_1}^{\epsilon_k\cdots\epsilon_1} X_{u_0,i}^{\epsilon_0} | k \geq 0,\ 1 \leq i \leq m,\ u_0,\ldots,u_k \in U, \epsilon_0,\ldots,\epsilon_k = \pm 1\}.
\end{aligned} \tag{9}$$

These previous definitions are the same as those given in [4].

For any finite sequence of controls $\mu = (v_1, \cdots, v_k)$, we will denote by $|\mu|$ the length k of this sequence. Now, for any such μ we define the following Lie algebras:

$$\begin{aligned}
L_\mu^+ &= \mathrm{Lie}\,\{X_{u,i}^+,\ \mathrm{Ad}_{v_1}\cdots \mathrm{Ad}_{v_l} X_{u,i}^+ \mid 1 \leq l \leq k,\ 1 \leq i \leq m,\ u \in U\},\\
L_\mu^- &= \mathrm{Lie}\,\{X_{u,i}^+,\ \mathrm{Ad}_{v_1}^{-1}\cdots \mathrm{Ad}_{v_l}^{-1} X_{u,i}^+ \mid 1 \leq l \leq k,\ 1 \leq i \leq m,\ u \in U\},\\
L_\mu &= \mathrm{Lie}\,\{X_{u,i}^+,\ \mathrm{Ad}_{v_1}^{\alpha}\cdots \mathrm{Ad}_{v_l}^{\alpha} X_{u,i}^+ \mid 1 \leq l \leq k,\ 1 \leq i \leq m,\ \alpha = \pm 1,\ u \in U\}.
\end{aligned} \tag{10}$$

Notice that, if $\mu = (0, \cdots, 0)$, then the previous Lie algebras coincide respectively with the Lie algebras L_k^+, L_k^-, and L_k defined in [4].

The way in which these algebras of vector fields will be used is as follows. We will show that the integral manifolds that they give rise to help in describing the geometry of the sets $R(x)$, $C(x)$, and $O(x)$, in the sense that the corresponding motions (forward and/or backwards) of our system lie in these orbits (see Propositions 4.4 and 4.5 given below). Moreover we will see that the notions of forward and backward accessibility, as well as transitivity, are related to the full dimensionality of the tangent subspaces at each point, corresponding to these families of vector fields (see Proposition 6.1).

Along these lines, the following two results were proved in [4], Corollary 4.4 and Theorem 2 part (a) respectively. We repeat their statements here for the convenience of the reader.

Proposition 3.1 Let Σ be a smooth invertible system. If $y \in \tilde{R}(x)$, and $\dim \mathrm{Lie}\,\Gamma^+(y) = n$, then Σ is forward accessible from x.

Proposition 3.2 Let Σ be a smooth invertible system, then the following conditions are equivalent:

1. Σ is forward accessible from all $x \in \mathcal{X}$,
2. $\dim \Gamma^+(x) = n$ for all $x \in \mathcal{X}$,
3. $\dim \mathrm{Lie}\,\Gamma^+(x) = n$ for all $x \in \mathcal{X}$.

4 Some General Properties

Definition 4.1 A map $h : \mathcal{X} \to \mathcal{X}$ is *balanced* (with respect to the given system Σ) if it can be written as $f_{u_1}^{\epsilon_1}\circ \cdots \circ f_{u_k}^{\epsilon_k}$ for some $u_i \in U$, $\epsilon_i = \pm 1$ and with $\sum_{i=1}^k \epsilon_i = 0$.

Lemma 4.2 Assume that $y = f_{u_1}^{\epsilon_1}\circ \cdots \circ f_{u_k}^{\epsilon_k}(x)$, with $u_i \in U$, $\epsilon_i = \pm 1$ for each $i = 1, \cdots, k$. Consider the partial sums:

$$l_i = \sum_{j=1}^{i} \epsilon_j \quad \text{for} \quad i = 1, \cdots, k.$$

Assume that $l_k = 0$, and let $l = \max \{ |l_i| \mid i = 1, \cdots, k\}$. Pick any sequence $v_1, \cdots, v_l$ of elements in U. Then one can write:

$$y = g_k \circ \cdots \circ g_1(x), \tag{11}$$

where each g_i is a balanced map of the form:

$$g_i = h^i \circ f_{u_i}^{\epsilon_i} \circ k^i,$$

and, for each $i = 1, \cdots, k$, the maps h^i and k^i have the form:

$$\begin{cases} h^i &= f_{v_1}^{\alpha(i)} \circ \cdots \circ f_{v_{\mu(i)}}^{\alpha(i)} \\ \\ k^i &= f_{v_{\nu(i)}}^{-\alpha(i)} \circ \cdots \circ f_{v_1}^{-\alpha(i)} \end{cases}$$

and $\alpha(i) = \pm 1$, $|\mu(i) - \nu(i)| = 1$.

Proof. Let $\alpha(i) = \operatorname{sgn}(l_i) \in \{0, 1, -1\}$ and write also $l_0 = 0$. For $i = 1, \cdots, k$, define:

$$g_i = h^i \circ f_{u_i}^{\epsilon_i} \circ k^i$$

as follows:

$$\begin{cases} k^i &= f_{v_{|l_{i-1}|}}^{\alpha_{i-1}} \circ \cdots \circ f_{v_1}^{\alpha_{i-1}} \\ \\ h^i &= f_{v_1}^{-\alpha_{i-1}} \circ \cdots \circ f_{v_{|l_i|}}^{-\alpha_{i-1}}. \end{cases}$$

The g_i are balanced maps, since $l_i = \epsilon_i + l_{i-1}$. Moreover, by definition, it holds that:

$$k^i = (h^{i-1})^{-1} \qquad i = 2, \cdots, k.$$

We now prove, by induction on i, that:

$$g_i \circ \cdots \circ g_1 = h^i \circ f_{u_i}^{\epsilon_i} \circ \cdots \circ f_{u_1}^{\epsilon_1}. \tag{12}$$

If $i = 1$, then $l_1 = \epsilon_1$ and $l_0 = 0$, so (12) is obvious. The induction step for $i > 1$ is provided by:

$$g_i \circ \cdots \circ g_1 \;=\; (h^i \circ f_{u_i}^{\epsilon_i} \circ k^i)(h^{i-1} \circ f_{u_{i-1}}^{\epsilon_{i-1}} \circ \cdots \circ f_{u_1}^{\epsilon_1}) \;=\; h^i \circ f_{u_i}^{\epsilon_i} \circ \cdots \circ f_{u_1}^{\epsilon_1}.$$

Now, since $l_k = 0$, $h^k =$ identity, so (12) gives the desired conclusion. ∎

If Δ is a set of smooth vector fields and $x \in \mathcal{X}$, we denote by $\mathrm{Orb}_\Delta(x)$ the orbit of Δ passing through x. Recall that, by definition, $y \in \mathrm{Orb}_\Delta(x)$ if and only if there exists an absolute continuous curve $\gamma : [a, b] \to \mathcal{X}$ such that $\gamma(a) = x$, $\gamma(b) = y$, and there exist t_i with $a = t_0 < t_1 < \cdots < t_r = b$, and vector fields $X_i \in \Delta$ such that γ restricted to $[t_i, t_{i+1}]$ is an integral curve of X_i or $-X_i$.

The following fact about continuos-time systems is well known, we repeat it here since it is needed in the proof of the next Proposition. Let Σ be a continuous-time system described by the following set of controlled differential equations:

$$\dot{x}(t) \ = \ f(x(t), u(t)). \tag{13}$$

(For a precise definition of continuous-time systems see for instance [7], [8], or [6].) Given two states $x_1,\ x_2$, we say that x_1 can be controlled to x_2 if there exist some interval $[0, T]$, $T \geq 0$, and an essentially bounded measurable map $u(\cdot)$ defined on $[0, T]$, such that the solution of the differential equation (13) with this $u(\cdot)$, and with $x(0) = x_1$, is defined on all the interval $[0, T]$, and $x(T) = x_2$. We say that x_2 is weakly reachable from x_1 if there exists a finite sequence of states $z_1 = x_1,\ z_2, \ldots, z_k = x_2$ such that for each $l = 2, \ldots, k$ either z_{l-1} can be controlled to z_l or z_l can be controlled to z_{l-1}. Let:

$$\Delta \ = \ \{f(\cdot, u)\}_{u \in U}.$$

Given these notations, the following fact holds (for the proof see Proposition 2.16 in [7]):

Lemma 4.3 $\quad x_1$ is weakly accessible from $x_2 \ \Leftrightarrow \ x_1 \in \mathrm{Orb}_{\Delta}(x_2)$.

The following result generalizes [4] Proposition 5.2, which deals with the very special case in which $\mu = \underbrace{0, \cdots, 0}_{k}$.

Proposition 4.4 Let k be any nonnegative integer. Assume that Σ is a smooth system with connected U. Then:

1. $R^k(x) \subseteq \mathrm{Orb}_{L_\mu^-}(y)$ for all $y \in R^k(x)$ and for all μ with $|\mu| \geq k$,
2. $C^k(x) \subseteq \mathrm{Orb}_{L_\mu^+}(y)$ for all $y \in C^k(x)$ and for all μ with $|\mu| \geq k - 1$.

Proof. Note that in the above statements, it is always sufficient to prove the inclusion for any particular y in $R^k(x)$ or, for the second part, in $C^k(x)$, since $\mathrm{Orb}_{\Delta}(z) = \mathrm{Orb}_{\Delta}(y)$ if $z \in \mathrm{Orb}_{\Delta}(y)$, for any set of vector fields Δ.

We prove now the first part. Let $z \in R^k(x)$. Then:

$$z = f_{u_1} \circ \cdots \circ f_{u_k}(x),$$

for some $(u_1, \cdots, u_k) \in U^k$. Now take any μ with $|\mu| \geq k$, $\mu = (v_1, \cdots, v_k, \cdots, v_{|\mu|})$, and consider the state:

$$y = f_{v_1} \circ \cdots \circ f_{v_k}(x) \in R^k(x).$$

We will prove that z is in $\mathrm{Orb}_{L_\mu^-}(y)$. We have:

$$z = f_{u_1} \circ \cdots \circ f_{u_k} \circ f_{v_k}^{-1} \circ \cdots \circ f_{v_1}^{-1}(y).$$

We can write the previous equation as:

$$z = g_1 \circ \cdots \circ g_k(y),$$

where:

$$\begin{cases} g_1 &= f_{u_1}\circ f_{v_1}^{-1} \\ g_i &= f_{v_1}\circ\cdots\circ f_{v_{i-1}}\circ f_{u_i}\circ f_{v_i}^{-1}\circ\cdots\circ f_{v_1}^{-1} \qquad \text{for} \quad i=2,\ldots,k. \end{cases}$$

Letting $z_0 = z$, and, for $i = 1, \cdots, k$, $z_i = g_i^{-1}(z_{i-1})$, we have $z_k = y$ (see Fig. 1).

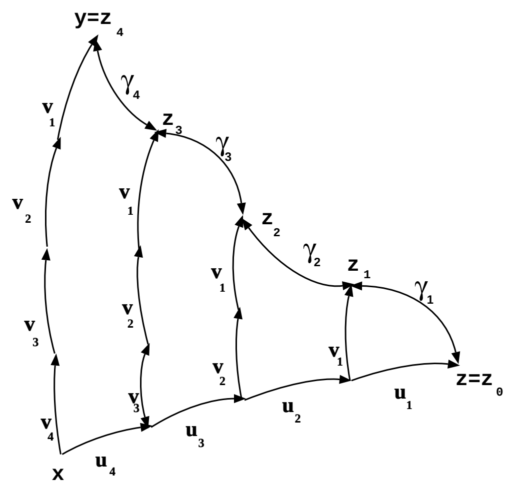

Figure 1: *case with* $k = 4$

We prove that $z_i \in \text{Orb}_{L_\mu^-}(z_{i-1})$, from which the desired conclusion follows by transitivity.

Given the assumptions on the set U, for $i = 1, \ldots, k$, there exist smooth curves $\alpha_i : [0,T] \to U$, such that $\alpha_i(0) = v_i$, $\alpha_i(T) = u_i$, and $\alpha_i(t) \in \text{int}\, U$. Let

$$\gamma_i(t) = f_{v_1}\circ\cdots\circ f_{v_i}\circ f_{\alpha_i(t)}^{-1}\circ f_{v_{i-1}}^{-1}\circ\cdots\circ f_{v_1}^{-1}(z_{i-1}), \tag{14}$$

for $i = 1, \cdots, k$, and $t \in [0,T]$.

Note that $\gamma_i(0) = z_{i-1}$, $\gamma_i(T) = z_i$, and, for any fixed t,

$$\gamma_i(t+\epsilon) = f_{v_1}\circ\cdots\circ f_{v_i}\circ f_{\alpha_i(t+\epsilon)}^{-1}(q_i),$$

where

$$q_i = f_{v_{i-1}}^{-1}\circ\cdots\circ f_{v_1}^{-1}(z_{i-1}) = f_{\alpha_i(t)}\circ f_{v_i}^{-1}\circ\cdots\circ f_{v_1}^{-1}(\gamma_i(t)).$$

Therefore:

$$\begin{aligned} \frac{\partial}{\partial t}\gamma_i(t) &= \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} f_{v_1}\circ\cdots\circ f_{v_i}\circ f_{\alpha_i(t+\epsilon)}^{-1}\circ f_{\alpha(t)}\circ f_{v_i}^{-1}\circ\cdots\circ f_{v_1}^{-1}(\gamma_i(t)) \\ &= -\sum_{j=1}^m \alpha_{ij}'(t)\text{Ad}_{v_1}^{-1}\cdots\text{Ad}_{v_i}^{-1} X_{\alpha_i(t),j}^+(\gamma_i(t)), \end{aligned} \tag{15}$$

where α_{ij} is the $j-$th component of the curve α_i. For each $i = 1, \ldots, k$, we may interpret equation (15) as the equation of a continuous-time system with state trajectory $\gamma_i(t)$ and control $(\alpha_i(t), \alpha_{i1}'(t), \ldots, \alpha_{im}'(t)$, so by Lemma 4.3 it follows that $z_i \in \text{Orb}_{L_\mu^-}(z_{i-1})$, as desired.

Now we prove the second part. The proof follows the same lines as the proof of the first part. Let $z \in C^k(x)$. Then:

$$z = f_{u_1}^{-1}\circ\cdots\circ f_{u_k}^{-1}(x),$$

for some $(u_1, \cdots, u_k) \in U^k$. Now take any μ with $|\mu| \geq k-1$, $\mu = (v_1, \cdots, v_{k-1}, v_k, \cdots, v_l)$. (If $|\mu| = k-1$, we choose v_k arbitrarily; it will be clear later that we need v_k only for technical reasons.) Let $y \in C^k(x)$ be the following state:

$$y = f_{v_1}^{-1} \circ \cdots \circ f_{v_k}^{-1}(x).$$

We will prove that z is in $\mathrm{Orb}_{L_\mu^+}(y)$. We have:

$$z = f_{u_1}^{-1} \circ \cdots \circ f_{u_k}^{-1} \circ f_{v_k} \circ \cdots \circ f_{v_1}(y).$$

We can write the previous equation as:

$$z = g_1 \circ \cdots \circ g_k(y),$$

where:

$$\begin{cases} g_1 &= f_{u_1}^{-1} \circ f_{v_1} \\ g_i &= f_{v_1}^{-1} \circ \cdots \circ f_{v_{i-1}}^{-1} \circ f_{u_i}^{-1} \circ f_{v_i} \circ \cdots \circ f_{v_1} \qquad \text{for } \ i = 2, \dots, k. \end{cases}$$

Letting $z_k = y$, and, for $i = 1, \cdots, k$, $z_{k-i} = g_{k-i+1}(z_{k-i+1})$, we have (see Fig. 2):

$$\begin{cases} z_0 &= z \\ z_{i-1} &= f_{v_1}^{-1} \circ \cdots \circ f_{v_{i-1}}^{-1} \circ f_{u_i}^{-1} \circ f_{v_i} \circ \cdots \circ f_{v_1}(z_i). \end{cases}$$

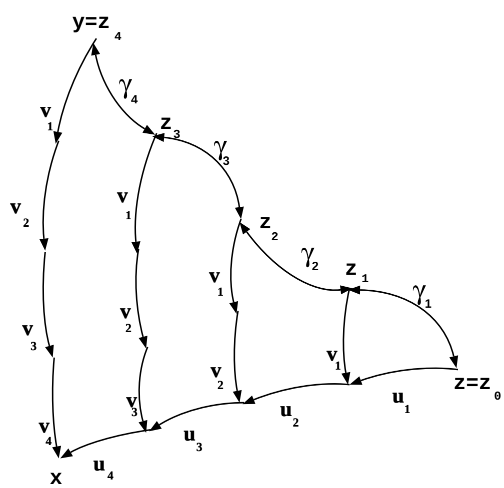

Figure 2: *case with $k = 4$*

To prove our statement, it is now sufficient to show that, for $i = 1, \cdots, k$, $z_{i-1} \in \mathrm{Orb}_{L_\mu^+}(z_i)$.

As before, for each $i = 1, \dots, k$, there exists a smooth curve $\alpha_i : [0, T] \to U$ such that $\alpha_i(0) = v_i$, $\alpha_i(T) = u_i$, and $\alpha_i(t) \in \mathrm{int}\, U$. Let

$$\gamma_i(t) = f_{v_1}^{-1} \circ \cdots \circ f_{v_{i-1}}^{-1} \circ f_{\alpha_i(t)}^{-1} \circ f_{v_i} \circ \cdots \circ f_{v_1}(z_i), \tag{16}$$

for $i = 1, \cdots, k$, and $t \in [0, T]$.

Arguing as in the first case, we conclude:

$$
\begin{aligned}
\frac{\partial}{\partial t}\gamma_i(t) &= \frac{\partial}{\partial\epsilon}|_{\epsilon=0} f_{v_1}^{-1}\circ\cdots\circ f_{v_{i-1}}^{-1}\circ f_{\alpha_i(t+\epsilon)}^{-1}\circ f_{\alpha_i(t)}\circ f_{v_{i-1}}\circ\cdots\circ f_{v_1}(\gamma_i(t)) \\
&= -\sum_{j=1}^{m}\alpha'_{ij}(t)\mathrm{Ad}_{v_1}\cdots\mathrm{Ad}_{v_{i-1}}X^{+}_{\alpha_i(t),j}(\gamma_i(t)).
\end{aligned}
\tag{17}
$$

∎

We now give a similar result for the *zero-time* orbit of a point x. We introduce the following notation:

$$
O_k^0(x) = \{y = f_{u_s}^{\epsilon_s}\circ\cdots\circ f_{u_1}^{\epsilon_1}(x) \mid s \geq 0,\ \sum_{i=1}^{s}\epsilon_i = 0,\ u_j\in U,\ \text{ and } \ |\sum_{i=1}^{j}\epsilon_i| \leq k \ \ \forall\ j=1,\cdots,s\},
\tag{18}
$$

with $k\in\mathbb{Z}$, $k\geq 0$.

Proposition 4.5 Let k be any nonnegative integer, and assume that Σ is a smooth invertible system with connected U. Then, for all $x\in\mathcal{X}$, all $y\in O_k^0(x)$, and all sequences of control values μ with $|\mu|\geq k$, we have:

$$
O_k^0(x)\subseteq \mathrm{Orb}_{L_\mu}(y).
$$

Proof. Notice first that $x\in O_k^0(x)$, by using the empty sequence. We will prove that for each x,

$$
O_k^0(x)\subseteq \mathrm{Orb}_{L_\mu}(x).
\tag{19}
$$

The general statement follows from this, since $y\in O_k^0(x)\subseteq \mathrm{Orb}_{L_\mu}(x)$ implies $\mathrm{Orb}_{L_\mu}(x) = \mathrm{Orb}_{L_\mu}(y)$.

Let $y\in O_k^0(x)$. Then:

$$
y = f_{u_s}^{\epsilon_s}\circ\cdots\circ f_{u_1}^{\epsilon_1}(x)
$$

with $\sum_{i=1}^{s}\epsilon_i = 0$ and $k\geq|\sum_{i=1}^{j}\epsilon_i|$ for any $j=1,\cdots,s$. Now take any μ with $|\mu|\geq k$, $\mu=(v_1,\cdots,v_k,\cdots,v_{|\mu|})$. We can now apply Lemma 4.2, with $l=k$, and we obtain:

$$
y = g_s\circ\cdots\circ g_1(x).
$$

Here, each g_i is a balanced map of the form:

$$
g_i = f_{v_1}^{\alpha(i)}\circ\cdots\circ f_{v_{\mu(i)}}^{\alpha(i)}\circ f_{u_i}^{\epsilon_i}\circ f_{v_{\nu(i)}}^{-\alpha(i)}\circ\cdots\circ f_{v_1}^{-\alpha(i)}
$$

with $\alpha(i)=\pm 1$, and $|\mu(i)-\nu(i)|=1$. Now let:

$$
\begin{cases}
z_0 = x \\
z_{i+1} = g_{i+1}(z_i) \qquad \text{for } \ i=0,\ldots,s-1;
\end{cases}
$$

thus $z_s = y$ (see Fig. 3).

Let $\rho(i)=\max\{\mu(i),\nu(i)\}$. Given the assumptions on the set U, for each $i=1,\ldots,s$, there exists a smooth curve $\beta_i:[0,T]\to U$, such that $\beta_i(0)=v_{\rho(i)}$, $\beta_i(T)=u_i$, and $\beta_i(t)\in\mathrm{int}\,U$. Let:

$$
\gamma_i(t) = f_{v_1}^{\alpha(i)}\circ\cdots\circ f_{v_{\mu(i)}}^{\alpha(i)}\circ f_{\beta_i(t)}^{\epsilon_i}\circ f_{v_{\nu(i)}}^{-\alpha(i)}\circ\cdots\circ f_{v_1}^{-\alpha(i)}(z_{i-1})
\tag{20}
$$

for $i=1,\cdots,s$, and $t\in[0,T]$. Thus it holds that $\gamma_i(0)=z_{i-1}$ and $\gamma_i(T)=z_i$. To prove (19), it is enough to establish that $z_{i-1}\in\mathrm{Orb}_{L_\mu}(z_i)$.

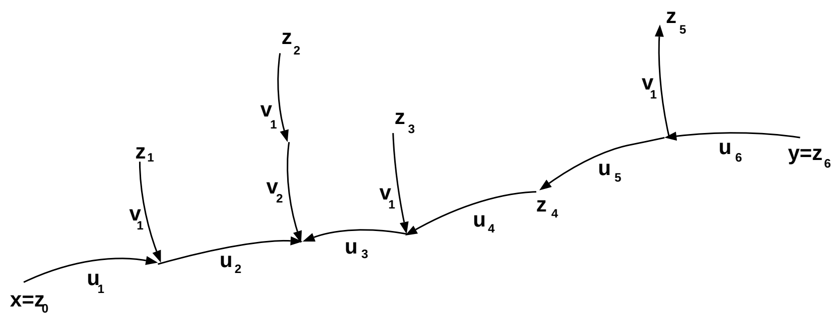

Figure 3: *case with* $s = 6, k = 3$

If $\alpha(i) = 1$ and $\rho(i) = \mu(i)$, then necessarily $\epsilon_i = -1$. Thus equation (20) is of the same type as equation (14), and, as in (15), we may conclude that:

$$\frac{\partial}{\partial t}\gamma_i(t) = -\sum_{j=1}^{m} \beta'_{ij}(t)\mathrm{Ad}_{v_1}^{-1}\cdots \mathrm{Ad}_{v_{\mu(i)}}^{-1} X^{+}_{\beta_i(t),j}(\gamma_i(t)),$$

where β_{ij} denotes the $j-$th component of the curve β_i. If $\alpha(i) = 1$ and $\rho(i) = \nu(i)$, then necessarily $\epsilon_i = 1$. In this case, instead of considering the curve $\gamma_i(t)$, we consider the following curve:

$$\gamma_i^{-1}(t) = f_{v_1}^{\alpha(i)}\circ\cdots\circ f_{v_{\nu(i)}}^{\alpha(i)}\circ f_{\beta_i(t)}^{-\epsilon_i}\circ f_{v_{\mu(i)}}^{-\alpha(i)}\circ\cdots\circ f_{v_1}^{-\alpha(i)}(z_i), \tag{21}$$

which joins z_i to z_{i-1}. Since equation (21) is again of the same type as equation (14), now we can conclude as before.

If $\alpha(i) = -1$ and $\rho(i) = \nu(i)$, then necesarily $\epsilon_i = -1$. Thus equation (20) is of the same type as equation (16), and, as in (17), we may conclude that:

$$\frac{\partial}{\partial t}\gamma_i(t) = -\sum_{j=1}^{m} \beta'_{ij}(t)\mathrm{Ad}_{v_1}\cdots \mathrm{Ad}_{v_{\mu(i)}} X^{+}_{\beta_i(t),j}(\gamma_i(t)).$$

Finally, if $\alpha(i) = -1$ and $\rho(i) = \mu(i)$, then necessarily $\epsilon_i = 1$. So we can argue as before by considering the curve $\gamma_i^{-1}(t)$ instead of γ_i. ∎

5 New Results on Accessibility

In this section we present some new results for systems with connected control space U.

Proposition 5.1 Let Σ be any smooth invertible system. If $y \in R^k(x)$, and $\mu = (v_1, \cdots, v_k)$ are such that:

$$y = f_{v_1}\circ\cdots\circ f_{v_k}(x),$$

then

$$\dim \mathrm{Lie}\,\Gamma^{+}(x) \geq \dim L_\mu(y). \tag{22}$$

Proof. Assume that $\dim L_\mu(y) = r$, and let $Y_1, \cdots, Y_r$ be vector fields in L_μ so that $\{ Y_1(y), \ldots, Y_r(y) \}$ is a basis of $L_\mu(y)$. Without loss of generality, we may assume that each Y_i is a vector field involving Lie brackets of a finite numbers of vector fields of the type:

$$X^{+}_{u,j_1}, \quad \text{or} \quad \mathrm{Ad}_{v_1}\cdots\mathrm{Ad}_{v_l} X^{+}_{u,j_2}, \quad \text{or} \quad \mathrm{Ad}_{v_1}^{-1}\cdots\mathrm{Ad}_{v_l}^{-1} X^{+}_{u,j_3},$$

with $l \leq k$, and $j_i \in \{1, \ldots, m\}$.

Consider, for $i = 1, \cdots, r$, the following vector fields:

$$Z_i = \mathrm{Ad}_{v_k} \cdots \mathrm{Ad}_{v_1} Y_i.$$

These are linearly independent at x; in fact:

$$\mathrm{Ad}_{v_k} \cdots \mathrm{Ad}_{v_1} Y_i(x) = d(f_{v_1} \circ \cdots \circ f_{v_k})^{-1}(x) Y_i(f_{v_1} \circ \cdots \circ f_{v_k}(x)) = d(f_{v_1} \circ \cdots \circ f_{v_k})^{-1}(x) Y_i(y).$$

Moreover, it follows recursively from the fact that, for any two vector fields X_1 and X_2,

$$\mathrm{Ad}_v [X_1, X_2] = [\mathrm{Ad}_v X_1, \mathrm{Ad}_v X_2], \quad \text{for any} \;\; v \in U,$$

that $Z_i \in \mathrm{Lie}\,\Gamma^+$ for each $i = 1, \cdots, r$. Thus (22) holds. ∎

Definition 5.2 For any nonnegative integer k, and for $x \in \mathcal{X}$ we say that:

1. x is $k-$forward accessible if $\mathrm{int}\, R^k(x) \neq \emptyset$,
2. x is $k-$backward accessible if $\mathrm{int}\, C^k(x) \neq \emptyset$,
3. x is $k-$transitive if $O_k^0(x)$ is open.

Recall that if $\mu = (v_1, \cdots, v_k)$ is any finite sequence of controls, then we will denote by $|\mu|$ the length k of this sequence.

Proposition 5.3 Let Σ be a smooth invertible system, with connected U. Then:

1. if x is $k-$forward accessible, then for all μ with $|\mu| \geq k-1$, $\mathrm{Orb}_{\,L_\mu^+}(x)$ is open;
2. if x is $k-$backward accessible, then for all μ with $|\mu| \geq k$, $\mathrm{Orb}_{\,L_\mu^-}(x)$ is open;
3. if x is $k-$transitive, then for all μ with $|\mu| \geq k$, $\mathrm{Orb}_{\,L_\mu}(x)$ is open.

Proof.

(1) Since x is $k-$forward accessible, there exists an open set V contained in $R^k(x)$. Let $\mu = (v_1, \cdots, v_{k-1}, v_k, \cdots, v_l)$, where if $|\mu| = k-1$, we choose v_k arbitrarily, since it will be clear later that v_k is needed only for technical reasons. Let

$$W = f_{v_1}^{-1} \circ \cdots \circ f_{v_k}^{-1}(V).$$

We will prove that $W \subset \mathrm{Orb}_{\,L_\mu^+}(x)$, from which (1) follows.

Pick any $y \in W$; then there exists $z \in V$ such that $y \in C^k(z)$. Moreover, since $z \in R^k(x)$, we also have $x \in C^k(z)$; thus, applying the second result in Proposition 4.4, we can conclude:

$$y \in \mathrm{Orb}_{\,L_\mu^+}(x)$$

as desired.

(2) We proceed as in (1). Thus if $\mu = (v_1, \cdots, v_k, \cdots, v_{|\mu|})$, we let $V,\ W$ be two open sets chosen so that:

$$\begin{aligned} V &\subset C^k(x), \\ W &= f_{v_1} \circ \cdots \circ f_{v_k}(V), \end{aligned}$$

and we will prove that $W \subset \text{Orb}_{L_\mu^-}(x)$.

Pick any $y \in W$, then $y \in R^k(z)$ for some $z \in V$. Moreover, we also have $x \in R^k(z)$; thus, applying the first result in Propositon 4.4, we can conclude:

$$y \in \text{Orb}_{L_\mu^-}(x)$$

as desired.

(3) This part is an obvious consequence of Proposition 4.5, since $x \in O_k^0(x)$ implies:

$$O_k^0(x) \subset \text{Orb}_{L_\mu}(x).$$

Thus $\text{Orb}_{L_\mu}(x)$ is open. ∎

6 Analytic Case

Throughout all this section we asssume that an analytic invertible system Σ , with connected control space U, is given. All the results presented here hold under these assumptions.

Proposition 6.1 Denote by μ a sequence of control values.

1. If x is $k-$forward accessible then for all μ with $|\mu| \geq k-1$, $\dim L_\mu^+(x) = n$.
2. If x is $k-$backward accessible then for all μ with $|\mu| \geq k$, $\dim L_\mu^-(x) = n$.
3. If x is $k-$transitive then for all μ with $|\mu| \geq k$, $\dim L_\mu(x) = n$.

Proof. We will prove only the first statement; the second and the third follow using the same arguments. Since Σ is analytic, applying a theorem of Nagano (see [9], section 9, or [7], Theorem 5), we know that the distribution associated to L_μ^+ is integrable. Moreover, if x is $k-$forward accessible then, by the first result in Proposition 5.3, we have that $\text{Orb}_{L_\mu^+}(x)$ is open. Thus we can conclude that $\dim L_\mu^+(x) = n$, as desired. ∎

Proposition 6.2 If $y \in R^l(x)$, and y is $k-$transitive for some $k \leq l$, then:

$$\dim \text{Lie}\, \Gamma^+(x) = n.$$

Proof. The statement is an immediate consequence of Proposition 5.1, and Proposition 6.1 part 3. ∎

We say that a system Σ is *bounded* transitive if there exists a nonnegative integer k such that $O^k(x)$ has non-empty interior for all $x \in \mathcal{X}$. Notice that if $O^k(x)$ has non-empty interior then, clearly, $O_{2k}^0(x)$ is open. Thus the following Corollary is a consequence of the previous Proposition and of Proposition 3.2:

Corollary 6.3 If Σ is bounded transitive, then Σ is also forward accessible. ∎

Remark 6.4 Notice that if the state space $\mathcal{X}$ is compact, Σ transitive implies Σ bounded transitive. To see this fact we argue as follows. For each $x \in \mathcal{X}$, denote by k_x any positive integer such that x is k_x-transitive. Then, by continuity, there exists an open neighborhood O_x of x such that if $y \in O_x$ then y is k_x-transitive. These open sets $\{O_x\}_{x\in\mathcal{X}}$ cover $\mathcal{X}$. Let $\{O_{x_i}\}_{i=1,\dots,l}$ be a finite subcover. Then, letting $k = \max\{k_{x_i}\}_{i=1,\dots,l}$, we have that $O^k(x)$ has nonempty interior for all $x \in \mathcal{X}$; thus Σ is bounded transitive.

So, in particular, the previous result implies that, if the state space $\mathcal{X}$ is compact, then a system Σ is transitive if and only if it is forward accessible. This result was proved in a very different way before, see [1] Theorem 2.

Definition 6.5 Given $x \in \mathcal{X}$, we let:

$$\omega(x) = \{y \in \mathcal{X} \mid \exists\, x_{n_k} \in R^{n_k}(x),\ x_{n_k} \to y,\ n_k \to \infty\}.$$

Note that, for control space consisting of only one element, $U = \{u\}$, the previous Definition is the standard ω-limit set.

Proposition 6.6 If $y \in \omega(x)$ and y is transitive, then $\dim \mathrm{Lie}\,\Gamma^+(x) = n$.

Proof. Since y is transitive, there exists a nonnegative integer k such that $O_k^0(y)$ is open. Thus there exists a neighborhood W of y, such that $O_k^0(z)$ is open for all $z \in W$. Since $y \in \omega(x)$, there exists $\tilde{y} \in R^l(x) \cap W$ with $l \geq k$; thus, by Proposition 6.2, we conclude the desired result. ∎

Recall that if Y is a vector field on a manifold $\mathcal{X}$, one says that $x \in \mathcal{X}$ is a *positively Poisson stable point* for Y if and only if for each neighbourhood V of x and each $T \geq 0$ there exists some $t > T$ such that $e^{tY}(x) \in V$, where $e^{tY}(\cdot)$ represents the flow of Y.

Analoguosly, one can define positive Poisson stability in discrete time, for a given diffeomorphism $f : \mathcal{X} \to \mathcal{X}$, as done in [1]. Next we define a generalization of this concept to systems.

Definition 6.7 Given a system Σ , the point $x \in \mathcal{X}$ is said to be *positively Poisson stable* for Σ if and only if for each neighbourhood V of x and each integer $N \geq 0$ there exist some integer $k > N$, and $(v_1, \cdots, v_k) \in U^k$ such that $f_{v_k} \circ \cdots \circ f_{v_1}(x) \in V$.

The following result generalizes [1] Theorem 1, which deals with the very special case of states which are positively Poisson stable for the diffeomorphism f_0. The proof in this case requires the full machinary just introduced.

Theorem 1 *Let $x \in \mathcal{X}$ be a positively Poisson stable point for Σ . Then x is transitive if and only if x is forward accessible.*

Proof. Notice that x positively Poisson stable for Σ means that $x \in \omega(x)$. Thus from Proposition 6.6, we know that if x is transitive then $\dim \mathrm{Lie}\,\Gamma^+(x) = n$. So there exists a neighbourhood W of x such that $\dim \mathrm{Lie}\,\Gamma^+(y) = n$ for all $y \in W$. Since $x \in \omega(x)$, and, since by the analyticity assumption $\tilde{R}(x)$ is dense in $R(x)$ (see (4), there exists some $y \in W \cap \tilde{R}(x)$. Thus we can conclude that x is forward accessible, using Proposition 3.1.

The other implication being obvious, the statement is proved. ∎

Definition 6.8 Given a system Σ , we say that Σ is *weakly asymptotically controllable* to a state $\bar{x}$ if $\bar{x} \in \operatorname{clos} R(x)$ for all $x \in \mathcal{X}$.

Remark 6.9 Saying that $\bar{x} \in \operatorname{clos} R(x)$ is *not* equivalent to saying that there exists a fixed infinite sequence u_i, $i > 0$, such that the sequence $x_i = f_{u_i,\ldots,u_1}(x)$ converges to $\bar{x}$, notion which is sometimes is called *asymptotic controllability.* Our notion is weaker, in fact, as it is proved in the next Lemma, it is equivalent to $\bar{x} \in \omega(x)$, which, in general, implies only that a subsequence of the x_i's converges to $\bar{x}$.

Lemma 6.10 Assume that $(\bar{x}, \bar{u}) \in \mathcal{X} \times U$ is not an equilibrium pair, that is, $f(\bar{x}, \bar{u}) \neq \bar{x}$. Then the following properties are equivalent:

1. Σ is weakly asymptotically controllable to $\bar{x}$,

2. $\bar{x} \in \omega(x)$ for all $x \in \mathcal{X}$.

Proof. Obviously (2) implies (1); thus we need only to establish the converse. Let W_n be a sequence of neighborhoods of $\bar{x}$ such that if $x_n \in W_n$ then $x_n \to \bar{x}$. We may assume that $W_0 = \mathcal{X}$. We will prove, by induction, that for any $x \in \mathcal{X}$, there exists a sequence $\{x_n\}_{n\geq 0}$ such that:

$$x_n \in R^{k_n}(x) \cap W_n \quad \text{ with } \; k_n \geq n. \tag{23}$$

Clearly, from (23), Property 2 follows.

Pick any $x \in \mathcal{X}$. Define $x_0 = x$, and $k_0 = 0$; then $x_0 \in R^{k_0}(x) \cap W_0$. Assume that we have already defined $x_0, \cdots, x_{n-1}$. We let:

$$\tilde{x}_n = \begin{cases} f(x_{n-1}, \bar{u}) & \text{if} \quad f(x_{n-1}, \bar{u}) \neq \bar{x} \\ \\ f(f(x_{n-1}, \bar{u}), \bar{u}) & \text{if} \quad f(x_{n-1}, \bar{u}) = \bar{x} \end{cases}$$

then $\tilde{x}_n \neq \bar{x}$. By assumption there exists some $x_n \in R^{\tilde{k}_n}(\tilde{x}_n) \cap W_n$ for some $\tilde{k}_n > 0$. Thus $x_n \in R^{k_n}(x) \cap W_n$, with $k_n = k_{n-1} + 1 + \tilde{k}_n$, if $\tilde{x}_n$ was defined in the first way, and $k_n = k_{n-1} + 2 + \tilde{k}_n$ otherwise. In any case, we have $k_n \geq n$ since $k_{n-1} \geq n-1$. Thus (23) holds. ∎

We now obtain one of our main results:

Theorem 2 *Let Σ be an analytic invertible system. If Σ is weakly asymptotically controllable to $\bar{x}$, and $\bar{x}$ is transitive then Σ is forward accessible from all $x \in \mathcal{X}$.*

Proof. Unless we are in the trivial case $\mathcal{X} = \{\bar{x}\}$, $\bar{x}$ transitive implies that there exists $\bar{u}$ such that $f(\bar{x}, \bar{u}) \neq \bar{x}$. Thus, by Lemma 6.10, $\bar{x} \in \omega(x)$ for all $x \in \mathcal{X}$, which, using Proposition 6.6, gives $\dim \operatorname{Lie} \Gamma^+(x) = n$ for all $x \in \mathcal{X}$. So the statement follows from Proposition 3.2. ∎

Example 6.11 Let's consider the following class of systems:

$$\Sigma: \; x(t+1) = Ax(t) + Bu_1(t) + g(x(t), u_2(t)),$$

with $\mathcal{X} = \mathbb{R}^n$, $U = \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, $u_i \in \mathbb{R}^{m_i}$, and where A, B are matrices with $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times m_1}$. Assume that g is an analytic function such that:

$$g(x, 0) = 0 \quad \forall \; x \in \mathcal{X}.$$

It is easy to see that the assumption that (A, B) is a stabilizable pair implies that Σ is weakly asymptotically controllable to 0. Recall that (A, B) stabilizable is equivalent to say that there exists a matrix $F \in \mathbb{R}^{m_1 \times n}$ such that the matrix $A + BF$ is Hurwitz, i.e. all its eigenvalues have negative real part. So the previous Theorem applies in this case, in particular, we have the following conclusion:

$$\text{If } A, B \text{ is a stabilizable pair and } \Sigma \text{ is transitive from 0 then}$$
$$\Sigma \text{ is forward accessible from all } x \in \mathcal{X}.$$

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LARGER COPY OF FIGURE 1:

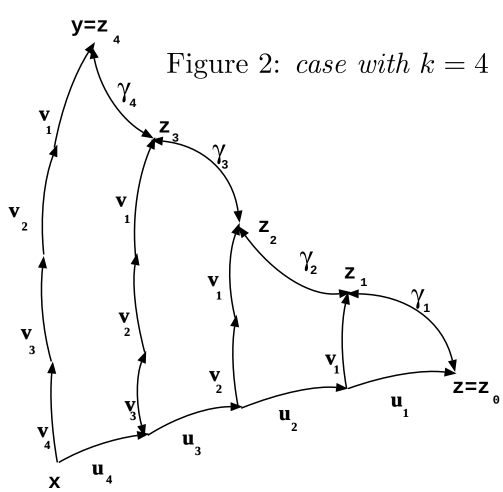

Figure 2: *case with $k = 4$*

LARGER COPY OF FIGURE 2:

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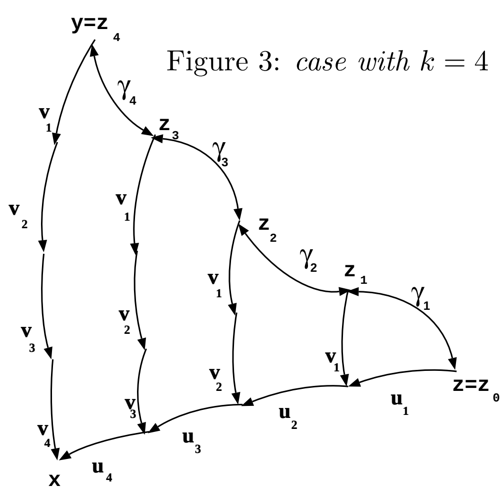

Figure 3: *case with* $k = 4$

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1.5

Figure 4: *case with* $s = 6, k = 3$