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Construction of extrapolation processes

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Abstract

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Let (S_n) be a sequence converging to S and such that $\forall n$, $S_n - S = a_n g_n$. The aim of this paper is to show how to construct extrapolation methods for accelerating the convergence of (S_n). Two cases are considered: (i) (a_n) unknown and (g_n) known, and (ii) (a_n) and (g_n) unknown. The first case covers the important particular cases $S_n - S = \mathrm{O}(g_n)$ or $\mathrm{o}(g_n)$. The iterated application of the processes is also studied. All the results are illustrated by numerical examples.

1. Introduction

In numerical analysis and in applied mathematics one often has to deal with a sequence (S_n) of approximations of the exact solution S of the problem. This is, for example, the case when constructing approximations depending on a parameter h (usually the step size as in quadrature methods or in numerical methods for solving ODEs and PDEs), in perturbation methods, in fixed point iterations (or more generally in iterative processes) or in the summation of convergent series. Quite often the convergence of the sequence (S_n) to S is slow and it needs to be accelerated. In many cases this can be done by using an extrapolation method. Various extrapolation methods are known [8], each of them being only able to accelerate the convergence of sequences whose error $S_n - S$ satisfies some particular assumptions since, as proved in [9], a universal convergence acceleration algorithm cannot exist. The most general extrapolation method actually known is the so-called E-algorithm [1,12] (see [6] for a survey) which was proved to be very effective if an asymptotic expansion of the error is known. However, in most of the practical cases sufficiently comprehensive information on the error is not available and the choice between the existing methods is a critical matter.

Thus, in this paper, instead of using pre-existing extrapolation methods, we shall describe how to construct well-adapted extrapolation processes based on restricted information about the error.

Let (S_n) be a sequence converging to S such that for all n

$$S_n - S = a_n g_n,$$

where (a_n) and (g_n) are sequences which are known or not and which may depend on (S_n). Of course, since (S_n) converges to S, $(a_n g_n)$ tends to zero. In the sequel we shall assume that (g_n) converges to zero (an assumption which does not restrict the generality) but we shall make no assumption on the convergence of (a_n). Thus our formalism covers many interesting situations such as:

- $S_n - S = O(g_n)$, which corresponds to $|a_n| \leqslant M$, $\forall n$;
- $S_n - S = o(g_n)$, which corresponds to $\lim_{n\to\infty} a_n = 0$;
- $S_n - S = a_1 g_1(n) + a_2 g_2(n) + \cdots$, which corresponds to $g_n = g_1(n)$ and $a_n = a_1 + a_2 g_2(n)/g_1(n) + \cdots = a_1 + \varepsilon_n$ with $\lim_{n\to\infty} \varepsilon_n = 0$;
- $S_n = c_0 f_0 + \cdots + c_n f_n$, which corresponds to $g_n = -c_{n+1}$ or $g_n = -f_{n+1}$ or $g_n = -c_{n+1} f_{n+1}$.

The first situation was treated for example in [19] where an extrapolation scheme similar to one of ours is used.

We shall now explain how to extrapolate the sequence (S_n) or, in other words, how to construct by extrapolation a new sequence (T_n) converging to S faster than (S_n) (that is $\lim_{n\to\infty}(T_n - S)/(S_n - S) = 0$) under some assumptions. Three cases can occur

(i) (a_n) and (g_n) known,
(ii) (a_n) unknown and (g_n) known (or the converse),
(iii) (a_n) and (g_n) unknown.

The first case is not interesting since $S = S_n - a_n g_n$ for all n. Let us now study the two other cases. Of course the construction of suitable extrapolation processes will need the knowledge of some information about (a_n) or about (a_n) and (g_n) and will depend on this information as we shall see below.

2. Second case: (a_n) unknown, (g_n) known

We shall study in this case, three different constructions of an extrapolation process according to the information known on the sequence (a_n). The more comprehensive the information, the more powerful the algorithm. Let us begin by the best case.

Let P be a linear operator on sequences, that is transforming any sequence (u_n) into $(v_n = P(u_n))$ and moreover such that $\forall (u_n)$, $\forall a$ and $\forall b$

$$P(au_n + b) = aP(u_n) + b \quad \text{for } n = 0, 1, \ldots.$$

We assume that a linear operator P on sequences is known such that $\forall n$, $P(a_n) = 0$. We have

$$(S_n - S)/g_n = a_n$$

and thus, applying P to both sides, we have for all n

$$P((S_n - S)/g_n) = P(a_n) = 0.$$

By the linearity property of P we have for all n

$$S = P(S_n/g_n)/P(1/g_n).$$

This is the approach followed by Weniger [20] for constructing some extrapolation processes. If such an operator P is known it leads to the exact result S. The main practical problem is to find P which is possible in some particular cases (for example if a_n is a polynomial of degree k in n, one can take $P = \Delta^{k+1}$) but not in the general case. Thus we shall now make use of a less complete information.

Let us assume that an approximation (b_n) of (a_n) is known. We set

$$T_n = S_n - b_n g_n, \quad n = 0, 1, \ldots.$$

Obviously $(T_n - S)/(S_n - S) = 1 - b_n/a_n$ and we have the following:

Theorem 2.1. *A necessary and sufficient condition that (T_n) converges to S faster than (S_n) is that $\lim_{n\to\infty} b_n/a_n = 1$.*

Thus, in this case it is also quite easy to accelerate the convergence of (S_n) if a sequence (b_n) satisfying the condition of Theorem 2.1 is known. If this is not the case, we shall try to build such a sequence (b_n). For that purpose we consider the linear function $f(x) = ax + b$ such that $S_n = f(g_n)$ and $S_{n+1} = f(g_{n+1})$ and we shall set $T_n = f(0)$. Thus we have extrapolated at zero and

$$T_n = S_n - \frac{\Delta S_n}{\Delta g_n} g_n, \quad n = 0, 1, \ldots,$$

which corresponds to the choice $(b_n = \Delta S_n/\Delta g_n)$. Let us study this transformation which is identical to the first column of the E-algorithm with $g_1(n) = g_n$ [1,12] or with the Θ-procedure [2].

We have

$$T_n - S = -\frac{\Delta a_n}{\Delta g_n} g_n g_{n+1}$$

and it follows that

$$\frac{T_n - S}{S_n - S} = \frac{1 - a_{n+1}/a_n}{1 - g_n/g_{n+1}} \quad \text{and} \quad \frac{T_n - S}{S_{n+1} - S} = \frac{1 - a_n/a_{n+1}}{1 - g_{n+1}/g_n}.$$

Thus two cases must be considered according whether or not (g_{n+1}/g_n) tends to one. We shall say that (g_n) is a logarithmic sequence if $\lim_{n\to\infty} g_{n+1}/g_n = 1$. We shall say that (g_n) is nonlogarithmic if

$$\exists \alpha < 1 < \beta, \quad \exists N, \quad \forall n \geqslant N, \quad g_{n+1}/g_n \notin [\alpha, \beta].$$

This assumption is equivalent to $g_n = O(\Delta g_n)$. It must be noticed that the nonlogarithmic case includes the interesting particular case $\lim_{n\to\infty} g_{n+1}/g_n = 0$. This particular case will be treated separately in order to give the best possible results. All the proofs are almost obvious and will be omitted.

Theorem 2.2. *If (g_n) is nonlogarithmic and if $\exists 0 < m \leqslant M$, $\exists N$ such that $\forall n \geqslant N$, $m \leqslant |g_{n+1}/g_n| \leqslant M$ then a necessary and sufficient condition that (T_n) converges to S faster than (S_n) and (S_{n+1}) is that $\lim_{n\to\infty} a_{n+1}/a_n = 1$.*

Let us give some remarks on this result.

Remark 2.3. If $\exists a \neq 0$ and finite such that (a_n) converges to a then the condition of Theorem 2.2 is satisfied.

Remark 2.4. We also have

$$\frac{T_n - S}{S_n - S} = -\frac{\Delta a_n}{a_n} \cdot \frac{1}{1 - g_n/g_{n+1}}$$

and

$$\frac{T_n - S}{S_{n+1} - S} = -\frac{\Delta a_n}{a_{n+1}} \cdot \frac{1}{g_{n+1}/g_n - 1}.$$

Thus instead of giving a necessary and sufficient condition involving the limit of the ratio a_{n+1}/a_n it is possible to formulate a result with a condition on the limit of Δa_n. However both results are not equivalent since $\lim_{n\to\infty} a_{n+1}/a_n = 1$ does not imply that $\lim_{n\to\infty} \Delta a_n = 0$ and vice versa. If $\lim_{n\to\infty} a_{n+1}/a_n = 1$ and if $\exists M$, $\exists N$ such that $\forall n \geqslant N$, $|a_n| \leqslant M$ then $\lim_{n\to\infty} \Delta a_n = 0$. Conversely if $\lim_{n\to\infty} \Delta a_n = 0$ and if $\exists m$, $\exists N$ such that $\forall n \geqslant N$, $m \leqslant |a_n|$ then $\lim_{n\to\infty} a_{n+1}/a_n = 1$. We can also use the fact that if $\lim_{n\to\infty} |a_n| = \infty$ and if $\exists M$, $\exists N$ such that $\forall n \geqslant N$, $|\Delta a_n| \leqslant M$ then $\lim_{n\to\infty} a_{n+1}/a_n = 1$.

Let us now study the particular case mentioned above. We have:

Theorem 2.5. *If* $\lim_{n\to\infty} g_{n+1}/g_n = 0$ *and if* $\exists M$, $\exists N$ *such that* $\forall n \geqslant N$, $|a_{n+1}/a_n| \leqslant M$ *or if* $\exists m$, $\exists M$, $\exists N$ *such that* $\forall n \geqslant N$, $m \leqslant |a_n|$ *and* $|\Delta a_n| \leqslant M$, *then* (T_n) *converges to S faster than* (S_n).

Since the computation of T_n involves S_{n+1} it would be better to compare the convergence of (T_n) and (S_{n+1}). Of course a stronger assumption will be required.

Theorem 2.6. *If* $\lim_{n\to\infty} g_{n+1}/g_n = 0$, *then a necessary an sufficient condition that* (T_n) *converges to S faster than* (S_{n+1}) *is that* $\lim_{n\to\infty} a_{n+1}/a_n = 1$.

The conditions of Theorems 2.5 and 2.6 are satisfied if $\exists a \neq 0$ such that $\lim_{n\to\infty} a_n = a$. As stated in Remark 2.4 this condition can be replaced by conditions on (Δa_n).

Let us now study the logarithmic case which is more difficult to treat. We have:

Theorem 2.7. *If* (g_n) *is logarithmic, then a necessary and sufficient condition that* (T_n) *converges to S faster than* (S_n) *is that*

$$\lim_{n\to\infty} \frac{1 - a_{n+1}/a_n}{1 - g_n/g_{n+1}} = 0.$$

A necessary and sufficient condition that (T_n) *converges to S faster than* (S_{n+1}) *is that*

$$\lim_{n\to\infty} \frac{1 - a_n/a_{n+1}}{1 - g_{n+1}/g_n} = 0.$$

As we can see the conditions involve both (a_n) and (g_n) and results similar to those of Remarks 2.3 and 2.4 cannot hold since in this case the denominator tends to zero and its study cannot be separated from that of the numerator. However, if (T_n) converges faster than (S_n) and if $\exists m$, $\exists N$ such that $\forall n \geqslant N$, $m \leqslant |a_{n+1}/a_n|$, then (T_n) also converges faster than (S_{n+1}). Conversely if (T_n) converges faster than (S_{n+1}) and if $\exists M$, $\exists N$ such that $\forall n \geqslant N$, $|a_{n+1}/a_n| \leqslant M$, then (T_n) also converges faster than (S_n).

If the conditions of the preceding theorems are not satisfied, then $(T_n - S)/(S_n - S)$ (or $(T_n - S)/(S_{n+1} - S)$) does not tend to zero and two cases can occur:

(i) $\exists c \neq 1$ such that $\lim_{n\to\infty}(T_n - S)/(S_n - S) = c$. Then it is possible to construct from (S_n) and (T_n) a new sequence (t_n) converging to S faster than (S_n) by using the so-called ACCES-algorithm [15] (see [8] for a subroutine).

(ii) $c = 1$ or the ratio $(T_n - S)/(S_n - S)$ has no limit. Then a contractive sequence transformation [5,7] can be used for improving the results.

3. Third case: (a_n) and (g_n) unknown

Replacing a_n by $a_n g_n$ and g_n by 1 for all n leads to the case treated in Section 2. However the assumptions of the theorems become too difficult to check and a separate treatment is more suitable.

We shall follow the same plan as in the preceding section and first we shall assume that an operator P such that $P(a_n) = 0$ is known. We again have $S = P(S_n/g_n)/P(1/g_n)$ but this relation cannot be used for computing S since (g_n) is an unknown sequence. Thus we shall replace (g_n) by an approximation (h_n) and set for all n

$$T_n = P(S_n/h_n)/P(1/h_n).$$

In the general case, that is without specifying the properties of P, nothing can be said about (T_n) and we shall now follow the second approach described in Section 2. Let (b_n) be an approximation of (a_n). We set

$$T_n = S_n - b_n h_n, \quad n = 0, 1, \ldots.$$

Theorem 2.1 still holds for this transformation by replacing in its condition b_n by $b_n h_n$ and a_n by $a_n g_n$. The condition is satisfied, for example, if $\exists a \neq 0$ such that

$$\lim_{n\to\infty} b_n/a_n = \lim_{n\to\infty} g_n/h_n = a.$$

In practical cases this condition is often difficult to check since it involves both (b_n) and (h_n). As we shall see now we can replace it, in some cases, by one single (but more complicated) condition.

Let us consider the case where (a_n) is unknown and $(g_n = S_{n-1} - S)$. We have, for all n

$$S = \frac{S_n - a_n S_{n-1}}{1 - a_n}.$$

Let us replace in this expression (a_n) by an approximation (b_n) and set

$$T_n = \frac{S_n - b_n S_{n-1}}{1 - b_n}, \quad n = 1, 2, \ldots.$$

Obviously we have:

Theorem 3.1. *A necessary and sufficient condition that* (T_n) *converges to* S *faster than* (S_{n-1}) *is that* $\lim_{n\to\infty}(a_n - 1)/(b_n - 1) = 1$.

Overholt's process [18] is based on this idea. The condition of Theorem 3.1 is satisfied if $\exists a \neq 1$ such that (a_n) and (b_n) converge to a. It also holds if $(a_n - b_n)$ converges to zero and if $\exists \alpha < 1 < \beta$, $\exists N$ such that $\forall n \geqslant N$, $b_n \notin [\alpha, \beta]$, a result proved by Lembarki [14]. By analogy with continued fractions, b_n can be called a converging factor.

Let us now consider the general case where (a_n) and (g_n) are arbitrary unknown sequences and again perform extrapolation at zero by a linear function $f(x) = ax + b$ such that $S_n = f(h_n)$ and $S_{n+1} = f(h_{n+1})$ where (h_n) is an approximation of (g_n). We obtain

$$T_n = S_n - \frac{\Delta S_n}{\Delta h_n} h_n, \quad n = 0, 1, \ldots$$

and we have

$$\frac{T_n - S}{S_n - S} = \left(1 - \frac{a_{n+1}}{a_n} \cdot \frac{g_{n+1}}{g_n} \cdot \frac{h_n}{h_{n+1}}\right) \cdot \left(1 - \frac{h_n}{h_{n+1}}\right)^{-1}$$

and

$$\frac{T_n - S}{S_{n+1} - S} = \left(\frac{a_n}{a_{n+1}} \cdot \frac{g_n}{g_{n+1}} \cdot \frac{h_{n+1}}{h_n} - 1\right) \cdot \left(\frac{h_{n+1}}{h_n} - 1\right)^{-1}.$$

Thus we obtain the following generalizations of the results given in the preceding section.

Theorem 3.2. *If* (h_n) *is nonlogarithmic and if* $\exists a \neq 0$ *and finite such that* $\lim_{n\to\infty} g_n/h_n = a$, *then a necessary and sufficient condition that* (T_n) *converges to* S *faster than* (S_n) *and* (S_{n+1}) *is that* $\lim_{n\to\infty} a_{n+1}/a_n = 1$.

Of course in that case we have a more general result.

Theorem 3.3. *If* (h_n) *is nonlogarithmic, then a necessary and sufficient condition that* (T_n) *converges to* S *faster than* (S_n) *and* (S_{n+1}) *is that*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \cdot \frac{g_{n+1}}{g_n} \cdot \frac{h_n}{h_{n+1}} = 1.$$

In the case of logarithmic sequences we obviously have:

Theorem 3.4. *If* (h_n) *is logarithmic, then a necessary and sufficient condition that* (T_n) *converges to* S *faster than* (S_n) *is that*

$$\lim_{n\to\infty} \left(1 - \frac{a_{n+1}}{a_n} \cdot \frac{g_{n+1}}{g_n} \cdot \frac{h_n}{h_{n+1}}\right) \cdot \left(1 - \frac{h_n}{h_{n+1}}\right)^{-1} = 0.$$

A necessary and sufficient condition that (T_n) *converges to* S *faster than* (S_{n+1}) *is that*

$$\lim_{n\to\infty} \left(\frac{a_n}{a_{n+1}} \cdot \frac{g_n}{g_{n+1}} \cdot \frac{h_{n+1}}{h_n} - 1\right) \cdot \left(\frac{h_{n+1}}{h_n} - 1\right)^{-1} = 0.$$

Of course, in these conditions $a_n g_n$ and $a_{n+1} g_{n+1}$ can be respectively replaced by $S_n - S$ and $S_{n+1} - S$. They can also be replaced respectively by

$$\lim_{n \to \infty} \frac{(S_{n+1} - S)/(S_n - S) - 1}{h_{n+1}/h_n - 1} = 1$$

and

$$\lim_{n \to \infty} \frac{(S_n - S)/(S_{n+1} - S) - 1}{h_n/h_{n+1} - 1} = 1.$$

Such conditions are difficult to check in practice.

4. Iteration of the procedures

We shall now study the iterated application of the procedures described in the preceding sections.

The first extrapolation method considered was the transformation given by

$$T_n = S_n - b_n g_n, \quad n = 0, 1, \ldots.$$

Obviously we have

$$T_n - S = (a_n - b_n) g_n.$$

If an approximation (c_n) of $(a_n - b_n)$ is known, then we can consider the sequence (U_n) given by

$$U_n = T_n - c_n g_n, \quad n = 0, 1, \ldots$$

and, by Theorem 2.1, a necessary and sufficient condition that (U_n) converges to S faster than (T_n) is that $\lim_{n \to \infty} c_n/(a_n - b_n) = 1$. The knowledge of such a sequence (c_n) is only possible in a limited number of cases, and we shall now consider the iteration of the second transformation of Section 2 which corresponds to the choice $b_n = \Delta S_n / \Delta g_n$. In that case we have

$$T_n - S = a'_n g'_n, \quad n = 0, 1, \ldots$$

with $a'_n = -\Delta a_n$ and $g'_n = g_n g_{n+1}/\Delta g_n$. Thus, in order to iterate the process, the sequences (a'_n) and (g'_n) must satisfy the same properties as the corresponding sequences (a_n) and (g_n). If (g_n) is nonlogarithmic, new and quite heavy assumptions have to be introduced for proving that (g'_n) is also nonlogarithmic in the general case. (The only simple case is when $\lim_{n \to \infty} g_{n+1}/g_n = a \neq 1$. See Theorem 4.3 below.) This is the reason why we shall now concentrate on the logarithmic case and study the assumptions under which (g'_n) is also logarithmic. We obviously have:

Theorem 4.1. *If*

$$\lim_{n \to \infty} \frac{g_{n+1}}{g_n} = \lim_{n \to \infty} \frac{\Delta g_{n+1}}{\Delta g_n} = 1,$$

then $\lim_{n \to \infty} g'_{n+1}/g'_n = 1$.

Remark 4.2. If (g_n) is monotone and logarithmic, then we cannot have $\lim_{n\to\infty} \Delta g_{n+1}/\Delta g_n = a$ with $a = 0$, $+\infty$ or $a \neq 1$. Thus either $(\Delta g_{n+1}/\Delta g_n)$ tends to 1 or has no limit.

This result was first obtained by Kowalewski [13] by a much longer proof. She sets

$$\frac{S_{n+1}-S}{S_n-S} = 1 - \lambda_n \quad \text{with} \quad \lim_{n\to\infty} \lambda_n = 0.$$

If (S_n) is monotone, then $\forall n$, $\lambda_n > 0$ and she proved that either $(\lambda_{n+1}/\lambda_n)$ tends to 1 or has no limit which is equivalent to our result.

The last result is:

Theorem 4.3. *If* $\lim_{n\to\infty} g_{n+1}/g_n = a \neq 1$, *then* $\lim_{n\to\infty} g'_{n+1}/g'_n = a$.

This result holds also if $a = 0$.

5. Numerical examples

We shall now give numerical examples illustrating all the procedures and theorems given in the preceding sections. In order to avoid tedious tables of numbers, the sequences obtained will be compared by using the kinematical notions introduced in [4]. Let (S_n) be a sequence converging to S. We set, $\forall n$,

$$d_n = -\log_{10} | S_n - S |.$$

The "speed" of the sequence (S_n) is defined by $(v_n = \Delta d_n)$ and its "acceleration" by $(\gamma_n = \Delta v_n)$. Let (v'_n) and (γ'_n) be respectively the speed and the acceleration of (T_n). It was proved in [4] that if $\exists k > 0$ such that $\forall n \geqslant N$, $v'_n \geqslant v_n + k$ then (T_n) converges faster than (S_n). Since this condition is only sufficient, it was also proved that the same conclusion also holds if $\forall n \geqslant N$, $v'_n > v_n$ and $\gamma'_n \geqslant \gamma_n$.

The following figures represent d_n as a function of n. We use dashed lines for (S_n) and solid lines for (T_n).

Example 5.1. We consider the sequence $S_n = n \sin(1/n)$, $n = 1, 2, \ldots$, which converges to $S = 1$. We have

$$S_n - S = \frac{1}{n^2}\left(-\frac{1}{6} + \frac{1}{5!n^2} - \cdots\right).$$

Thus, in order to illustrate Theorem 2.1, we shall take $g_n = n^{-2}$ and $b_n = -\frac{1}{6} + e_n$ where (e_n) is an arbitrary sequence converging to zero. For $e_n = 1/n$, $e_n = 0.9^n$, and $e_n = 1/n^2$, we obtain Figs. 1–3 respectively.

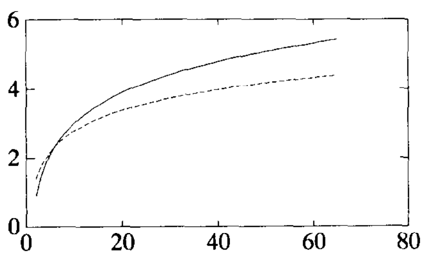

Fig. 1.

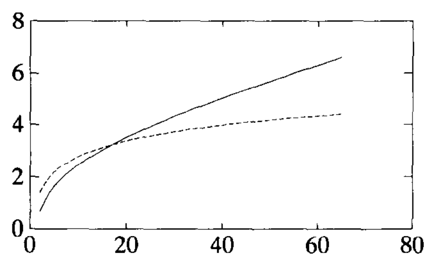

Fig. 2.

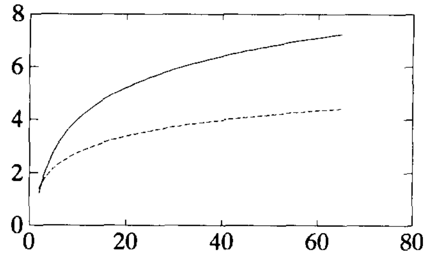

Fig. 3.

Example 5.2. Let us consider the sequence

$$S_n = \sum_{i=1}^{n} \tfrac{1}{2} i(i+1) x^{i-1}, \qquad n = 1, 2, \ldots, \qquad |x| < 1.$$

It can be proved [11] that

$$S_n = \frac{1}{(1-x)^3} - x^n \left[\frac{1}{(1-x)^3} + \frac{n(n+3) + xn(n+1)}{2(1-x)^2} \right],$$

which shows that $S = (1-x)^{-3}$. We shall take $g_n = x^n$ and thus (g_n) is nonlogarithmic. We have

$$a_n = \frac{1}{2(1-x)^3} [2 + (1-x) n(n+3+x(n+1))]$$

and $\lim_{n\to\infty} a_{n+1}/a_n = 1$, which shows, by Theorem 2.2, that (T_n) converges faster than (S_n) and (S_{n+1}). For $x = 0.5$, we have Fig. 4.

Example 5.3. We consider the sequence [11]

$$S_n = \frac{1 \cdot 2}{3!} + \cdots + \frac{n \cdot 2^n}{(n+2)!} = 1 - \frac{2^{n+1}}{(n+2)!},$$

which converges to $S = 1$. We take $g_n = 1/(n+2)!$ and $a_n = 2^{n+1}$. Thus $\lim_{n\to\infty} g_{n+1}/g_n = 0$ and $a_{n+1}/a_n = 2$ which shows by Theorem 2.5 and 2.6 that (T_n) converges faster than (S_n) but not faster than (S_{n+1}) and we obtain Fig. 5.

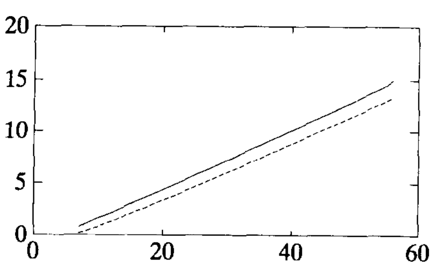

Fig. 4.

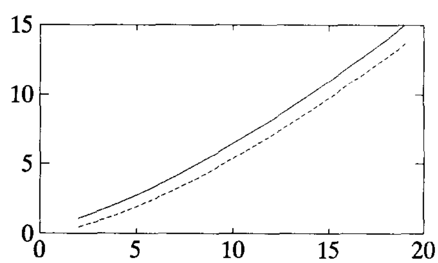

Fig. 5.

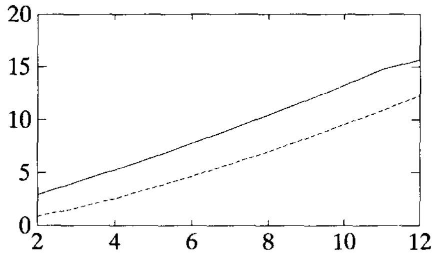

Fig. 6.

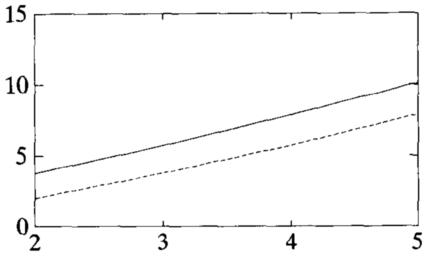

Fig. 7.

Example 5.4. We have

$$e^t = 1 + \frac{t}{1!} + \cdots + \frac{t^n}{n!} + \frac{t^{n+1}}{(n+1)!} e^{t\tau_n}$$

with $0 < \tau_n < 1$ if $t > 0$ and $\lim_{n\to\infty} \tau_n = 0$ [10].

Setting

$$S_n = 1 + \frac{t}{1!} + \cdots + \frac{t^n}{n!}, \qquad S = e^t,$$

$$g_n = \frac{t^{n+1}}{(n+1)!} \quad \text{and} \quad a_n = -e^{t\tau_n},$$

we have

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

Thus, by Theorems 2.5 and 2.6, (T_n) converges faster than (S_n) and (S_{n+1}) and we obtain Fig. 6 with $t = 0.5$.

Example 5.5. Let us consider the integral

$$S = \int_{-1}^{1} \frac{e^{2x}}{\sqrt{1-x^2}} dx = \pi I_0(2)$$

and compute approximate values by the Gauss–Chebyshev method:

$$S_n = \sum_{i=0}^{n} A_i f(x_i)$$

with

$$f(x) = e^{2x}, \qquad A_i = \pi/(n+1), \qquad x_i = \cos\frac{2i+1}{2n+2}\pi.$$

We know that

$$S - S_n = \frac{2\pi f^{(2n+2)}(\zeta_n)}{4^{n+1}(2n+2)!}, \quad \zeta_n \in [-1, +1].$$

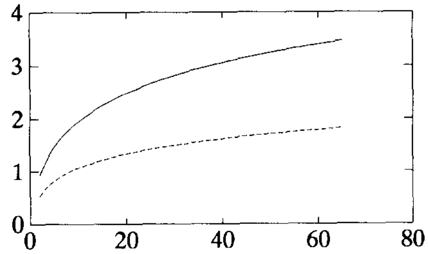

Fig. 8.

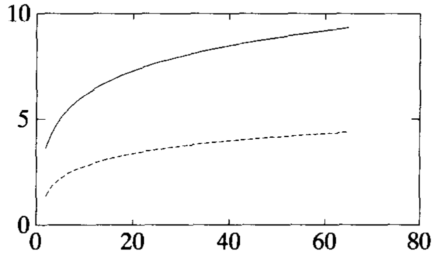

Fig. 9.

If we take

$$g_n = \frac{1}{4^{n+1}(2n+2)!},$$

then $\lim_{n\to\infty} g_{n+1}/g_n = 0$ and the condition of Theorem 2.5 is satisfied but nothing can be said about the condition of Theorem 2.6. The exact value of $I_0(2) = J_0(2i)$ is computed by the algorithm given in [16] which ensures a precision of $8.6 \cdot 10^{-18}$ (see Fig. 7).

Example 5.6. Let us consider the sequence [11]

$$S_n = \frac{3}{1 \cdot 2 \cdot 4} + \cdots + \frac{n+2}{n(n+1)(n+3)}$$
$$= \frac{29}{36} - \frac{1}{n+3}\left[1 + \frac{3}{2(n+2)} + \frac{4}{3(n+1)(n+2)}\right],$$

which converges to 29/36. Taking $g_n = (n+3)^{-1}$, it is easy to see that the conditions of Theorem 2.7 are satisfied and that (T_n) converges faster than (S_{n+1}). We obtain Fig. 8.

Example 5.7. Let us take again $S_n = n \sin(1/n)$ and $g_n = n^{-2}$. We have

$$\frac{1 - a_n/a_{n+1}}{1 - g_{n+1}/g_n} \sim \frac{6}{5!} n^{-2},$$

which shows, by Theorem 2.7, that (T_n) converges faster than (S_{n+1}) and we obtain Fig. 9.

Example 5.8. We consider the sequence (S_n) generated by

$$S_n - S = (S_{n-1} - S) \cdot \tfrac{1}{2}\sin(an + b/n), \quad n = 1, 2, \ldots.$$

We have $g_n = S_{n-1} - S$ and $a_n = \frac{1}{2}\sin(an + b/n)$. Choosing $b_n = \frac{1}{2}\sin an$. We obtain

$$a_n - b_n = \cos\frac{2an + b/n}{2} \sin\frac{b}{2n}.$$

Thus $\lim_{n\to\infty}(a_n - b_n) = 0$. Moreover $\exists \alpha < 1 < \beta$ such that $\forall n \geqslant N$, $b_n \notin [\alpha, \beta]$ and the conditions of Theorem 3.1 are satisfied.

With $S_0 = 0$, $S = 1$, $a = 2$ and $b = 1$ we get Fig. 10.

Thus (T_n) converges faster than (S_{n-1}) but it cannot be proved that it converges faster than (S_n) which explains why these results do not exhibit a better improvement.

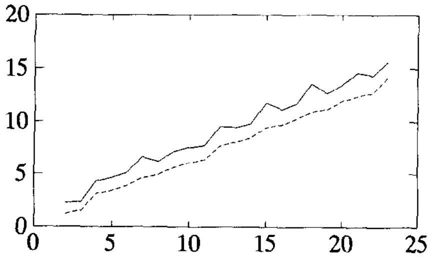

Fig. 10.

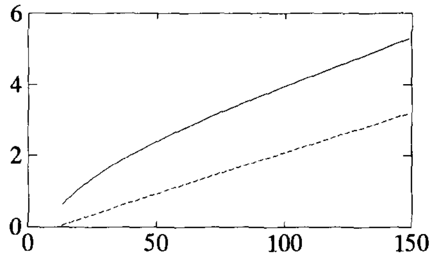

Fig. 11.

Example 5.9. We consider the sequence (S_n) given by

$$S_n = S + a_n g_n, \quad n = 0, 1, \ldots$$

with

$$g_n = \lambda^{n+1}\left(0.5 + \frac{1}{\ln(n+2)}\right), \qquad a_n = 2 + (n+1)^{-1}, \qquad |\lambda| < 1.$$

Choosing $h_n = \lambda^{n+1}$, we see that the conditions of Theorem 3.2 are satisfied and we obtain, with $\lambda = 0.95$ and $S = 1$, Fig. 11.

Example 5.10. We consider the sequence (S_n) given by

$$S_n = S + a_n g_n, \quad n = 0, 1, \ldots$$

with

$$g_n = \frac{\lambda^{n+1}}{n+1}, \qquad a_n = 2 + (n+1)^{-1}, \qquad |\lambda| < 1.$$

Choosing $h_n = \lambda^{n+1}$, we see that Theorem 3.3 holds. With $\lambda = 0.95$ and $S = 1$ we get Fig. 12.

Example 5.11. We consider the sequence (S_n) given by

$$S_n = S + a(n+b)^{-1}, \qquad n = 0, 1, \ldots, b \geqslant 1.$$

If $h_n = (n+1)^{-1}$ then the conditions of Theorem 3.4 are satisfied. With $S = 1$, $a = 2$ and $b = 4$ we obtain Fig. 13.

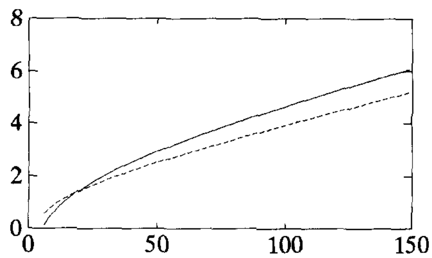

Fig. 12.

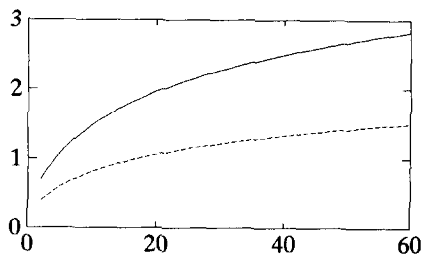

Fig. 13.

Three of the transformations studied in the preceding sections are of the form

$$T_n = \frac{S_{n+1} - b_n S_n}{1 - b_n}, \quad n = 0, 1, \ldots.$$

When (S_n) is a strictly monotone sequence it is of interest to know conditions on the sequence (b_n) such that (T_n) is also monotone either in the same or in the opposite direction as (S_n). In particular if the monotonicity is reversed, then intervals containing S are obtained. Such conditions were given by Opfer [17]. Other methods for constructing intervals containing S can be found in [3].

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