A short survey on the Cauchy problem in sheaf theory

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Abstract

This short expository paper shows how sheaf theory applies to the study of initial value problems. More precisely, starting with the classical Cauchy-Kowalevski theorem as the only tool borrowed from P.D.E., the methods of the microlocal theory of sheaves of [7] are utilized to obtain "more difficult" results such as the well poseness of the hyperbolic Cauchy problem for hyperfunctions, or of the non-characteristic Cauchy problem with holomorphic ramified data.

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1 Foreword

In the following I will present a short survey on the Cauchy problem in the language of sheaf theory. This should be considered as an appendix to Schneiders's lectures [11], sketching the interplay between $\mathcal{D}$-modules and the microlocal theory of sheaves by Kashiwara and Schapira [7].

To my knowledge, there does not exist any reference book on this subject. A good survey (from which I benefitted) may be found in Chapter 11 of [7].

2 Tools from P.D.E.

Let me begin by recalling two basic tools in the study of analytic partial differential equations: the Cauchy-Kowalevski theorem and the Zerner propagation lemma.

Let X be an open neighborhood of 0 in $\mathbb{C}^n$ with coordinates $(z) = (z_1, z')$, and let Y be the hypersurface of X defined by the equation $z_1 = 0$. Let

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$P(z,\partial_z)=\sum_{|\alpha|=0}^{m} a_\alpha(z)\partial_z^\alpha$ be a differential operator on X with holomorphic coefficients, and denote by $\sigma(P)(z;\zeta)=\sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha$ its principal symbol.

Recall that Y is called non-characteristic for the operator P if,

$$\text{for every } z\in Y,\quad \sigma(P)(z,dz_1)\neq 0, \tag{2.1}$$

which is equivalent to the requirement $a_{(m,0\ldots,0)}(0,z')\neq 0$.

If u is a holomorphic function defined in a neighborhood of Y, one denotes by $\gamma_Y(u)$ its first m traces on Y, $\gamma_Y(u)=(u|_Y,\ldots,\partial_{z_1}^{m-1}u|_Y)$.

Theorem 2.1. *[Cauchy-Kowalevski] Let $v\in\mathcal{O}_X$, $(w)\in\mathcal{O}_Y^m$, and consider the Cauchy problem*

$$\begin{cases} Pu=v,\\ \gamma_Y(u)=(w). \end{cases} \tag{2.2}$$

Assume that Y is non-characteristic for P. Then (2.2) has a unique solution $u\in\mathcal{O}_X|_Y$.

Leray [8] proved a sharper version of the previous result. Denote by $B(0,\rho)$ and $B'(0,\rho)$ the open balls of center 0 and radius $\rho>0$ in $\mathbb{C}^n$ and $\mathbb{C}^{n-1}$ respectively.

Theorem 2.2. *[Cauchy-Kowalevski-Leray] Assume that Y is non-characteristic for P. Then there exist constants $r,\rho_0,\delta>0$ such that for every ρ with $0<\rho\leq\rho_0$, every x with $|x|\leq r$, and every data $v\in\mathcal{O}_X(B(0,\rho))$, $(w)\in\mathcal{O}_Y^m(B'(0,\rho))$, the Cauchy problem (2.2) has a unique solution $u\in\mathcal{O}_X(B(0,\delta\rho))$.*

A useful corollary of the last theorem is the following propagation result:

Lemma 2.3. *[Zerner] Let ψ be a C^1 function and set $\Omega=\{x;\psi(x)<0\}$. Assume that $\sigma(P)(x_0;d\psi(x_0))\neq 0$ and let $u\in\mathcal{O}_X(\Omega)$ be such that Pu extends holomorphically to a neighborhood of $x_0\in\partial\Omega$. Then u extends holomorphically to a neighborhood of x_0.*

3 Cauchy-Kowalevski-Kashiwara theorem

Let $f:Y\longrightarrow X$ be a morphism of complex manifolds, and consider the associated correspondence of cotangent bundles:

$$T^*Y \underset{{}^tf'}{\longleftarrow} Y\underset{X}{\times}T^*X \underset{f_\pi}{\longrightarrow} T^*X.$$

Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and recall that f is called non-characteristic for $\mathcal{M}$ if

$$f_\pi{}^{-1}(\mathrm{char}(\mathcal{M})) \cap {}^t f'^{-1}(T^*_Y Y) \subset Y \underset{X}{\times} T^*_X X, \tag{3.1}$$

where $T^*_X X$ denotes the zero-section of T^*X.

The following formulation of the Cauchy-Kowalevski theorem is due to Kashiwara [4].

Theorem 3.1. *[Cauchy-Kowalevski-Kashiwara] Assume that f is non characteristic for $\mathcal{M}$. Then the natural morphism*

$$f^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \tag{3.2}$$

is an isomorphism.

As recalled in [11], Kashiwara's original proof is based on Theorem 2.1, plus some algebraic arguments. In a sense, the difficult part is the switch of language from the statement of Theorem 2.1, to that of Theorem 3.1.

Let $f : Y \longrightarrow X$ be the immersion of the hypersurface defined in the previous section, and let $\mathcal{M}$ be the $\mathcal{D}_X$-module associated to the operator P:

$$0 \longrightarrow \mathcal{D}_X \xrightarrow{\cdot P} \mathcal{D}_X \longrightarrow \mathcal{M} \longrightarrow 0. \tag{3.3}$$

In this case,

$$\mathrm{char}(\mathcal{M}) = \{(z;\zeta) \in T^*X;\ \sigma(P)(z;\zeta) = 0\},$$

and conditions (2.1) and (3.1) are thus equivalent. Recall moreover that Theorems 2.1 and 3.1 are also equivalent. In fact, the free $\mathcal{D}_X$ resolution (3.3) gives:

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \simeq [\mathcal{O}_X \xrightarrow{P\cdot} \mathcal{O}_X],$$

and one easily checks that

$$R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y) \simeq (\mathcal{O}_Y)^m.$$

Taking the zero-th and first cohomology group of the isomorphism (3.2), one then has

$$\begin{cases} \ker P \simeq (\mathcal{O}_Y)^m, \\ \mathrm{coker}\, P = 0. \end{cases}$$

In other words, for every $v \in \mathcal{O}_X|_Y$, $w \in (\mathcal{O}_Y)^m$, one can solve the problems:

$$\begin{cases} Pu = 0 \\ \gamma_Y(u) = (w) \end{cases}, \qquad Pu = v, \tag{3.4}$$

and it is evident that (2.2) and (3.4) are equivalent.

4 A statement for Cauchy problems

Let $\mathcal{M}$ be a left coherent $\mathcal{D}_X$-module, let $f : Y \longrightarrow X$ be a morphism of complex manifolds, and assume that f is non-characteristic for $\mathcal{M}$. Let $\mathcal{S}$ and $\mathcal{T}$ be two left $\mathcal{D}_X$ and $\mathcal{D}_Y$-modules respectively, and let

$$\underline{\phi} : \underline{f}^{-1}\mathcal{S} \longrightarrow \mathcal{T} \tag{4.1}$$

be a given $\mathcal{D}_Y$-linear morphism.

The discussion in the previous section was meant to show the reader that the isomorphism (3.2) is equivalent to the solution of the associated Cauchy problem with holomorphic data. On the same lines, we may say that we will have solved the Cauchy problem with data in $\mathcal{S}$, $\mathcal{T}$, if we establish that the natural morphism:

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{S}) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{T}), \tag{4.2}$$

induced by $\underline{\phi}$, is an isomorphism.

Let K and L be sheaves on X and Y respectively (or, more precisely, objects of the derived categories). As we will see in sections 6.2, 6.3, interesting classes of solution spaces are obtained by considering the situation:

(i) $\mathcal{S} = R\mathcal{H}om(K,\mathcal{O}_X)$,

(ii) $\mathcal{T} = R\mathcal{H}om(L,\mathcal{O}_Y)$,

(iii) the morphism $\underline{\phi}$ is induced by a morphism $\phi : f^{-1}K \longleftarrow L$.

Notice that one has the isomorphisms:

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{S}) &\simeq R\mathcal{H}om(K, R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)),\\ R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{T}) &\simeq R\mathcal{H}om(L, R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{O}_Y)),\end{aligned}$$

and recall that Theorem 3.1 gives:

$$R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y,\mathcal{O}_Y) \simeq f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X).$$

Setting:

$$F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X),$$

the Cauchy problem (4.2) is thus reduced to giving conditions on f, F, K, L, and ϕ so that the natural morphism:

$$f^{-1}R\mathcal{H}om(K,F) \longrightarrow R\mathcal{H}om(L,f^{-1}F) \tag{4.3}$$

induced by ϕ, is an isomorphism.

5 Propagation and micro-support

A statement like (4.3) is almost purely sheaf-theoretical: what is lacking is a sheaf-theoretical counterpart to the notion of non-characteristicity. It is here that the notion of micro-support by Kashiwara and Schapira [7] appears.

With the notations of section 2, Lemma 2.3 says that given $g \in (\mathcal{O}_X)_{x_0}$ and $f \in (\Gamma_\Omega \mathcal{O}_X)_{x_0}$ such that $Pf = g|_\Omega$, there exists $\widetilde{f} \in (\mathcal{O}_X)_{x_0}$ such that $f = \widetilde{f}|_\Omega$ and $P\widetilde{f} = g$. This is equivalent to the vanishing of the first cohomology group of the simple complex associated to the double complex:

$$\begin{array}{ccc} (\Gamma_\Omega \mathcal{O}_X)_{x_0} & \xrightarrow{P} & (\Gamma_\Omega \mathcal{O}_X)_{x_0} \\ \uparrow & & \uparrow \\ (\mathcal{O}_X)_{x_0} & \xrightarrow{P} & (\mathcal{O}_X)_{x_0}. \end{array}$$

Hence, Zerner's lemma is implied by the requirement:

$$(\mathrm{R}\Gamma_{X \setminus \Omega} F)_{x_0} = 0,$$

where $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \mathcal{O}_X)$.

One is now in a position to forget the complex structure, and to forget P.D.E.

Let X be a real analytic manifold (actually, for most results C^1 manifolds would suffice), and denote by $\mathbf{D}^{\mathrm{b}}(X)$ the derived category of the category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on X.

Definition 5.1. The micro-support $SS(F)$ of an object F of $\mathbf{D}^{\mathrm{b}}(X)$ is the closed conic involutive subset of T^*X given by: $p = (x_0; \xi_0) \notin SS(F)$ if there exists an open neighborhood U of p such that for every $x_1 \in X$ and for every C^1 function ψ verifying $\psi(x_1) = 0$ and $d\psi(x_1) \in U$, one has:

$$(\mathrm{R}\Gamma_{\psi \geq 0} F)_{x_1} = 0.$$

It is not possible to review the microlocal theory of sheaves here. I shall simply list some elementary results that will be useful in the following.

Let A be a locally closed subset of X, and let $M \subset X$ be a closed submanifold. One denotes by $C_M(A)$ the Whitney normal cone of A along M. This is a closed conic subset of $T_M X$, the normal bundle to M in X.

One denotes by $\mathbb{C}_A$ the sheaf which is zero on $X \setminus A$, and which is the constant sheaf with stalk $\mathbb{C}$ on A. One denotes by $D'F = R\mathcal{H}om(F, \mathbb{C}_X)$ the dual of F.

Let $f : Y \longrightarrow X$ be a morphism of real analytic manifolds. Similarly to the case of $\mathcal{D}$-modules, one says that f is non-characteristic for F if

$${f_\pi}^{-1} SS(F) \cap {}^t f'^{-1}(T^*_Y Y) \subset Y \underset{X}{\times} T^*_X X.$$

One denotes by $\omega_{Y/X} = f^!\mathbb{C}_X$ the relative dualizing complex. Recall that $\omega_{Y/X} \simeq or_{Y/X}[\dim Y - \dim X]$, where $or_{Y/X}$ denotes the relative orientation sheaf.

Proposition 5.2. *(i) Let $F \in Ob(D^b(X))$ and assume that f is non-characteristic for F. Then the following estimate holds:*

$$SS(f^{-1}F) \subset {}^t f' f_\pi{}^{-1}(SS(F)),$$

and one has a natural isomorphism:

$$f^{-1}F \otimes \omega_{Y/X} \xrightarrow{\sim} f^!F.$$

(ii) Let $G \in Ob(D^b(Y))$ and assume that f is proper on $supp(G)$. Then:

$$SS(Rf_*G) \subset f_\pi {}^t f'^{-1}(SS(G)).$$

(iii) Let $F, G \in Ob(D^b(X))$ and assume that G has $\mathbb{R}$-constructible cohomology groups, and that $SS(F) \cap SS(G) \subset T^*_X X$. Then:

$$D'G \otimes F \xrightarrow{\sim} R\mathcal{H}om(G, F).$$

(iv) Let $F \in Ob(D^b(X))$ and let $M \subset X$ be a closed submanifold. Then:

$$SS(\mathrm{R}\Gamma_M F) \subset T^*M \cap C_{T^*_M X}(SS(F))$$

(cf. section 6.2 for the identification $T^*M \subset T_{T^*_M X}T^*X$).

6 Back to P.D.E.

Let X be a complex analytic manifold, let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and set $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. Using Theorem 2.2 it is easy to prove the inclusion:

$$SS(F) \subset \mathrm{char}(\mathcal{M}), \tag{6.1}$$

and in fact equality holds.

As we will now see, using (6.1) and the results of the previous section, it is possible to recover classical results in P.D.E.

Let M be a real analytic manifold and let X be a complexification of M. Recall that the sheaf of Sato's hyperfunctions is defined by:

$$\mathcal{B}_M = R\mathcal{H}om(D'_X\mathbb{C}_M, \mathcal{O}_X).$$

Notice that the natural morphism $D'_X F \otimes G \longrightarrow R\mathcal{H}om(F,G)$ applied to $F = D'_X \mathbb{C}_M$, $G = \mathcal{O}_X$ gives the embedding

$$\mathcal{A}_M \hookrightarrow \mathcal{B}_M,$$

where $\mathcal{A}_M = \mathbb{C}_M \otimes \mathcal{O}_X$ is the sheaf of real analytic functions on M.

Remark 6.1. More generally, if M is a generic submanifold of a complex manifold X, then $R\mathcal{H}om(D'_X\mathbb{C}_M, \mathcal{O}_X)$ is isomorphic to the complex of CR-hyperfunctions (i.e. the complex of hyperfunction solutions on M to the tangential $\overline{\partial}$ system).

6.1 Elliptic operators

The short exact sequence:

$$0 \longrightarrow T^*_M X \longrightarrow M \underset{X}{\times} T^*X \longrightarrow T^*M \longrightarrow 0,$$

defining the conormal bundle $T^*_M X$, and the isomorphism:

$$M \underset{X}{\times} T^*X \simeq T^*M \oplus iT^*M,$$

which is due to the complex structure, give an identification:

$$T^*_M X \simeq iT^*M.$$

Recall that an operator $P \in \mathcal{D}_X$ is called elliptic if

$$\sigma(P)(x; i\xi) \neq 0 \quad \text{for every } \xi \neq 0.$$

More generally, a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is called elliptic if

$$\mathrm{char}(\mathcal{M}) \cap T^*_M X \subset T^*_X X.$$

Applying Proposition 5.2 (iii) to $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ and $G = D'\mathbb{C}_M$, one thus gets the classical regularity theorem:

Theorem 6.2. *Let $\mathcal{M}$ be an elliptic coherent $\mathcal{D}_X$-module. Then:*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{A}_M) \xrightarrow{\sim} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M).$$

6.2 Hyperbolic operators

One of the keys to understand hyperbolic operators is the identification of T^*M to a submanifold of $T^*_{T^*_MX}T^*X$.

Let $-H : T^*T^*X \xrightarrow{\sim} TT^*X$ be the opposite of the Hamiltonian isomorphism H. If $\Lambda \subset T^*X$ is a Lagrangian submanifold, $-H$ induces an identification $T^*\Lambda \simeq T_\Lambda T^*X$. In particular, for $\Lambda = T^*_MX$ one gets:

$$T^*T^*_MX \simeq T_{T^*_MX}T^*X \simeq T^*_{T^*_MX}T^*X. \tag{6.2}$$

By the zero section $M \longrightarrow T^*_MX$ of $\pi : T^*_MX \longrightarrow M$, and by the induced map:

$${}^t\pi' : T^*_MX \underset{M}{\times} T^*M \longrightarrow T^*T^*_MX,$$

one gets a map:

$$T^*M \simeq M \underset{M}{\times} T^*M \longrightarrow T^*T^*_MX. \tag{6.3}$$

Composing (6.2) and (6.3) one gets the identification:

$$T^*M \hookrightarrow T^*_{T^*_MX}T^*X. \tag{6.4}$$

Let N be a real submanifold of M, let $Y \subset X$ be a complexification of N, and consider the embeddings:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle i} \\ N & \xrightarrow{g} & M. \end{array}$$

One says that N is hyperbolic for a coherent $\mathcal{D}_X$-module $\mathcal{M}$ if

$$T^*_NM \cap C_{T^*_MX}(\mathrm{char}(\mathcal{M})) \subset T^*_MM \tag{6.5}$$

via the identification (6.4).

Of course, if $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_XP$ is the $\mathcal{D}_X$-module associated to a single differential operator $P \in \mathcal{D}_X$, one recovers the usual definition of (weak) hyperbolicity.

As proved by [1], [7], the good solution space for the study of hyperbolic Cauchy problems is the space of Sato's hyperfunctions.

Theorem 6.3. *Assume that N is hyperbolic for $\mathcal{M}$. Then the natural morphism:*

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N) \tag{6.6}$$

is an isomorphism.

Notice that (6.6) is of the form (4.2). It also fit with (4.3) if one sets $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, $K = D'\mathbb{C}_M$, $L = D'\mathbb{C}_N$, and let ϕ be the morphism induced by the natural morphism $\mathbb{C}_M \longrightarrow \mathbb{C}_N$.

Proof. By Proposition 5.2 (iv), hypothesis (6.5) implies that g is non-characteristic for $\mathrm{R}\Gamma_M(F)$. One then has the chain of isomorphisms:

$$\begin{aligned} g^{-1}\mathrm{R}\Gamma_M(F) &\simeq g^!\mathrm{R}\Gamma_M(F) \otimes \omega_{N/M}^{\otimes -1} \\ &\simeq g^! i^! F \otimes \omega_{N/M}^{\otimes -1} \\ &\simeq j^! f^! F \otimes \omega_{N/M}^{\otimes -1} \\ &\simeq \mathrm{R}\Gamma_N(F) \otimes \omega_{N/M}^{\otimes -1}. \end{aligned}$$

Recall that $D'\mathbb{C}_M \cong \omega_{M/X}$. We have thus proved the isomorphism:

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{B}_M) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N\mathcal{B}_M) \otimes \omega_{N/M}^{\otimes -1}.$$

The proof is then achieved by following "division" lemma. □

Lemma 6.4. *Assume that Y is non-characteristic for $\mathcal{M}$. Then there is an isomorphism:*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \Gamma_N\mathcal{B}_M) \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{B}_N) \otimes \omega_{N/M}. \tag{6.7}$$

Proof. Setting $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, one has the isomorphisms:

$$\begin{aligned} \mathrm{R}\Gamma_N(F) &\simeq j^! f^! F \\ &\simeq \mathrm{R}\Gamma_N(f^{-1}F) \otimes \omega_{Y/X}. \end{aligned}$$

By Theorem 3.1, $f^{-1}F \simeq R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_Y)$, and one concludes. □

6.3 Ramified functions

An instance of naturally occurring solution spaces when studying non-characteristic Cauchy problems with singular data (or characteristic problems with holomorphic data) is the space of ramified holomorphic functions.

Let $p : \widetilde{\mathbb{C}^*} \longrightarrow \mathbb{C}$ be the universal covering of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Recall that one can choose a coordinate $t \in \mathbb{C} \simeq \widetilde{\mathbb{C}^*}$ so that $p(t) = \exp(2\pi i t)$.

Let Y be a germ of smooth hypersurface of a complex manifold X, and choose a locally defined complex analytic function $g : X \longrightarrow \mathbb{C}$, with $dg \neq 0$, such that $Y = g^{-1}(0)$. Set $\widetilde{X^*} = \widetilde{\mathbb{C}^*} \times_{\mathbb{C}} X$, and consider the Cartesian diagram:

$$\begin{array}{ccc} \widetilde{X^*} & \longrightarrow & \widetilde{\mathbb{C}^*} \\ \downarrow^{p_X} & & \downarrow^{p} \\ X & \xrightarrow{g} & \mathbb{C}. \end{array}$$

The complex $\mathcal{O}^{\mathrm{ram}}_{Y/X}$ of ramified holomorphic function along Y in X is defined as $\mathcal{O}^{\mathrm{ram}}_{Y/X} = Rp_{X*}p_X^{-1}\mathcal{O}_X$.

Notice that $p_X^!$ is an exact functor and that $p_X^! = p_X^{-1}$. By the Poincaré-Verdier duality one gets:

$$\begin{aligned} Rp_{X*}p_X^{-1}\mathcal{O}_X &= R\mathcal{H}om(Rp_{X!}\mathbb{C}_{\widetilde{X^*}}, \mathcal{O}_X) \\ &= R\mathcal{H}om(g^{-1}p_!\mathbb{C}_{\widetilde{\mathbb{C}^*}}, \mathcal{O}_X). \end{aligned}$$

Summarizing up, one can write

$$\mathcal{O}^{\mathrm{ram}}_{Y/X} = R\mathcal{H}om(K, \mathcal{O}_X), \tag{6.8}$$

where $K = g^{-1}p_!\mathbb{C}_{\widetilde{\mathbb{C}^*}}$.

Remark 6.5. More generally, let $T \subset X$ be a closed subset, $\widetilde{p} : \widetilde{X} \longrightarrow X \setminus T$ be a covering map, and set $p = i \circ \widetilde{p}$, where $i : X \setminus T \longrightarrow X$ is the inclusion map. The complex $R\mathcal{H}om(K, \mathcal{O}_X)$, for $K = p_!\mathbb{C}_{\widetilde{X}}$, may then be considered as a complex of "ramified" holomorphic functions along T.

Let X be an open subset of $\mathbb{C}^n$ with $0 \in X$, let $z = (z_1, z') = (z_1, \dots, z_n)$ be the coordinates on X and let $(z;\zeta)$ be the associated coordinates in T^*X. Consider the Cauchy problem:

$$\begin{cases} Pu = 0, \\ \gamma_Y(u) = (w), \end{cases} \tag{6.9}$$

where P is a differential operator of order m, the hyperplane $Y = \{z \in X; z_1 = 0\}$ is non-characteristic for P, and (w) are holomorphic functions on Y ramified along the hypersurface $Z = \{z \in X; z_1 = z_2 = 0\}$. Let $f : Y \longrightarrow X$ be the embedding. Suppose that P has characteristics with constant multiplicities transversal to $Y \times_X T^*X$ at ${}^tf'^{-1}(T^*_ZY) \cap \mathrm{char}(\mathcal{M})$.

Let $Z_1, \dots, Z_r$ be the (locally uniquely determined) smooth hypersurfaces of X whose conormal bundles are the union of the bicharacteristics of P issued from ${}^tf'^{-1}(\dot{T}^*_ZY)$. Notice that the Z_i are transversal to each other, and transversal to Y.

In [3], Hamada, Leray and Wagschal proved that the holomorphic solution of (6.9), defined in a neighborhood of $Y \setminus Z$, extends holomorphically as a sum of ramified functions along the Z_j's.

Following [2], let us reduce this statement to the form (4.2), (4.3).

Choose analytic functions $g : Y \longrightarrow \mathbb{C}$, $g_i : X \longrightarrow \mathbb{C}$ with $dg \neq 0$, $dg_i \neq 0$, such that $g_i \circ f = g$ and $Z = g^{-1}(0)$, $Z_i = g_i^{-1}(0)$. Set $L = g^{-1}p_!\mathbb{C}_{\widetilde{X}}$, $K_i = g_i^{-1}p_!\mathbb{C}_{\widetilde{X}}$, and let K be the third term of a distinguished triangle:

$$K \longrightarrow \oplus_{i=1}^r K_i \xrightarrow{h} \oplus_{i=1}^{r-1}\mathbb{C}_X \xrightarrow{+1},$$

where h is the composite of the map $\oplus_{i=1}^r \tau_j$ and the map $\oplus_{i=1}^r \mathbb{C}_X \longrightarrow \oplus_{i=1}^{r-1} \mathbb{C}_X$, given by $(a_1, \ldots, a_r) \mapsto (a_2 - a_1, \ldots, a_r - a_{r-1})$. Notice that the natural morphisms $L \longrightarrow \mathbb{C}_Y$, $K_i \longrightarrow \mathbb{C}_X$, induce a morphism

$$\phi : L \longrightarrow f^{-1}K.$$

The sheaf of holomorphic functions on Y ramified along Z is given by:

$$\mathcal{O}_{Z/Y}^{\mathrm{ram}} = R\mathcal{H}om(L, \mathcal{O}_Y),$$

and one may check that the complex

$$\sum_i \mathcal{O}_{Z_i/X}^{\mathrm{ram}} = R\mathcal{H}om(K, \mathcal{O}_X)$$

is concentrated in degree zero, and that its sections are sums of ramified functions along the Z_j's.

Theorem 6.6. *([2]) The natural morphism:*

$$f^{-1}R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sum_i \mathcal{O}_{Z_i/X}^{\mathrm{ram}})|_Z \longrightarrow R\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{O}_{Z/Y}^{\mathrm{ram}})|_Z \qquad (6.10)$$

is an isomorphism.

Proof. With the above notations, we may rewrite (6.10) as

$$f^{-1}R\mathcal{H}om(K, F)|_Z \xrightarrow{\sim} R\mathcal{H}om(L, f^{-1}F)|_Z,$$

and it is possible to give a purely sheaf-theoretical proof of this isomorphism for which we refer to [2]. □

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