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Poisson structures on moduli spaces of sheaves over Poisson surfaces

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Summary. We introduce and study the notion of Poisson surface. We prove that the choice of a Poisson structure on a surface S canonically determines a Poisson structure on the moduli space $\mathcal{M}$ of stable sheaves on S. This result generalizes previous results obtained by Mukai [14], for abelian or $K3$ surfaces, and by Tyurin [16].

Introduction

In [14], Mukai proved that the moduli space $\mathcal{M}$ of sheaves on an abelian or $K3$ surface S has a natural symplectic structure ω. However, in Mukai's paper, a symplectic structure is defined as a nowhere degenerate holomorphic 2-form, hence a natural question arises: can we prove directly that ω is closed?

We may remark that the reason for considering only abelian or $K3$ surfaces is that their canonical bundle is trivial, i.e., they are symplectic surfaces, and it is actually the choice of a symplectic structure on the surface S that induces a symplectic structure on the moduli space $\mathcal{M}$.

In a later paper [16], Tyurin generalized this result by showing that the choice of a 2-form $\omega \in H^0(S, \wedge^2 T^*S)$, which he calls a 'symplectic structure' on S (resp. of a bivector field $\theta \in H^0(S, \wedge^2 TS)$, which he calls a 'Poisson structure'), determines in a canonical way a 2-form $\tilde{\omega} \in H^0(\mathcal{M}, \wedge^2 T^*\mathcal{M})$ (resp. a bivector field $\tilde{\theta} \in H^0(\mathcal{M}, \wedge^2 T\mathcal{M})$), i.e., a symplectic structure (resp. a Poisson structure) on $\mathcal{M}$. In this case, again, no mention is made of the closure condition $d\tilde{\omega} = 0$ for the symplectic structure $\tilde{\omega}$, nor of the analogous condition that the Poisson bracket associated to a Poisson structure must satisfy the Jacobi identity.

In this paper we consider the general case of a Poisson surface S and show that, in correspondence to the choice of a Poisson structure on S, there is a canonically defined Poisson structure on the moduli space $\mathcal{M}$ of stable sheaves on S, i.e., there is a bilinear antisymmetric bracket $\{\cdot,\cdot\}$, defined on the sheaf of regular functions on $\mathcal{M}$, that is a derivation in each entry and satisfies the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0,$$

for any functions f,g,h on $\mathcal{M}$.

If S is an abelian or a $K3$ surface, i.e., in the symplectic case, our proof shows that the symplectic form defined by Mukai is actually closed.

This paper is organized as follows: in §1 we recall some basic definitions and results of symplectic geometry, then, in §2, we introduce and study Poisson surfaces. In §3 we collect some results on moduli spaces of sheaves on a Poisson surface S, and, in §4, we define the Poisson structure on the moduli space $\mathcal{M}$ canonically associated to the choice of a Poisson structure on S. In §5, we shall prove that θ satisfies a certain closure condition, equivalent to the Jacobi identity for the Poisson bracket defined by θ. Finally, in §6, we conclude with some remarks on the rank of the Poisson structure θ, i.e., on the dimension of the symplectic leaves of the Poisson variety $\mathcal{M}$.

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1. Symplectic and Poisson structures

We recall here some definitions and results of symplectic geometry.

Let X be a smooth algebraic variety over the complex field $\mathbb{C}$. A (holomorphic) symplectic structure on X is a closed nondegenerate 2-form $\omega \in H^0(X,\Omega_X^2)$. Given a symplectic structure ω, we define the Hamiltonian vector field H_f of a regular function f by requiring that $\omega(H_f,v)=\langle df,v\rangle$, for every tangent field v. Then, for $f,g\in\Gamma(U,\mathcal{O}_X)$, we define the Poisson bracket $\{f,g\}$ of f and g by setting $\{f,g\}=\langle H_f,dg\rangle=\omega(H_g,H_f)$. The map $g\mapsto\{f,g\}$ is a derivation of $\Gamma(U,\mathcal{O}_X)$ whose corresponding vector field is precisely H_f. The pairing $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ is a bilinear antisymmetric map that is a derivation in each entry and satisfies the Jacobi identity

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}=0, \tag{1.1}$$

for any $f,g,h\in\Gamma(U,\mathcal{O}_X)$. This implies that $[H_f,H_g]=H_{\{f,g\}}$, where $[u,v]=uv-vu$ is the commutator of the vector fields u and v.

A natural generalization of symplectic structures is given by the notion of Poisson structure.

A Poisson structure on X is a Lie algebra structure $\{\cdot,\cdot\}$ on $\mathcal{O}_X$ satisfying the identity $\{f,gh\}=\{f,g\}h+g\{f,h\}$. Equivalently, this is given by an antisymmetric contravariant 2-tensor $\theta\in H^0(X,\wedge^2TX)$, where we set $\{f,g\}=\langle\theta,df\wedge dg\rangle$. Then θ is a Poisson structure if the bracket it defines satisfies the Jacobi identity (1.1). For any $f\in\Gamma(U,\mathcal{O}_X)$, the map $g\mapsto\{f,g\}$ is a derivation of $\Gamma(U,\mathcal{O}_X)$, hence corresponds to a vector field H_f on U, called the hamiltonian vector field associated to f.

Note that giving $\theta \in H^0(X, \wedge^2 TX)$ is equivalent to giving a homomorphism of vector bundles $B : T^*X \to TX$, with $\langle \theta, \alpha \wedge \beta \rangle = \langle B(\alpha), \beta \rangle$ (or $\langle \alpha, B(\beta) \rangle$, up to a sign), for 1-forms α, β.

Let us define an operator $\tilde{d} : H^0(X, \wedge^2 TX) \to H^0(X, \wedge^3 TX)$ as follows:

$$\begin{aligned}\tilde{d}\theta(\alpha,\beta,\gamma) &= B(\alpha)\theta(\beta,\gamma) - B(\beta)\theta(\alpha,\gamma) + B(\gamma)\theta(\alpha,\beta)\\ &\quad - \langle [B(\alpha), B(\beta)], \gamma\rangle + \langle [B(\alpha), B(\gamma)], \beta\rangle - \langle [B(\beta), B(\gamma)], \alpha\rangle,\end{aligned}$$

for 1-forms α, β, γ, where $[\cdot,\cdot]$ denotes the usual commutator of vector fields.

By a straightforward computation (using local coordinates), it is easy to prove the following

Proposition 1.1. *The bracket $\{\cdot,\cdot\}$, defined by an element $\theta \in H^0(X, \wedge^2 TX)$, satisfies the Jacobi identity if and only if $\tilde{d}\theta = 0$.*

Remark 1.2. A condition classically known to be equivalent to the Jacobi identity for the bracket $\{\cdot,\cdot\}$ is the vanishing of the so-called Schouten bracket $[\theta,\theta] = 0$ (see [15], for example). This is equivalent to the condition expressed by the preceding proposition.

When θ has maximal rank everywhere, to give θ is equivalent to giving its inverse 2-form $\omega \in H^0(X, \Omega_X^2)$, and the condition $\tilde{d}\theta = 0$ is equivalent to $d\omega = 0$. In this case the Poisson structure induces a symplectic structure.

2. Poisson surfaces

In this section we consider Poisson structures on smooth algebraic surfaces. Let S be a smooth algebraic surface over the complex field $\mathbb{C}$. We shall denote by ω_S its canonical line bundle, by K_S its canonical divisor, and by $q = \dim H^1(S, \mathcal{O}_S)$ the irregularity of S. We have the following

Proposition 2.1. *A Poisson structure on S is given by a global section s of the anticanonical line bundle ω_S^{-1}.*

Proof. A Poisson structure on S is, by definition, an element $s \in H^0(S, \wedge^2 TS) = H^0(S, \omega_S^{-1})$ that satisfies the condition $\tilde{d}s = 0$. But S is a surface, hence the map $\tilde{d}$ is identically zero. □

Definition 2.2. *A Poisson surface S is a smooth algebraic surface which admits a non-zero Poisson structure, i.e., such that $H^0(S, \omega_S^{-1}) \neq 0$.*

We have the following

Proposition 2.3. *Let S be a connected Poisson surface. Then S is either a $K3$ surface (if $\omega_S \cong \mathcal{O}_S$ and $q = 0$) or an abelian surface (if $\omega_S \cong \mathcal{O}_S$ and $q = 2$) or a ruled surface (if ω_S is not trivial).*

Proof . Let s be a non-zero section of ω_S^{-1}; we have an exact sequence

$$0 \to \omega_S \xrightarrow{s} \mathcal{O}_S \to \mathcal{O}_D \to 0,$$

where D is the divisor of s. It follows that either $H^0(S, \omega_S) = 0$ or $H^0(S, \omega_S) = \mathbb{C}$. In the second case ω_S has a non-vanishing global section, hence it is trivial; S is then either a $K3$ or an abelian surface, according to the value of q. If $H^0(S, \omega_S) = 0$, by considering the exact sequence

$$0 \to \omega_S^{n+1} \xrightarrow{s} \omega_S^n \to \omega_S^n|_D \to 0,$$

and using induction on n, it follows that $H^0(S, \omega_S^n) = 0$ for all $n \geq 1$. By recalling now a theorem of Enriques (cf., for example, [3, Chap. VI]), we conclude that S is a ruled surface. □

Remark 2.4. Note that not every ruled surface is a Poisson surface. Let, for example, $S = C \times \mathbb{P}^1$; then we have, for every integer n,

$$H^0(S, \omega_S^n) = H^0(C, \omega_C^n) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2n)).$$

It follows that $H^0(S, \omega_S^{-1}) \neq 0$ if and only if C is a rational or an elliptic curve.

Remark 2.5. If S is a Poisson surface and ω_S^{-1} is ample, then it follows that either $S = \mathbb{P}^1 \times \mathbb{P}^1$ or S is obtained from $\mathbb{P}^2$ by blowing-up n distinct points in general position, with $n \leq 8$. It follows, in particular, that S is rational ([3, Ch. V, Ex. 1]).

Remark 2.6. Let $s \in H^0(S, \omega_S^{-1})$ be a Poisson structure on S, and denote by D the divisor of s. Then the rank of the Poisson structure is 2 on the open subset $S \setminus D$, and is 0 on D. Hence the Poisson structure induces a symplectic structure on $S \setminus D$.

Example 2.7. Let $S = \mathbb{P}^2$ and take as the anticanonical divisor D a triple line. We may fix homogeneous coordinates (x_0, x_1, x_2) on $\mathbb{P}^2$ such that the section s defining the divisor D is given by $s = x_0^3$. The Poisson structure defined by s on $\mathbb{P}^2$ induces a symplectic structure on $\mathbb{C}^2 = \mathbb{P}^2 \setminus D$. If we consider the coordinates (X, Y) on $\mathbb{C}^2$ given by $X = x_1/x_0$ and $Y = x_2/x_0$, it is immediate to see that this symplectic structure is given by the 2-form $dX \wedge dY$. In other words, the Poisson structure determined by s on $\mathbb{P}^2$ induces on $\mathbb{C}^2$ the standard symplectic structure.

3. The moduli space of semistable sheaves on S

Let S be a connected Poisson surface, and let us fix a Poisson structure $s \in H^0(S, \omega_S^{-1})$ on S. Let us denote by D the divisor of s, $D \in |-K_S|$.

We recall now some basic definitions and results on moduli spaces of semistable sheaves on surfaces.

Let H be a very ample divisor on S. For every coherent torsion-free $\mathcal{O}_S$-module E, we set

$$p_E(n) = \frac{\chi(E(n))}{\mathrm{rk}(E)},$$

where $\mathrm{rk}(E)$ is the rank at the generic point of S, and $E(n) = E \otimes \mathcal{O}_S(nH)$.

From now on, by a sheaf on S we will always mean an $\mathcal{O}_S$-module.

Definition 3.1. *A sheaf E on S is said to be H-stable (resp. H-semistable) if it is a coherent torsion-free $\mathcal{O}_S$-module and, for every proper coherent subsheaf F of E, we have*

$$p_F(n) < p_E(n), \quad (\text{resp. } p_F(n) \leq p_E(n)),$$

for all sufficiently large integers n.

Let us denote by r, c_1 and c_2, respectively the rank and the Chern classes of a coherent torsion-free sheaf on S. We have the following well-known result:

Theorem 3.2. *For fixed r, c_1, c_2, there exists a coarse moduli space $\overline{\mathcal{M}} = \overline{\mathcal{M}}(r, c_1, c_2)$ parametrizing S-equivalence classes of H-semistable sheaves of rank r and Chern classes c_1 and c_2 on S. $\overline{\mathcal{M}}$ is a projective variety and it contains an open subset $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ parametrizing isomorphism classes of H-stable sheaves.*

In the sequel we shall denote by E either an H-semistable (resp. H-stable) sheaf on S, or the point of $\overline{\mathcal{M}}$ (resp. $\mathcal{M}$) corresponding to the S-equivalence class (resp. isomorphism class) of E.

From infinitesimal deformation theory, it follows that there is a canonical isomorphism

$$T_E\overline{\mathcal{M}} \cong \mathrm{Ext}^1(E, E), \tag{3.1}$$

where $T_E\overline{\mathcal{M}}$ denotes the tangent space to $\overline{\mathcal{M}}$ at E.

Then, from Grothendieck-Serre duality, it follows that the cotangent space to $\overline{\mathcal{M}}$ at E is given by

$$T_E^*\overline{\mathcal{M}} \cong \mathrm{Ext}^1(E, E \otimes \omega_S). \tag{3.2}$$

We now turn to the problem of smoothness of moduli spaces. We have the following result:

Proposition 3.3. *The moduli space $\mathcal{M} = \mathcal{M}(r, c_1, c_2)$ is a smooth quasi-projective variety of dimension $(1 - r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$.*

Proof. If S is a symplectic surface ($\omega_S \cong \mathcal{O}_S$), this is proved in [14, Theorem 0.1]. If S is a Poisson surface and ω_S is not trivial, then the divisor $D \in |-K_S|$ of a Poisson structure s is effective. It follows that $(D \cdot H) > 0$, hence $(K_S \cdot H) < 0$. Under this condition, the smoothness of $\mathcal{M}$ is proved in [13, Corollary 6.7.3]. The computation of the dimension of $\mathcal{M}$ is an easy application of the Riemann-Roch theorem, and is done in [13, Proposition 6.9]. □

Since $\mathcal{M}$ is a smooth variety, we may give global versions of (3.1) and (3.2). First we need a definition:

Definition 3.4. *Let $f : X \to T$ be a T-scheme, and E, F two coherent $\mathcal{O}_X$-modules. The i-th relative* Ext*-sheaf $\mathcal{E}xt^i_{\mathcal{O}_T}(E, F)$ is the sheaf associated to the presheaf $U \mapsto \operatorname{Ext}^i_{f^{-1}(U)}(E_U, F_U)$ for every open subset U of T.*

Now we note that, in general, a universal family $\mathcal{E}$ on $\mathcal{M}$ does not exist (not even on any Zariski open subset of $\mathcal{M}$); however it does exist locally in the étale topology (or in the complex topology). As shown by Mukai, this is enough to ensure that the i-th relative Ext-sheaves $\mathcal{E}xt^i_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E})$ on $\mathcal{M}$ are well defined, for any integer i.

Then we have:

Proposition 3.5. *Let $p : \mathcal{M} \times S \to \mathcal{M}$ and $q : \mathcal{M} \times S \to S$ be the canonical projections. There are canonical isomorphisms*

$$T\mathcal{M} \cong \mathcal{E}xt^1_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E}), \tag{3.3}$$

and

$$T^*\mathcal{M} \cong \mathcal{E}xt^1_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)). \tag{3.4}$$

Let us denote by $\mathcal{M}^0$ the open subset of $\mathcal{M}$ parametrizing isomorphism classes of H-stable locally free sheaves on S. Then, for $E \in \mathcal{M}^0$, we have

$$T_E\mathcal{M}^0 \cong H^1(S, \mathcal{E}nd(E)), \tag{3.5}$$

and

$$T^*_E\mathcal{M}^0 \cong H^1(S, \mathcal{E}nd(E) \otimes \omega_S). \tag{3.6}$$

Now, even if a universal family $\mathcal{E}$ on $\mathcal{M}$ does not exist, the sheaf $\mathcal{E}nd(\mathcal{E})$ on $\mathcal{M} \times S$ is well defined, and we have

$$\mathcal{E}nd(\mathcal{E})|_{\{E\}\times S} \cong \mathcal{E}nd(E), \quad \forall E \in \mathcal{M}.$$

Hence we may rewrite (3.3) and (3.4) for $\mathcal{M}^0$ as follows:

$$T\mathcal{M}^0 \cong R^1p_*(\mathcal{E}nd(\mathcal{E})), \tag{3.7}$$

and

$$T^*\mathcal{M}^0 \cong R^1p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)). \tag{3.8}$$

This may also be proved directly by noting that, since $\mathcal{M}^0$ is smooth, the function $E \mapsto \dim H^1(S, \mathcal{E}nd(E))$ is constant on $\mathcal{M}^0$. It follows that $R^1p_*(\mathcal{E}nd(\mathcal{E}))$ is locally free and we have isomorphisms $R^1p_*(\mathcal{E}nd(\mathcal{E}))\otimes\mathbb{C}(E) \cong H^1(S, \mathcal{E}nd(E))$, for every $E \in \mathcal{M}^0$. A similar argument works for the cotangent bundle.

4. Poisson structures on $\mathcal{M}$

Let S be a Poisson surface and choose a Poisson structure $s \in H^0(S, \omega_S^{-1})$ on S. We shall define an element $\theta = \theta_s \in H^0(\mathcal{M}, \otimes^2 T\mathcal{M})$ as follows: for any $E \in \mathcal{M}$, $\theta(E) : T_E^*\mathcal{M} \times T_E^*\mathcal{M} \to \mathbb{C}$ is defined by

$$\begin{aligned}\theta(E) : \mathrm{Ext}^1(E, E \otimes \omega_S) \times \mathrm{Ext}^1(E, E \otimes \omega_S) &\xrightarrow{\circ} \\ \mathrm{Ext}^2(E, E \otimes \omega_S^2) \xrightarrow{s} \mathrm{Ext}^2(E, E \otimes \omega_S) &\xrightarrow{\mathrm{Tr}} \mathbb{C},\end{aligned} \tag{4.1}$$

where the first map is the composition map, the second is induced by the multiplication by the section s, and the third is the trace map.

Note that, by Grothendieck-Serre duality and the stability hypothesis on E, it follows that the trace map $\mathrm{Tr} : \mathrm{Ext}^2(E, E \otimes \omega_S) \to \mathbb{C}$ is an isomorphism.

A method analogous to the one used by Mukai in [14] may be used to prove the following

Proposition 4.1. *The sheaf $\mathcal{L} = \mathcal{E}xt^2_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S))$ is a trivial invertible sheaf on $\mathcal{M}$, and there is a bilinear map*

$$\begin{aligned}\theta : &\mathcal{E}xt^1_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)) \otimes \mathcal{E}xt^1_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S)) \\ &\xrightarrow{\circ} \mathcal{E}xt^2_{\mathcal{O}_\mathcal{M}}(\mathcal{E}, \mathcal{E} \otimes q^*(\omega_S^2)) \xrightarrow{s} \mathcal{L}\end{aligned}$$

such that, for every $E \in \mathcal{M}$, $\theta \otimes \mathbb{C}(E)$ coincides with the map $\theta(E)$ defined in (4.1).

As we have previously stated, giving θ is equivalent to giving a homomorphism of vector bundles

$$B : T^*\mathcal{M} \to T\mathcal{M},$$

where we set $\theta(\alpha \otimes \beta) = \langle B(\alpha), \beta \rangle$.

It is easy to see that in this situation the homomorphism B is the map induced on Ext-sheaves by the multiplication by the section s. On the fibers over a point $E \in \mathcal{M}$, we have

$$B(E) : \mathrm{Ext}^1(E, E \otimes \omega_S) \xrightarrow{s} \mathrm{Ext}^1(E, E). \tag{4.2}$$

From now on we shall restrict ourselves to the open subset $\mathcal{M}^0$ of $\mathcal{M}$, and we shall use indifferently the expressions 'locally-free sheaf' or 'vector bundle'.

If E is an H-stable locally-free sheaf, the map $\theta(E)$ may be written as

$$\begin{aligned}\theta(E) : H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \times H^1(S, \mathcal{E}nd(E) \otimes \omega_S) &\xrightarrow{\circ} \\ H^2(S, \mathcal{E}nd(E) \otimes \omega_S^2) \xrightarrow{s} H^2(S, \mathcal{E}nd(E) \otimes \omega_S) &\xrightarrow{\mathrm{Tr}} \mathbb{C}.\end{aligned} \tag{4.3}$$

This is essentially the cup-product of two cohomology classes, followed by the multiplication by s. From the graded commutativity of the usual cup-product it follows that $\theta(E)$ is skew-symmetric (cf., for example, [10, p. 707]), hence to prove that θ defines a Poisson structure on $\mathcal{M}^0$ we have only to prove that it satisfies the closure condition $\tilde{d}\theta = 0$. Note that if we prove that θ defines a

Poisson structure on the open subset $\mathcal{M}^0$, then the same holds on the closure $\overline{\mathcal{M}^0}$.

As a final remark note that, for $E \in \mathcal{M}^0$, the global map θ of Proposition 4.1 may be written as

$$\theta : R^1 p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)) \otimes R^1 p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S)) \xrightarrow{\circ} R^2 p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S^2)) \xrightarrow{s} \mathcal{L},$$

where the trivial invertible sheaf $\mathcal{L}$ is given by $\mathcal{L} = R^2 p_*(\mathcal{E}nd(\mathcal{E}) \otimes q^*(\omega_S))$, while the map B is the map induced on cohomology by the multiplication by the section s:

$$B(E) : H^1(S, \mathcal{E}nd(E) \otimes \omega_S) \xrightarrow{\mathrm{id} \otimes s} H^1(S, \mathcal{E}nd(E)). \tag{4.4}$$

5. The closure of θ

In this section we shall prove that $\tilde{d}\theta = 0$, thus proving that θ defines a Poisson structure on $\mathcal{M}^0$. We note that this proof is original even in the symplectic case.

We start by recalling some preliminaries (see [5] for a detailed description of what follows).

Let $\mathbb{C}[\epsilon]/(\epsilon^2)$ be the ring of dual numbers over $\mathbb{C}$. By convenience of notations, in the sequel it will be denoted simply by $\mathbb{C}[\epsilon]$. Let us denote by S_ϵ the fiber product $S \times \mathrm{Spec}(\mathbb{C}[\epsilon])$. If $p_\epsilon : S_\epsilon \to S$ is the natural morphism and F is a vector bundle on S, we shall denote by $F[\epsilon]$ its trivial infinitesimal deformation, $F[\epsilon] = p_\epsilon^*(F)$.

Let $\pi : X \to Y$ be a morphism (locally of finite presentation) of schemes, and F, G two locally free sheaves on X. We denote by $\mathcal{D}iff^1_{X/Y}(F, G)$ the sheaf of relative differential operators from F to G of order ≤ 1. There is an exact sequence ([11, Ch. IV, §16.8])

$$0 \to \mathcal{H}om_X(F, G) \to \mathcal{D}iff^1_{X/Y}(F, G) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \otimes \mathcal{H}om_X(F, G) \to 0,$$

where σ is the *symbol* morphism. Then, if $F = G$ and we restrict to differential operators with 'scalar symbol', written $\mathcal{D}^1_{X/Y}(F)$, we get the exact sequence

$$0 \to \mathcal{E}nd_X(F) \to \mathcal{D}^1_{X/Y}(F) \xrightarrow{\sigma} \mathcal{D}er_Y(\mathcal{O}_X) \to 0. \tag{5.1}$$

Let $p : \mathcal{M}^0 \times S \to \mathcal{M}^0$ and $q : \mathcal{M}^0 \times S \to S$ denote the canonical projections. To apply (5.1) to $q : \mathcal{M}^0 \times S \to S$, note that, as previously said, even if there is no Poincaré bundle $\mathcal{E}$ on $\mathcal{M}^0 \times S$, the sheaf $\mathcal{E}nd(\mathcal{E})$ is well defined. By a similar argument it follows easily that the sheaf $\mathcal{D}^1_S(\mathcal{E}) = \mathcal{D}^1_{\mathcal{M}^0 \times S/S}(\mathcal{E})$ of first-order differential operators with scalar symbols that are $q^*(\mathcal{O}_S)$-linear, is also well defined. Then we have an exact sequence

$$0 \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{D}^1_S(\mathcal{E}) \to p^* T\mathcal{M}^0 \to 0.$$

By applying p_*, and noting that $p_*p^*T\mathcal{M}^0 \cong T\mathcal{M}^0$ since p is a proper morphism, we get a long exact cohomology sequence, a piece of which is

$$\cdots \to T\mathcal{M}^0 \to R^1p_*(\mathcal{E}nd(\mathcal{E})) \to R^1p_*(\mathcal{D}_S^1(\mathcal{E})) \to \cdots .$$

It is easy to prove that the map $T\mathcal{M}^0 \to R^1p_*(\mathcal{E}nd(\mathcal{E}))$ is precisely the isomorphism (3.7), hence the image of $R^1p_*(\mathcal{E}nd(\mathcal{E}))$ in $R^1p_*(\mathcal{D}_S^1(\mathcal{E}))$ is zero. It follows that, for every section $\{\eta_{ij}\}$ of $R^1p_*(\mathcal{E}nd(\mathcal{E}))$, there exist sections $\dot{D}_i$ of $\mathcal{D}_S^1(\mathcal{E})$ over suitable open subsets V_i, such that

$$\eta_{ij} = \dot{D}_j - \dot{D}_i, \tag{5.2}$$

where, by simplicity of notations, we have not explicitly indicated the restrictions to the intersection $V_{ij} = V_i \cap V_j$.

If $E \in \mathcal{M}^0$, and we consider restrictions to $\{E\} \times S$, it follows that (5.2) is valid with $\{\eta_{ij}\}$ being a 1-cocycle with values in $\mathcal{E}nd(E)$, and $\dot{D}_i$ being sections of $\mathcal{D}_S^1(E)$.

Let now X be a k-scheme. A tangent vector field on X is a k-linear map of sheaves $D : \mathcal{O}_X \to \mathcal{O}_X$ such that the induced map $D(U) : \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ is a k-derivation, for every open subset U of X. Equivalently, a vector field on X can be expressed by an automorphism over $\operatorname{Spec} k[\epsilon]$

$$\begin{array}{ccc} X \times \operatorname{Spec}(k[\epsilon]) & \xrightarrow{\tilde{D}} & X \times \operatorname{Spec}(k[\epsilon]) \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec}(k[\epsilon]) & \end{array}$$

that restricts to the identity morphism of X when one looks at the fibers over $\operatorname{Spec} k$.

Over an open affine subset $U = \operatorname{Spec} A$ of X the tangent field $D : \mathcal{O}_X \to \mathcal{O}_X$ is given equivalently by a k-derivation $D(U) : A \to A$. In this situation the automorphism $\tilde{D}$ is determined by the k-algebra homomorphism $\tilde{D}(U) : A[\epsilon] \to A[\epsilon]$ given by $\tilde{D}(U) = 1 + \epsilon D(U)$.

Let now $D : \mathcal{O}_{\mathcal{M}^0} \to \mathcal{O}_{\mathcal{M}^0}$ be a tangent vector field on $\mathcal{M}^0$ and denote by $\tilde{D}$ the corresponding automorphism of $\mathcal{M}^0 \times \operatorname{Spec} \mathbb{C}[\epsilon]$. Let $\mathcal{E}$ be a local universal family for stable vector bundles (the local existence of $\mathcal{E}$ in the étale topology is enough for our purposes), and $\mathcal{E}[\epsilon]$ be its pull-back to $\mathcal{M}^0 \times \operatorname{Spec} \mathbb{C}[\epsilon]$. The vector field D (or the automorphism $\tilde{D}$) may be described locally by giving the infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^* \mathcal{E}[\epsilon]$ of the local universal family $\mathcal{E}$. At a point $\mathcal{E} \in \mathcal{M}^0$ the corresponding tangent vector is given by $E_\epsilon = \mathcal{E}_\epsilon|_{\{E\} \times S_\epsilon}$, which is an infinitesimal deformation of the vector bundle E.

From what we have previously seen, the tangent field $\mathcal{E}_\epsilon$ corresponds to a global section $\eta = \{\eta_{ij}\}$ of $R^1p_*(\mathcal{E}nd(\mathcal{E}))$, which can be described in terms of first-order differential operators. Let us give another useful interpretation of this fact.

Let $D : \mathcal{O}_{\mathcal{M}^0} \to \mathcal{O}_{\mathcal{M}^0}$ be the derivation corresponding to the infinitesimal deformation $\mathcal{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^* \mathcal{E}$ determined by the global section $\eta = \{\eta_{ij}\}$ of $R^1p_*(\mathcal{E}nd(E))$. Let $(V_i)_{i \in I}$, $V_i = \operatorname{Spec}(A_i)$, be an open affine covering of

$\mathscr{M}^0 \times S$. The vector field D is locally described by giving, for each $i \in I$, a $\mathbb{C}[\epsilon]$-automorphism of $A_i[\epsilon]$ of the form $1 + \epsilon D_i$, where $D_i : A_i \to A_i$ is the $\mathbb{C}$-derivation determined by the restriction of D to V_i. Let $M_i = \Gamma(V_i, \mathscr{E})$ and $M_i[\epsilon] = \Gamma(V_i, \mathscr{E}[\epsilon])$. The infinitesimal deformation $\mathscr{E}_\epsilon = (\tilde{D} \times \mathrm{id}_S)^* \mathscr{E}[\epsilon]$ may be described as obtained by gluing the sheaves $\widetilde{M_i[\epsilon]}$ by means of suitable isomorphisms.

Let us denote by

$$\widetilde{1 + \epsilon \dot{D}_i} : \mathscr{E}_\epsilon|_{V_i \times \operatorname{Spec} \mathbb{C}[\epsilon]} \xrightarrow{\sim} \widetilde{M_i[\epsilon]}$$

the trivialization isomorphisms, where $\dot{D}_i : M_i \to M_i$ is a first-order differential operator with associated $\mathbb{C}$-derivation $D_i : A_i \to A_i$. By what we have previously seen, the gluing isomorphism on the intersection $V_i \cap V_j$ is given by $1 + \epsilon\eta_{ij} = (1 + \epsilon\dot{D}_j)(1 + \epsilon\dot{D}_i)^{-1} = 1 + \epsilon(\dot{D}_j - \dot{D}_i)$. It follows that $\eta_{ij} = \dot{D}_j - \dot{D}_i$, which is precisely (5.2).

Now let us consider two tangent vector fields $D^1, D^2 : \mathscr{O}_{\mathscr{M}^0} \to \mathscr{O}_{\mathscr{M}^0}$, corresponding to the global sections $\eta^1 = \{\eta^1_{ij}\}$ and $\eta^2 = \{\eta^2_{ij}\}$ of $R^1 p_*(\mathscr{E}nd(E))$. By what we have seen, there exist first-order differential operators $\dot{D}^1_i$ and $\dot{D}^2_i$ such that

$$\eta^1_{ij} = \dot{D}^1_j - \dot{D}^1_i,$$
$$\eta^2_{ij} = \dot{D}^2_j - \dot{D}^2_i.$$

Let us denote by

$$\begin{array}{ccc} \mathscr{M}^0 \times \operatorname{Spec} \mathbb{C}[\epsilon, \epsilon'] \times S & \xrightarrow{\tilde{D}^h \times \mathrm{id}_S} & \mathscr{M}^0 \times \operatorname{Spec} \mathbb{C}[\epsilon, \epsilon'] \times S \\ & \searrow \quad \swarrow & \\ & \operatorname{Spec} \mathbb{C}[\epsilon, \epsilon'] & \end{array}$$

the automorphism corresponding to D^h, for $h = 1, 2$. The vector field D^h is given equivalently by the infinitesimal deformation $\mathscr{E}^h_\epsilon = (\tilde{D}^h \times \mathrm{id}_S)^* \mathscr{E}[\epsilon]$ of $\mathscr{E}$, described by the global section $\{\eta^h_{ij}\}$ of $R^1 p_*(\mathscr{E}nd(E))$. Let us recall that if $f^h_i : \mathscr{E}^h_\epsilon|_{V_i} \to \widetilde{M^h_i[\epsilon]}$ are isomorphisms, then the sheaf $\mathscr{E}^h_\epsilon$ is constructed by gluing the sheaves $\widetilde{M^h_i[\epsilon]}$ and $\widetilde{M^h_j[\epsilon]}$ along the open sets V_{ij} by means of the isomorphisms $\widetilde{1 + \epsilon\eta^h_{ij}} = f^h_j|_{V_{ij}} \circ f^{h^{-1}}_i|_{V_{ij}}$.

By applying the same reasoning, the second-order differential operator $D^1 D^2$ is equivalent to $(\tilde{D}^1 \times \mathrm{id}_S)^* (\tilde{D}^2 \times \mathrm{id}_S)^* \mathscr{E}$. We have isomorphisms

$$(\tilde{D}^1 \times \mathrm{id}_S)^* (\tilde{D}^2 \times \mathrm{id}_S)^* \mathscr{E}|_{V_i \times \operatorname{Spec} \mathbb{C}[\epsilon, \epsilon']} \xrightarrow{(1+\epsilon\dot{D}^1_i)\circ(1+\epsilon'\dot{D}^2_i)} \widetilde{M_i[\epsilon, \epsilon']},$$

hence the gluing isomorphisms are given by $((1 + \epsilon\dot{D}^1_j) \circ (1 + \epsilon'\dot{D}^2_j)) \circ ((1 + \epsilon\dot{D}^1_i) \circ (1 + \epsilon'\dot{D}^2_i))^{-1} = 1 + \epsilon(\dot{D}^1_j - \dot{D}^1_i) + \epsilon'(\dot{D}^2_j - \dot{D}^2_i) + \epsilon\epsilon'(\dot{D}^1_j \dot{D}^2_j - \dot{D}^2_j \dot{D}^1_i - \dot{D}^1_j \dot{D}^2_i + \dot{D}^2_i \dot{D}^1_i)$, that we can write in a simpler form as $1 + \epsilon\eta^1_{ij} + \epsilon'\eta^2_{ij} + \epsilon\epsilon'(\dot{D}^1_j \eta^2_{ij} - \eta^2_{ij} \dot{D}^1_i)$.

In conclusion, we have proved that the second-order differential operator $D^1 D^2$, or, in other words the 'infinitesimal deformation' of $\eta^2 = \{\eta^2_{ij}\}$ in the direction given by $\eta^1 = \{\eta^1_{ij}\}$, is described by giving gluing isomorphisms of the form

$$1+\epsilon\eta^1_{ij}+\epsilon'\eta^2_{ij}+\epsilon\epsilon'(\dot{D}^1_j\eta^2_{ij}-\eta^2_{ij}\dot{D}^1_i). \tag{5.3}$$

Analogously, we find that D^2D^1 is equivalent to the data of

$$1+\epsilon\eta^1_{ij}+\epsilon'\eta^2_{ij}+\epsilon\epsilon'(\dot{D}^2_j\eta^1_{ij}-\eta^1_{ij}\dot{D}^2_i). \tag{5.4}$$

Finally, it is easy to see that the vector field $[D^1,D^2]$ is determined by gluing isomorphisms of the form

$$1+\epsilon\epsilon'([\dot{D}^1,\dot{D}^2]_j-[\dot{D}^1,\dot{D}^2]_i).$$

Now we are able to prove the following

Theorem 5.1. *Let S be a Poisson surface and $s\in H^0(S,\omega_S^{-1})$ a Poisson structure on S. The antisymmetric contravariant 2-tensor $\theta=\theta_s\in H^0(\mathscr{M}^0,\wedge^2T\mathscr{M}^0)$ defines a Poisson structure on the moduli space $\mathscr{M}^0$ of H-stable vector bundles on S.*

Proof. We have to prove that $\tilde{d}\theta=0$. Let η^1,η^2,η^3 be three 1-forms on $\mathscr{M}^0$, i.e., three global sections of $R^1p_*(\mathscr{E}nd(\mathscr{E})\otimes q^*(\omega_S))$. To compute $(B(\eta^1))(\theta(\eta^2,\eta^3))=(B(\eta^1))(\langle s\eta^2,\eta^3\rangle)$, i.e., the derivative of the function $\langle s\eta^2,\eta^3\rangle$ along the vector field $B(\eta^1)$, we shall use first-order Taylor series expansions of η^2 and η^3, i.e., we shall compute $\langle s\eta^2_\epsilon,\eta^3_\epsilon\rangle$, where η^2_ϵ and η^3_ϵ are 'infinitesimal deformations' along the vector field $B(\eta^1)$ of η^2 and η^3 respectively.

Let us represent the cohomology classes η^h by 1-cocycles $\eta^h=\{\eta^h_{ij}\}$, for $h=1,2,3$, so that $B(\eta^h)$ is represented by the 1-cocycle $\{s\eta^h_{ij}\}$. We know that there exist first-order differential operators $\dot{D}^h_i$ such that

$$s\eta^h_{ij}=\dot{D}^h_j-\dot{D}^h_i,\quad \text{for } h=1,2,3. \tag{5.5}$$

If we apply what we have previously seen to the second-order differential operator $B(\eta^h)B(\eta^k)$, we find that the infinitesimal deformation of $B(\eta^k)=s\eta^k$ along the vector field $B(\eta^h)$ is given by

$$s\eta^k_\epsilon=\{s\eta^k_{ij}+\epsilon(\dot{D}^h_j(s\eta^k_{ij})-s\eta^k_{ij}\dot{D}^h_i)\}=\{s\eta^k_{ij}+s\epsilon(\dot{D}^h_j\eta^k_{ij}-\eta^k_{ij}\dot{D}^h_i)\},$$

because of the $\mathscr{O}_S$-linearity of the differential operators.

Since the multiplication by s is injective at the level of cocycles, it follows that

$$\eta^k_\epsilon=\{\eta^k_{ij}+\epsilon(\dot{D}^h_j\eta^k_{ij}-\eta^k_{ij}\dot{D}^h_i)\}, \tag{5.6}$$

for $h,k=1,2,3$.

Then we have:

$$\begin{aligned}\langle s\eta^2_\epsilon,\eta^3_\epsilon\rangle&=\langle\{s\eta^2_{ij}+s\epsilon(\dot{D}^1_j\eta^2_{ij}-\eta^2_{ij}\dot{D}^1_i)\},\{\eta^3_{ij}+\epsilon(\dot{D}^1_j\eta^3_{ij}-\eta^3_{ij}\dot{D}^1_i)\}\rangle\\&=\{s\eta^2_{ij}\circ\eta^3_{jk}\}+\epsilon\{(s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk}-\eta^3_{jk}\dot{D}^1_j)+s(\dot{D}^1_j\eta^2_{ij}-\eta^2_{ij}\dot{D}^1_i)\eta^3_{jk})\},\end{aligned}$$

from which it follows that

$$(B(\eta^1))(\theta(\eta^2,\eta^3))=\{s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk}-\eta^3_{jk}\dot{D}^1_j)+s(\dot{D}^1_j\eta^2_{ij}-\eta^2_{ij}\dot{D}^1_i)\eta^3_{jk}\}. \tag{5.7}$$

Now, by using the decomposition (5.5), and by recalling that the antisymmetry of θ implies that $\langle s\eta^h, \eta^l\rangle = \{s\eta^h_{ij} \circ \eta^l_{jk}\} = -\{\eta^h_{ij} \circ s\eta^l_{jk}\} = -\langle \eta^h, s\eta^l\rangle$, we may decompose the various terms of (5.7) in different ways as follows:

$$\begin{aligned} s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) &= \dot{D}^2_j\dot{D}^1_k\eta^3_{jk} - \dot{D}^2_j\eta^3_{jk}\dot{D}^1_j - \dot{D}^2_i\dot{D}^1_k\eta^3_{jk} + \dot{D}^2_i\eta^3_{jk}\dot{D}^1_j \\ &= -\eta^2_{ij}\dot{D}^1_k\dot{D}^3_k + \eta^2_{ij}\dot{D}^1_k\dot{D}^3_j + \eta^2_{ij}\dot{D}^3_k\dot{D}^1_j - \eta^2_{ij}\dot{D}^3_j\dot{D}^1_j, \end{aligned}$$

and

$$\begin{aligned} s(\dot{D}^1_j\eta^2_{ij} - \eta^2_{ij}\dot{D}^1_i)\eta^3_{jk} &= \dot{D}^1_j\dot{D}^2_j\eta^3_{jk} - \dot{D}^1_j\dot{D}^2_i\eta^3_{jk} - \dot{D}^2_j\dot{D}^1_i\eta^3_{jk} + \dot{D}^2_i\dot{D}^1_i\eta^3_{jk} \\ &= -\dot{D}^1_j\eta^2_{ij}\dot{D}^3_k + \eta^2_{ij}\dot{D}^1_i\dot{D}^3_k + \dot{D}^1_j\eta^2_{ij}\dot{D}^3_j - \eta^2_{ij}\dot{D}^1_i\dot{D}^3_j. \end{aligned}$$

The last expression we have to compute is the following:

$$\begin{aligned} \langle [B_s(\eta^1), B_s(\eta^2)], \eta^3\rangle &= \{\langle([\dot{D}^1, \dot{D}^2]_j - [\dot{D}^1, \dot{D}^2]_i), \eta^3\rangle\} \\ &= \{(\dot{D}^1_j\dot{D}^2_j - \dot{D}^2_j\dot{D}^1_j - \dot{D}^1_i\dot{D}^2_i + \dot{D}^2_i\dot{D}^1_i)\eta^3_{jk}\} \\ &= \{\dot{D}^1_j\dot{D}^2_j\eta^3_{jk} - \dot{D}^2_j\dot{D}^1_j\eta^3_{jk} - \dot{D}^1_i\dot{D}^2_i\eta^3_{jk} + \dot{D}^2_i\dot{D}^1_i\eta^3_{jk}\}. \end{aligned}$$

As for the last term, we may use the following decomposition:

$$\begin{aligned} \langle [B_s(\eta^2), B_s(\eta^3)], \eta^1\rangle &= \langle \eta^1, [B_s(\eta^2), B_s(\eta^3)]\rangle \\ &= \{\eta^1_{ij}([\dot{D}^2, \dot{D}^3]_k - [\dot{D}^2, \dot{D}^3]_j)\} \\ &= \{\eta^1_{ij}\dot{D}^2_k\dot{D}^3_k - \eta^1_{ij}\dot{D}^3_k\dot{D}^2_k - \eta^1_{ij}\dot{D}^2_j\dot{D}^3_j + \eta^1_{ij}\dot{D}^3_j\dot{D}^2_j\}. \end{aligned}$$

Now, by collecting all the pieces, we find

$$\begin{aligned} \tilde{d}\theta(\eta^1, \eta^2, \eta^3) &= B_s(\eta^1)(\theta(\eta^2, \eta^3)) - B_s(\eta^2)(\theta(\eta^1, \eta^3)) + B_s(\eta^3)(\theta(\eta^1, \eta^2)) \\ &\quad - \langle [B_s(\eta^1), B_s(\eta^2)], \eta^3\rangle + \langle [B_s(\eta^1), B_s(\eta^3)], \eta^2\rangle \\ &\quad - \langle [B_s(\eta^2), B_s(\eta^3)], \eta^1\rangle \\ &= \{s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) - \dot{D}^3_j\dot{D}^1_i\eta^2_{jk} - \dot{D}^1_j\dot{D}^3_i\eta^2_{jk} \\ &\quad + \dot{D}^1_j\dot{D}^3_j\eta^2_{jk} + \dot{D}^3_i\dot{D}^1_i\eta^2_{jk}\} \\ &= \{s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) \\ &\quad + (\dot{D}^1_j\dot{D}^3_j - \dot{D}^1_j\dot{D}^3_i - \dot{D}^3_j\dot{D}^1_i + \dot{D}^3_i\dot{D}^1_i)\eta^2_{jk}\} \\ &= \{s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) + (\dot{D}^1_j(s\eta^3_{ij}) - (s\eta^3_{ij})\dot{D}^1_i)\eta^2_{jk}\} \\ &= \{s\eta^2_{ij}(\dot{D}^1_k\eta^3_{jk} - \eta^3_{jk}\dot{D}^1_j) + s(\dot{D}^1_j\eta^3_{ij} - \eta^3_{ij}\dot{D}^1_i)\eta^2_{jk}\} \\ &= \{s\eta^2_{ij} \circ \xi^3_{jk} + s\xi^3_{ij} \circ \eta^2_{jk}\} \\ &= 0, \end{aligned}$$

where, for the infinitesimal deformation η^3_ϵ of η^3 along $B_s(\eta^1)$, we have set $\eta^3_{\epsilon\, ij} = \eta^3_{ij} + \epsilon\xi^3_{ij}$. Note that the last equality follows from the antisymmetry of θ. □

Remark 5.2. We note here that a perfectly analogous result holds for the moduli spaces of H-stable sheaves with fixed determinant; the only difference being that, in this case, the tangent (resp. cotangent) space to the moduli variety at a point E is given by $\mathrm{Ext}^1_0(E,E)$ (resp. $\mathrm{Ext}^1_0(E,E\otimes\omega_S)$), where the subscript 0 means that we are considering only trace-free classes.

6. The rank of θ

Let us turn now to the computation of the rank of the Poisson structure θ, i.e., of the dimension of the symplectic leaves of the Poisson variety $\mathcal{M}^0$.

Let S be a Poisson surface and let us fix a Poisson structure $s\in H^0(S,\omega_S^{-1})$ on S. If s is a symplectic structure, i.e., if S is an abelian or a $K3$ surface, then θ actually defines a symplectic structure on $\mathcal{M}^0$, hence it has everywhere maximal rank (equal to the dimension of $\mathcal{M}^0$).

If S is not symplectic, let us denote by D the divisor of s and suppose, for simplicity, that D is an irreducible nonsingular curve (note that from the adjunction formula it follows that the canonical line bundle of D is trivial, hence D is an elliptic curve). For any vector bundle E on S, we have an exact sequence

$$0\to\mathcal{E}nd(E)\otimes\omega_S\xrightarrow{s}\mathcal{E}nd(E)\to\mathcal{E}nd(E|_D)\to 0. \tag{6.1}$$

Lemma 6.1. *If E is H-semistable, then $H^0(S,\mathcal{E}nd(E)\otimes\omega_S)=0$.*

Proof. Note that an H-semistable sheaf is also μ-H-semistable, where a coherent sheaf E is said to be μ-H-semistable (resp. μ-H-stable) if it torsion-free and for any proper coherent torsion-free subsheaf F of E, we have

$$\mu(F)=\frac{(c_1(F)\cdot H)}{\mathrm{rk}(F)}\le\frac{(c_1(E)\cdot H)}{\mathrm{rk}(E)}=\mu(E)\qquad(\text{resp. }\mu(F)<\mu(E)).$$

Now we recall that, in the proof of Proposition 3.3, we have shown that $(K_S\cdot H)<0$, hence, if $\phi\in\mathrm{Hom}(E,E\otimes\omega_S)$ is not zero, we have $\mu(E)\le\mu(\phi(E))\le\mu(E\otimes\omega_S)=\mu(E)+(K_S\cdot H)<\mu(E)$, which is a contradiction. □

Now, by considering the long exact cohomology sequence of (6.1), we get

$$\begin{aligned}0\to\mathbb{C}\to H^0(D,\mathcal{E}nd(E|_D))&\to H^1(S,\mathcal{E}nd(E)\otimes\omega_S)\\ &\xrightarrow{B(E)} H^1(S,\mathcal{E}nd(E))\to\cdots.\end{aligned}$$

We have thus proved the following

Proposition 6.2. *The kernel of the hamiltonian morphism $B(E)$ is given by*

$$\mathrm{Ker}\,B(E)=H^0(D,\mathcal{E}nd(E|_D))/\mathbb{C}.$$

Hence

$$\mathrm{rk}(B(E))=\dim\mathcal{M}^0-\dim H^0(D,\mathcal{E}nd(E|_D))+1.$$

This shows how the rank of the Poisson structure θ at a point $E\in\mathcal{M}^0$ is determined by the restriction $E|_D$ of E to the curve D. On the open subset of

$\mathcal{M}^0$ consisting of vector bundles E that restrict to a simple bundle on D, the rank of θ is equal to the dimension of $\mathcal{M}^0$, i.e., θ is nondegenerate, or, in other terms, it induces a symplectic structure.

Remark 6.3. If $\dim \mathcal{M}^0 = 2$, then $\mathcal{M}^0$ is, like S, a Poisson surface, i.e., an abelian or a $K3$ surface in the symplectic case, or a ruled surface with an effective anticanonical divisor in the general case.

Remark 6.4. By recalling that the dimension of $\mathcal{M}^0 = \mathcal{M}^0(r, c_1, c_2)$ is given by $(1-r)c_1^2 + 2rc_2 - r^2\chi(\mathcal{O}_S) + 1$ (if it is non-empty), and that the dimension of a symplectic variety is even, we see that, if r and c_1 are both even, then $\dim \mathcal{M}^0$ is odd, hence the symplectic leaves of $\mathcal{M}^0$ have dimension strictly less then the dimension of $\mathcal{M}^0$. It follows that there are no H-stable vector bundles $E \in \mathcal{M}^0$ such that $E|_D$ is simple. The same conclusion holds, for example, if r is odd and $\chi(\mathcal{O}_S)$ is even.

Remark 6.5. If $S = \mathbb{P}^2$, much more is known on the structure of the moduli space $\mathcal{M}$. If we denote by $\mathcal{M}^0_\mu$ the moduli space of μ-H-stable locally free sheaves on $\mathbb{P}^2$, then we know that $\mathcal{M}^0_\mu$ is irreducible and that, under some technical conditions on the rank and Chern classes, it is everywhere dense in $\overline{\mathcal{M}}$ (see [8] for details). It follows that, in this case, θ defines a Poisson structure on all of $\mathcal{M}$.

Remark 6.6. Let us denote now by $\mathcal{M}_D$ the moduli space of semistable vector bundles on the curve D and by $\mathcal{M}_S$ the open subset of $\mathcal{M}^0$ such that

$$E \in \mathcal{M}_S \Rightarrow E|_D \in \mathcal{M}_D.$$

Let us denote by $\rho : \mathcal{M}_S \to \mathcal{M}_D$ the restriction map.

By recalling the Poisson structure of $\mathcal{M}_S$, for every $E \in \mathcal{M}_S$ we get the following commutative diagram

$$\begin{array}{ccc} T_E\mathcal{M}_S & \xrightarrow{T_E\rho} & T_{E|_D}\mathcal{M}_D \\ \uparrow{\scriptstyle B(E)} & & \uparrow{\scriptstyle \tilde{B}(E)} \\ T^*_E\mathcal{M}_S & \xleftarrow{T^*_E\rho} & T^*_{E|_D}\mathcal{M}_D, \end{array}$$

where $\tilde{B}(E) = T_E\rho \circ B(E) \circ T^*_E\rho$.

Since the canonical bundle of the curve D is trivial, the preceding diagram is canonically identified with

$$\begin{array}{ccc} H^1(S, \mathcal{E}nd(E)) & \longrightarrow & H^1(D, \mathcal{E}nd(E|_D)) \\ \uparrow{\scriptstyle B(E)} & & \uparrow{\scriptstyle \tilde{B}(E)} \\ H^1(S, \mathcal{E}nd(E) \otimes \omega_S) & \longleftarrow & H^0(D, \mathcal{E}nd(E|_D)). \end{array}$$

Now, the long exact cohomology sequence of (6.1), gives

$$H^0(D, \mathscr{E}nd(E|_D)) \xrightarrow{\delta} H^1(S, \mathscr{E}nd(E) \otimes \omega_S) \xrightarrow{B(E)} H^1(S, \mathscr{E}nd(E)) \xrightarrow{\alpha} H^1(D, \mathscr{E}nd(E|_D)),$$

and it is easy to see that $\tilde{B}(E)$ is precisely the composition $\alpha \circ B(E) \circ \delta$, hence it is zero.

This proves that the Poisson structure of $\mathscr{M}_S$ induces on the image $\rho(\mathscr{M}_S) \subset \mathscr{M}_D$ a Poisson structure that is identically zero.

We would like to end with some remarks on the relations between the moduli space of stable vector bundles on $\mathbb{P}^2$ and the moduli space of anti self-dual connections on S^4.

Let P be a principal SU(r)-bundle on $S^4 = \mathbb{R}^4 \cup \{\infty\}$, and k be minus its Pontryagin index. Let us denote by $\widetilde{M}(\mathrm{SU}(r), k)$ the moduli space parametrizing pairs consisting of an anti self-dual SU(r)-connection on P and an isomorphism $P_\infty \cong \mathrm{SU}(r)$. This is a manifold of (real) dimension $4rk$, if k is sufficiently large. The usual moduli space of anti self-dual SU(r)-connections on P is then given by $M(\mathrm{SU}(r), k) = \widetilde{M}(\mathrm{SU}(r), k)/\,\mathrm{SU}(r)$.

Now let us fix a line $D \subset \mathbb{P}^2$ (so that we get isomorphisms $\mathbb{P}^2 \setminus D \cong \mathbb{C}^2 \cong \mathbb{R}^4$) and denote by $\widetilde{\mathscr{M}}(r, 0, k)$ the moduli space of pairs consisting of a rank r holomorphic vector bundle E on $\mathbb{P}^2$ with $c_1(E) = 0$ and $c_2(E) = k$ and a trivialization of $E|_D$.

Donaldson proved in [6] that there is a natural isomorphism

$$\widetilde{M}(\mathrm{SU}(r), k) \cong \widetilde{\mathscr{M}}(r, 0, k).$$

If $\mathscr{M}^0(r, 0, k)$ denotes the moduli space of stable rank r vector bundles on $\mathbb{P}^2$ with Chern classes $c_1 = 0$ and $c_2 = k$, it is known that there are Zariski open subsets $U \subset \widetilde{\mathscr{M}}(r, 0, k)$ and $V \subset \mathscr{M}^0(r, 0, k)$ such that U fibers over V. As a consequence we find that an open subset of the moduli space $M(\mathrm{SU}(r), k)$ fibers over V with fiber $\mathrm{SL}(r, \mathbb{C})/\,\mathrm{SU}(r)$. Hence, after factoring out by this group, we get a complex structure and a (complex) Poisson structure on the real moduli space of anti self-dual connections.

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