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CONTRACTIBILITY AND FIXED POINT PROPERTY: THE CASE OF DECOMPOSABLE SETS

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INTRODUCTION

SCHAUDER's fixed point theorem [1] asserts that any compact mapping of a convex set into itself has a fixed point. Various attempts have been made to do fixed point theory in spaces more general than convex sets, but no general theory is known.

A theorem by Lefschetz [2, corollary, p. 359] leads one to believe that each compact, homologically trivial and locally connected set has the fixed point property, but Borsuk [3] gave an example to refute this conjecture. His set is neither contractible nor locally contractible, hence it leaves open the following version of Lefschetz's corollary.

CONJECTURE. Any contractible, locally contractible set has the fixed point property for compact mappings.

Borsuk showed in [4] that this conjecture is true in finite dimensional metric spaces (see theorem 1). Some particular cases of the conjecture—for decomposable subsets of L^1—were proved by Cellina [5] and Fryszkowski [6]. Furthermore, it has been proved that each decomposable subset of L^1 is a retract of L^1 [7]. As Smart pointed out in a recent paper [8], these authors do not mention or use the contractibility and local contractibility properties that these spaces possess: Smart proved a weaker version of their theorem—an L^p-continuous map of a decomposable set M into an L^∞-compact subset of M has a fixed point—by a method related to the contraction process in decomposable sets. Notice that this requirement is quite strong: it means that no "cut and paste" of functions of $T(M)$ is possible; as an example the L^1-precompact subset of $L^1[0, 1]$ $\{\chi_{[0,\alpha]}, \alpha \in [0, 1]\}$ is not L^∞-precompact.

Here I generalize Smart's arguments in order to prove that each L^p-compact mapping $(1 \le p < \infty)$ of a decomposable subset of L^p into itself has a fixed point. This can be regarded as a further step of the project aimed at the conjecture. The main tool is a consequence of Liapunov's theorem on the range of a vector measure as it is formulated in [9]: it enables the construction of a nonconvex analogue of Schauder's projection onto a nonsymmetrical nonconvex analogue of the convex hull of a finite number of points. Its advantage with respect to the more familiar decomposable hull is that the first is compact whereas the second is not, unless it is trivial (see for instance [10]). As a consequence, each compact map of a topological space into a decomposable set can be approximated by a sequence of functions having a finite dimensional range. This property is in common with compact maps into a convex set [11, Chapter VI].

NOTATIONS AND PRELIMINARY RESULTS

Definition 1. A subset A of a topological space X is said to be contractible in the space X to a set $B \subset X$ provided that the inclusion $A \hookrightarrow X$ is homotopic (in the space X^A) to a map with values belonging to B. If $A = X$ and B consists of a single point, then X is said to be contractible.

Definition 2. A topological space X is said to be locally contractible at a point $x_0 \in X$ provided that each neighbourhood U of x_0 contains a neighbourhood U' of x_0 which is contractible in U to a point. The space X is said to be locally contractible if it is locally contractible at each of its points.

It is clear that any convex set (and, as a consequence, any absolute retract) is contractible and locally contractible (see [4] for details). The next theorem shows that the converse is true for compact and finite dimensional spaces. By the dimension of a compact space X we understand the covering dimension of X [12, Chapter V]. Specifically, we say $\dim X \leq n$ provided that each open covering of X has a refinement of order $\leq n$ (the order of a covering being the largest integer n such that there are $n + 1$ members of the covering which have a nonempty intersection); if $\dim X \leq n$ is true and $\dim X \leq n - 1$ is false, then we say $\dim X = n$.

THEOREM 1 [4, V(10.5)]. A finite dimensional compact metric space is an absolute retract if and only if it is contractible and locally contractible.

Definition 3. Let (X, d) be a compact metric space and r be a nonnegative real number. The space X is said to have Hausdorff r-measure zero if for each $\varepsilon > 0$ there exists a finite decomposition of X

$$X = A_1 \cup \cdots \cup A_k$$

such that

$$(\operatorname{diam} A_1)^r + \cdots + (\operatorname{diam} A_k)^r < \varepsilon$$

where $\operatorname{diam} A = \sup\{d(a, b) : a, b \in A\}$.

The connection between the topological concept of dimension and the metrical concept of Hausdorff r-measure, given by the following theorem, will be used later.

THEOREM 2 [12, theorem VII.3]. Let $n \in \mathbb{N}$ and X be a compact space of $(n + 1)$-measure zero. Then $\dim X \leq n$.

In what follows, we consider a measure space $(I, \mathfrak{M}, \mu)$ where $\mathfrak{M}$ is a σ-algebra of subsets of I (measurable sets), μ is a complete positive nonatomic measure such that $\mu(I) < +\infty$, $(E, |\cdot|)$ is a Banach space, $L^p = L^p(I)$, $1 \leq p < +\infty$, is the space of Bochner integrable functions with values in E, endowed with the norm $\|h\|_p = (\int_I |h|^p \, d\mu)^{1/p}$. The characteristic function of A is denoted by χ_A.

Definition 4. A nonempty subset S of L^p is said to be decomposable if whenever f and g are in S and A is in $\mathfrak{M}$ then $f\chi_A + g\chi_{I \setminus A}$ is in S.

Decomposable sets have many properties in common with convex sets: see, for instance [5, 7, 13]. In particular, Smart [14] pointed out that the decomposable subsets of $L^p[0, 1]$ are contractible and locally contractible. We prove here a more general statement.

PROPOSITION 1. Each decomposable subset S of $L^p(I)$ is contractible and locally contractible.

Proof. Liapunov's theorem on the range of a vector measure [9, proposition 1.1] provides the existence of a family $(A_t)_{t\in[0,1]}$ of measurable subsets of I such that

$$s \le t \Rightarrow A_s \subset A_t, \qquad A_0 = \varnothing, \qquad A_1 = I, \qquad \mu(A_t) = t\mu(I) \qquad (t \in [0, 1]). \tag{1}$$

I claim that for any fixed f in S the map

$$\Phi\colon (t, g) \mapsto f\chi_{A_t} + g\chi_{I\setminus A_t}$$

describes a contraction of S to f and of any ε-neighbourhood of S to f. Clearly, the claim is proved if we show that the previous map is continuous.

For this purpose, fix a positive ε, and let $t_0 \in [0, 1]$, $g_0 \in S$. Let δ be such that

$$\forall A \in \mathfrak{M} : \mu(A) < \delta \Rightarrow \int_A |f|^p \,\mathrm{d}\mu < \left(\frac{\varepsilon}{3}\right)^p \quad \text{and} \quad \int_A |g_0|^p \,\mathrm{d}\mu < \left(\frac{\varepsilon}{3}\right)^p. \tag{2}$$

Let t and g be such that $\|g - g_0\|_p < \varepsilon/3$ and $|t - t_0| < \delta/\mu(I)$. Since we have:

$$\begin{aligned}\Phi(t, g) - \Phi(t_0, g_0) &= f(\chi_{A_t} - \chi_{A_{t_0}}) + g\chi_{I\setminus A_t} - g_0\chi_{I\setminus A_{t_0}} \\ &= f(\chi_{A_t} - \chi_{A_{t_0}}) + (g\chi_{I\setminus A_t} - g_0\chi_{I\setminus A_t}) + (g_0\chi_{I\setminus A_t} - g_0\chi_{I\setminus A_{t_0}})\end{aligned}$$

then it turns out that

$$\|\Phi(t, g) - \Phi(t_0, g_0)\|_p \le \left(\int_{A_t \Delta A_{t_0}} |f|^p \,\mathrm{d}\mu\right)^{1/p} + \|g - g_0\|_p + \left(\int_{A_t \Delta A_{t_0}} |g_0|^p \,\mathrm{d}\mu\right)^{1/p}$$

where $A \,\Delta B = (A\setminus B) \cup (B\setminus A)$. Hence, by (1) and (2)

$$\|\Phi(t, g) - \Phi(t_0, g_0)\|_p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The claim is proved.

Definition 5. A family $(A_t)_{t\in[0,1]}$ of measurable sets is called increasing if the following properties hold:

$$A_0 = \varnothing; \qquad A_1 = I; \qquad s \le t \Rightarrow A_s \subset A_t.$$

Let us consider a family of positive vector-valued measures $\mu_x = (\mu_x^1, \ldots, \mu_x^n)$ which are absolutely continuous with respect to μ. The space M^n of such vector measures is endowed with the topology induced by the norm $\|\mu_x\|$, the total variation of μ_x. The proposition formulated below is [9, proposition 1.2] where μ_0 is replaced by (μ, ν).

PROPOSITION 2. Let X be a compact topological space and ν be an absolutely continuous positive measure with respect to μ. Assume that the map $x \mapsto \mu_x \in M^n$ is continuous. Then, for every $\varepsilon > 0$ there exists an increasing family $(A_t)_{t\in[0,1]}$ of measurable sets with the following properties:

$$|\mu_x(A_t) - t\mu_x(I)| < \varepsilon \qquad (t \in [0, 1],\ x \in X); \tag{3}$$

$$\mu(A_t) = t\mu(I), \qquad \nu(A_t) = t\nu(I) \qquad (t \in [0, 1]). \tag{4}$$

MAIN RESULTS

THEOREM 3. Let S be a decomposable subset of $L^p(I)$, T be an L^p-continuous mapping into an L^p-compact subset K of S. Then T has a fixed point.

The proof is modelled on a standard proof of Schauder's second theorem, given for instance in [1] and in Smart's proof of theorem 3 in the case where K is an L^∞-compact subset of S [8]. A replacement for the convex combinations used in Schauder's projection is studied: let $f_1, \ldots, f_n$ be fixed in L^p and $\mathcal{A} = (A_t)_{t \in [0,1]}$ be an increasing family of measurable sets. Analogous to what is described in [14], if $c_i \geq 0$ and $c_1 + \cdots + c_n = 1$ we use the notation

$$c_1 f_1 \oplus_{\mathcal{A}} \cdots \oplus_{\mathcal{A}} c_n f_n$$

for the function

$$f_1 \chi_{A_{c_1}} + \cdots + f_2 \chi_{(A_{c_1+c_2} \setminus A_{c_1})} + \cdots + f_n \chi_{(I \setminus A_{c_1+\cdots+c_{n-1}})}$$

which equals f_i on $A_{c_1+\cdots+c_i} \setminus A_{c_1+\cdots+c_{i-1}}$. We regard this as a "nonlinear convex combination" (with respect to $\mathcal{A}$) of $f_1, \ldots, f_n$ with weights $c_1, \ldots, c_n$.

These combinations make up the "nonlinear convex hull" of $f_1, \ldots, f_n$ with respect to the increasing family $\mathcal{A}$; it will be denoted by

$$\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n).$$

As is evident, if $f_1, \ldots, f_n$ belong to a decomposable set S then $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ is contained in S. The rest of the proof follows after theorems 4 and 5 and lemma 1 below.

Definition 6. Let $\phi \in L^1$. A family $\mathcal{A} = (A_t)_{t \in [0,1]}$ of measurable sets is called refining ϕ (with respect to $(I, \mathfrak{M}, \mu)$) if and only if the following equalities hold:

$$\int_{A_t} \phi \, d\mu = t \int_I \phi \, d\mu \qquad (t \in [0, 1]).$$

THEOREM 4. Let $f_1, \ldots, f_n \in L^p$ and $\mathcal{A} = (A_t)_{t \in [0,1]}$ be an increasing family of measurable sets. Then $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ is (i) compact, (ii) contractible and (iii) locally contractible. Furthermore, if $\mathcal{A}$ is refining $(|f_1| + \cdots + |f_n|)^p$ then (iv) $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ is finite dimensional.

Proof. Let C be the convex compact subset of $\mathbb{R}^{n-1}$ defined by

$$C = \{(c_1, \ldots, c_{n-1}) \in \mathbb{R}^{n-1} : 0 \leq c_1 \leq \cdots \leq c_{n-1} \leq 1\}$$

and let $\Psi: C \to \mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ be the function defined by

$$\Psi(c_1, \ldots, c_{n-1}) = f_1 \chi_{A_{c_1}} + f_2 \chi_{(A_{c_2} \setminus A_{c_1})} + \cdots + f_n \chi_{(I \setminus A_{c_{n-1}})}.$$

Since the integral operator is absolutely continuous with respect to the measure μ, then Ψ is continuous hence $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n) = \Psi(C)$ is compact. Furthermore $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ can be contracted to f_1 by the process described in the proof of proposition 1. The proof of local contractibility is essentially the proof of [14, theorem 2(iii)] taking into account that we have $\Psi(c)$ instead of f^c.

In (iv) let $\mathcal{A} = (A_t)_{t \in [0,1]}$ be an increasing family refining $(|f_1| + \cdots + |f_n|)^p$. I claim that Ψ satisfies a Hölder condition. With a view to prove the above assumption, set for any $x = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ the norm of x to be $\|x\| = |x_1| + \cdots + |x_{n-1}|$. Since $\Psi(c)$ and $\Psi(d)$

agree on $\bigcup_i (A_{c_i}\backslash A_{c_{i-1}}) \cap (A_{d_i}\backslash A_{d_{i-1}})$ then we have:

$$\|\Psi(c) - \Psi(d)\|_p^p \leq \sum_i \int_{A_{c_i}\Delta A_{d_i}} |\Psi(c) - \Psi(d)|^p \,\mathrm{d}\mu$$
$$\leq \sum_i \int_{A_{c_i}\Delta A_{d_i}} 2^p(|f_1| + \cdots + |f_n|)^p \,\mathrm{d}\mu.$$

Furthermore, $\mathcal{A}$ is refining $(|f_1| + \cdots + |f_n|)^p$, hence the following equalities hold:

$$\int_{A_{c_i}\Delta A_{d_i}} (|f_1| + \cdots + |f_n|)^p \,\mathrm{d}\mu = |d_i - c_i| \int_I (|f_1| + \cdots + |f_n|)^p \,\mathrm{d}\mu \qquad (i = 1, \ldots, n-1).$$

As a consequence, if we set $Q = (\int_I (|f_1| + \cdots + |f_n|)^p \,\mathrm{d}\mu)^{1/p}$, we have

$$\|\Psi(c) - \Psi(d)\|_p \leq 2Q\|c - d\|^{1/p}. \tag{5}$$

Now, we prove that $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ has np-measure zero.

Let $\varepsilon > 0$ and $B_1, \ldots, B_k$ be such that

$$(\operatorname{diam} B_1)^n + \cdots + (\operatorname{diam} B_k)^n < \frac{\varepsilon}{2Q}. \tag{6}$$

Such a decomposition exists since the $(n-1)$ dimensional polytope C has n-measure zero. Then

$$\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n) = \Psi(C) = \Psi(B_1) \cup \cdots \cup \Psi(B_k).$$

Since, by (5),

$$\operatorname{diam} \Psi(B_i) \leq 2Q(\operatorname{diam} B_i)^{1/p} \qquad (i = 1, \ldots, k) \tag{7}$$

then (6) and (7) imply

$$\operatorname{diam} \Psi(B_1)^{np} + \cdots + \operatorname{diam} \Psi(B_k)^{np} < \varepsilon$$

and this proves the claim. Theorem 2 yields the conclusion.

The fixed point property for $\mathrm{noca}_{\mathcal{A}}(f_1, \ldots, f_n)$ now follows immediately from theorem 1.

THEOREM 5. Let $f_1, \ldots, f_n \in L^p$ and $A = (A_t)_{t \in [0,1]}$ be an increasing family refining $(|f_1| + \cdots + |f_n|)^p$. Then $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ has the fixed point property.

The proof of theorem 3 is based on the following lemma.

LEMMA 1 (a nonconvex "Schauder's projection"). Let K be a compact subset of L^p. Then for every $\varepsilon > 0$ there exist $f_1, \ldots, f_n \in K$, an increasing family $\mathcal{A} = (A_t)_{t \in [0,1]}$ refining $(|f_1| + \cdots + |f_n|)^p$ and a continuous mapping P_ε of K into $\mathrm{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ such that $\|P_\varepsilon f - f\|_p \leq \varepsilon$ for all f in K.

Proof. Let $f_1, \ldots, f_n$ in K be such that their $\varepsilon/2^{1/p}$ neighbourhoods cover K. Let, for $1 \leq i \leq n$ and $f \in K$, μ_f^i, ν be the absolutely continuous measures (with respect to μ) defined by

$$\mu_f^i(A) = \int_A |f - f_i|^p \,\mathrm{d}\mu \qquad (i = 1, \ldots, n);$$

$$\nu(A) = \int_A (|f_1| + \cdots + |f_n|)^p \,\mathrm{d}\mu.$$

Clearly the map $f \in K \mapsto (\mu_f^1, \ldots, \mu_f^n)$ is continuous, hence by proposition 2 there exists an increasing family $\mathcal{A} = (A_t)_{t \in [0,1]}$ with the properties:

$$|\mu_f^i(A_t) - t\mu_f^i(I)| < \frac{\varepsilon^p}{4n} \qquad (t \in [0,1], f \in K, i = 1, \ldots, n); \tag{8}$$

$$\int_{A_t} (|f_1| + \cdots + |f_n|)^p \, d\mu = t \int_I (|f_1| + \cdots + |f_n|)^p \, d\mu \qquad (t \in [0,1]); \tag{9}$$

$$\mu(A_t) = t\mu(I) \qquad (t \in [0,1]). \tag{10}$$

For $1 \leq i \leq n$ and $f \in K$ set

$$r_i(f) = \max\left(0, \frac{\varepsilon}{2^{1/p}} - \|f - f_i\|_p\right);$$

thus for each $f \in K$ some $r_i(f)$ is nonzero. Then set

$$c_i(f) = \frac{r_i(f)}{\sum_j r_j(f)}$$

so that for each f, $\sum_i c_i(f) = 1$ and the map $f \mapsto (c_1(f), \ldots, c_n(f))$ is continuous. I claim that the map

$$P_\varepsilon : K \to \operatorname{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$$

defined by

$$P_\varepsilon(f) = c_1(f) f_1 \oplus_{\mathcal{A}} \cdots \oplus_{\mathcal{A}} c_n(f) f_n$$

has the required properties. In fact, set for any $f \in K$ and $i = 1, \ldots, n-1$ the function d_i to be $d_i(f) = c_1(f) + \cdots + c_i(f)$. Then we have:

$$\|P_\varepsilon(f) - f\|_p^p = \sum_i \int_{A_{d_i(f)} \setminus A_{d_{i-1}(f)}} |f - f_i|^p \, d\mu. \tag{11}$$

By (8), for each $i = 1, \ldots, n-1$ the following inequality holds:

$$\int_{A_{d_i(f)} \setminus A_{d_{i-1}(f)}} |f - f_i|^p \, d\mu \leq c_i(f) \int_I |f - f_i|^p \, d\mu + \frac{\varepsilon^p}{2n}.$$

Since $c_i(f) \neq 0$ if and only if $\|f - f_i\|_p < \varepsilon/2^{1/p}$ then

$$\|P_\varepsilon(f) - f\|_p^p \leq \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{2} = \varepsilon^p.$$

The continuity of P_ε follows from the fact that the family $\mathcal{A}$ refines $(|f_1| + \cdots + |f_n|)^p$: in this situation, the proof of theorem 4 (iv) shows that the map Ψ is continuous.

Since, for any f in K, we have $P_\varepsilon(f) = \Psi(d(f))$, then the conclusion follows from the continuity of the function $d = (d_1, \ldots, d_n)$.

Proof of theorem 3. Fix $k \in \mathbb{N}$. Then, by lemma 1, there exist $f_1, \ldots, f_n \in K$, an increasing family $\mathcal{A} = (A_t)_{t \in [0,1]}$ refining $(|f_1| + \cdots + |f_n|)^p$ and a continuous function $P_{1/k}$ of K into $\operatorname{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ such that $\|P_{1/k}(f) - f\|_p \leq 1/k$ for all f in K. Let i be the inclusion of $\operatorname{noco}_{\mathcal{A}}(f_1, \ldots, f_n)$ into S. Let us consider the composed map $P_{1/k} \circ T \circ i$.

$$\begin{array}{ccc} S & \xrightarrow{\quad T \quad} & K \\ \uparrow{\scriptstyle i} & & {\scriptstyle P_{1/k}}\downarrow \\ \mathrm{noco}_{\mathfrak{a}}(f_1, \ldots, f_n) & \xrightarrow{P_{1/k} \circ T \circ i} & \mathrm{noco}_{\mathfrak{a}}(f_1, \ldots, f_n) \end{array}$$

This function maps the set $\mathrm{noco}_{\mathfrak{a}}(f_1, \ldots, f_n)$ into itself, hence, by theorem 5, it has a fixed point x_k. Thus

$$\|x_k - Tx_k\|_p = \|P_{1/k}(T(x_k)) - T(x_k)\|_p \leq \frac{1}{k}.$$

By the compactness of K, T has a fixed point.

The following consequence of the nonconvex "Schauder's projection" of lemma 1 provides a further property of decomposable sets which is in common with convex sets.

THEOREM 6. Let X be a topological space, S be a decomposable subset of $L^p(I)$ and $T: X \to S$ be a continuous compact mapping. Then for each $\varepsilon > 0$ there exists a continuous map $T_\varepsilon: X \to S$ whose range has a finite dimension and such that

$$\|T_\varepsilon(x) - T(x)\|_p < \varepsilon \qquad \forall x \in X.$$

Proof. Let K be a compact subset of S such that $T(X) \subset K$ and fix $\varepsilon > 0$. Let

$$P_\varepsilon: K \to \mathrm{noco}_{\mathfrak{a}}(f_1, \ldots, f_n)$$

be the nonconvex "Schauder's projection" of lemma 1. Then, by theorem 4(iv), the map $T_\varepsilon = P_\varepsilon \circ T$ fulfils the requirements of the claim.

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