

Oriented Measures with Continuous Densities and the Bang–Bang Principle

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We introduce the notion of an oriented measure. For such a measure μ , given v in $L^1([a, b])$, $0 < v < 1$, there exist two sets $E \subset [a, b]$ whose characteristic functions have less than n discontinuity points and such that $\int v d\mu = \mu(E)$. Given a solution x to the control problem $L(x) = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' + a_0(t)x \in [\phi_1, \phi_2]$ there exist two bang–bang solutions y, z having a contact of order n with x at a and b such that $y \leq x \leq z$. Reachable sets of bang–bang constrained solutions are convex; an application to the calculus of variations yields a density result. © 1994 Academic Press, Inc.

INTRODUCTION

A classical theorem of Liapunov [8] states that given a finite dimensional vector measure μ on an interval $[a, b]$ which admits a density function $f = (f_1, \dots, f_n)$ and given a measurable function v defined on $[a, b]$ with values in $[0, 1]$, there exists a measurable subset E of $[a, b]$ such that

$$\forall i \in \{1, \dots, n\}, \quad \int_a^b f_i \chi_E = \int_a^b f_i v. \quad (*)$$

However, the proofs of this theorem are not constructive and thus do not give any information about the set E .

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Halkin [9] showed that if for each vector $p \in \mathbb{R}^n$ the set

$$\{t \in [a, b] : p \cdot f(t) > 0\}$$

(where \cdot is the usual scalar product) is a finite (resp. countable) union of intervals then there exists a set E satisfying (*) which is a finite (resp. countable) union of intervals. As far as we know this condition has not been applied apart from the case of piecewise analytical functions [9, 10, 12]. The results we present here are based on the following which is new.

Orientation Condition A. We say that n real functions f_1, \dots, f_n verify condition A on an interval $[a, b]$ if for each k in $\{1, \dots, n\}$, the determinant

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_k) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_k) \end{vmatrix}$$

is not equal to zero whenever the $x_i \in [a, b]$ are distinct and its sign is constant on the k -tuples (x_1, \dots, x_k) such that $a \leq x_1 < x_2 < \dots < x_k \leq b$.

A measure μ whose components μ_1, \dots, μ_n admit continuous density functions f_1, \dots, f_n which satisfy the orientation condition A is said to be oriented. Although this condition implies Halkin's, it possesses various advantages:

- it allows us to build a set E satisfying (*) whose characteristic function has at most n points of discontinuity;
- in the case where $0 < v < 1$ there exist exactly two such sets E_1 and E_2 and in addition the associated characteristic functions χ_{E_1} and χ_{E_2} have exactly n discontinuity points; moreover, one set is a neighbourhood of a , whereas the other is not.

We give two proofs of this result, neither of which uses the traditional convexity-extremal points arguments. Both use algebraic tricks directly related to condition A ; the first one is based on the implicit function theorem and the second one on the Caccioppoli global inversion theorem.

A consequence of our theorem is that if the interval $[a, b]$ can be partitioned as a finite (resp. countable) union of intervals on which the orientation condition A holds then we can build a set E satisfying (*) which is a finite (resp. countable) union of intervals. We also point out an operational criterion which ensures the validity of the orientation condition A : if f_1, \dots, f_n are of class \mathcal{C}^{n-1} on $[a, b]$, it is enough that the Wronskians $W(f_1), \dots, W(f_1, \dots, f_n)$ do not vanish on $[a, b]$ for A to hold.

This allows us to formulate a new result concerning bang-bang solutions to linear control systems described by a generic linear differential equation

$$L(x) = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' + a_0(t)x \in [\phi_1, \phi_2],$$

where ϕ_1 and ϕ_2 belong to L^1 . More precisely we show that given a solution x to the above problem there exist two bang-bang solutions y and z (i.e., $L(y), L(z) \in \{\phi_1, \phi_2\}$) such that

$$\forall t \in [a, b], \quad y(t) \leq x(t) \leq z(t)$$

$$\forall k \in \{0, \dots, n-1\}, \quad y^{(k)}(a) = x^{(k)}(a) = z^{(k)}(a), \quad y^{(k)}(b) = x^{(k)}(b) = z^{(k)}(b),$$

and $L(y)$ and $L(z)$ are of the form $\chi_E \phi_1 + (1 - \chi_E) \phi_2$, where the set E is a finite union of intervals; i.e., y and z are solutions associated to relay controls. The relay principle was studied by Andreini and Bacciotti in [4] under the strong assumption that $\phi_1, \phi_2, a_0, \dots, a_{n-1}$ be analytical. In order to apply our Liapunov's type theorem we make the solutions to

$$L(x) = v \in [0, 1], \quad x(a) = \dots = x^{(n-1)}(a) = 0$$

explicit through the integral representation formulas

$$\forall k \in \{0, \dots, n-1\} \quad x^{(k)}(t) = \int_a^t \frac{\partial^k R}{\partial t^k}(t, s) v(s) ds,$$

where $R(t, s)$ is the resolvent of the operator L . Our Wronskian criterion then applies directly to

$$R(b, \cdot), \frac{\partial R}{\partial t}(b, \cdot), \dots, \frac{\partial^{n-1} R}{\partial t^{n-1}}(b, \cdot)$$

and thus our main theorem yields a bang-bang solution,

$$y(t) = \int_a^t R(t, s) \chi_E(s) ds,$$

satisfying the required tangency conditions; moreover, the set E is a finite union of intervals which does not contain the point a .

Surprisingly the same Wronskian conditions allow us to apply an extended version of Pólya's generalized Rolle theorem for linear differential operators of order n and functions whose n th derivative are only piecewise continuous. We obtain that if $0 < v < 1$ then the graphs of x and y do not intersect. Since $y^{(n)}(a) < x^{(n)}(a)$ then $y < x$ on the whole interval $]a, b[$.

We give two applications of this result:

- The reachable set of solutions which are constrained by a given obstacle and subject to prescribed initial conditions coincides with the reachable set of bang-bang solutions submitted to the same conditions, so that this last one is convex.

- We consider the problem of minimizing the integral functionals

$$I(x, u) = \int_a^b f(t, x(t), u(t)) dt,$$

where $x: [a, b] \rightarrow \mathbb{R}^n$ is such that $x^{(k)}(a), x^{(k)}(b)$ ($0 \leq k \leq n-1$) are fixed and u is a control belonging to $U(x, t) \subset \mathbb{R}^n$. The classical approach to obtain existence of a minimum is to impose conditions in order to have the lower semicontinuity of I with respect to u (for instance convexity of $u \mapsto f(t, x, u)$).

Recently in an effort to provide existence criteria other than convexity in u some sufficient conditions have been given: for problems of the calculus of variations ($x' = u$ in the above setting) and for maps of the form $f(t, x, x') = g(t, x) + h(t, x')$, existence of solutions has been obtained by requiring that the real map $x \mapsto g(t, x)$ be monotonic [11] or, for x in \mathbb{R}^n , that the same function be concave [5]. Optimal control problems escaping to convexity conditions have been handled in [14].

It has been proved further in [6] that there exists a dense subset \mathcal{G} of $\mathcal{C}(\mathbb{R})$ such that, for g in it, the problem

$$\text{minimize } \int_a^b g(x(t)) dt + \int_a^b h(x'(t)) dt; \quad x(a) = x_0, \quad x(b) = x_1$$

admits a solution for every lower semicontinuous h satisfying growth conditions. Our theorem gives a straightforward generalization of the above result.

Let us remark that the elementary case $n=1$ of our n -dimensional Liapunov's type theorem appeared as a technical tool in [1, Lemma 3.4]; the case $n=2$ was handled in our previous paper [7] with very different techniques which are not applicable to higher dimensions. This work deals only with measures having continuous densities; the general case will be treated in a forthcoming paper.

PRELIMINARY RESULTS

One of the two proofs of Theorem 1 relies on the following powerful, but not enough appreciated, theorem.

CACCIOPPOLI GLOBAL INVERSION THEOREM. *Let E be an arcwise connected metric space, let F be a simply connected metric space, let f be a proper map from E with values in F . If f is a local homeomorphism at each point of E then f is a global homeomorphism between E and F .*

Proof. The proof and several applications of this theorem can be found in [2, 3]. ■

Let us introduce some notations. Let A be an $n \times n$ matrix with real coefficients. By $\det A$ or $|A|$ we denote its determinant. For each $i, j \in \{1, \dots, n\}$, by A_{ij} we mean the $(n-1) \times (n-1)$ matrix obtained by removing the i th row and the j th column from A . Surprisingly, the following simple algebraic trick will play an essential role in the existence part of the proof of Theorem 1 which does not involve the Caccioppoli theorem.

LEMMA S. *Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ matrix with real coefficients. Let x_1, \dots, x_n be such that*

$$\begin{aligned} a_{1,1}x_1 + \dots + a_{1,n-1}x_{n-1} + a_{1,n}x_n &= 0 \\ a_{2,1}x_1 + \dots + a_{2,n-1}x_{n-1} + a_{2,n}x_n &= 0 \\ \vdots & \\ a_{n-1,1}x_1 + \dots + a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= 0 \end{aligned}$$

If $\det A_{nn} \neq 0$ then

$$a_{n1}x_1 + \dots + a_{nn}x_n = \frac{|A|}{|A_{nn}|}x_n.$$

Proof. The Cramer rule applied to the above system yields

$$\forall i \in \{1, \dots, n-1\}, \quad x_i = \frac{(-1)^{n+i} |A_{ni}|}{|A_{nn}|} x_n$$

so that

$$a_{n1}x_1 + \dots + a_{nn}x_n = \frac{\sum_{i=1}^{n-1} a_{ni} (-1)^{n+i} |A_{ni}|}{|A_{nn}|} x_n = \frac{|A|}{|A_{nn}|} x_n$$

since $|A| = \sum_{i=1}^{n-1} a_{ni} (-1)^{n+i} |A_{ni}|$ is the development of the determinant of $|A|$ along the first row. ■

The main tool in the inductive proof of Theorem 1 is the existence and uniqueness of maximal implicit functions passing through a prescribed point.

LEMMA M. Let Ω be an open subset of $\mathbb{R}^{n-1} \times [a, b]$ and F a continuously differentiable map from Ω into \mathbb{R}^{n-1} such $\partial F/\partial(x_1, \dots, x_{n-1})$ is invertible everywhere. Let $(\bar{x}_1, \dots, \bar{x}_n)$ verify that $F(\bar{x}_1, \dots, \bar{x}_n) = 0$. Then there exists a unique couple (I, ψ) verifying Property P below such that I is maximal for the set inclusion with respect to this property.

Property P. I is an interval containing \bar{x}_n , Ψ is a continuous map from I into \mathbb{R}^{n-1} , $\Psi(\bar{x}_n) = (\bar{x}_1, \dots, \bar{x}_{n-1})$, $F(\Psi(x_n), x_n) = 0$ for every x_n in I .

Proof of lemma. Suppose first that (I, Ψ_I) and (J, Ψ_J) both satisfy Property P. Put

$$Z = \{x_n \in I \cap J : \Psi_I(x_n) = \Psi_J(x_n)\}.$$

This set is not empty (since $\bar{x}_n \in Z$) and is closed because Ψ_I and Ψ_J are continuous. Let $x_n^* \in Z$ and

$$(x_1^*, \dots, x_{n-1}^*) = \Psi_I(x_n^*) = \Psi_J(x_n^*),$$

so that $F(x_1^*, \dots, x_n^*) = 0$. We have

$$\left| \frac{\partial F}{\partial(x_1, \dots, x_{n-1})}(x_1^*, \dots, x_n^*) \right| \neq 0$$

and we can thus apply the implicit function theorem at the point (x_1^*, \dots, x_n^*) . There exist an open interval $]x_n^* - \varepsilon, x_n^* + \varepsilon[$, a neighbourhood \mathcal{O} of $(x_1^*, \dots, x_{n-1}^*)$, and a function ϕ from $]x_n^* - \varepsilon, x_n^* + \varepsilon[$ into \mathcal{O} such that $\phi(x_n^*) = (x_1^*, \dots, x_{n-1}^*)$ and

$$\forall x_n \in]x_n^* - \varepsilon, x_n^* + \varepsilon[, \forall (x_1, \dots, x_{n-1}) \in \mathcal{O},$$

$$F(x_1, \dots, x_n) = 0 \Leftrightarrow (x_1, \dots, x_{n-1}) = \phi(x_n).$$

Thus for every x_n in $I \cap J \cap]x_n^* - \varepsilon, x_n^* + \varepsilon[$

$$\phi(x_n) = \Psi_I(x_n) = \Psi_J(x_n),$$

whence Z is also open. Since $I \cap J$ is connected then $Z = I \cap J$. Put

$$\mathcal{F} = \{(I, \Psi_I) \text{ satisfying property P}\}$$

and let

$$I_M = \bigcup_{(I, \Psi_I) \in \mathcal{F}} I.$$

The previous uniqueness property allows us to define a function Ψ_M on I_M such that $\Psi_M = \Psi_I$ on I . The couple (I_M, Ψ_M) solves our problem. ■

THE ORIENTATION CONDITION Δ AND SOME RELATED FACTS

Orientation Condition Δ . We say that n real functions f_1, \dots, f_n verify condition Δ on an interval $[a, b]$ if for each k in $\{1, \dots, n\}$, the determinant

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \cdots & f_1(x_k) \\ f_2(x_1) & f_2(x_2) & \cdots & f_2(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(x_1) & f_k(x_2) & \cdots & f_k(x_k) \end{vmatrix}$$

is not equal to zero whenever the $x_i \in [a, b]$ are distinct and its sign is constant on the k -tuples (x_1, \dots, x_k) such that $a \leq x_1 < x_2 < \cdots < x_k \leq b$.

EXAMPLE 1. For $n=1$, condition Δ states that the function f_1 does not vanish. For $n=2$, the functions f_1, f_2 satisfy Δ if and only if $f_1 \neq 0$ and f_2/f_1 is strictly monotonic.

EXAMPLE 2. The functions $f_i(t) = t^{i-1}$ ($i \geq 1$) satisfy condition Δ on \mathbb{R} (the corresponding determinants are Vandermonde determinants).

Our interest in condition Δ relies on the following nice facts.

LEMMA 1. Let f_1, \dots, f_n be n measurable bounded functions satisfying Δ on $[a, b]$. Let v_1, \dots, v_n be n positive functions in $L^1([a, b])$. Then for each $(n-1)$ -tuple $(\gamma_1, \dots, \gamma_{n-1})$ such that $a < \gamma_1 < \cdots < \gamma_{n-1} < b$ the determinant

$$\begin{vmatrix} \int_a^{\gamma_1} f_1 v_1 & \int_{\gamma_1}^{\gamma_2} f_1 v_2 & \cdots & \int_{\gamma_{n-1}}^b f_1 v_n \\ \int_a^{\gamma_1} f_2 v_1 & \int_{\gamma_1}^{\gamma_2} f_2 v_2 & \cdots & \int_{\gamma_{n-1}}^b f_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^{\gamma_1} f_n v_1 & \int_{\gamma_1}^{\gamma_2} f_n v_2 & \cdots & \int_{\gamma_{n-1}}^b f_n v_n \end{vmatrix}$$

is not equal to zero.

Proof. Since the determinant is a multilinear continuous form, we can write

$$\begin{aligned} & \left| \begin{array}{ccc} \int_a^{\gamma_1} f_1 v_1 & \cdots & \int_{\gamma_{n-1}}^b f_1 v_n \\ \vdots & \ddots & \vdots \\ \int_a^{\gamma_1} f_n v_1 & \cdots & \int_{\gamma_{n-1}}^b f_n v_n \end{array} \right| \\ &= \int_a^{\gamma_1} ds_1 \cdots \int_{\gamma_{n-1}}^b ds_n \left| \begin{array}{ccc} f_1(s_1) v_1(s_1) & \cdots & f_1(s_n) v_n(s_n) \\ \vdots & \ddots & \vdots \\ f_n(s_1) v_1(s_1) & \cdots & f_n(s_n) v_n(s_n) \end{array} \right| \\ &= \iint \cdots \int_{[a, \gamma_1] \times [\gamma_1, \gamma_2] \times \cdots \times [\gamma_{n-1}, b]} v_1(s_1) v_2(s_2) \cdots v_n(s_n) \omega(s_1, s_2, \dots, s_n) \\ & \quad \times ds_1 ds_2 \cdots ds_n, \end{aligned}$$

where

$$\omega(s_1, \dots, s_n) = \left| \begin{array}{ccc} f_1(s_1) & \cdots & f_1(s_n) \\ \vdots & \ddots & \vdots \\ f_n(s_1) & \cdots & f_n(s_n) \end{array} \right|.$$

However, the function

$$(s_1, \dots, s_n) \mapsto v_1(s_1) \cdots v_n(s_n) \omega(s_1, \dots, s_n)$$

is either positive a.e. or negative a.e. on the open non-empty domain

$$]a, \gamma_1[\times]\gamma_1, \gamma_2[\times \cdots \times]\gamma_{n-1}, b[$$

so that its integral over $[a, \gamma_1] \times [\gamma_1, \gamma_2] \times \cdots \times [\gamma_{n-1}, b]$ cannot vanish. ■

LEMMA 2. Let f_1, \dots, f_m be m measurable bounded functions satisfying condition Δ on $[a, b]$. Let v_1, \dots, v_m be m positive functions in $L^1([a, b])$. Let $(\gamma_1, \dots, \gamma_{m-1})$ be an $(m-1)$ -tuple such that $(\gamma_0 =) a < \gamma_1 < \cdots < \gamma_{m-1} < b (= \gamma_m)$. If x_1, \dots, x_m are m real numbers not all equal to zero then there exists k in $\{1, \dots, m\}$ such that

$$\sum_{i=1}^m x_i \int_{\gamma_{i-1}}^{\gamma_i} f_k(s) v_i(s) ds \neq 0.$$

Proof. Assume that

$$\forall k \in \{1, \dots, m\}, \quad \sum_{i=1}^m x_i \int_{\gamma_{i-1}}^{\gamma_i} f_k(s) v_i(s) ds = 0.$$

Then the determinant whose elements are the coefficients of x_1, \dots, x_m in the above system,

$$\begin{vmatrix} \int_a^{\gamma_1} f_1 v_1 & \int_{\gamma_1}^{\gamma_2} f_1 v_2 & \dots & \int_{\gamma_{m-1}}^b f_1 v_m \\ \int_a^{\gamma_1} f_2 v_1 & \int_{\gamma_1}^{\gamma_2} f_2 v_2 & \dots & \int_{\gamma_{m-1}}^b f_2 v_m \\ \vdots & \vdots & \ddots & \vdots \\ \int_a^{\gamma_1} f_m v_1 & \int_{\gamma_1}^{\gamma_2} f_m v_2 & \dots & \int_{\gamma_{m-1}}^b f_m v_m \end{vmatrix}$$

is necessarily equal to zero, thus contradicting Lemma 1. ■

LEMMA 3. Let f_1, \dots, f_n be n measurable bounded functions satisfying condition Δ on the interval $[a, b]$. Let $\alpha_1, \dots, \alpha_n$ be such that $(\alpha_0 =) a < \alpha_1 < \dots < \alpha_n < b (= \alpha_{n+1})$. Then, given a positive ε , there exist $n+1$ positive real numbers $\lambda_0, \dots, \lambda_n$, such that

$$\forall l \in \{0, \dots, n\}, \quad 0 < \lambda_l < \varepsilon,$$

$$\forall k \in \{1, \dots, n\}, \quad \sum_{l=0}^n (-1)^l \lambda_l \int_{\alpha_l}^{\alpha_{l+1}} f_k = 0.$$

Proof. Consider the $n \times n$ linear system

$$\begin{aligned} \lambda_0 \int_a^{\alpha_1} f_1 - \lambda_1 \int_{\alpha_1}^{\alpha_2} f_1 + \dots + (-1)^{n-1} \lambda_{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_1 &= (-1)^{n-1} \lambda_n \int_{\alpha_n}^b f_1 \\ \lambda_0 \int_a^{\alpha_1} f_2 - \lambda_1 \int_{\alpha_1}^{\alpha_2} f_2 + \dots + (-1)^{n-1} \lambda_{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_2 &= (-1)^{n-1} \lambda_n \int_{\alpha_n}^b f_2 \\ \vdots & \vdots \quad \ddots \quad \vdots \\ \lambda_0 \int_a^{\alpha_1} f_n - \lambda_1 \int_{\alpha_1}^{\alpha_2} f_n + \dots + (-1)^{n-1} \lambda_{n-1} \int_{\alpha_{n-1}}^{\alpha_n} f_n &= (-1)^{n-1} \lambda_n \int_{\alpha_n}^b f_n, \end{aligned}$$

where λ_n is a parameter. The determinant of the system is

$$\omega_n = (-1)^{n(n-1)/2} \begin{vmatrix} \int_a^{x_1} f_1 & \cdots & \int_{\alpha_{n-1}}^{x_n} f_1 \\ \vdots & \ddots & \vdots \\ \int_a^{x_1} f_n & \cdots & \int_{\alpha_{n-1}}^{x_n} f_n \end{vmatrix}.$$

By condition \mathcal{A} , its sign does not depend on $(\alpha_1, \dots, \alpha_n)$. Moreover, for each i in $\{0, \dots, n-1\}$,

$$\omega_i = \begin{vmatrix} \int_a^{x_i} f_1 & \cdots & (-1)^{i-2} \int_{\alpha_{i-2}}^{x_{i-1}} f_1 & (-1)^{i-1} \int_{x_i}^b f_1 & (-1)^i \int_{x_i}^{\alpha_{i+1}} f_1 & \cdots & (-1)^{n-1} \int_{\alpha_{n-1}}^{x_n} f_1 \\ \int_a^{x_i} f_2 & \cdots & (-1)^{i-2} \int_{\alpha_{i-2}}^{x_{i-1}} f_2 & (-1)^{i-1} \int_{x_i}^b f_2 & (-1)^i \int_{x_i}^{\alpha_{i+1}} f_2 & \cdots & (-1)^{n-1} \int_{\alpha_{n-1}}^{x_n} f_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \int_a^{x_i} f_n & \cdots & (-1)^{i-2} \int_{\alpha_{i-2}}^{x_{i-1}} f_n & (-1)^{i-1} \int_{x_i}^b f_n & (-1)^i \int_{x_i}^{\alpha_{i+1}} f_n & \cdots & (-1)^{n-1} \int_{\alpha_{n-1}}^{x_n} f_n \end{vmatrix},$$

i.e.,

$$\omega_i = (-1)^{n(n-1)/2} \begin{vmatrix} \int_a^{x_i} f_1 & \cdots & \int_{\alpha_{i-2}}^{x_{i-1}} f_1 & \int_{x_i}^{\alpha_{i+1}} f_1 & \cdots & \int_{\alpha_n}^b f_1 \\ \int_a^{x_i} f_2 & \cdots & \int_{\alpha_{i-2}}^{x_{i-1}} f_2 & \int_{x_i}^{\alpha_{i+1}} f_2 & \cdots & \int_{\alpha_n}^b f_2 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \int_a^{x_i} f_n & \cdots & \int_{\alpha_{i-2}}^{x_{i-1}} f_n & \int_{x_i}^{\alpha_{i+1}} f_n & \cdots & \int_{\alpha_n}^b f_n \end{vmatrix}$$

Thus λ_i which by the Cramer formula equals $\lambda_n \omega_i / \omega_n$ has, by condition \mathcal{A} and Lemma 1, the sign of λ_n ; choosing λ_n such that

$$0 < \lambda_n < \min \left(\frac{\omega_n}{\omega_0} \varepsilon, \dots, \frac{\omega_n}{\omega_{n-1}} \varepsilon, \varepsilon \right),$$

we obtain an $(n+1)$ -tuple which solves the problem. ■

We give now a criterion for the fulfilment of the orientation condition \mathcal{A} . If f_1, \dots, f_{k+1} are of class \mathcal{C}^k on $[a, b]$ we will denote their Wronskian by

$$W(f_1, \dots, f_{k+1})(t) = \begin{vmatrix} f_1(t) & \cdots & f_{k+1}(t) \\ \vdots & \ddots & \vdots \\ f_1^{(k)}(t) & \cdots & f_{k+1}^{(k)}(t) \end{vmatrix}.$$

PROPOSITION 1. Let $h_1, \dots, h_n \in \mathcal{C}^{n-1}([a, b])$ be such that

$$\forall t \in [a, b], \quad W(h_1)(t) \neq 0, \dots, W(h_1, \dots, h_n)(t) \neq 0.$$

Then h_1, \dots, h_n satisfy the orientation condition Δ .

Proof. By [13, Theorem V], for each k -tuple (t_1, \dots, t_k) , such that $a \leq t_1 < \dots < t_k \leq b$, there exists $\xi \in]t_1, t_k[$ such that $W(h_1(\xi), \dots, h_k(\xi))$ has the same sign as the determinant

$$\begin{vmatrix} h_1(t_1) & h_1(t_2) & \dots & h_1(t_k) \\ h_2(t_1) & h_2(t_2) & \dots & h_2(t_k) \\ \vdots & \vdots & \ddots & \vdots \\ h_k(t_1) & h_k(t_2) & \dots & h_k(t_k) \end{vmatrix}$$

It follows that the above determinant does not vanish and, by continuity, it keeps a constant sign on the connected set of the k -tuples (t_1, \dots, t_k) such that $a \leq t_1 < \dots < t_k \leq b$. ■

Remark 1. It is easy to prove that if h_1, \dots, h_n satisfy the orientation condition Δ on $[a, b]$ and are of class \mathcal{C}^{n-1} then $W(h_1), \dots, W(h_1, \dots, h_n)$ are either non-negative or non-positive on the whole interval $[a, b]$.

Remark 2. For $n=2$, the Wronskian conditions on f_1, f_2 state exactly that $f_1 \neq 0$ and $f_1 f_2' - f_1' f_2 \neq 0$, whence f_2/f_1 is strictly monotonic. However, these conditions are not necessary for property Δ to hold (a function may be strictly increasing without having a positive derivative).

THE RANGE OF A FINITE DIMENSIONAL ORIENTED MEASURE

In this section we study the range of a finite dimensional measure μ whose components μ_1, \dots, μ_n admit continuous density functions f_1, \dots, f_n which satisfy the orientation condition Δ : such a measure is said to be oriented.

THEOREM 1. Let $v \in L^1([a, b])$ be such that $0 \leq v \leq 1$. Let f_1, \dots, f_n be n real valued continuous functions on $[a, b]$ satisfying condition Δ on $[a, b]$. Then there exist a n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ and a n -tuple $\beta = (\beta_1, \dots, \beta_n)$, such that

$$a \leq \alpha_1 \leq \dots \leq \alpha_n \leq b, \quad a \leq \beta_1 \leq \dots \leq \beta_n \leq b,$$

and if we define

$$E_{\alpha}^{-} = \bigcup_{\substack{0 \leq i \leq n \\ i \text{ odd}}} [\alpha_i, \alpha_{i+1}], \quad E_{\beta}^{+} = \bigcup_{\substack{0 \leq i \leq n \\ i \text{ even}}} [\beta_i, \beta_{i+1}]$$

(where $\beta_0 = a$, $\alpha_{n+1} = \beta_{n+1} = b$), then we have

$$\begin{aligned} \forall k \in \{1, \dots, n\}, \quad \int_a^b f_k(s) \chi_{E_{\alpha}^{-}}(s) ds &= \int_a^b f_k(s) v(s) ds \\ &= \int_a^b f_k(s) \chi_{E_{\beta}^{+}}(s) ds. \quad (*) \end{aligned}$$

If, in addition, $0 < v < 1$, then $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$ are unique and verify

$$a < \alpha_1 < \dots < \alpha_n < b, \quad a < \beta_1 < \dots < \beta_n < b.$$

Remark. This theorem has already been proved for $n=2$ in [7], but the orientation condition \mathcal{A} was not formulated in such a precise way (see Remark 1 after Proposition 1).

EXAMPLE. There exists a non-oriented measure μ on an interval, a measurable subset A which is not a finite union of intervals, such that for every measurable subset E

$$\mu(A) = \mu(E) \Rightarrow A = E \quad \text{a.e.}$$

Consider, for instance, the measure $\mu = (\mu_1, \mu_2)$ whose density functions are

$$f_1(t) = 1, \quad f_2(t) = 1 + t \sin(1/t)$$

and the set $A = \{t \in [0, 1] : t \sin(1/t) > 0\}$ (in this case the measure μ is positive but condition \mathcal{A} is not fulfilled).

We will deal only with the situation where $0 < v < 1$: the fact that the number of intervals corresponding to v does not depend on v , together with a classical approximation argument, yields the general case (this is done explicitly in the proof of Theorem 5). We will give two proofs of the theorem. The first one relies on an induction, whereas the second one is based on Caccioppoli global inversion theorem. The following lemma will be used in both proofs.

LEMMA. Assume that $0 < v < 1$ and let l be an integer smaller than n . Then if the l -tuple $\alpha = (\alpha_1, \dots, \alpha_l)$ (resp. $\beta = (\beta_1, \dots, \beta_l)$) and its corresponding

set E_x^- (resp. E_β^+) satisfy (*) with $a \leq \alpha_1 \leq \dots \leq \alpha_l \leq b$ (resp. $a \leq \beta_1 \leq \dots \leq \beta_l \leq b$) then $l=n$ and $a < \alpha_1 < \dots < \alpha_n < b$ (resp. $a < \beta_1 < \dots < \beta_n < b$).

Proof of the lemma. We first show that under the above assumption there exists an m -tuple $\gamma = (\gamma_1, \dots, \gamma_m)$, $m \leq l$, such that $a < \gamma_1 < \dots < \gamma_m < b$ and either E_γ^- or E_γ^+ satisfy (*). Assume, for instance, that there exists $i \in \{0, \dots, l\}$, such that $\alpha_i = \alpha_{i+1}$ (where possibly $\alpha_0 = a$ and $\alpha_{l+1} = b$). We have the following cases:

• $i=0$ so that $a = \alpha_1$. Put $m=l-1$, $\gamma = (\alpha_2, \dots, \alpha_l)$; then E_γ^+ satisfies (*).

• $0 < i < l$. Put $m=l-2$, $\gamma = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+2}, \dots, \alpha_n)$; then E_γ^- satisfies (*).

• $i=l$. Put $m=l-1$, $\gamma = (\alpha_1, \dots, \alpha_{l-1})$; then E_γ^- satisfies (*).

If two components of the m -tuple γ are equal we iterate the above operation on γ until after a finite number of steps we obtain a tuple having distinct components and one of whose associated sets satisfies (*).

We are thus led to prove the result for an l -tuple α such that $a < \alpha_1 < \dots < \alpha_l < b$; similar arguments hold for an l -tuple of type β . Suppose that $l < n$. Then by (*) we have

$$\forall k \in \{1, \dots, n\},$$

$$\sum_{\substack{0 \leq i \leq l \\ i \text{ even}}} \int_{\alpha_i}^{\alpha_{i+1}} f_k(s) v(s) ds - \sum_{\substack{0 \leq i \leq l \\ i \text{ odd}}} \int_{\alpha_i}^{\alpha_{i+1}} f_k(s) (1-v(s)) ds = 0$$

(where $\alpha_0 = a$, $\alpha_{l+1} = b$).

We restrict our attention on the first $l+1$ equations of the above system; i.e., k belongs to $\{1, \dots, l+1\}$. Application of Lemma 2 with $m=l+1$ and $\gamma_1 = \alpha_1, \dots, \gamma_l = \alpha_l$,

$$x_i = (-1)^{i+1}, \quad v_i = \begin{cases} v & i \text{ odd} \\ 1-v & i \text{ even} \end{cases}, \quad 1 \leq i \leq l+1,$$

shows that these equations cannot hold simultaneously, thus yielding a contradiction. ■

First Proof of Theorem 1. Consider the case $n=1$. Let $f_1 \in \mathcal{C}([a, b])$ satisfy Δ ; i.e., f_1 does not vanish on $[a, b]$. Since f_1 has a constant sign on $[a, b]$ there exist unique reals α, β in $[a, b]$ such that

$$\int_x^b f_1(s) ds = \int_a^b f_1(s) v(s) ds = \int_a^\beta f_1(s) ds.$$

Since $0 < \nu < 1$ we have $a < \alpha < \beta < b$. Clearly $E_\alpha^- = [\alpha, b]$ and $E_\beta^+ = [a, \beta]$ satisfy (*). Assume the theorem is true at rank $n - 1$.

Let f_1, \dots, f_n, ν be functions satisfying the hypothesis of the theorem. By the induction assumption there exist $(n - 1)$ tuples $(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1})$ and $(\bar{\beta}_1, \dots, \bar{\beta}_{n-1})$ satisfying (*). Define for each k in $\{1, \dots, n\}$ and n -tuple $(\alpha_1, \dots, \alpha_n)$ such that $a \leq \alpha_1 \leq \dots \leq \alpha_n \leq b$,

$$F_k(\alpha_1, \dots, \alpha_n) = \sum_{\substack{0 \leq l \leq n \\ l \text{ odd}}} \int_{\alpha_l}^{\alpha_{l+1}} f_k(s) ds - \int_a^b f_k(s) \nu(s) ds$$

and put

$$\mathcal{S} = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : a \leq \alpha_1 < \alpha_2 < \dots < \alpha_n \leq b, \\ \forall k \in \{1, \dots, n - 1\}, F_k(\alpha_1, \dots, \alpha_n) = 0\}.$$

The set \mathcal{S} is not empty; $(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b)$ and $(a, \bar{\beta}_1, \dots, \bar{\beta}_{n-1})$ belong to \mathcal{S} :

(i) Existence of $(\alpha_1, \dots, \alpha_n)$. Let D be the open subset of $\mathbb{R}^{n-1} \times [a, b]$ defined by

$$D = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : a < \alpha_1 < \dots < \alpha_n \leq b\}$$

and define $F : D \rightarrow \mathbb{R}^{n-1}$ by

$$F(\alpha_1, \dots, \alpha_n) = (F_1(\alpha_1, \dots, \alpha_n), \dots, F_{n-1}(\alpha_1, \dots, \alpha_n)).$$

The map F is \mathcal{C}^1 on D and its jacobian matrix is

$$\text{Jac } F(\alpha_1, \dots, \alpha_n) = \begin{pmatrix} -f_1(\alpha_1) & +f_1(\alpha_2) & \dots & (-1)^n f_1(\alpha_n) \\ -f_2(\alpha_1) & +f_2(\alpha_2) & \dots & (-1)^n f_2(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ -f_{n-1}(\alpha_1) & +f_{n-1}(\alpha_2) & \dots & (-1)^n f_{n-1}(\alpha_n) \end{pmatrix}.$$

We see that

$$\left| \frac{\partial F}{\partial(\alpha_1, \dots, \alpha_{n-1})}(\alpha_1, \dots, \alpha_n) \right| = (-1)^{n(n-1)/2} \begin{vmatrix} f_1(\alpha_1) & \dots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \dots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}$$

which, by the orientation condition A , does not vanish and keeps a constant sign when $\alpha_1 < \alpha_2 < \dots < \alpha_{n-1}$. Consider the equation

$$F((\alpha_1, \dots, \alpha_{n-1}), \alpha_n) = (0, \dots, 0). \tag{†}$$

Let $(\xi_1, \dots, \xi_n) \in D$ verify (\dagger) , i.e.,

$$(\xi_1, \dots, \xi_n) \in \mathcal{S} \setminus \{(a, \bar{\beta}_1, \dots, \bar{\beta}_{n-1})\}.$$

Such a point exists: for instance, $(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b)$. We apply the implicit function theorem at (ξ_1, \dots, ξ_n) . There exists an open interval I containing ξ_n , an open neighbourhood U of $(\xi_1, \dots, \xi_{n-1})$, a continuous function

$$\psi: I \rightarrow U \\ \alpha_n \mapsto (\alpha_1(\alpha_n), \dots, \alpha_{n-1}(\alpha_n)),$$

such that

$$\forall ((\eta_1, \dots, \eta_{n-1}), \eta_n) \in D \cap (U \times I), \\ F(\eta_1, \dots, \eta_n) = (0, \dots, 0) \Leftrightarrow (\eta_1, \dots, \eta_{n-1}) = \psi(\eta_n).$$

Moreover, ψ is \mathcal{C}^1 and we have

$$\alpha'_j(\alpha_n) = \frac{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{j-1}) & f_1(\alpha_{j+1}) & \cdots & f_1(\alpha_n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{j-1}) & f_{n-1}(\alpha_{j+1}) & \cdots & f_{n-1}(\alpha_n) \end{vmatrix}}{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}}$$

so that $\alpha'_j(\alpha_n) > 0$ on I and the functions α_j are increasing. Lemma M yields a maximal interval I_M on which ψ can be extended. Let $\xi_n^* = \inf I_M$. The functions $\alpha_1, \dots, \alpha_{n-1}$, being increasing on I_M , admit the limits

$$\xi_j^* = \lim_{\substack{\eta_n \rightarrow \xi_n^* \\ \eta_n > \xi_n^*}} \alpha_j(\eta_n).$$

We remark that $\xi_n^* < b$ since $\xi_n \leq b$. By continuity,

$$F(\xi_1^*, \dots, \xi_n^*) = (0, \dots, 0).$$

We claim that $\xi_1^* = a$. Suppose $\xi_1^* > a$. By the maximality of I_M , $(\xi_1^*, \dots, \xi_n^*)$ belongs to $\bar{D} \setminus D$ so that there exists $i \in \{1, \dots, n-2\}$ such that $\xi_i^* = \xi_{i+1}^*$. The $(n-1)$ -tuple $(\xi_1^*, \dots, \xi_{i-1}^*, \xi_{i+2}^*, \dots, \xi_n^*, b)$ and its associated set E_{ξ^*} satisfy

$$\forall k \in \{1, \dots, n-1\}, \quad \int_a^b f_k(s) \chi_{E_{\xi^*}}(s) ds = \int_a^b f_k(s) v(s) ds$$

so that the induction hypothesis implies that

$$a < \xi_1^* < \dots < \xi_{l-1}^* < \xi_{l+2}^* < \dots < \xi_n^* < b < b,$$

which is absurd.

Since $\xi_l^* = a$, the $(n-1)$ -tuple $(\xi_1^*, \dots, \xi_{n-1}^*)$ is the one given by the theorem at rank $n-1$ so that $\xi_i^* = \beta_{i-1}$ for each i in $\{2, \dots, n\}$. Thus for each point (ξ_1, \dots, ξ_n) of $\mathcal{S} \setminus \{(a, \beta_1, \dots, \beta_{n-1})\}$ there exists a continuous arc in \mathcal{S} joining (ξ_1, \dots, ξ_n) to $(a, \beta_1, \dots, \beta_{n-1})$. This proves that \mathcal{S} is arcwise connected.

At this stage we prove that $F_n(a, \beta_1, \dots, \beta_{n-1})$ and $F_n(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b)$ have opposite signs. Since $F(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b) = 0$ then for each k in $\{1, \dots, n-1\}$,

$$-\sum_{\substack{0 \leq i \leq n-1 \\ i \text{ even}}} \int_{\bar{\alpha}_i}^{\bar{\alpha}_{i+1}} f_k(s) v(s) ds + \sum_{\substack{0 \leq i \leq n-1 \\ i \text{ odd}}} \int_{\bar{\alpha}_i}^{\bar{\alpha}_{i+1}} f_k(s) (1-v(s)) ds = 0$$

(where $\bar{\alpha}_0 = a, \bar{\alpha}_n = b$).

For k, j in $\{1, \dots, n\}$ put

$$x_j^z = (-1)^j, \quad a_{kj}^z = \int_{\bar{\alpha}_{j-1}}^{\bar{\alpha}_j} f_k v_j^z, \quad A^z = (a_{kj}^z)_{1 \leq k, j \leq n},$$

where

$$v_j^z = \begin{cases} v & \text{if } j \text{ is odd} \\ 1-v & \text{if } j \text{ is even,} \end{cases}$$

so that the above equations become

$$\forall k \in \{1, \dots, n-1\}, \quad \sum_{j=1}^n a_{kj}^z x_j^z = 0.$$

Since $F_n(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b) = \sum_{j=1}^n a_{nj}^z x_j^z$, application of Lemma S gives

$$F_n(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b) = \frac{|A^z|}{|A_{nn}^z|} (-1)^n.$$

Similarly if we define for k, j in $\{1, \dots, n\}$ ($\beta_0 = a, \beta_n = b$),

$$x_j^\beta = (-1)^{j+1}, \quad a_{kj}^\beta = \int_{\beta_{j-1}}^{\beta_j} f_k v_j^\beta, \quad A^\beta = (a_{kj}^\beta)_{1 \leq k, j \leq n},$$

where

$$v_j^\beta = \begin{cases} v & \text{if } j \text{ is even} \\ 1-v & \text{if } j \text{ is odd,} \end{cases}$$

then we have

$$F_n(a, \bar{\beta}_1, \dots, \bar{\beta}_{n-1}) = \sum_{j=1}^n a_{nj}^\beta x_j^\beta = \frac{|A^\beta|}{|A_{nn}^\beta|} (-1)^{n+1}.$$

By condition A on f_1, \dots, f_n and Lemma 1, $|A^\alpha|$ and $|A^\beta|$ have the same sign, as do $|A_{nn}^\alpha|$ and $|A_{nn}^\beta|$. It follows that $F_n(a, \bar{\beta}_1, \dots, \bar{\beta}_{n-1})$ and $F_n(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b)$ have opposite signs. Moreover, the set \mathcal{S} is connected; the map F_n is continuous on \mathcal{S} and thus must vanish at a point $(\alpha_1, \dots, \alpha_n)$ of \mathcal{S} . By the very definition of \mathcal{S} we have also that

$$\forall k \in \{1, \dots, n-1\}, \quad F_k(\alpha_1, \dots, \alpha_n) = 0,$$

so that $(\alpha_1, \dots, \alpha_n)$ solves the problem.

(ii) Uniqueness of $(\alpha_1, \dots, \alpha_n)$. Let $(\alpha_1, \dots, \alpha_n)$ be in \mathcal{S} with $a < \alpha_1 < \dots < \alpha_n < b$ and build (I_M, ψ) as in the existence part. The maximal interval I_M is in fact $[\bar{\beta}_{n-1}, b]$ so that $(\alpha_1, \dots, \alpha_n)$ belongs to a continuous path in \mathcal{S} joining $(a, \bar{\beta}_1, \dots, \bar{\beta}_{n-1})$ and $(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b)$. By local unicity of ψ near $(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b)$, the arc does not depend on $(\alpha_1, \dots, \alpha_n)$ (recall that we apply the implicit function theorem on the space $\mathbb{R}^{n-1} \times [a, b]$ and that b is an interior point of the topological space $[a, b]$). For each $\alpha_n \in]\bar{\beta}_{n-1}, b[$, we have

$$\frac{d}{d\alpha_n} F_n(\psi(\alpha_n), \alpha_n) = \sum_{i=1}^n \frac{\partial F_n}{\partial \alpha_i} \alpha'_i(\alpha_n)$$

$$= \sum_{i=1}^n (-1)^i f_n(\alpha_i) \frac{\begin{vmatrix} f_1(\alpha_1) & \dots & f_1(\alpha_{i-1}) & f_1(\alpha_{i+1}) & \dots & f_1(\alpha_n) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \dots & f_{n-1}(\alpha_{i-1}) & f_{n-1}(\alpha_{i+1}) & \dots & f_{n-1}(\alpha_n) \end{vmatrix}}{\begin{vmatrix} f_1(\alpha_1) & \dots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \dots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}}$$

$$= (-1)^n \frac{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ f_n(\alpha_1) & \cdots & f_n(\alpha_n) \end{vmatrix}}{\begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_{n-1}) \\ \vdots & \ddots & \vdots \\ f_{n-1}(\alpha_1) & \cdots & f_{n-1}(\alpha_{n-1}) \end{vmatrix}}$$

Thus F_n is strictly monotonic along the arc joining $(a, \bar{\beta}_1, \dots, \bar{\beta}_{n-1})$ and $(\bar{\alpha}_1, \dots, \bar{\alpha}_{n-1}, b)$ so that F_n vanishes only for one value α_n . Since this path is unique, then the n -tuple $(\alpha_1, \dots, \alpha_n)$ is unique. ■

Second Proof of Theorem 1. Let

$$\Omega = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : a < \alpha_1 < \dots < \alpha_n < b\}$$

and

$$F = \left\{ \left(\int_a^b f_1 v, \dots, \int_a^b f_n v \right) : v \in L^1([a, b]), 0 < v < 1 \right\}$$

For each $(\alpha_1, \dots, \alpha_n) \in \Omega$ put

$$\theta(\alpha_1, \dots, \alpha_n) = \left(\int_a^b f_1 \chi_{E_x^-}, \dots, \int_a^b f_n \chi_{E_x^-} \right),$$

where

$$E_x^- = \bigcup_{\substack{0 \leq i \leq n \\ i \text{ odd}}} [\alpha_i, \alpha_{i+1}] \quad (\alpha_{n+1} = b).$$

We first show that θ takes its values in F .

Let $(\alpha_1, \dots, \alpha_n)$ in Ω ; applying Lemma 3 to (f_1, \dots, f_n) , $(\alpha_1, \dots, \alpha_n)$ and $\varepsilon = \frac{1}{4}$, we obtain an $(n+1)$ -tuple $(\lambda_0, \dots, \lambda_n)$ such that

$$\forall l \in \{0, \dots, n\}, \quad 0 < \lambda_l < \varepsilon,$$

$$\forall k \in \{1, \dots, n\}, \quad \sum_{l=0}^n (-1)^l \lambda_l \int_{\alpha_l}^{\alpha_{l+1}} f_k = 0.$$

Put

$$v = \sum_{\substack{0 \leq i \leq n \\ i \text{ even}}} \lambda_i \chi_{[\alpha_i, \alpha_{i+1}]} + \sum_{\substack{0 \leq i \leq n \\ i \text{ odd}}} (1 - \lambda_i) \chi_{[\alpha_i, \alpha_{i+1}]}$$

By construction we have $0 < v < 1$ and

$$\forall k \in \{1, \dots, n\}, \quad \int_a^b f_k v = \int_a^b f_k \chi_{E_x^-}$$

so that $\theta(\alpha_1, \dots, \alpha_n)$ belongs to F .

The purpose of what follows is to show that the map $\theta: \Omega \rightarrow F$ satisfies the hypotheses of the Caccioppoli theorem.

- (1) Obviously Ω is arcwise connected.
- (2) The set F , being convex, is simply connected.
- (3) The map θ is a local homeomorphism at each point of Ω . In fact, θ is differentiable at each point $(\alpha_1, \dots, \alpha_n)$ of Ω and its jacobian

$$\det \text{Jac } \theta(\alpha_1, \dots, \alpha_n) = (-1)^{n(n+1)/2} \begin{vmatrix} f_1(\alpha_1) & \cdots & f_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ f_n(\alpha_1) & \cdots & f_n(\alpha_n) \end{vmatrix}$$

does not vanish on Ω by the condition Δ .

- (4) Finally θ is proper. Let K be a compact subset of F and let $\alpha^k = (\alpha_1^k, \dots, \alpha_n^k)$ be a sequence of points in $\theta^{-1}(K)$. Since the sequence $(\theta(\alpha^k))_{k \in \mathbb{N}}$ is contained in $K \subset F$ then by compactness we may assume that there exists $v^* \in L^1([a, b])$, $0 < v^* < 1$, such that

$$\lim_{k \rightarrow \infty} \theta(\alpha^k) = \left(\int_a^b f_1 v^*, \dots, \int_a^b f_n v^* \right). \quad (\circ)$$

The closure $\bar{\Omega} = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n : a \leq \alpha_1 \leq \dots \leq \alpha_n \leq b\}$ of Ω is compact and therefore $(\alpha^k)_{k \in \mathbb{N}}$ admits a subsequence which converges to $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*) \in \bar{\Omega}$. By (\circ) we have

$$\forall k \in \{1, \dots, n\}, \quad \int_a^b f_k \chi_{E_{\alpha^*}^-} = \int_a^b f_k v^*$$

and the initial lemma implies that $a < \alpha_1^* < \dots < \alpha_n^* < b$. Thus α^* belongs to Ω and $\theta^{-1}(K)$ is compact. By the Caccioppoli theorem, θ is a global homeomorphism. ■

As a consequence of Theorem 1, we deduce the following.

THEOREM 2. *Let v be a measurable function on $[a, b]$ such that $0 \leq v \leq 1$. Let f_1, \dots, f_n be n continuous functions on $[a, b]$. Assume that the interval $[a, b]$ is a finite (resp. countable) union of intervals on which the*

orientation condition Δ for f_1, \dots, f_n holds. Then there exists a set E which is finite (resp. countable) union of intervals such that

$$\forall k \in \{1, \dots, n\}, \quad \int_a^b f_k \chi_E = \int_a^b f_k v.$$

Proof. Under the hypothesis of the theorem, there exists a finite (resp. countable) family of disjoint open intervals $(I_j)_{j \in J}$ included in $[a, b]$ such that $[a, b] \setminus \bigcup_{j \in J} I_j$ is a negligible set (with respect to Lebesgue measure) and the functions f_1, \dots, f_n satisfy condition Δ on each interval $I_j, j \in J$. We apply Theorem 1 to f_1, \dots, f_n and v on the interval I_j ; there exists a set E_j included in I_j whose characteristic function has less than n discontinuity points such that

$$\forall k \in \{1, \dots, n\}, \quad \int_{I_j} f_k \chi_{E_j} = \int_{I_j} f_k v.$$

The set $E = \bigcup_{j \in J} E_j$ solves the problem. ■

Proposition 1 shows that the hypotheses of the theorem are fulfilled as soon as

- f_1, \dots, f_n are of class \mathcal{C}^{n-1} on $[a, b]$,
- the set $Z = \{t \in [a, b] : \exists k \in \{1, \dots, n\}, W(f_1, \dots, f_k)(t) = 0\}$ is finite (resp. is negligible).

This result weakens Halkin's condition [9] that the interval is a countable union of intervals on which the functions f_1, \dots, f_n are analytical.

SOME RESULTS ON LINEAR DIFFERENTIAL EQUATIONS

We consider a linear differential operator,

$$L(D) = D^n + a_{n-1}(t) D^{n-1} + \dots + a_1(t) D + a_0(t),$$

where D is the derivative operator $D = d/dt$ and a_0, \dots, a_{n-1} are n real-valued continuous functions on an interval $[a, b]$.

A GENERALIZED ROLLE THEOREM. If f_1, \dots, f_{k+1} are of class \mathcal{C}^k on $[a, b]$ we will denote their Wronskian by

$$W(f_1, \dots, f_{k+1})(t) = \begin{vmatrix} f_1(t) & \dots & f_{k+1}(t) \\ \vdots & \ddots & \vdots \\ f_1^{(k)}(t) & \dots & f_{k+1}^{(k)}(t) \end{vmatrix}$$

DEFINITION (see [13]). The operator L possesses property W on $[a, b]$ if there exist $n-1$ functions h_1, \dots, h_{n-1} satisfying

$$\begin{aligned} \forall i \in \{1, \dots, n-1\}, \quad L(D)(h_i) = 0 \quad \text{on } [a, b] \\ \forall t \in [a, b], \quad W(h_1)(t) > 0, \dots, W(h_1, \dots, h_{n-1})(t) > 0. \end{aligned}$$

We will use the fact that property W always holds locally: for each fixed t_0 in $[a, b]$, the $n-1$ solutions h_1, \dots, h_{n-1} to the $n-1$ Cauchy problems ($1 \leq i \leq n-1$)

$$L(D)(h_i) = 0, \quad h_i^{(k)}(t_0) = \delta(i-1, k), \quad 0 \leq k \leq n-1$$

(where $\delta(j, k) = 0$ if $j \neq k$ and $\delta(j, k) = 1$ if $j = k$) are such that

$$\forall i \in \{1, \dots, n-1\}, \quad W(h_1, \dots, h_i)(t_0) = 1.$$

Therefore, the inequalities

$$W(h_1)(t) > 0, \dots, W(h_1, \dots, h_{n-1})(t) > 0$$

hold in a neighbourhood of t_0 .

The interest in property W is that it allows us to decompose the linear differential operator L into a "product" of differential expressions of first order.

THEOREM 3 (See [13]). Let the linear differential operator $L(D)$ possess property W on $[a, b]$. Then there exist $n+1$ functions u_0, \dots, u_n such that for each i in $\{0, \dots, n\}$, u_i is of class \mathcal{C}^{n-i} on $[a, b]$ and

$$\forall y \in \mathcal{C}^n([a, b]), \quad L\left(\frac{d}{dt}\right)y = u_n \frac{d}{dt} u_{n-1} \frac{d}{dt} u_{n-2} \cdots u_2 \frac{d}{dt} u_1 \frac{d}{dt} u_0 y.$$

As a consequence of this decomposition, we derive a generalized Rolle theorem. We say that the function f has N zeroes on $[a, b]$ if there exist l distinct points t_1, \dots, t_l and l positive integers m_1, \dots, m_l such that $m_1 + \dots + m_l = N$, f is at least $m_k - 1$ times differentiable at t_k ($1 \leq k \leq l$), and

$$\forall k \in \{1, \dots, l\}, \forall i \in \{0, \dots, m_k - 1\}, \quad f^{(i)}(t_k) = 0.$$

THEOREM 4. Let the differential operator $L(D)$ possess property W on $[a, b]$. Let f be a piecewise \mathcal{C}^n function of class \mathcal{C}^{n-1} defined on $[a, b]$ and let k be the number of discontinuity points of $f^{(n)}$ in $]a, b[$. If f vanishes at $(n+1) + k$ points in the interval $[a, b]$ then there exists ξ in $]a, b[$ such that $L(D)f(\xi) = 0$.

Proof. Let $t_1 < \dots < t_k$ be the distinct zeroes of f in $[a, b]$ with multiplicities m_1, \dots, m_k . By applying the Rolle theorem successively on the intervals $[t_1, t_2], \dots, [t_{k-1}, t_k]$, we obtain $k-1$ points t_1^1, \dots, t_{k-1}^1 , such that $t_1 < t_1^1 < t_2 < \dots < t_{k-1} < t_{k-1}^1 < t^k$ and

$$\forall i \in \{1, \dots, k-1\}, \quad \frac{d}{dt}(u_0 f)(t_i^1) = 0.$$

Taking into account the multiple zeroes of f we see that $D(u_0 f)$ admits $n+k$ zeroes on $[a, b]$. At step $n-1$, this process yields the existence of $k+2$ zeroes for the function

$$g = u_{n-1} \frac{d}{dt} u_{n-2} \dots u_2 \frac{d}{dt} u_1 \frac{d}{dt} u_0 f.$$

Either one of these zeroes is double or g possesses $k+2$ distinct roots; in this situation, at least two of them must lie in one of the $k+1$ intervals on which g is \mathcal{C}^1 and the Rolle theorem yields a zero of Dg . In both cases, we obtain the existence of a zero of the function $u_n D(g) = L(D)(f)$. ■

We will use the following straightforward corollary of this theorem.

COROLLARY. *Let the differential operator $L(D)$ possess property W on $[a, b]$. Let f be a piecewise \mathcal{C}^n function of class \mathcal{C}^{n-1} defined on $[a, b]$, $f^{(n)}$ having at most n discontinuity points. Moreover, assume that*

$$\forall i \in \{0, \dots, n-1\}, \quad f^{(i)}(a) = f^{(i)}(b) = 0.$$

If $L(D)(f)$ does not vanish then f has no roots in $]a, b[$.

Let us remark that property W is essential for Theorem 4 to hold.

Example. Let $f(t) = \sin t - \alpha t$ where α is chosen so that f admits three zeroes on the interval $[\pi/2, 3\pi]$. If we set $L(D) = D^2 + 1$ we have $L(D)(f)(t) = -\alpha t$ which does not vanish on $[\pi/2, 3\pi]$. However, it is easy to check that the operator L possesses property W on the interval $[a, b]$ if and only if $b - a < \pi$.

DEFINITION OF THE RESOLVENT. We say that $R: [a, b] \times [a, b] \mapsto \mathbb{R}$ is the resolvent of the operator L if for each fixed s in $[a, b]$ the function $t \mapsto R(t, s)$ solves the Cauchy problem

$$L(D)y = 0, \quad y(s) = \dots = y^{(n-2)}(s) = 0, \quad y^{(n-1)}(s) = 1.$$

As it is well known, R is of class \mathcal{C}^{n+k} on $[a, b] \times [a, b]$ whenever the functions a_0, \dots, a_{n-1} are of class \mathcal{C}^k .

PROPOSITION 2. Let $R(t, s)$ be a function in $\mathcal{C}^{2n-2}([a, b] \times [a, b])$ satisfying

$$\forall s \in [a, b], \quad R(s, s) = \frac{\partial R}{\partial t}(s, s) = \dots = \frac{\partial^{n-2} R}{\partial t^{n-2}}(s, s) = 0,$$

$$\frac{\partial^{n-1} R}{\partial t^{n-1}}(s, s) = 1.$$

Then, for each t_0 in $[a, b]$, there exists $\delta > 0$ such that for every t in $[t_0 - \delta, t_0 + \delta] \cap [a, b]$, the functions

$$h'_i(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(t, s), \quad 1 \leq i \leq n,$$

satisfy condition Δ on the interval $[t_0 - \delta, t_0 + \delta] \cap [a, b]$.

Proof. For each $t_0 \in [a, b]$ and for each $k \in \{1, \dots, n\}$, we have $W(h'_1(t_0), \dots, h'_k(t_0)) = 1$; in fact,

$$W(h'_1(t_0), \dots, h'_k(t_0)) = \begin{vmatrix} \frac{\partial^{n-1} R}{\partial t^{n-1}}(t_0, t_0) & \dots & \frac{\partial^{n-k} R}{\partial t^{n-k}}(t_0, t_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial^{n+k-2} R}{\partial t^{n+k-2}}(t_0, t_0) & \dots & \frac{\partial^{n-1} R}{\partial t^{n-1}}(t_0, t_0) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & \dots & \dots & 0 \\ * & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ * & \dots & * & 1 & 0 \\ * & \dots & \dots & * & 1 \end{vmatrix} = 1.$$

By continuity, there exists $\delta > 0$ such that

$$\forall t, s \in [t_0 - \delta, t_0 + \delta] \cap [a, b], \forall k \in \{1, \dots, n\}, \quad W(h'_1(s), \dots, h'_k(s)) > 0.$$

Proposition 1 yields the conclusion. ■

BANG-BANG CONSTRAINED SOLUTIONS

We consider the n -dimensional linear control system

$$L(D)x = x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' + a_0(t)x \in [\phi_1, \phi_2] \quad \text{a.e. on } [a, b], \quad (\text{P})$$

where the n functions a_0, \dots, a_{n-1} belong to $\mathcal{C}^{n-2}([a, b])$ and ϕ_1, ϕ_2 in $L^1([a, b])$ verify $\phi_1 \leq \phi_2$. The function y is said to be a bang-bang solution to (P) if it solves (P) and, moreover,

$$L(D)y \in \{\phi_1, \phi_2\}.$$

Given a solution x to (P), the existence of a bang-bang solution y satisfying

$$\forall k \in \{0, \dots, n-1\}, \quad y^{(k)}(a) = x^{(k)}(a), \quad y^{(k)}(b) = x^{(k)}(b)$$

has been proven, for instance, by Cesari [8] and Olech [12].

THEOREM 5. *Let x in $W^{n,1}([a, b])$ be a solution to the control problem (P). Then there exist two bang-bang solutions y and z satisfying the tangency conditions*

$$\begin{aligned} \forall k \in \{0, \dots, n-1\}, \quad y^{(k)}(a) = x^{(k)}(a) = z^{(k)}(a), \\ y^{(k)}(b) = x^{(k)}(b) = z^{(k)}(b), \end{aligned}$$

and the inequalities

$$\forall t \in [a, b], \quad y(t) \leq x(t) \leq z(t).$$

Moreover, $L(D)y$ and $L(D)z$ are of the form $\chi_E \phi_1 + (1 - \chi_E) \phi_2$, where the set E is a finite union of intervals; i.e., y, z are solutions associated to relay controls (see [4]).

Proof. We will only prove the existence of the function y ; similar arguments hold for z . Let $R(t, s) \in \mathcal{C}^{2n-2}([a, b] \times [a, b])$ be the resolvent of the operator L . By Proposition 2, there exists $\delta > 0$ such that the functions

$$h_i'(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(t, s), \quad 1 \leq i \leq n,$$

satisfy condition \mathcal{A} on $[a, a + \delta]$ for each t in $[a, a + \delta]$. Choosing δ small enough, we may assume that the operator L possesses property \mathcal{W} on $[a, a + \delta]$.

Suppose first that conditions \mathcal{W} and \mathcal{A} hold in the whole interval $[a, b]$. It is not restrictive to assume that $\phi_1 = 0$, $\phi_2 = \phi \geq 0$, $x(a) = x'(a) = \dots = x^{(n-1)}(a) = 0$. In fact, let x satisfy $L(D)x \in [\phi_1, \phi_2]$. Then, if we set

$$x_a(t) = x(a) + \frac{x'(a)}{1!} (t-a) + \dots + \frac{x^{(n-1)}(a)}{(n-1)!} (t-a)^{n-1},$$

the function \bar{x} defined by $\bar{x} = x - x_a$ verifies

$$L(D)\bar{x} \in [\psi_1, \psi_2], \quad \forall k \in \{0, \dots, n-1\}, \bar{x}^{(k)}(a) = 0,$$

where $\psi_i = \phi_i - L(D)x_a$, $i = 1, 2$. Clearly the function \bar{x} defined by

$$\bar{x}(t) = \bar{x}(t) - \int_a^t R(t, s) \psi_1(s) ds$$

satisfies

$$L(D)\bar{x} = L(D)\bar{x} - \psi_1 \in [0, \psi_2 - \psi_1], \quad \forall k \in \{0, \dots, n-1\}, \bar{x}^{(k)}(a) = 0.$$

If we assume that the theorem holds in this situation, there exists a function \bar{y} such that $L(D)\bar{y}$ has the desired form and

$$\begin{aligned} \forall k \in \{0, \dots, n-1\} \quad & \bar{y}^{(k)}(a) = 0, \bar{y}^{(k)}(b) = \bar{x}^{(k)}(b), \\ \forall t \in [a, b], \quad & \bar{y}(t) \leq \bar{x}(t). \end{aligned}$$

It is now easy to check that the function y , defined by

$$y(t) = \bar{y}(t) + \int_a^t R(t, s) \psi_1(s) ds + x_a(t),$$

solves our problem.

We assume now that

$$0 \leq \rho \leq \phi, \quad L(D)x = \rho, \quad \forall k \in \{0, \dots, n-1\}, x^{(k)}(a) = 0,$$

so that, with the notations of Proposition 2, we have

$$\begin{aligned} \forall k \in \{0, \dots, n-1\}, \quad x^{(k)}(t) &= \int_a^t \frac{\partial^k R}{\partial t^k}(t, s) \rho(s) ds \\ &= \int_a^t h_{n-k}'(s) \rho(s) ds. \end{aligned}$$

Let $(\rho_m)_{m \in \mathbb{N}}$ and $(\phi_m)_{m \in \mathbb{N}}$ be two sequences of continuous functions such that

$$\forall t \in [a, b], 0 < \rho_m(t) < \phi_m(t), \quad \rho_m \xrightarrow{L^1} \rho, \phi_m \xrightarrow{L^1} \phi$$

and set

$$x_m(t) = \int_a^t R(t, s) \rho_m(s) ds.$$

Clearly,

$$\begin{aligned} \forall k \in \{0, \dots, n-1\}, \quad x_m^{(k)}(t) &= \int_a^t \frac{\partial^k R}{\partial t^k}(t, s) \rho_m(s) ds \\ &= \int_a^t h_{n-k}^t(s) \rho_m(s) ds. \end{aligned}$$

Since each ϕ_m is positive then the functions

$$f_i(s) = h_i^b(s) \phi_m(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(b, s) \phi_m(s), \quad 1 \leq i \leq n,$$

satisfy condition \mathcal{A} on $[a, b]$. Then by Theorem 1 applied to f_1, \dots, f_n and $v = \rho_m/\phi_m$, corresponding to each m there exists a unique n -tuple $(\alpha_1^m, \dots, \alpha_n^m)$ such that

$$(\alpha_0^m =) a < \alpha_1^m < \dots < \alpha_n^m < b (= \alpha_{n+1}^m)$$

and if we set

$$E_m^- = \bigcup_{\substack{i \text{ odd} \\ 0 \leq i \leq n}} [\alpha_i^m, \alpha_{i+1}^m], \quad y_m(t) = \int_a^t R(t, s) \phi_m(s) \chi_{E_m^-}(s) ds$$

then we have

$$\forall i \in \{1, \dots, n\}, \quad \int_a^b f_i(x) \chi_{E_m^-}(s) ds = \int_a^b f_i(s) v(s) ds;$$

i.e.,

$$\forall k \in \{0, \dots, n-1\},$$

$$\int_a^b \frac{\partial^k R}{\partial t^k}(b, s) \phi_m(s) \chi_{E_m^-}(s) ds = \int_a^b \frac{\partial^k R}{\partial t^k}(b, s) \rho_m(s) ds \quad (*)$$

so that

$$\forall k \in \{0, \dots, n-1\}, \quad D^k(y_m - x_m)(a) = D^k(y_m - x_m)(b) = 0$$

and $L(D)(y_m - x_m) = \phi_m \chi_{E_m^-} - \rho_m$ does not vanish on $[a, b]$; since L possesses property W on $[a, b]$, the corollary to Theorem 4 then implies that

$$\forall t \in]a, b[, \quad y_m(t) \neq x_m(t). \quad (**)$$

Since by construction,

$$\begin{aligned} y_m^{(n)}(a) &= L(D)y_m(a) - (a_{n-1}(a)D^{n-1}y_m(a) + \dots + a_1(a)Dy_m(a) \\ &\quad + a_0(a)y_m(a)) = 0 \end{aligned}$$

and, analogously, $x_m^{(n)}(a) = \rho(a) > 0$, then by continuity $y_m < x_m$ in a neighbourhood $]a, a + \varepsilon[$ of a , which together with (**), yields the global inequality

$$\forall t \in]a, b[, \quad y_m(t) < x_m(t). \quad (***)$$

By compactness we may assume that

$$\forall i \in \{1, \dots, n\}, \quad \lim_{m \rightarrow \infty} \alpha_i^m = \alpha_i,$$

Clearly,

$$(\alpha_0 =)a \leq \alpha_1 \leq \dots \leq \alpha_n \leq b (= \alpha_{n+1}).$$

Put

$$E^- = \bigcup_{\substack{i \text{ odd} \\ 0 \leq i \leq n}} [\alpha_i, \alpha_{i+1}], \quad y(t) = \int_a^t R(t, s) \phi(s) \chi_{E^-}(s) ds.$$

Obviously, for all k in $\{0, \dots, n-1\}$, $y^{(k)}(a) = 0$. Moreover, by passing through the limit in (*) and (***), we obtain

$$x^{(k)}(b) = \int_a^b \frac{\partial^k R}{\partial t^k}(b, s) \rho(s) ds = \int_a^b \frac{\partial^k R}{\partial t^k}(b, s) \phi(s) \chi_{E^-}(s) ds = y^{(k)}(b)$$

for all k in $\{0, \dots, n-1\}$ and

$$\forall t \in [a, b], \quad y(t) \leq x(t).$$

The function y solves our problem.

The General Case. Since the functions

$$h'_i(s) = \frac{\partial^{n-i} R}{\partial t^{n-i}}(t, s), \quad 1 \leq i \leq n,$$

satisfy locally condition Δ and the operator $L(D)$ possesses property W locally then by compactness there exists a subdivision

$$c_0 = a < c_1 < \dots < c_l < b = c_{l+1}$$

of $[a, b]$ such that properties W and Δ hold on each interval $I_j = [c_j, c_{j+1}]$, $0 \leq j \leq l$. Let x solve $L(D)x = \rho$, $\phi_1 \leq \rho \leq \phi_2$; by the first case, for each j in $\{0, \dots, l\}$ there exists a function y_j in $W^{n,1}(I_j)$ such that $L(D)y_j$ has the desired form,

$$\forall k \in \{0, \dots, n-1\}, \quad y_j^{(k)}(c_j) = x^{(k)}(c_j), \quad y_j^{(k)}(c_{j+1}) = x^{(k)}(c_{j+1}),$$

and

$$\forall t \in I_j, \quad y_j(t) \leq x(t).$$

The function $y \in W^{n,1}([a, b])$ obtained by gluing together the functions y_0, \dots, y_l is a solution to our problem. ■

Remark. The proof of the theorem shows that if there exist n solutions h_1, \dots, h_n to $L(D)y = 0$ on $[a, b]$ satisfying

$$W(h_1) \neq 0, \dots, W(h_1, \dots, h_n) \neq 0 \quad \text{on } [a, b]$$

then the resolvent of the operator L satisfies condition Δ on $[a, b]$ and therefore the bang-bang solutions y and z can be built in such a way that $L(D)y$ and $L(D)z$ are of the form $\chi_E \phi_1 + (1 - \chi_E) \phi_2$, where the characteristic function of the set E has less than n discontinuity points on $[a, b]$. For instance, this is the case when $L(D) = D^n$ (see Example 2 following the definition of condition Δ) or when

$$L(D) = D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0$$

and the algebraic equation

$$X^n + a_{n-1}X^{n-1} + \dots + a_1X + a_0 = 0$$

has n distinct real roots.

THE REACHABLE SET OF BANG-BANG CONSTRAINED SOLUTIONS

Let $a_0, \dots, a_{n-1} \in \mathcal{C}^{n-2}([a, b])$ and ϕ_1, ϕ_2 in $L^1([a, b])$ verify $\phi_1 \leq \phi_2$. Consider the control problem (P_a)

$$\begin{aligned} L(D)x &= x^{(n)} + a_{n-1}(t)x^{(n-1)} + \dots + a_1(t)x' \\ &+ a_0(t)x \in [\phi_1, \phi_2] \quad \text{a.e. on } [a, b] \end{aligned}$$

with the initial conditions

$$\forall k \in \{0, \dots, n-1\}, \quad x^{(k)}(a) = x_k,$$

where x_0, \dots, x_{n-1} are n fixed real numbers. Let c be an arbitrary function defined on $I = [a, b]$ and consider the reachable sets \mathcal{X}_I^c and \mathcal{Y}_I^c associated to (P_a) defined by

$$\begin{aligned} \mathcal{X}_I^c &= \{(x(b), x'(b), \dots, x^{(n-1)}(b)) \\ &: \forall t \in I \ x(t) \leq c(t), \ x \text{ solution to } (P_a)\}, \\ \mathcal{Y}_I^c &= \{(y(b), y'(b), \dots, y^{(n-1)}(b)) \\ &: \forall t \in I \ y(t) \leq c(t), \ y \text{ bang-bang solution to } (P_a)\}. \end{aligned}$$

Then Theorem 5 yields the following result.

THEOREM 6. *The sets \mathcal{X}_I^c and \mathcal{Y}_I^c coincide; in particular, the reachable set associated to the bang-bang constrained solutions \mathcal{Y}_I^c is convex.*

AN APPLICATION TO THE CALCULUS OF VARIATIONS

THEOREM 7. *Let $a_0, \dots, a_{n-1} \in \mathcal{C}^{n-2}([a, b])$, $\phi_1, \phi_2 \in L^1([a, b])$ verify that $\phi_1 \leq \phi_2$ and let L be the linear differential operator of order n defined by*

$$L(D) = D^n + a_{n-1}(t)D^{n-1} + \dots + a_1(t)D + a_0(t).$$

Let x_0^1, \dots, x_{n-1}^1 and x_0^2, \dots, x_{n-1}^2 be $2n$ fixed real numbers. Then there exists a dense subset \mathcal{D} of $\mathcal{C}(\mathbb{R})$ for the uniform convergence such that for g in \mathcal{D} the problem

$$\begin{aligned} \min \left\{ \int_a^b g(x(t)) dt + \int_a^b h(L(D)x(t)) dt : x \in W^{n,1}([a, b]), \right. \\ \left. \forall k \in \{0, \dots, n-1\}, \ x^{(k)}(a) = x_k^1, \ x^{(k)}(b) = x_k^2 \right\} \end{aligned}$$

admits at least one solution for every lower semicontinuous function h satisfying the growth condition $h(u) \geq c\psi(|u|)$, ψ being l.s.c. and convex, $\lim_{r \rightarrow +\infty} \psi(r)/r = +\infty$.

Proof. With our Theorem 5 and the preceding application, the proof is a direct adaptation of the proof given in [6] for the case $L(D) = D$. ■

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