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Plane stress plasticity in periodic composites

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Abstract

In this paper a homogenised constitutive relation for the global behaviour of periodic composite structures is derived in the case of elasto-plastic material components. In principle, the representative volume element could be affected by any kind of non-linear material behaviour respecting the complementarity rule, therefore special emphasis is put on the description of the generality of the algorithm. The method is currently restricted to small strains, plane problems and monotonic proportional loading conditions.

1. Introduction

Modern technical applications require more and more the use of artificial heterogeneous materials which may be characterised by good mechanical properties (for example high stiffness and low weight) or may be designed to satisfy special technological purposes [1]. Often these composite materials are periodic, i.e. if a mechanical or geometric property a (for example the constitutive tensor) of a periodic body Ω (see Fig. 1) is taken into consideration it is possible to write

$$\text{if } \mathbf{x} \in \Omega \text{ and } (\mathbf{x}+\mathbf{Y}) \in \Omega \Rightarrow a(\mathbf{x}+\mathbf{Y}) = a(\mathbf{x}). \tag{1}$$

In Eq. (1) $\mathbf{Y}$ is the (geometric) period of the structure.

The characteristic size of the single cell of periodicity is assumed much smaller than the geometrical dimensions of the structure: a classical example of periodic structure is given by masonry walls but many other modern applications utilise periodic composite structures. To reduce the enormous computational cost required by a finite element discretization of such kind of structures, some homogenisation techniques were introduced, with acceptable results, to solve linear problems [2–6]. Those methods are based on the possibility of writing an asymptotic expansion of the mechanical quantities in two variables linked with two different length scales [7]: the macroscopic scale (called X in Fig. 1), relative to the whole structure, in which the dimensions of the heterogeneities are very small, and the microscopic scale (called Y in Fig. 1), relative to the single cell of periodicity, which is the scale of the heterogeneities.

The asymptotic expansions of the displacement, strain and stress fields in the two scale variables, are introduced in the governing equations. In this way a set of independent differential problems can be defined and each of them can be solved. The solution of the original linear problem can be found by superposing the solutions of the independent problems. In the simplest case, when only the first order terms are retained, displacement,

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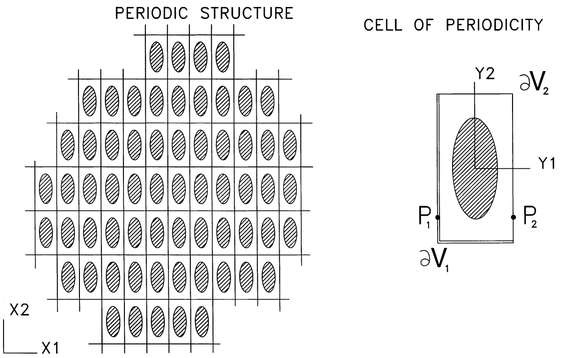

Fig. 1. Periodic structure (macroscopic reference (X1, X2)) and single cell of periodicity (microscopic reference (Y1, Y2)) and periodic boundary conditions.

strain and stress distributions, in heterogeneous media, can be treated as the sum of two contributions: an averaged term function of X, regarding the global structure, and a periodic fluctuating contribution function of Y and with zero mean value in the cell of periodicity, regarding the local heterogeneity. In more advanced numerical applications the first two or three terms of the expansion are taken into account [8]. This approach, based on the principle of superposition, is not applicable in the case of non-linear problems.

This paper aims at defining a homogenised constitutive relation for the global non-linear behaviour of periodic composite bodies. The proposed method allows for the description of a large class of different constitutive behaviours, all those which obey the complementarity rule.

The method is applicable with the following restrictions:

plane situations;
monotonic proportional loading;
small strains;

but we believe that it can be extended to more generic situations. The approach presented in this paper is clearly different from those presented in literature [9–13].

2. Basic concepts

Local quantities are indicated with lower case letters while global quantities are indicated with capital letters. In particular: σ_{ij} is the local (or microscopic) stress tensor, e_{ij} the local (or microscopic) strain tensor, Σ_{ij} the global (or macroscopic) stress tensor, E_{ij} the global (or macroscopic) strain tensor.

The asymptotic analysis shows that global stresses and global strains are equal to the mean values of the corresponding local quantities [7]

$$\Sigma_{ij} = \frac{1}{V}\int_V \sigma_{ij}\,\mathrm{d}V, \tag{2a}$$

$$E_{ij} = \frac{1}{V}\int_V e_{ij}\,\mathrm{d}V, \tag{2b}$$

where V is the volume of the cell of periodicity.

Eqs. (2a) and (2b) define macroscopic stresses and strains which are assumed to be linked by a macroscopic constitutive law. Such a law can be constructed starting from the constitutive relations of the single components and the geometry of the unit cell.

We assume that for each constituent of the composite body the stress field is constrained by the usual relation

$$\boldsymbol{\sigma}(\mathbf{y}) \in \mathbf{P}(\mathbf{y}), \tag{3}$$

where $\mathbf{P}(\mathbf{y})$ is the set of stress states that the material can admit in the six-dimensional space of stresses (or in the three-dimensional space of principal stresses). $\mathbf{P}(\mathbf{y})$ depends on the single material and hence on the position $\mathbf{y}$ in the cell of periodicity.

In many cases $\mathbf{P}(\mathbf{y})$ is defined by means of a yield function $f(\mathbf{y},\boldsymbol{\sigma})$

$$\mathbf{P}(\mathbf{y}) = \{\boldsymbol{\sigma} | f(\mathbf{y},\boldsymbol{\sigma}) \leqslant 0\} \tag{4}$$

In the case of metals $\mathbf{P}(\mathbf{y})$ assumes the von Mises form

$$\mathbf{P}(\mathbf{y}) = \left\{\boldsymbol{\sigma} \left| \sqrt{\frac{3}{2}s_{ij}s_{ij}} \leqslant \sigma_0(\mathbf{y}) \right.\right\} \tag{5}$$

where s_{ij} denotes the deviatoric part of the stress tensor and $\sigma_0(\mathbf{y})$ the yield stress at point $\mathbf{y}$.

Since microscopic stress states $\boldsymbol{\sigma}$ must lie within the set given in Eq. (3), it seems reasonable that all physical macroscopic stress states $\boldsymbol{\Sigma}$ are contained in a *macroscopic* (or *effective*) *domain* $\mathbf{P}^{\text{eff}}$ whose frontier is called *macroscopic* (or *effective*) *extremal yield surface* [7]:

$$\Sigma \in \mathbf{P}^{\text{eff}}. \tag{6}$$

The constitutive behaviour of the composite material presents an initial range in which there exists a linear relation between global strains and global stresses followed by a non-linear range. In any case the global stresses have to be contained in the effective extremal yield surface. The object of this paper is to find an algorithm to describe the non-linear range of the behaviour.

The constitutive relation between the global stress $\mathbf{\Sigma}$ and the global strain $\mathbf{E}$ is determined taking into account the geometry of the cell of periodicity and the non-linearities due to the elasto-plastic properties of the single constituents. Moreover it is clear that the global behaviour of the cell can present some kind of hardening even if the single components are perfectly plastic because the unit cell is a statically undetermined structure.

3. Method

3.1. Boundary conditions

The basic idea of the method presented in this paper is to assume the unit cell as mechanical element which determines the global constitutive law of the material. Therefore if the relation which links $\mathbf{\Sigma}$ to $\mathbf{E}$ in the unit cell is known the global constitutive behaviour of the composite material is known and given by the relation $\mathbf{\Sigma} = \mathbf{\Sigma}(\mathbf{E})$. If we want to define the constitutive behaviour of the material in the case of monotonic proportional loading we may decide to simulate a large number of different loading paths on the unit cell in such a way that any loading case of the same class can be approximated reasonably well by an interpolation between some paths previously simulated. There are several approximations involved: first of all the unit cell on which the different cases are simulated is constrained in a way which cannot represent all the possible 'in situ' conditions. There are three classical boundary conditions applied to the RVE in the composite material literature:

(1) uniform strains on ∂V:

$$u_i = E_{ij} y_j, \tag{7}$$

(2) uniform stresses on ∂V:

$$\sigma_{ij} n_j = \Sigma_{ij} n_j, \tag{8}$$

(3) periodic boundary conditions on ∂V:

$$u_i = E_{ij} y_j + u_i^* \quad u_i^* \text{ periodic on } \partial V, \tag{9}$$

$$\sigma_{ij} n_j \text{ anti-periodic on } \partial V. \tag{10}$$

In this last case the surface of the unit cell is decomposed in two parts

$$\partial V = \partial V_1 + \partial V_2 \tag{11}$$

and each point $P_1 := P_1 \in \partial V_1$ has a corresponding point $P_2 := P_2 \in \partial V_2$ (see Fig. 1). Boundary conditions (1) and (2) are justified when the RVE is large with respect to the heterogeneity size in such a way that the fluctuations of the stress tensor σ_{ij} or of the displacement vector u_i occur with a wavelength small compared with the dimensions of the RVE. This is not the case for periodic composite materials, for which the boundary conditions (3) are generally preferable [7]. However it is apparent that these boundary conditions can adequately represent the in situ configuration only in a region of the structure far from the real boundary of the structure where arbitrary restraint conditions are in general applied.

If we consider now two corresponding points P_1, P_2 of the unit cell, Eq. (9) implies the following relation between their total displacements:

$$u_i(P_2) - u_i(P_1) = E_{ij}(y_j(P_2) - y_j(P_1)). \tag{12}$$

Eq. (12) expresses a restraint condition which can be easily imposed in a finite element code to take into account the periodic boundary conditions.

3.2. Numerical experiments

Given a unit cell on which the periodic boundary conditions (12) are imposed, the problem to be solved is to find the relation

$$\mathbf{\Sigma} = \mathbf{\Sigma}(\mathbf{E}) \tag{13}$$

which will be assumed as constitutive law of the homogenised material. The unknown relation (13) is numerically obtained by solving a 'large' number of 'local problems' given on the unit cell [14,15]

$$\begin{cases} \text{microscopic constitutive laws} \\ \operatorname{div} \boldsymbol{\sigma} = 0 \text{ micro-equilibrium} \\ E_{ij} = \dfrac{1}{V}\displaystyle\int_V \varepsilon_{ij}\, \mathrm{d}V = (\alpha_0 + \alpha_1 t)E^0_{ij} \text{ given.} \end{cases} \tag{14}$$

A global strain tensor E^0_{ij} is imposed to the cell and it is monotonically increased to generate a kinematic loading path: this means that, numerically, a large number of equal kinematic steps is applied to the unit cell. The homogenised stress tensor Σ_{ij} is then computed, by means of Eq. (2a) for each step of the load history. The sequence of steps characterised by a fixed E^0_{ij}, generates a sequence of points in the stress space. Therefore we have one point, in the stress space, for each load step. These points are called *interpolation points*: here the behaviour of the homogenised material (and precisely the value of the homogenised strains E_{ij} and stresses Σ_{ij} is known.

Repeating the procedure for several different given tensors E^0_{ij} we know the behaviour of the homogenised material in a discrete number of points and for a discrete variety of load situations.

At this point we introduce a simplifying hypothesis: we assume that all the interpolation points, on different loading paths, characterised by the same number of steps, are on the same 'plastic surface', i.e. they are labelled by the same value of an internal variable k. In this manner by connecting points relative to the corresponding steps of different loading paths it is possible to construct a series of 'plastic surfaces' generated by the numerical experiments (see Fig. 2).

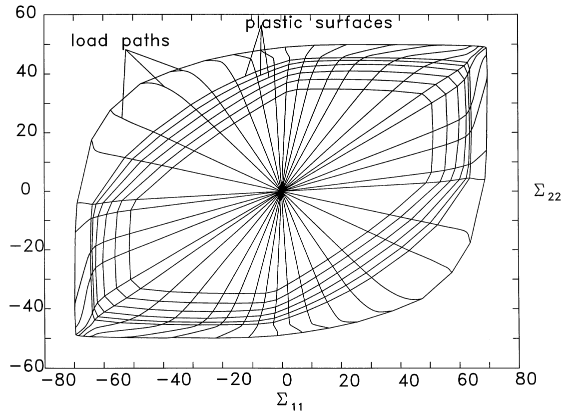

Fig. 2. Load paths and global yield surfaces.

Remark 3.1. It is possible to show that in the case of symmetric unit cells the periodic displacement u^* is zero on the corners of the cell. Therefore the global strain tensor E_{ij} can be imposed by applying, to the corners, the displacements $u_i = E_{ij}y_j$ and, to the other points, of the borders the periodic boundary conditions.

Remark 3.2. The definition of 'plastic surfaces' given in this paragraph might be misleading. These surfaces are simply the geometrical locus of points which are in some sense 'equivalent' along different loading paths [14,15]. They do not limit a region of the stress space in which the behaviour of the material is linear: in fact, in the case of more general loading conditions, starting from a point C on the mth plastic surface along the nth loading path and reversing the kinematic loading, the material behaves linearly in an interval smaller than the 'diameter' of the mth plastic surface in the direction of the nth loading path. In Fig. 3 we show a generic load case: starting from an unloaded cell (point A in the stress space) a displacement which brings the cell to the first yielding (point B) and to a generic level of yielding

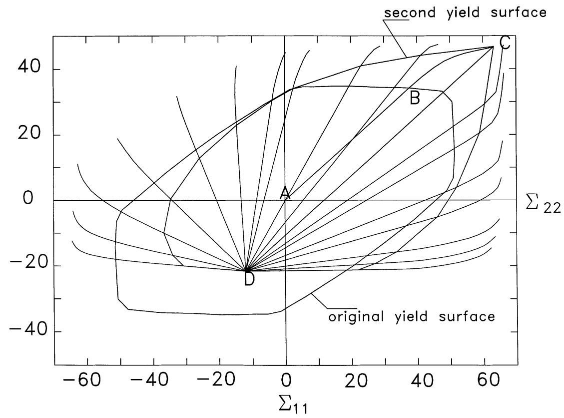

Fig. 3. A generic load case – isotropic and kinematic hardening.

(point C) is applied, then a reverse load is applied up to the point D where a new yielding occurs.

Starting from point D a number of load paths is applied again to the cell so that a new surface is built. It is apparent that the new elastic domain has changed shape, size and position in the stress space, therefore a very complex hardening mechanism can be inferred.

This indicates that there exists a kinematic component of the global hardening which is neglected by our method. The plastic surfaces previously described could be interpreted as effective yield surfaces in the case of isotropic hardening with no kinematic hardening, but this hypothesis does not reflect the behaviour of the material, this is also the reason why our method cannot be applied to more general loading conditions.

Remark 3.3. If the number of loading paths is sufficiently large it is reasonable to assume that the final point of the loading path lies in a position very close to the extremal yield surface [7]. Since, if the material components are characterised by an associative flow rule, it is possible to show that the plastic strain increment is normal to the extremal yield surface, the method presented in this paper may also be used to find such a surface. In fact it gives position and orientation of the surface in a discrete number of points. The method may fail in the case of surfaces with corners because in such a case the corner would attract a set of loading paths, all those characterised by a strain increment oriented along a direction internal to the cone of the surface corner [16].

Remark 3.4. The associative plasticity of the components implies the normality rule for the extremal yield surface but, to the best of our knowledge, it does not give any information on the global elasto-plastic behaviour which precedes that surface. Therefore the elasto-plastic algorithm should not rely on any kind of normality but it should be able to cope with both associative and non-associative flow rules. This is also true because the shape of any intermediate yield surface is not known and therefore the definition of the flow rule has to be independent of it.

3.3. Elasto-plastic constitutive law

The behaviour of the material is linear up to the most interior curve of Fig. 2, flow rule and hardening law have to be defined for the homogenised material beyond it. Each point in the space Σ_{11}–Σ_{22}–Σ_{12} in the elasto-plastic range, can be indicated by the values of three variables: a value of k and the values of the ratios E_{11}/E_{22} and E_{11}/E_{12}. By means of such values we define a flow rule and a hardening law for the effective material. Since the stress in the material components is limited also the global stress has to be contained in a region of the space Σ_{11}–Σ_{22}–Σ_{12}. For the homogenised elasto-plastic material this *consistency condition* is assumed as

$$\mathbf{\Sigma}(\mathbf{x}) \in \mathbf{S}^{\mathrm{eff}}(\mathbf{x}), \tag{15}$$

where $\mathbf{S}^{\mathrm{eff}}$ is the effective elastic domain. As discussed in Remark 3.2, the size, shape and position of $\mathbf{S}^{\mathrm{eff}}$ in the stress space continuously varies in an unknown manner. In a way similar to the homogeneous case we assume that $\mathbf{S}^{\mathrm{eff}}$ is contained in a yield surface

$$\mathbf{S}^{\mathrm{eff}}(\mathbf{x}) = \{\mathbf{\Sigma} | f(\mathbf{\Sigma}) \leqslant \mathbf{\Sigma}_0(\mathbf{x})\} \tag{16}$$

$\mathbf{\Sigma}_0(\mathbf{x})$ is the global yield stress. Once the function $f(\mathbf{\Sigma})$ has been defined, the yield stress $\mathbf{\Sigma}_0(\mathbf{x})$ is known at the interpolation points and its value, in a generic point of the stress space, is obtained by linear interpolation among the eight points in the stress space at the corners of the region (*patch*) where the current stress state lies (see Fig. 4). For the sake of simplicity Fig. 4 and the following Fig. 5 are drawn in the plane Σ_{11}–Σ_{22} but the procedure has been developed for the full plane stress case and therefore a three-dimensional interpolation is carried out. In principle it would seem possible to compute a six-dimensional interpolation for a completely generic stress case.

The *flow rule* is written in the usual form:

$$\mathbf{R} := -\mathbf{\Sigma}_{\mathrm{tr}} + \mathbf{\Sigma}_i + \dot{k}\ \mathbf{Dm} = \mathbf{0}, \quad \dot{k} \geqslant 0, \tag{17}$$

where $\mathbf{D}$ is the elastic constitutive tensor obtained with the elastic homogenisation [2,3], $\dot{k}$ the increment of the plastic flow, $\mathbf{m}$ the 'flow direction', $\mathbf{\Sigma}_{\mathrm{tr}}$, $\mathbf{\Sigma}_i$ the global stresses.

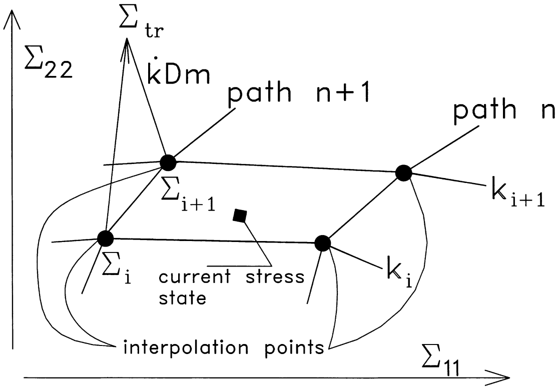

Fig. 4. Interpolation of the stresses and of the flow direction.

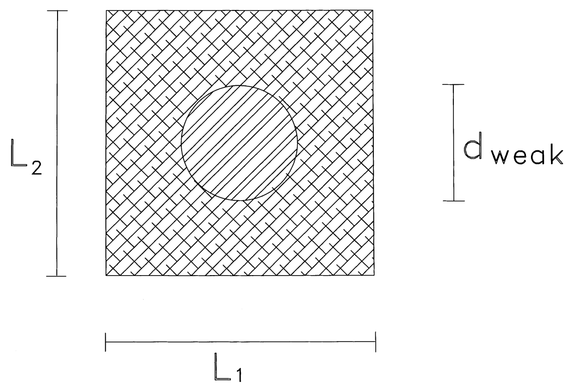

Fig. 5. Geometrical data of the cell of periodicity.

It is clearly possible to evaluate the flow direction at each interpolation point in the following way (see Fig. 4): starting from any interpolation point at the level $(i-1)$ one multiplies the fixed strain increment $\Delta\mathbf{E}$ by the elastic effective tensor $\mathbf{D}$, in this way the trial stress $\mathbf{\Sigma}_{tr}$ is computed. Since there are some plastic deformations in some part of the unit cell, the strain $\Delta\mathbf{E}$ generates the new stress $\mathbf{\Sigma}_i$. The flow direction at the interpolation point at the level $(i-1)$ can be computed as

$$\dot{k}\mathbf{Dm} = \mathbf{\Sigma}_{tr} - \mathbf{\Sigma}_i$$
$$\Rightarrow \mathbf{m} = \frac{1}{\dot{k}}\mathbf{D}^{-1}[\mathbf{\Sigma}_{tr} - \mathbf{\Sigma}_i] = \frac{1}{\dot{k}}[\Delta\mathbf{E} - \mathbf{D}^{-1}\mathbf{\Sigma}_i]. \qquad (18)$$

The value of $\dot{k}$ has been arbitrarily chosen to be 0.1, hence the quantity $\mathbf{m}$ does not give only an information about the flow direction but also about the entity of the plastic flow.

Remark 3.5. For $\mathbf{m}$ the interpolated value is taken and not the derivative of the consistency condition, so that, with this method, both associative and non-associative plasticity can be indifferently taken into account. Once consistency condition and flow rule have been determined, the global constitutive law is fully defined and can be assumed as constitutive law of the homogenised material.

3.4. Global solution

With the global constitutive law determined, a global problem can be solved: given a macroscopic structure composed of a large number of unit cells and subjected to a given system of external loads $\mathbf{F}$ and boundary conditions the relevant displacements can be found solving the problem

$$\begin{cases} \text{macroscopic constitutive law} \\ \operatorname{div} \mathbf{\Sigma} = \mathbf{F} \text{ macro-equilibrium} \\ \text{global boundary conditions.} \end{cases} \qquad (19)$$

The unknown of the problem (19) is the global displacement field on which the global stresses $\mathbf{\Sigma}$ depend.

3.5. Stress recovery

Finally the global solution is used to evaluate the local distribution of micro-stresses by solving a local problem (14) in which the imposed global strains are the ones computed in the solution of the problem (19): the macro-strains computed at the Gauss points of the homogenised structure are imposed to the unit cell and the micro-stresses are obtained by the solution of the corresponding local problem.

4. Validation

In this paragraph we carry out a computation adopting the same discretization for both the heterogeneous and the homogeneous case. In this way we evaluate the error caused by all the approximations involved in the method. We apply the procedure described in the preceding paragraphs to the cell of periodicity of Fig. 5. The quadrangular cell is composed of two different materials: a weak matrix and a strong inclusion. The two materials are, in this case, isotropic and elastic perfectly-plastic with von Mises associate plasticity.

Geometric data:
$L_1 = 20$ mm, $L_2 = 20$ mm, $d_{weak} = 8$ mm.
Mechanical characteristics

Inclusion (strong):	Matrix (weak):
$E = 2.1 \times 10^5$ MPa	$E = 2.1 \times 10^4$ MPa
$v = 0.18$	$v = 0.18$
$\sigma_Y = 220$ MPa	$\sigma_Y = 100$ MPa

In Fig. 6(a–c) we show three numerical experiments which corresponds to the following cases:

Case 1	Case 2	Case 3
(Fig. 6(a))	(Fig. 6(b))	(Fig. 6(c))
$E_{11} \neq 0$	$E_{11} = 0$	$E_{12} = 0$
$E_{22} = 0$	$E_{22} \neq 0$	$E_{22} = 0$
$E_{12} = 0$	$E_{12} = 0$	$E_{12} \neq 0$

In Fig. 7 the load paths and the first five surfaces are represented in the plane Σ_{11}, Σ_{22}.

Fig. 8 shows a periodic structure with 15 cells. The single cell is equal to that of Fig. 5. The vertical displacements of the bottom edge and the horizontal displacement of the bottom left corner are restrained to zero. A constant distribution of

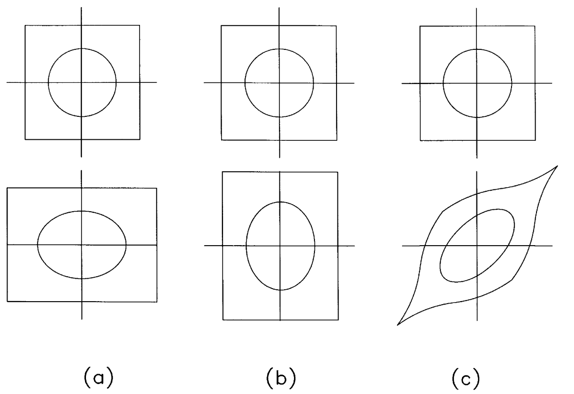

Fig. 6. (a) Numerical experiment with $E_{11} \neq 0$, $E_{22} = 0$, $E_{12} = 0$; (b) Numerical experiment with $E_{11} = 0$, $E_{22} \neq 0$, $E_{12} = 0$; (c) Numerical experiment with $E_{11} = 0$, $E_{22} = 0$, $E_{12} \neq 0$.

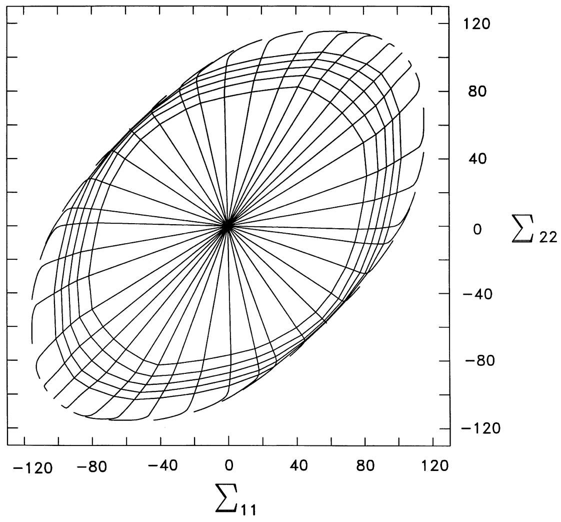

Fig. 7. Load paths and global yield surfaces.

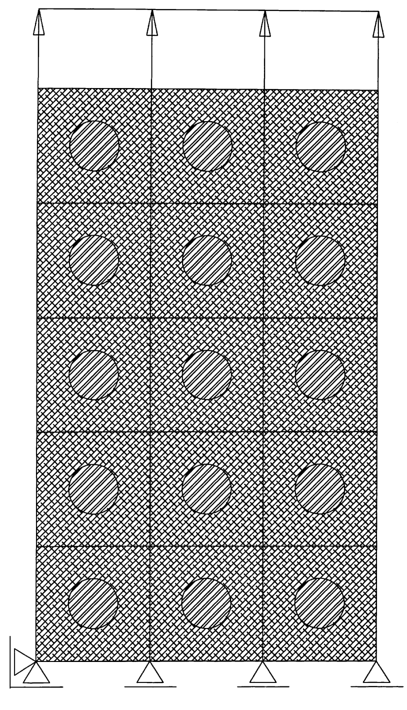

Fig. 8. Geometry and boundary conditions of the periodic structure.

monotonically growing vertical displacements is applied to the top edge. The problem is solved using a discretization with 160 nodes and 135 elements with the numerical constitutive law, described in the previous sections, which takes into account the non-linearities due to the elasto-plastic behaviour of the single materials.

The sum of the vertical reactions at the bottom edge is compared to that of an equivalent model with a finite element discretization, with the same number of nodes and elements, which describes the real material distribution and the real mechanical characteristics of the single components. The comparison between the vertical reactions is shown in Fig. 9 where we can note that the error induced by the homogenisation procedure never overcomes the value of 5%.

5. Numerical example

Fig. 10 shows a periodic structure with 48 cells. The single cell is rectangular. It is described in Fig. 11 and has the following characteristics.

Geometric data:

$l_1 = 30$ mm,	$t_{\text{steel}} = 3$ mm
$l_2 = 21.3$ mm,	$t_{\text{epoxy}} = 0.9$ mm

Mechanical characteristics:

Steel:	Epoxy:
$E = 2.1 \times 10^5$ MPa	$E = 2.1 \times 10^4$ MPa
$v = 0.18$	$v = 0.18$
$\sigma_Y = 220$ MPa	$\sigma_Y = 100$ MPa

The vertical displacements of the bottom edge and the horizontal displacement of the bottom right corner of the periodic structure are restrained to zero. A linear distribution of vertical displacements is applied to the top edge: the absolute value of ratio between the left and the right displacement is equal to 4. The problem is solved using a rough discretization (63 nodes and 48 elements) with the numerical constitutive law previously defined.

The vertical reactions at the bottom edge are compared to those of an equivalent model with a finite element discretization which describes the real material distribution and the real mechanical characteristics of the single components. This

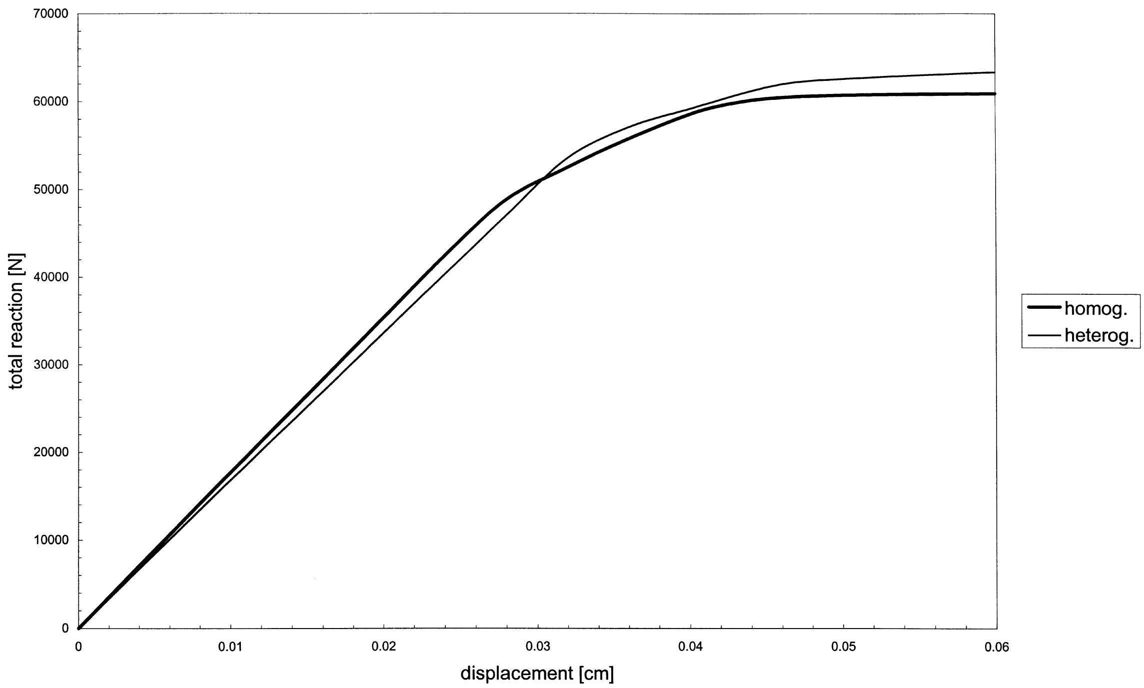

Fig. 9. Comparison between vertical reactions of homogeneous and heterogeneous model.

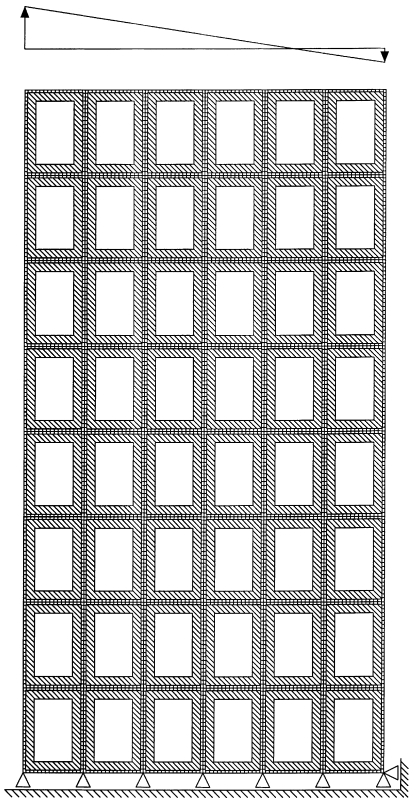

Fig. 10. Geometry and boundary conditions of the periodic structure.

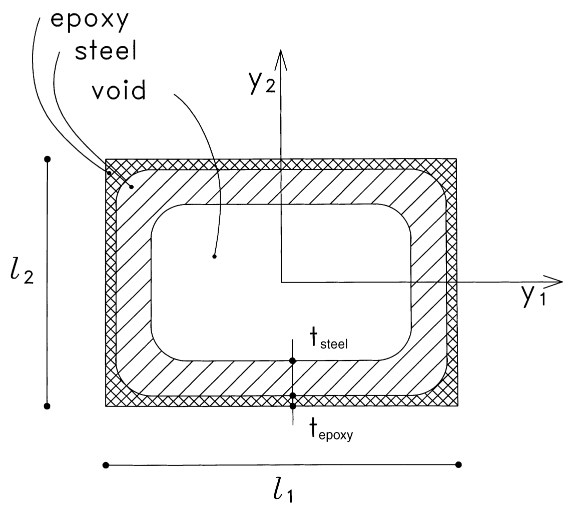

Fig. 11. Geometrical data of the cell of periodicity.

discretization consists of 13,122 nodes and 9492 elements. In Fig. 12(a) and (b) we show the discretization of the heterogeneous and of the homogenised model. The comparison between the vertical reactions in the two models is shown in Fig. 13 (the reactions of the heterogeneous model are integrated along the restrained boundary in order to have quantities comparable with those obtained with the homogeneous model).

6. Conclusions

A homogenised constitutive relation for periodic composite media with non-linear material components has been defined. The method is valid for small strains and for plane problems with monotonic proportional loading. Such a procedure allows for considerable reduction of the computational effort needed for finite element analyses of real composite structures and it is applicable both to associative and non-associative plasticity. Moreover it does not assume any a priori form of yield surface or hardening mechanism but it closely follows the behaviour of the material. The main drawbacks is the large number

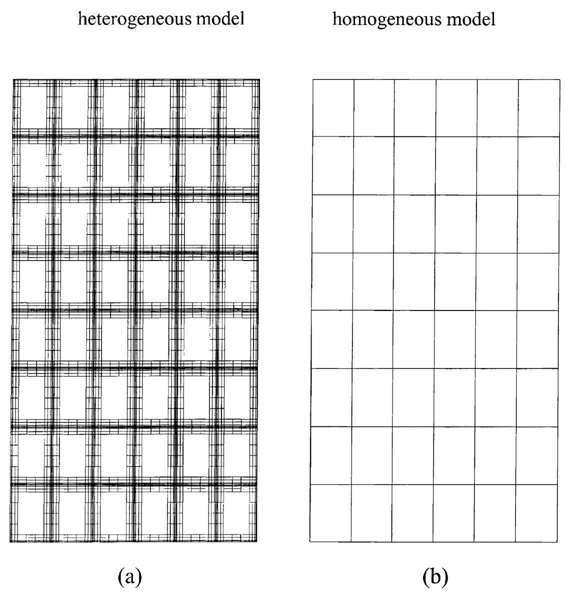

Fig. 12. (a) Heterogeneous model; (b) Homogeneous model.

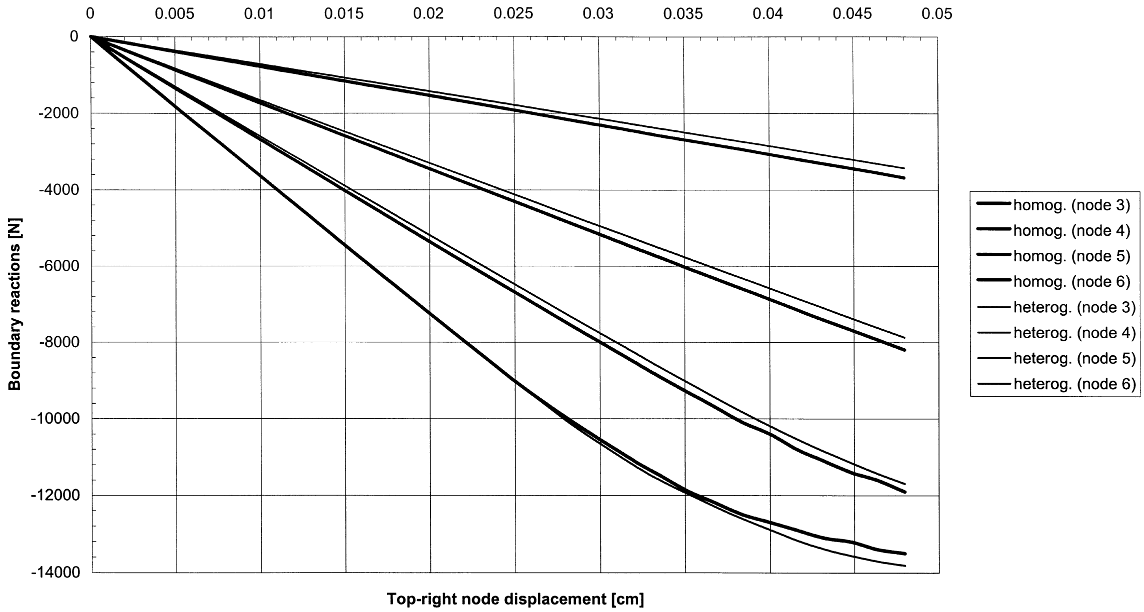

Fig. 13. Comparison between vertical reactions.

of numerical experiments required to obtain a sufficiently dense grid of interpolation points and the hypothesis of monotonic proportional loading which restricts the possibility of application of the proposed algorithm.

Acknowledgements

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