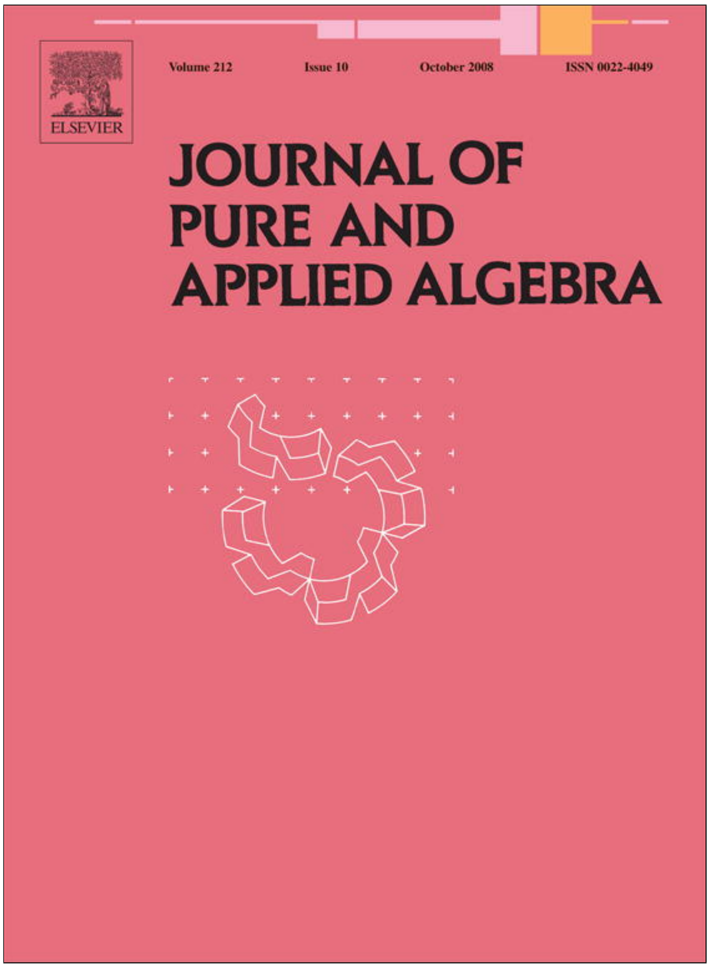
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The class semigroup of local one-dimensional domains☆

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ABSTRACT

Let R be a local one-dimensional domain. We investigate when the class semigroup $\mathcal{S}(R)$ of R is a Clifford semigroup. We make use of the Archimedean valuation domains which dominate R, as a main tool to study its class semigroup. We prove that if $\mathcal{S}(R)$ is Clifford, then every element of the integral closure $\bar{R}$ of R is quadratic. As a consequence, such an R may be dominated by at most two distinct Archimedean valuation domains, and $\bar{R}$ coincides with their intersection. When $\mathcal{S}(R)$ is Clifford, we find conditions for $\mathcal{S}(R)$ to be a Boolean semigroup. We derive that R is almost perfect with Boolean class semigroup if, and only if R is stable. We also find results on $\mathcal{S}(R)$, through examination of $[V/P : R/\mathfrak{M}]$ and $v(\mathfrak{M})$, where V dominates R, and P, $\mathfrak{M}$ are the respective maximal ideals.

0. Introduction

The class semigroup $\mathcal{S}(R)$ of a (commutative) integral domain R is a natural notion, derived from the classical definition of class group of R. Namely, $\mathcal{S}(R)$ is the multiplicative semigroup of the nonzero fractional ideals of R, modulo the subgroup of the principal ones. Equivalently, $\mathcal{S}(R)$ consists of the isomorphy classes of the nonzero ideals of R, endowed with the naturally induced multiplication.

In recent years, several papers have been dedicated to the investigation of the instance where the class semigroup is a Clifford semigroup, thus having a particularly attractive structure. However, this problem was first studied, using a somehow different terminology, in an old paper by Dade, Taussky and Zassenhaus [11], which was unnoticed by the authors of more recent papers. The author thanks F. Halter-Koch for drawing his attention to [11], where, *inter alia*, it is proved that orders in quadratic number fields have Clifford class semigroups, but this property is no longer true for field extensions of $\mathbb{Q}$ of degree larger than 2. Zanardo and Zannier [34] re-proved these facts, and added results on the structure of $\mathcal{S}(R)$. Bazzoni and Salce [7] proved that, if R is a valuation domain, then $\mathcal{S}(R)$ is a Clifford semigroup, and determined its structure. Thereafter, Bazzoni [4–6] made a thorough investigation on this topic. Kabbaj and Mimouni [17] studied the case when $\mathcal{S}(R)$ is Boolean. Franz Halter-Koch [14] extended the investigation of [11], considering the semigroup of lattices over Noetherian one-dimensional domains, and their corresponding class semigroups. We recall one main result in [6], namely, an integrally closed domain R has Clifford class semigroup if and only if R is a Prüfer domain of finite character.

The purpose of the present paper is to investigate conditions for the class semigroup of a local one-dimensional domain R to be a Clifford semigroup, or, more generally, for an element of $\mathcal{S}(R)$ to be regular, in the Van Neumann sense. Of course, in view of the above recalled result by Bazzoni [6], we are interested in the case when R is not integrally closed.

One motivation for confining ourselves to the local one-dimensional case is the recently acknowledged importance of the class of the so-called almost perfect domains. Let us recall that a commutative ring R is said to be *almost perfect* if R/I is perfect (in the sense of Bass [2]) for every nonzero ideal I of R. The first investigation of almost perfect local integral domains was made by Smith [32], under the name of "domains with TTN". Bazzoni and Salce [8,10] studied strongly flat modules

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and the existence of strongly flat covers. They rediscovered the class of almost perfect domains, through the following characterization, reminiscent of a classical result by Bass [2] for perfect rings: a commutative integral domain R is almost perfect if and only if all R-modules have a strongly flat cover. Almost perfect domains form a large and interesting class of rings, containing one-dimensional Noetherian domains. The papers [9,33,28,26,27] have been dedicated to the study of local almost perfect domains and their modules.

Any local one-dimensional domain R is dominated by at least one Archimedean (i.e., one-dimensional) valuation domain V. We make use of the Archimedean valuation domains which dominate a local one-dimensional domain R, as a main tool to investigate the class semigroup $\mathcal{S}(R)$.

We prove that if $\mathcal{S}(R)$ is Clifford, then every element of the integral closure $\bar{R}$ of R is quadratic over R (Theorem 2.1). As a consequence, such an R may be dominated by at most two distinct Archimedean valuation domains, and $\bar{R}$ coincides with their intersection. Moreover, if R is almost perfect, then the dominating valuation domains are necessarily discrete. In Theorem 2.10 we prove that if $\mathcal{S}(R)$ is a Clifford semigroup, then every finitely generated ideal I of R is isomorphic to I^2; moreover, if R is almost perfect, then $\mathcal{S}(R)$ is a Boolean semigroup. Using this result, we find a connection with the important and widely studied class of stable rings (a notion going back to Bass [3] and Lipman [18]). Namely, in Theorem 2.12 we prove that a local one-dimensional domain R is almost perfect with Boolean (or, equivalently, Clifford) class semigroup if, and only if R is stable.

In the final part of the paper we look for information on the class semigroup of the local one-dimensional domain R, through examination of the degree of the residue field extension $[V/P : R/\mathfrak{M}]$ and of $v(\mathfrak{M})$. Here $\mathfrak{M}$ is the maximal ideal of R, V is an Archimedean valuation domain which dominates R, v is the valuation of V and P its maximal ideal; $v(\mathfrak{M})$ denotes the infimum of the values of the elements of $\mathfrak{M}$. We show that $\mathcal{S}(R)$ cannot be a Clifford semigroup if either $[V/P : R/\mathfrak{M}] \geq 3$, or $v(\mathfrak{M}) \geq 3$, or both $[V/P : R/\mathfrak{M}]$ and $v(\mathfrak{M})$ equal 2. In Theorem 3.5 we examine the situation when R is a local one-dimensional domain dominated by a discrete valuation ring V, and R contains a nonzero ideal of V. Under these circumstances, if $[V/P : R/\mathfrak{M}] = 2$, then $\mathcal{S}(R)$ is Clifford if and only if $v(\mathfrak{M}) = 1$; if $[V/P : R/\mathfrak{M}] = 1$, then $\mathcal{S}(R)$ is Clifford if and only if $v(\mathfrak{M}) \leq 2$. In Proposition 3.7 we show that this result cannot be extended to the case when V is not discrete. Throughout the paper, several examples illustrate the various situations examined.

To conclude this introduction, we observe that some results of the present paper have been proved, for R Noetherian, in [6], and few others may be derived from Olberding's results in [22], and related papers. However, the use of the Archimedean valuation domains dominating R allows us to find direct and rather neat proofs, not needing those Noetherian conditions which were basic in the arguments of previous papers (see, e.g., [11]). It also helps to clarify some delicate points.

1. Preliminaries

In what follows R and $\mathfrak{M}$ will denote a local one-dimensional domain, and its maximal ideal, respectively. We denote by Q the field of fractions of R and by $U(R)$ the set of the units of R. Let us also remark that, for us, a local ring is not necessarily Noetherian.

A valuation v defined on a field Q is said to be *real* if its value group is an ordered subgroup of the real numbers. The valuation domain V of a real valuation v is called *Archimedean*. Of course, V is Archimedean if and only if it is one-dimensional. As usual, we say that V is a DVR, if V is a discrete Archimedean valuation domain. For general definitions and results on valuation theory we refer the reader to the texts by Schilling [31] and Nagata [20].

Let R be a local one-dimensional domain, and Q its field of fractions. We say that a valuation v on Q dominates R if the valuation domain V of v dominates R. Recall that this means $P \cap R = \mathfrak{M}$, where P is the maximal ideal of V.

Every local one-dimensional domain is dominated by some Archimedean valuation domain (see, for instance, [32]).

As observed in the introduction, local almost perfect domains form a subclass of local one-dimensional domains of particular importance. Recall that an integral domain R is almost perfect if R/I is a perfect ring, for every nonzero ideal I of R. However, to verify that a local domain R is almost perfect, we use the following characterizing property:

(*) let $\mathfrak{M}$ be the maximal ideal of R. For any nonzero $b \in \mathfrak{M}$ and for every sequence $\{a_n\}_n$ of elements of $\mathfrak{M}$, there exists a positive integer m such that $a_1 a_2 \ldots a_m \in bR$.

Let R be a local one-dimensional domain and let I be a nonzero ideal of R. If v is a real valuation dominating R, we define $v(I) = \inf\{v(r) : r \in I\}$. Recall the following fact, proved by Smith [32]. If R is local almost perfect and v is any real valuation dominating R, then $v(\mathfrak{M}) > 0$.

The main purpose of the present paper is to investigate conditions for the class semigroup $\mathcal{S}(R)$ of a local one-dimensional domain R to be a Clifford semigroup.

Recall that a semigroup $\mathcal{S}$ is said to be a Clifford semigroup if one of the following equivalent conditions holds:

(1) every element of $\mathcal{S}$ is contained in a subgroup G of $\mathcal{S}$;
(2) every $x \in \mathcal{S}$ is (Van Neumann) regular, i.e., there is $y \in \mathcal{S}$ such that $x = xyx$.
(3) $\mathcal{S}$ is a semilattice of groups.

For basic results on Clifford semigroups we refer the reader to [16] (see also [34]).

A semigroup $\mathcal{S}$ is called Boolean if every element of $\mathcal{S}$ is idempotent. Of course, a Boolean semigroup is automatically a Clifford semigroup.

An integral domain R is said to be stable if every ideal I of R is projective over its endomorphism ring $\mathrm{End}_R(I) = (I : I)$. The concepts of stable ideals and domains have been widely investigated by many authors, starting with Bass [3] and Lipman [18]; for instance, see [29,30,15,25,6,22–24,17]. Extensive references on stable domains may be found in Olberding's survey paper [21].

Let I be a finitely generated ideal of a ring R. We denote by $\mathrm{gen}(I)$ the minimal number of generators of I. When $\mathrm{gen}(I) = n$, we also say that I is n-generated.

By abuse of language, we will say that an ideal I of R is regular if the isomorphy class of I is regular as an element of the semigroup $\mathcal{S}(R)$. Recall that an ideal I is regular if and only if $I = I^2(I : I^2)$ (see [4], Lemma 1.1).

The content of the following proposition is well known. For instance, it may be derived from the results in Bass [2], Sally–Vasconcelos [30], and Kabbaj–Mimouni [17]. We give a proof for the sake of completeness.

Proposition 1.1. *Let R be a local one-dimensional Noetherian domain, such that $\mathrm{gen}(I) \le 2$ for every ideal I of R. Then $\mathcal{S}(R)$ is a Boolean semigroup.*

Proof. We have to show that $I^2 \cong I$ for every ideal I of R. Since this fact is obvious when I is principal, we may assume that $\mathrm{gen}(I) = 2$, say $I = \langle r, s\rangle$. Let v be any real valuation dominating R. Possibly replacing r with $r + s$, we may assume that $v(r) = v(s)$. By hypothesis $I^2 = \langle r^2, rs, s^2\rangle$ may be generated by two elements. Hence we have

$$a_1 r^2 + a_2 rs + a_3 s^2 \in \mathfrak{M} I^2$$

for suitable $a_i \in R$ not all in $\mathfrak{M}$, the maximal ideal of R. Now, at least two among the a_i must be units, since every element of $\mathfrak{M} I^2$ has v-value strictly larger than $2v(r) = 2v(s)$ and hence $r^2, rs, s^2 \notin \mathfrak{M} I^2$. We assume, without loss of generality, that $a_1 \in U(R)$. Then, since R is a local ring, from the above relation we readily get $r^2 + brs + cs^2 = 0$, for suitable $b, c \in R$. We conclude that $I^2 = \langle r^2, rs, s^2\rangle = \langle rs, s^2\rangle = sI$, as desired. □

Let us remark at once that the above proposition is not reversible. Specific examples were given by Sally–Vasconcelos [30] and by Heinzer–Lantz–Shah [15] (see our next Remark 4). Very recently, B. Olberding, in the still unpublished paper [24], has found a method for constructing a large class of stable local one-dimensional domains, either Noetherian or not. Being stable, all these domains have Boolean class semigroup (see the next section).

The following is a typical example of an almost perfect domain R which satisfies the hypothesis of Proposition 1.1.

Example 1.2. Let $R = K + X^2K[[X]]$. Obviously, R is dominated by the DVR $V = K[[X]]$. It is well known that R is a local one-dimensional Noetherian domain, such that $\mathrm{gen}(I) \le 2$ for every ideal I (for instance, see [1]). Actually, this property is proved also in our Theorem 3.5 (ii), since R satisfies the hypothesis of that result. By Proposition 1.1, $\mathcal{S}(R)$ is Boolean, so, in particular, it is a Clifford semigroup.

Since $v(\mathfrak{M}) > 0$ whenever R is almost perfect ([32]), one could ask if, conversely, the condition $v(\mathfrak{M}) > 0$ is enough to ensure that R is almost perfect. This is not the case, not even assuming that R is dominated by a unique DVR, as shown by the following example.

Example 1.3. We want to show that there exist local one-dimensional domains R, with maximal ideal $\mathfrak{M}$, which are not almost perfect and satisfy the following property: "R is dominated by a unique real valuation v, where v is discrete and $v(\mathfrak{M}) > 0$".

For the sake of simplicity, our example has characteristic 2. Straightforward adaptations provide examples of any characteristic $p > 0$. Let K be a field of characteristic 2, such that $[K : K^2]$ is countable. Let $\{u_n : n > 0\} \subset K$ be a 2-basis of K over K^2. Recall that it means $K = K^2(u_n : n > 0)$ and $[K^2(u_1, u_2, \ldots, u_n, u_{n+1}) : K^2(u_1, u_2, \ldots, u_n)] = 2$, for all $n > 0$. Let x be an indeterminate. We consider the ring $D = K^2[x, xu_n : n > 0] \subset K(x)$, and set $R = D_{\mathfrak{M}}$, where $\mathfrak{M} = \langle x, xu_n : n > 0\rangle$. It is clear that $K(x)$ is the field of fractions of R, and that R is dominated by the discrete valuation ring $V = K[x]_{\langle x\rangle}$. Let us now observe that, for all $\eta \in V$, η^2 lies in the DVR $W = K^2[x]_{\langle x\rangle}$, which is a subring of R. It readily follows that V coincides with the integral closure of R, hence V is the unique Archimedean valuation domain that dominates R, by [33], Theorem 3.2. In particular, R is one-dimensional, since its integral closure V is one-dimensional. By construction, $v(\mathfrak{M}) = 1$, where v is the valuation of V. Let us also note that, for any given $\eta \in R$, since $\eta^2 \in W$, we may write $\eta = f/g$, where $f, g \in D$ and $g \in U(W)$. In particular, f is a sum of monomials of the form $ax^m u_{n_1} \ldots u_{n_k}$, where $0 \ne a \in K^2$ and $k \le m$.

It remains to show that R is not almost perfect. We consider the sequence $\{a_n = xu_n : n > 0\} \subset \mathfrak{M}$, and the element $b = x$. We will prove that $a_1 \ldots a_k \notin bR$, for every $k > 0$. Assume for a contradiction that $a_1 \ldots a_m = x(f/g)$, for suitable $m > 0$, $f, g \in D$ and $g \in U(W)$. We get the equality $gx^m u_1 \ldots u_m = xf$ in $K[x]$. The coefficient of x^m in the first term is $cu_1 \ldots u_m$, for a suitable nonzero $c \in K^2$. On the other hand, the coefficient of x^m in the second member is the coefficient α of x^{m-1} in f. As observed above, α is a sum of terms of the form $au_{n_1} \ldots u_{n_k}$, where $0 \ne a \in K^2$ and $k \le m - 1$. But $cu_1 \ldots u_m = \alpha$ is impossible, since $u_1 \ldots u_m$ is independent over K^2 on the products of the u_i with less than m factors.

Remark 1. Recall that Bazzoni and Salce [7] proved that valuation domains have Clifford class semigroup, even when the dimension is larger than one. The Archimedean valuation domains provide typical examples of local one-dimensional domains whose class semigroups are Clifford but not necessarily Boolean (see [7], point (2) of Theorem 3). To give a concrete example, let V be a valuation domain with value group $\mathbb{Q}$, v the valuation of V, and $I = \{r \in V : v(r) > \sqrt{2}\}$. Since $\sqrt{2}$ is not in the value group of V, it is readily seen that the ideal I is not isomorphic to I^2. Then $\mathcal{S}(V)$ is not a Boolean semigroup.

2. Clifford and Boolean class semigroups

A main tool for our discussion is provided by the next theorem, which relates the regularity of two-generated ideals to integrality over R.

We will use the following terminology: if $y \in Q$ is integral over R, we say that y has degree k, if the positive integer k is the smallest possible degree of a monic polynomial in $R[X]$ which annihilates y.

Theorem 2.1. *Let R be a local one-dimensional domain, let v be a real valuation dominating R and let $I = \langle r, s\rangle$ be a two-generated ideal of R with $v(r) = v(s)$ (nonrestrictive hypothesis). If I is regular, then either r/s or s/r is integral of degree 2 over R. In particular, if $\mathcal{S}(R)$ is a Clifford semigroup, and $I = \langle r, s\rangle$ is a two-generated ideal of R, then the generators r and s can be chosen such that r/s is integral of degree 2 over R. Moreover $I \cong I^2$.*

Proof. If $I = \langle r, s\rangle$ is a regular ideal, we have $I = I^2(I : I^2)$. We first observe that there exists an element $f \in (I : I^2)$ such that $v(f) \le -v(s)$. In fact, the equality $I = I^2(I : I^2)$ implies

$$s = f_1 r^2 + f_2 rs + f_3 s^2, \qquad f_i \in (I : I^2),$$

and the above relation cannot hold if $v(f_i) > -v(s) = -v(r)$ for all $i \le 3$.

Then we take $f \in (I : I^2)$ such that $v(f) \le -v(s)$. We have

$$fr^2 = a_1 r + a_2 s \qquad a_1, a_2 \in R, \tag{1}$$

$$frs = b_1 r + b_2 s \qquad b_1, b_2 \in R, \tag{2}$$

$$fs^2 = c_1 r + c_2 s \qquad c_1, c_2 \in R. \tag{3}$$

Since $v(r) = v(s)$, we have $v(c_1 r + c_2 s) \ge v(s)$, and so (3) implies $v(f) + 2v(s) \ge v(s)$, whence $v(f) \ge -v(s)$. It follows that $v(f) = -v(s) = -v(r)$.

From (2) we get $f = b_1/s + b_2/r$, and therefore at least one of the b_i ($i = 1, 2$) satisfies $v(b_i) = 0$, hence it is a unit of R.

Now, using (1), we get

$$fr^2 = b_1(r/s)^2 s + b_2 r = a_1 r + a_2 s,$$

and hence

$$b_1(r/s)^2 + (b_2 - a_1)r/s - a_2 = 0.$$

Thus, if $b_1 \in U(R)$, we get r/s integral of degree ≤ 2 over R. Otherwise, from (3) we get

$$fs^2 = b_1 s + b_2 s^2/r = c_1 r + c_2 s,$$

which yields

$$b_2(s/r)^2 + (b_1 - c_2)s/r - c_1 = 0.$$

Since $b_1 \in \mathfrak{M}$ implies $b_2 \in U(R)$, we conclude that s/r is integral of degree ≤ 2 over R, as desired. Note that, since I is not principal, neither r/s nor s/r lies in R, hence the degree is exactly 2.

It remains to show that $I \cong I^2$. We assume, without loss of generality, that $(r/s)^2 + e_1(r/s) + e_0 = 0$, $e_i \in R$. Then $r^2 = -e_1 rs - e_0 s^2$ yields $I^2 = \langle rs, s^2\rangle = sI$, hence $I^2 \cong I$. □

Corollary 2.2. *Let R be a local one-dimensional domain dominated by a unique Archimedean valuation domain V. If $\mathcal{S}(R)$ is a Clifford semigroup, then V coincides with the integral closure of R, and every element of V has degree ≤ 2 over R.*

Proof. Let $\bar{R}$ denote the integral closure of R. Then the uniqueness of V implies that $\bar{R}$ is local and its maximal ideal coincides with the maximal ideal P of V (see Proposition 3.1 of [33]). In order to show that $\bar{R} = V$, it suffices to verify that $U(V) \subset \bar{R}$. Take any $u \in U(V)$; then $u = r/s$ for suitable $r, s \in R$ such that $v(r) = v(s)$ (v being the valuation of V). Since $\mathcal{S}(R)$ is a Clifford semigroup, the preceding theorem implies that either u or u^{-1} is integral of degree ≤ 2 over R. Assume for a contradiction that $u \notin \bar{R}$. Then u^{-1} is integral of degree two over R, and cannot be a unit of $\bar{R}$. It follows that u^{-1} lies in the maximal ideal of $\bar{R}$, which is P. This is impossible. Then $u \in \bar{R}$, and, since $u \in U(V)$ was arbitrary, we get $\bar{R} = V$. Moreover, u must have degree ≤ 2. In fact, a straightforward argument shows that a unit u of $\bar{R}$ has the same degree over R as u^{-1}, and, again by the preceding theorem, either u or u^{-1} is integral of degree ≤ 2. Finally, every element $\eta \in P$ is integral of degree ≤ 2 over R, since the unit $\eta + 1$ has this property. □

Example 2.3. Let $R = K + X^3K[[X]]$. The integral closure of R is $V = K[[X]]$, and, obviously, $\mathcal{S}(V) = 0$ is Boolean. However, $\mathcal{S}(R)$ is not a Clifford semigroup by the preceding corollary, since X is integral over R of degree 3.

Now let $R = F + XK[[X]]$, where F, K are fields and K is a purely transcendental extension of F. Then R is an integrally closed almost perfect domain (see [9] or [33]), and $V = K[[X]]$ is the unique DVR which dominates R. Since R is not a Prüfer domain, $\mathcal{S}(R)$ cannot be a Clifford semigroup, by Proposition 3 of [34] (or, more generally, by [6], Theorem 4.5).

Corollary 2.4. *Let R be a local one-dimensional domain, dominated by two distinct Archimedean valuation domains V_1 and V_2. If $\mathcal{S}(R)$ is a Clifford semigroup, then the integral closure $\bar{R}$ of R coincides with $V_1 \cap V_2$, and every element of $\bar{R}$ has degree ≤ 2 over R.*

Proof. We have to show that every $t \in V_1 \cap V_2$, not in R, is integral of degree two over R. Let v_1, v_2 be the valuations determined by V_1, V_2, respectively. Without loss of generality, we assume that $v_1(t) \geq v_2(t)$. We have three possibilities.

(1) $v_1(t) > 0$ and $v_2(t) = 0$. Since $\mathcal{S}(R)$ is a Clifford semigroup, from Theorem 2.1 it follows that either t or t^{-1} is integral over R of degree two. But t^{-1} is not integral over R, since it does not lie in V_1.

(2) $v_1(t) = 0 = v_2(t)$. In this case t is a unit of $V_1 \cap V_2$. From Theorem 2.1 we know that either t or t^{-1} is integral of degree two over R. It suffices to show that t, t^{-1} are both integral over R, since then they must have the same degree. Since v_1, v_2 are independent valuations, we apply the Approximation Theorem (see, e.g., [20]), to get $d \in Q$ such that $v_1(d - t) > 0$ and $v_2(d) > 0$. Since, necessarily, $v_1(d) = 0$, we argue as in (1), exchanging the roles of V_1 and V_2, to prove that d is integral over R. Moreover, since $v_2(d - t) = 0$, using (1) again, we see that $d - t$ is integral over R, as well. It follows that $t = d - (d - t)$ is integral over R. In the same way we can prove that t^{-1} is integral.

(3) $v_1(t), v_2(t) > 0$. Then $t + 1$ is a unit of $V_1 \cap V_2$, and hence it is integral of degree two, by (2). It follows that also t is integral over R of degree two. □

It is worth stating another useful consequence of Theorem 2.1.

Proposition 2.5. *Let R be a local one-dimensional domain dominated by at least three distinct Archimedean valuation domains. Then $\mathcal{S}(R)$ is not a Clifford semigroup.*

Proof. Let us denote by V_i and v_i, $1 \leq i \leq 3$, three distinct Archimedean valuation domains which dominate R and their respective valuations. Since the v_i are independent valuations, applying the Approximation Theorem we may find $x \in Q$ such that $v_1(x) > 0$, $v_2(x) = 0$ and $v_3(x) < 0$. In fact, let us pick any $c \in P_1$, where P_1 is the maximal ideal of V_1. Then there exists $d \in Q$ such that $v_1(d - 1) > 0$, $v_2(d - c) > v_2(c)$ and $v_3(d) > v_3(c)$. Thus necessarily $v_1(d) = 0$ and $v_2(d) = v_2(c)$, hence $x = c/d$ is the required element. Since the integral closure of R is contained in $V_1 \cap V_2 \cap V_3$, and $x \notin V_3$, the element $x \in V_1 \cap V_2$ is not integral over R. Since $x \notin R$, we have $x = r/s$, for suitable $r, s \in \mathfrak{M}$, and $v_2(x) = v_2(r) - v_2(s) = 0$. We are thus in the position to apply Theorem 2.1, once we prove that $1/x = s/r$ is not integral over R. This last fact follows from $v_1(1/x) < 0$, hence $1/x \notin V_1$. □

Note that an alternative, indirect proof of the above proposition may be derived from Corollary 2.4.

Corollary 2.6. *Let R be a local one-dimensional domain such that its integral closure $\bar{R}$ is also local. If $\mathcal{S}(R)$ is a Clifford semigroup, then $\bar{R}$ is a valuation domain, and hence, in particular, $\mathcal{S}(\bar{R})$ is also a Clifford semigroup.*

Proof. The preceding proposition shows that R, and hence $\bar{R}$, is dominated by at most two distinct Archimedean valuation domains. From Proposition 3.2 of [33] it follows that $\bar{R}$ is dominated by a unique Archimedean valuation domain, say V. Since the same is true for R, in view of Corollary 2.2 we get $\bar{R} = V$, as desired. □

The next result has an interesting consequence for almost perfect domains, namely, if R is local almost perfect, and $\mathcal{S}(R)$ is Clifford, then all the real valuations dominating R are necessarily discrete.

Proposition 2.7. *Let R be a local one-dimensional domain with maximal ideal $\mathfrak{M}$. If R is dominated by a nondiscrete real valuation v, and $v(\mathfrak{M}) > 0$, then $\mathcal{S}(R)$ is not a Clifford semigroup.*

Proof. Let $\Gamma(V)$ be the value group of V, the valuation domain of v. Then $\Gamma(V)$ is dense in $\mathbb{R}$, since v is nondiscrete. Without loss of generality, we assume that $v(\mathfrak{M}) = 1$. Pick $\delta \in \Gamma(V)$ with $0 < \delta < 1/2$, and choose $t \in V$ such that $v(t) = \delta$. Since $t \notin R$, we have $t = r/s$, where $r, s \in \mathfrak{M}$ and $v(s) < v(r)$. Let us consider the ideal $I = \langle r, s \rangle$. We will show that I is not regular, hence the assertion.

We must prove that $I \neq I^2(I : I^2)$. Note that $v(I) = v(s)$. Take any $f \in (I : I^2)$. We have $fs^2 \in I$, hence $v(fs^2) = v(f) + 2v(s) \geq v(I) = v(s)$. It follows that $v(f) \geq -v(s)$. Let us first suppose that $v(f) = -v(s)$. Since $fr^2 \in I$, we may write

$$fr^2 = ar + bs, \quad a, b \in R.$$

We distinguish various cases:

(1) both a and b are in $\mathfrak{M}$. Then $v(ar + bs) \geq v(s) + 1$, while $v(fr^2) = -v(s) + 2v(r) < v(s) + 1$, since $v(r) - v(s) = \delta < 1/2$. We have obtained a contradiction, hence this case is impossible.

(2) b is a unit of R. Then $v(ar + bs) = v(s)$ and $v(fr^2) = 2v(r) - v(s) > v(s)$. This is impossible.

(3) a is a unit of R and $b \in \mathfrak{M}$. Then $v(ar + bs) = v(r)$, since $v(r) - v(s) < 1/2$ implies $v(bs) \geq 1 + v(s) > v(ar) = v(r)$. Then we get $v(fr^2) = 2v(r) - v(s) > v(r) = v(ar + bs)$, again impossible.

We have thus seen that $v(f) = -v(s)$ is impossible, hence $v(f) > -v(s)$. Since $v(I^2) = 2v(s)$, and $f \in (I : I^2)$ was arbitrary, we conclude that any $z \in I^2(I : I^2)$ satisfies $v(z) > v(s)$. It follows that $s \notin I^2(I : I^2)$, and therefore $I \neq I^2(I : I^2)$. The desired conclusion follows. □

Example 2.8. Let K be a field, and let D be the ring of formal polynomials with exponents in $\mathbb{R}^+$. For $0 \neq f = \sum_{i=1}^n c_i X^{\alpha_i} \in D$, where $c_i \in K$ and $0 \leq \alpha_1 < \cdots < \alpha_n$ are non-negative real numbers, we define $v(f) = \alpha_1$. As is well known, the map $v(f/g) = v(f) - v(g)$ (for nonzero $f, g \in D$) defines a real valuation on the field of fractions of D. Let V be the valuation domain of v. We consider the ring $R = K + \mathfrak{M}$, where $\mathfrak{M} = \{f \in V : v(f) \geq 1\}$. Then R is an almost perfect domain, and V is the unique Archimedean valuation domain which dominates R (see [33] Theorem 3.2). Proposition 2.7 shows that $\mathcal{S}(R)$ is not Clifford. On the other hand, $\mathcal{S}(\bar{R})$ is Clifford, since it is easy to verify that $\bar{R}$ coincides with V.

The standard examples of almost perfect domains with Clifford class semigroup (for instance, that of Example 1.2) are dominated by a unique DVR. It is worth giving explicit examples where such domains are dominated by two distinct DVRs.

Example 2.9. We give two examples of local almost perfect domains R, dominated by two distinct DVRs, such that $\mathcal{S}(R)$ is a Boolean semigroup.

(I). Let F be a field, and let $Q = F(x)$ be the field of the rational functions over F. Let V_1, V_2 denote the valuation domains of Q whose maximal ideals are generated, respectively, by $x_1 = x$ and $x_2 = x - 1$. Denote by $v_i, P_i = x_i V_i$ the respective valuations and maximal ideals. Let $z = x_1 x_2 = x(x-1)$. Let $R = F + z(V_1 \cap V_2)$. Then R is a local almost perfect domain, with maximal ideal $\mathfrak{M} = z(V_1 \cap V_2)$, dominated exactly by V_1 and V_2 (see Theorem 4.1 and Example 4.2 of [33]). In view of Proposition 1.1, to show that $\mathcal{S}(R)$ is Boolean, it suffices to prove that every ideal of R is either principal, or isomorphic to the fractional ideal $\langle 1, x\rangle$.

We start with an easy observation. Since $x\mathfrak{M} \subseteq \mathfrak{M}$ and $x(x-1) \in \mathfrak{M}$, we get $x^m \equiv x$ modulo $\mathfrak{M}$ for all $m > 0$. It easily follows that any polynomial $f \in F[x]$ is congruent, modulo $\mathfrak{M}$, either to 0, or to a, or to bx, for suitable $0 \neq a, b \in F$. Note that $f \in U(R)$ whenever $f \equiv a$ modulo $\mathfrak{M}$, $0 \neq a \in F$.

Now let I be any nonzero ideal of R, say $I = \langle r_i : i < \lambda\rangle$, for some ordinal number λ. Possibly multiplying the generators by units of R, we may assume that all the r_i lie in $F[x]$. We may also assume that the value $v_1(r_0) = m$ is minimal. Then we have $r_i = x^m f_i$, where $f_i \in F[x]$ and x does not divide f_0. As observed above, for all $i < \lambda$, f_i is associated to either 1, or to $x + \eta_i$, or to η_i, for suitable $\eta_i \in \mathfrak{M}$. So, multiplying by units of R, we can suppose that the f_i have one of the preceding forms. Thus, in particular, $f_0 = 1$. Now we have $I = x^m\langle 1, f_i : 1 \leq i < \lambda\rangle$. An easy exercise shows that the fractional ideal $\langle 1, f_i : 1 \leq i < \lambda\rangle$ either coincides with $\langle 1\rangle$, or with $\langle 1, x\rangle$. The desired conclusion follows.

(II). Let $R = \mathbb{C}[x, y]_{\langle x,y\rangle}$, where x, y are subject to the relation $y^2 = x^2 + x^3$. Of course, R is the coordinate ring of the classical singular cubic, localized at the origin. Since it is Noetherian and one-dimensional, R is automatically an almost perfect domain. Every ideal I of R may be generated by two elements, hence $\mathcal{S}(R)$ is Boolean. This may be proved directly. More generally, the property follows, for instance, from Corollary 2.2 of [29], since the maximal ideal $\mathfrak{M} = \langle x, y\rangle$ of R satisfies $\mathfrak{M}^2 = x\mathfrak{M}$

It is also well known that R is dominated exactly by two distinct DVRs. For the sake of completeness, we verify this fact by using our preceding results. Since R is almost perfect, by Proposition 2.7 any Archimedean valuation domain that dominates R must be discrete, and there are at most two of them, by Proposition 2.5. Thus we must show that R is not dominated by a unique DVR. It suffices to show that the integral closure $\bar{R}$ of R is not a local domain (see, e.g., Proposition 3.1 of [33]). Let $\eta = 1 + y/x$; we have $\eta^2 - 2\eta - x = 0$, and so $\eta \in \bar{R}$. It is easy to verify that η^{-1} is not integral over R, so that η is not a unit of $\bar{R}$. In a similar way, one sees that $\beta = 1 - y/x$ lies in $\bar{R}$ and is not a unit. Then, since $\eta + \beta = 2 \in U(R)$, we conclude that $\bar{R}$ is not a local domain, as desired.

Let us now prove a main result of our paper.

Theorem 2.10. *Let R be a local one-dimensional domain such that $\mathcal{S}(R)$ is a Clifford semigroup. Then $I \cong I^2$ for every finitely generated ideal I of R. Moreover, if the real valuations dominating R are all discrete, then $\mathcal{S}(R)$ is a Boolean semigroup.*

Proof. Let V be a valuation domain that dominates R. Recall that, by Proposition 2.5, either V is unique, or there exists exactly another distinct valuation domain, say W, which dominates R. Accordingly, the integral closure $\bar{R}$ of R either coincides with V or with $V \cap W$ (Corollaries 2.2 and 2.4). Let I be a finitely generated ideal of R. For our convenience, we write $I = \langle x_1, x_i : i \in \Lambda\rangle$, where we assume that the set of indices Λ is finite of cardinality ≥ 2, by including redundant generators, if necessary. Since Λ is finite, we may assume that x_1 has minimum v-value among the generators, where v is the valuation of V. Hence, possibly modifying the generators, we may assume that $v(x_1) = v(x_i)$ for all $i \in \Lambda$. Now, if $W \neq V$ exists, we may also assume that $w(x_1) \leq w(x_i)$ for all $i \in \Lambda$, where w is the valuation of W. Hence, in any case, it follows that $x_i/x_1 \in V \cap W = \bar{R}$, for all $i \in \Lambda$. Then Theorem 2.1 implies that x_i/x_1 is integral of degree ≤ 2 over R.

Our aim is to show that $I^2 = x_1 I$, which yields our statement. We easily see that it suffices to show that $x_i^2, x_j^2, x_i x_j \in x_1\langle x_1, x_i, x_j\rangle = x_1 H$, for all indices $i \neq j$. Since x_i/x_1 is integral of degree ≤ 2, we get

$$x_i^2 + \alpha x_i x_1 + \beta x_1^2 = 0,$$

for suitable $\alpha, \beta \in R$. We derive that $x_i^2 \in \langle x_1^2, x_1 x_i\rangle \subseteq x_1 H$. In a similar way x_j/x_1 integral of degree ≤ 2 implies $x_j^2 \in x_1 H$.

It remains to show that $x_i x_j \in x_1 H$. Now H is regular, since $\mathcal{S}(R)$ is a Clifford semigroup, and hence $H = H^2(H : H^2)$. Note that $v(H^2) = v(x_1^2)$, so, as already seen in the proofs of Theorem 2.1 and Proposition 2.7, there must exist $f \in (H : H^2)$ such that $v(f) = -v(x_1)$. For such a choice of f we have

$$fx_1^2 = a_1x_1 + a_ix_i + a_jx_j$$
$$fx_1x_i = b_1x_1 + b_ix_i + b_jx_j$$
$$fx_1x_j = c_1x_1 + c_ix_i + c_jx_j$$
$$fx_ix_j = d_1x_1 + d_ix_i + d_jx_j$$

for suitable coefficients lying in R. We note that $v(fx_1^2) = v(x_1)$, and therefore one among a_1, a_i, a_j must be a unit of R. Eliminating fx_1^2 in the above equalities, we derive

$$(a_1x_1 + a_ix_i + a_jx_j)x_i = b_1x_1^2 + b_ix_ix_1 + b_jx_jx_1$$
$$(a_1x_1 + a_ix_i + a_jx_j)x_j = c_1x_1^2 + c_ix_ix_1 + c_jx_jx_1$$
$$(a_1x_1 + a_ix_i + a_jx_j)x_ix_j = d_1x_1^3 + d_ix_ix_1^2 + d_jx_jx_1^2.$$

Assume now, for a contradiction, that $x_ix_j \notin x_1H$. Then the first equality above implies that $a_j \in \mathfrak{M}$ and the second equality implies $a_i \in \mathfrak{M}$ (since $x_i^2, x_j^2 \in x_1H$). Then, as noted above, necessarily $a_1 \in U(R)$. Recall now that $x_i^2 = -\alpha x_1x_i - \beta x_1^2$. It follows that $x_i^2x_j + \alpha x_1x_ix_j \in x_1^2H$. In a similar way we find that $x_ix_j^2 + \gamma x_1x_ix_j \in x_1^2H$, for a suitable $\gamma \in R$.

It is now easy to check that the third equality above yields

$$(a_1 - a_i\alpha - a_j\gamma)x_1x_ix_j \in x_1^2H.$$

Since R is local, we have $a_1 - a_i\alpha - a_j\gamma \in U(R)$, and thus we obtain $x_ix_j \in x_1H$. We have thus reached the required contradiction, hence our conclusion follows.

Now we make the further assumption that the real valuations dominating R are all discrete. Hence V is a DVR, and the same goes for W, if it exists. We have to prove that also any ideal $I = \langle x_1, x_i : i \in \Lambda\rangle$, not finitely generated, satisfies $I \cong I^2$. In the present case the set of indices Λ is no longer finite. However, since the dominating valuations are discrete, we may still assume that $v(x_1) = v(x_i)$ and $w(x_1) \leq w(x_i)$, for all $i \in \Lambda$. Then the above argument works *verbatim*, and we get $I^2 = x_1I$, as desired. □

Remark 2. When $\mathcal{S}(R)$ is not Clifford, it is possible for a local almost perfect domain R to contain an ideal which is regular but not Boolean. For instance, consider a valuation domain V which contains a field K, and whose value group is the group of rational numbers $\mathbb{Q}$. Let $\mathfrak{M} = \{x \in V : v(x) > \sqrt{2}\}$ and let $R = K + \mathfrak{M}$. Then R is a local almost perfect domain, and V is the unique Archimedean valuation domain that dominates R (see [33]). It is an easy exercise to check that $\mathfrak{M}$ is regular but not Boolean (since $\sqrt{2}$ is not in the value group of V). Of course, $\mathcal{S}(R)$ is not Clifford, since V is not a DVR and dominates R (Proposition 2.7).

If the one-dimensional local domain R is stable, it is well known that its class semigroup $\mathcal{S}(R)$ is Clifford (for instance, see [6], Proposition 2.2). Then R is dominated by at most two distinct Archimedean valuation domains, by Proposition 2.5, and these are DVRs. In fact, their intersection is the integral closure of R, which is a Dedekind domain by the next lemma (which is Corollary 2.5 of [22]).

We want to show that a local one-dimensional domain R with Clifford class semigroup is almost perfect if and only if it is stable. In such a case $\mathcal{S}(R)$ is necessarily Boolean.

For convenience, we state the next lemma, which is Corollary 2.5 of Olberding's paper [22] (for R Noetherian, this result was proved by Rush [25]).

Lemma 2.11 ([22]). *An integral domain R is a one-dimensional stable domain if and only if $\bar{R}$ is a Dedekind domain, every R-submodule of $\bar{R}$ containing R is a ring, and there are at most two maximal ideals of $\bar{R}$ lying over each maximal ideal of R.*

We have seen in Example 1.3 that, for a local one-dimensional domain R, the condition $v(\mathfrak{M}) > 0$, for some real valuation v dominating R, does not imply that R is almost perfect. However, this condition is sufficient when $\mathcal{S}(R)$ is a Clifford semigroup. Let us also remark that, for R Noetherian, the equivalence (i) ⇔ (iii) in the next result follows from [6], Theorem 3.1.

Theorem 2.12. *Let R be a local one-dimensional domain. The following are equivalent:*

(i) *R is almost perfect and $\mathcal{S}(R)$ is a Clifford semigroup;*
(ii) *$\mathcal{S}(R)$ is a Clifford semigroup and every real valuation dominating R is discrete;*
(iii) *R is stable.*

Moreover, if these conditions hold, then $\mathcal{S}(R)$ is necessarily a Boolean semigroup.

Proof. (i) ⇒ (ii). As remarked just before Proposition 2.7, if R is almost perfect and $\mathcal{S}(R)$ is Clifford, then every real valuation dominating R is discrete.

(ii) ⇒ (iii) In view of Proposition 2.5, Corollary 2.2 and 2.4, and Proposition 2.7, we know that the integral closure $\bar{R}$ is either a DVR, or the intersection of two distinct DVRs. Then $\bar{R}$ is a Dedekind domain (actually a principal ideal domain) with at most two maximal ideals. To be in the position to apply Lemma 2.11, showing that R is stable, it remains to prove that every R-submodule of $\bar{R}$ containing R is a ring. It suffices to show that $a_1a_2 \in R+Ra_1+Ra_2$, for every given $a_1, a_2 \in \bar{R}$. Presently, we assume that $\bar{R} = V_1 \cap V_2$. We choose $x \in \mathfrak{M}$ such that $xa_j \in \mathfrak{M}$ $(j = 1, 2)$, and consider the ideal $I = \langle x, xa_1, xa_2 \rangle$ of R. Note that $v_i(x) \leq v_i(xa_j)$, $i, j \leq 2$, where v_i is the valuation of V_i. Then the proof of Theorem 2.10 shows that $I^2 = xI$. In particular, $x^2a_1a_2 \in xI$ yields $a_1a_2 \in R + Ra_1 + Ra_2$, as desired. The case where $\bar{R} = V$ is similar.

(iii) ⇒ (i). If R is stable then, as recalled above, $\mathcal{S}(R)$ is a Clifford semigroup. We have to show that R is almost perfect. Let $\bar{R}$ be the integral closure of R. Then either $\bar{R} = V$, or $\bar{R} = V_1 \cap V_2$, for suitable DVRs that dominate R. We denote by $\mathfrak{M}$ the maximal ideal of R. To see that R is almost perfect, we must show that, for every sequence $\{a_n\}_n$ contained in $\mathfrak{M}$ and every $b \in \mathfrak{M}$, there exists $m > 0$ such that $a_1 \ldots a_m \in bR$. We prove this fact in the case where $\bar{R} = V_1 \cap V_2$ (when $\bar{R} = V$ the proof is similar but easier). Let v_i be the valuation of V_i $(i = 1, 2)$. For $k > 0$ we set $a = a_1 \ldots a_k$ and $a' = a_{k+1} \ldots a_{2k}$. Since $v_i(a_n) \geq 1$ for all $n > 0$, choosing $k > 0$ large enough we get $v_i(a) \geq v_i(b)$ and $v_i(a') \geq v_i(b)$, for $i = 1, 2$. It follows that $a/b, a'/b \in \bar{R}$. Since R is stable, by Lemma 2.11 we have $aa'/b^2 \in R + R(a/b) + R(a'/b)$, and therefore $aa' = a_1 \ldots a_{2k} \in bR$, as desired.

Finally, condition (ii) implies that $\mathcal{S}(R)$ is Boolean, by Theorem 2.10. □

Remark 3. It is important to note that the condition that $\mathcal{S}(R)$ is a Boolean semigroup does not suffice to ensure that R is stable (or, equivalently, almost perfect). For instance, let V be a valuation domain with value group $\mathbb{R}$. Then the maximal ideal P is idempotent. Now let I be a nonmaximal ideal of R. Since $\mathbb{R}$ is the value group of V, there exists $a \in V$ such that $v(a) = v(I)$. It follows that either $I = aV$, when $a \in I$, or $I = aP$ otherwise. In both cases $I \cong I^2$, and we conclude that $\mathcal{S}(V)$ is Boolean, although V is not stable, since it is not a DVR.

This easy example shows that the implications (i)⇒(ii) and (i)⇒(iii) of [17] Theorem 3.2 are not valid. Hence that result, in the present form, is incorrect.

The contents and references of the following remark have been suggested by Bruce Olberding. The author thanks him for his kind help and for other useful discussions.

Remark 4. There exist local one-dimensional domains R which are stable and contain ideals that cannot be generated by two elements. In particular, the examples where R is Noetherian show that Proposition 1.1 is not reversible. Sally and Vasconcelos [30] constructed such an R, with $\text{gen}(\mathfrak{M}) = 3$. Their example was improved by Heinzer, Lantz and Shah [15]; they obtained, for all $n > 0$, a stable R such that $\text{gen}(\mathfrak{M}) = n$. Moreover, Olberding [22] proved that there even exist one-dimensional local stable domains which are not Noetherian. All these examples are based on a technical and rather deep construction made by Ferrand and Raynaud [12]. We observe that these examples all have characteristic 2, for a technical reason. In the still unpublished paper [24], Olberding has much generalized the above results, constructing local stable one-dimensional domains R of any characteristic, and with arbitrary $\text{gen}(\mathfrak{M})$, finite or not. He has also made a thorough study of their $\mathfrak{M}$-adic completions.

In the special case when 2 is a unit of the ring, we get some more information.

Proposition 2.13. *Let R be a local one-dimensional domain such that $\bar{R}$ is either a DVR or the intersection of two distinct DVRs, and $2 \in U(R)$. Then $\mathcal{S}(R)$ is Boolean if and only if every element of $\bar{R}$ has degree ≤ 2 over R.*

Proof. The necessity follows from Theorem 2.1, even if 2 is not a unit. Let us prove the sufficiency. Let $I = \langle x, y_i : i \in \lambda \rangle$ be any ideal of R. Possibly changing the generators, we may always assume that $y_i/x \in \bar{R}$ for all $i \in \Lambda$, since either $\bar{R} = V$, of $\bar{R} = V \cap W$, where V and W are DVRs. We want to show that $I^2 = xI$. Arguing as in the proof of Theorem 2.10, we see that it suffices to verify $I^2 = xI$ when I is generated by three elements, say $I = \langle x, y, z \rangle$, where $y/x, z/x \in \bar{R}$. Equivalently, we must show that $y^2, z^2, yz \in \langle x^2, xy, xz \rangle$. Since y/x is integral of degree two over R, we get $(y/x)^2 \in R + R(y/x)$, whence $y^2 \in Rx^2 + Rxy$. In a similar way, we get $z^2 \in Rx^2 + Rxz$. Finally, since $(y + z)/x$ is integral of degree ≤ 2, we get $(y + z)^2 = a(y + z)x + bx^2$, for suitable $a, b \in R$, and hence $2yz \in \langle x^2, xy, xz, y^2, z^2 \rangle = \langle x^2, xy, xz \rangle$. The desired conclusion follows, since $2 \in U(R)$. □

In the statement of the preceding proposition, we cannot avoid the somehow technical assumption that $\bar{R}$ is either a DVR or the intersection of two DVRs. For instance, a local almost perfect domain R can be integrally closed and dominated by a unique DVR $V \neq R$. In this case $\mathcal{S}(R)$ cannot be a Clifford semigroup, by Proposition 3 of [34]. A typical example of such an integrally closed R has been given in our preceding Example 2.3.

Example 2.14. It is easy to show that the conclusion of Proposition 2.13 fails if 2 is not a unit. For instance, let K be a field of characteristic 2 such that $[K : K^2] \geq 4$, and let $R = K^2 + XK[[X]]$. It is clear that R is dominated by the DVR $V = K[[X]]$. Since $z^2 \in R$ for every $z \in V$, we have $\bar{R} = V$ and every element of $\bar{R}$ has degree ≤ 2 over R. However, $\mathcal{S}(R)$ is not a Clifford semigroup. This follows from our next Proposition 3.1. However, let us exhibit an ideal of R which is not Boolean. Let us pick $\alpha, \beta \in K$ such that $1, \alpha, \beta, \alpha\beta$ are linearly independent over K^2 (this is possible since $[K : K^2] \geq 4$). Let $I = \langle X, X\alpha, X\beta \rangle$. As in the proof of Proposition 3.1, one shows that $I^2 \ncong I$.

3. Degree of the residue field extensions and $v(\mathfrak{M})$

In this final section we take a different point of view. We fix an Archimedean valuation domain V dominating R, possibly not unique, and we look for information on the class semigroup of R, through examination of $v(\mathfrak{M})$ and of $[V/P : R/\mathfrak{M}]$, the degree of the residue field extension (v the valuation of V, P its maximal ideal).

Our first result shows that if $\mathcal{S}(R)$ is a Clifford semigroup, then the degree of the residue field extensions is at most two.

Proposition 3.1. *Let R be a local one-dimensional domain dominated by an Archimedean valuation domain V, with maximal ideal P. If $[V/P : R/\mathfrak{M}] \geq 3$, then $\mathcal{S}(R)$ is not a Clifford semigroup.*

Proof. Let us first suppose that there exists $u \in U(V)$ such that $u + P$ has degree ≥ 3 over the field $R/\mathfrak{M}$. Since $u \notin R$, we have $u = r/s$, where $r, s \in \mathfrak{M}$. Let us consider the ideal $I = \langle r, s\rangle$. Then I is not regular, and hence $\mathcal{S}(R)$ is not a Clifford semigroup. In fact, since $v(r) = v(s)$, by Theorem 2.1 it suffices to see that neither u nor u^{-1} is integral over R of degree 2. In fact $u^2 + a_1u + a_0 = 0$, $a_i \in R$ would imply that $1 + P, u + P, u^2 + P$ are linearly dependent over $R/\mathfrak{M}$, against our assumption on u. The same argument works for u^{-1}.

Let us now assume that all the elements of V/P have degree two over $R/\mathfrak{M}$. Under the present circumstances, since $[V/P : R/\mathfrak{M}] \geq 3$, there exist $\alpha, \beta \in V/P$ such that $1, \alpha, \beta, \alpha\beta$ are linearly independent over $R/\mathfrak{M}$. We have $\alpha = u_1 + P$, $\beta = u_2 + P$, where $u_1, u_2 \in U(V)$. Let us now assume, for a contradiction, that $\mathcal{S}(R)$ is a Clifford semigroup. Then, possibly replacing u_i with u_i^{-1}, we may assume that $u_1, u_2 \in \bar{R}$, the integral closure of R (see Corollaries 2.2 and 2.4). Choose $s \in R$ such that $su_1, su_2 \in \mathfrak{M}$, and consider the ideal $I = \langle s, u_1s, u_2s\rangle$. Examination of the proof of Theorem 2.10 shows that this choice of the generators yields $I^2 = sI$. In particular, we have $s^2u_1u_2 = s(a_0s + a_1su_1 + a_2su_2)$, for suitable $a_i \in R$. But $u_1u_2 = a_0 + a_1u_1 + a_2u_2$ implies that $1, \alpha, \beta, \alpha\beta$ are linearly dependent over $R/\mathfrak{M}$, which is impossible. □

In the situation described in the second part of the above proof, we may have $\mathcal{S}(R)$ not Clifford, although every two-generated ideal is regular (see, for instance the above Example 2.14).

We anticipate a typical example of a local almost perfect domain R such that $[V/P : R/\mathfrak{M}] = 2$ and $\mathcal{S}(R)$ is Boolean. These kinds of domains will be examined in our next Theorem 3.5.

Example 3.2. Let $F \subset K$ be fields with $[K : F] = 2$, let x be an indeterminate, and let us consider the almost perfect domain $R = F + xK[[x]]$, which is dominated by the DVR $V = K[[x]]$. Then $[V/P : R/\mathfrak{M}] = [K : F] = 2$; moreover $\mathfrak{M} = P$, hence $v(\mathfrak{M}) = 1$. Thus R satisfies the hypotheses of our next Theorem 3.5, hence $\mathrm{gen}(I) \leq 2$ for every ideal I of R and $\mathcal{S}(R)$ is Boolean.

Now we find an upper bound for $v(\mathfrak{M})$, in case $\mathcal{S}(R)$ is a Clifford semigroup.

Proposition 3.3. *Let R be a local one-dimensional domain, and let v be a real valuation dominating R. If $v(\mathfrak{M}) \geq 3$, then $\mathcal{S}(R)$ is not a Clifford semigroup.*

Proof. Let V be the valuation domain of v. We pick $x \in V$ such that $0 < v(x) \leq 1$. Then $x \notin R$, since $v(\mathfrak{M}) \geq 3$. We make use of Theorem 2.1. Let $z = x + 1$. Let us show that z is not integral of degree 2 over R; equivalently, x is not integral of degree 2. Assume for a contradiction that $x^2 + a_1x + a_0 = 0$, $a_i \in R$. Now, if $a_1 \in U(R)$, then $v(x) = v(a_1x) = v(x^2 + a_0)$, which is impossible, since $0 < v(x) < v(x^2) \leq 2$, while $v(a_0)$ is either zero or ≥ 3. Thus, necessarily, $a_1 \in \mathfrak{M}$, hence $v(a_1) \geq 3$, and we readily see that $v(a_1x) > v(x^2 + a_0)$, again impossible. Now let us see that also z^{-1} is not integral of degree 2 over R. Since $z^{-1} = 1 - x/(x + 1)$, it suffices to show the same for $x/(x + 1)$. An argument similar to that used for x works. The desired conclusion follows from Theorem 2.1. □

Now we combine the conditions on $v(\mathfrak{M})$ and on the degree of residue field extensions.

Proposition 3.4. *Let R be a local one-dimensional domain, v a real valuation dominating R, V the valuation domain of v, and P its maximal ideal. If $v(\mathfrak{M}) = 2$ and $[V/P : R/\mathfrak{M}] = 2$, then $\mathcal{S}(R)$ is not a Clifford semigroup.*

Proof. Let us choose $x \in P$ with $v(x) \leq 1$; then $x \notin R$, since $v(\mathfrak{M}) = 2$. Let $u \in U(V)$ be such that $1 + P, u + P$ are linearly independent over $R/\mathfrak{M}$. Let us now choose $r, s \in \mathfrak{M}$ such that $s/r = x$; possibly multiplying both r and s by an element of R, we may assume that $ru \in \mathfrak{M}$. Let $I = \langle r, ru, s\rangle$; then $I^2 = r^2\langle 1, u^2, x^2, u, x, xu\rangle$. Let us show that the minimal number of generators of the R-module $X = \langle 1, u^2, x^2, u, x, xu\rangle$ is ≥ 4. It suffices to show that if

$$a_0 + a_1u + a_2x + a_3xu = y \in \mathfrak{M}X, \quad a_i \in R,$$

then all the a_i lie in $\mathfrak{M}$. Note that $v(y) \geq 2$, since $v(\mathfrak{M}) = 2$. Now, since $a_0 + a_1u \in P$, we get $a_0, a_1 \in \mathfrak{M}$, and therefore $v(a_0), v(a_1) \geq 2$. It follows that $v(a_0 + a_1u - y) \geq 2$, and hence $v(a_2x + a_3ux) \geq 2$. Thus we get $v(a_2 + a_3u) > 0$, and therefore $a_2, a_3 \in \mathfrak{M}$. We conclude that $\mathrm{gen}(I^2) \geq 4$, as well. Since $\mathrm{gen}(I) \leq 3$, we get $I \not\cong I^2$, and therefore $\mathcal{S}(R)$ is not a Clifford semigroup, in view of Theorem 2.10. □

Remark 5. Let $R = F + z(V_1 \cap V_2)$ be as in Example 2.9 (I). Then R provides a nontrivial example of an almost perfect domain such that $[V_i/P_i : R/\mathfrak{M}] = 1$ and $v_i(\mathfrak{M}) = 1$, for every Archimedean valuation domain V_i dominating it (note that $z \in \mathfrak{M}$ and $v_i(z) = 1$, for $i = 1, 2$). Recall that $\mathcal{S}(R)$ is a Boolean semigroup.

Not even the simplest possible case, namely when $[V/P : R/\mathfrak{M}] = 1$ and $v(\mathfrak{M}) = 1$, ensures that $\mathcal{S}(R)$ is Clifford, as discussed in the next remark.

Remark 6. Using the techniques of [33], it is easy to construct an almost perfect domain R, dominated by $n \geq 3$ distinct DVRs V_i $(1 \leq i \leq n)$, such that $[V_i/P : R/\mathfrak{M}] = 1$ and $v_i(\mathfrak{M}) = 1$ for all $i \leq n$ (see [33], Theorem 4.1). Then $\mathcal{S}(R)$ cannot be

Boolean, in view of Proposition 2.5. More interestingly, we may also give an example where R is dominated by exactly two distinct Archimedean valuation domains V_1, V_2, the conditions $[V_i/P : R/\mathfrak{M}] = 1$ and $v_i(\mathfrak{M}) = 1$ hold for $i = 1, 2$, and $\mathcal{S}(R)$ is not Boolean.

Let x_1, x_2 be indeterminates over the field K, and, for $i = 1, 2$, let V_i be the valuation domain of $K(x_1, x_2)$ determined by the valuation v_i such that $v_i(x_j) = \delta_{ij}$ (Kronecker's delta). We set $y = x_1x_2$, and consider the ring $R = K + y(V_1 \cap V_2)$. From [33] Theorem 4.1, it follows that R is an almost perfect domain dominated exactly by V_1 and V_2. Since $v_i(y) = 1$ and $V_i/x_iV_i = K = R/\mathfrak{M}$, we readily see that R satisfies the required conditions. It remains to verify that $\mathcal{S}(R)$ is not Boolean. We prove it indirectly, showing that $V_1 \cap V_2$ does not coincide with the integral closure of R, and then invoking Corollary 2.4. In fact, $x_1 \in V_1 \cap V_2$, and a straightforward exercise shows that x_1 is not integral over R. It is also worth noting that R is not Noetherian. For instance, it is easy to verify that the ideal $I = \langle yx_1^n : n \geq 0\rangle$ is not finitely generated.

The preceding example deals with local one-dimensional domains dominated by at least two distinct real valuations. Now we want to investigate $\mathcal{S}(R)$, assuming that R is dominated by a unique real valuation.

Recall that R is dominated by a unique Archimedean valuation domain V if, and only if the integral closure $\bar{R}$ of R contains a nonzero ideal of V ([33] Theorem 3.2). Examples where $\bar{R}$ contains a nonzero ideal of V, but R does not, are rather technical and difficult to find (see [9,33]). We will give a new interesting example of such an R, with $\mathcal{S}(R)$ not Clifford, in our final Example 3.6.

The problem of establishing when $\mathcal{S}(R)$ is Clifford has a complete answer, in the favorable but fairly general situation where R contains a nonzero ideal of a DVR V.

Theorem 3.5. *Let R be a local one-dimensional domain dominated by a DVR V, and assume that R contains a nonzero ideal of V.*

(i) If $[V/P : R/\mathfrak{M}] = 2$, then $\mathcal{S}(R)$ is Clifford if and only if $v(\mathfrak{M}) = 1$.

(ii) If $[V/P : R/\mathfrak{M}] = 1$, then $\mathcal{S}(R)$ is Clifford if and only if $v(\mathfrak{M}) \leq 2$. When $v(\mathfrak{M}) = 1$, we have $R = V$.

In both cases, if $\mathcal{S}(R)$ is Clifford, then every ideal of R may be generated by two elements, hence $\mathcal{S}(R)$ is Boolean.

Proof. Recall that V is necessarily the unique Archimedean valuation domain to dominate R ([33], Theorem 3.2). Let $P = xV$; then, by hypothesis, we have $R \supset x^kV$, for some $k > 0$.

(i) If $v(\mathfrak{M}) \geq 2$, then $\mathcal{S}(R)$ cannot be a Clifford semigroup, by Proposition 3.4.

Conversely, let $v(\mathfrak{M}) = 1$; then we may assume that the generator x of P lies in R. We need a preliminary observation. Let us fix any $\alpha \in U(V) \setminus R$. Then for every $\beta \in U(V)$ there exist $a, b \in R$ such that $\beta = a + b\alpha$. In fact, since $[V/P : R/\mathfrak{M}] = 2$, there exist $a_0, b_0 \in R$ and $\eta \in P$ such that

$$\beta = a_0 + b_0\alpha + \eta = a_0 + b_0\alpha + x^{t_1}\beta_1,$$

where $\beta_1 \in U(V)$ and $t_1 > 0$. We have $\beta_1 \equiv a_1 + b_1\alpha$ modulo P, for suitable $a_1, b_1 \in R$. Then, replacing β_1 in the above formula, we get a relation of the form $\beta = a_2 + b_2\alpha + x^{t_2}\beta_2$, where $t_2 > t_1$. Possibly repeating this procedure, we may also assume that $t_2 \geq k$. But then $x^{t_2}\beta_2 \in x^kV \subset R$, and we are done.

To reach the desired conclusion, we show that every nonzero fractional ideal I of R may be generated by two elements, and then we apply Proposition 1.1. Let I be a nonzero fractional ideal. Up to isomorphism, we may assume that zero is the minimal v-value of the elements of I, and that $1 \in I$. Since every fractional ideal is generated by its elements of minimal v-value, I is generated by a set $\mathcal{X}$ of units of V. If all these units lie in R, then $I = R$. Otherwise, $\mathcal{X}$ contains $\alpha \in U(V) \setminus R$. Then, as observed above, for any $\beta \in \mathcal{X}$ there exists $a, b \in R$ such that $\beta = a + b\alpha$, and hence $\beta \in \langle 1, \alpha\rangle$. It follows that $I = \langle 1, \alpha\rangle$, hence it is two-generated, as desired.

(ii) If $v(\mathfrak{M}) \geq 3$, then $\mathcal{S}(R)$ cannot be a Clifford semigroup, by Proposition 3.3.

Conversely, let us first consider the more difficult case when $v(\mathfrak{M}) = 2$. Since $P = xV$ and $v(\mathfrak{M}) = 2$, there exists $z \in \mathfrak{M}$ of the form $z = x^2u$, for a suitable $u \in U(V)$.

We first need to prove the following fact:

Every unit $\alpha \in U(V) \setminus R$ has the form $\alpha = a_0 + xz^na_1$, for suitable $a_0, a_1 \in U(R)$ and $n \geq 0$.

Let $\alpha \in U(V) \setminus R$. Since, presently, $[V/P : R/\mathfrak{M}] = 1$, there exists $a \in U(R)$ such that $v(\alpha - a) = m > 0$. Let us choose a such that $v(\alpha - a) = m$ is maximum. This is possible, since $x^kV \subset R$ implies $v(\alpha - a) < k$. Let us first show that we cannot have $m = 2n$ even. In fact, in such a case we should have $\alpha - a \in z^nV$, say $\alpha = a + z^n\beta$, for a suitable $\beta \in U(V) \setminus R$. Since $\beta - b \in P$, for some $b \in U(R)$, it follows that $v(\alpha - a - z^nb) > m$, against the choice of a. Then, necessarily, $m = 2n + 1$ is odd. Then $\alpha - a \in xz^nV$, say $\alpha = a_0 + xz^n\beta$, where $\beta \in U(V)$. Since $\beta \equiv b$ modulo P, for some $b \in U(R)$, we conclude that $v(\alpha - a - xz^nb) > 2n + 1$. Assume now, for a contradiction, that $\alpha \neq a_0 + xz^na_1$, for any choice of $a_0, a_1 \in U(R)$. Then there exist $c, d \in U(R)$ such that $v(\alpha - c - xz^nd) = t$ is maximum. In fact, $x^kV \subset R$, and our assumption implies $\alpha - c - xz^nd \notin R$. We have seen above that, necessarily, $t > 2n + 1$. Arguing as above, we see that $t = 2s$ even is not possible, otherwise we could find $e \in U(R)$ such that $v(\alpha - a - z^se - xz^nd) > t$. Then $t = 2s + 1$ is odd, and we may write $\alpha - c - xz^nd = xz^s\eta$, with $\eta \in U(V)$. Since $\eta \equiv e$ modulo P, for some $e \in U(R)$, we get $v(\alpha - c - xz^n(d + z^{s-n}e)) > t$, which contradicts t to be maximum, since $s > n$ implies $d + z^{s-n}e \in U(R)$. In both cases we have reached a contradiction, hence our claim follows.

Now we may show that every nonzero fractional ideal I of R is generated by two elements, which yields our conclusion, by Proposition 1.1. As in (i), we may assume, up to isomorphism, that I is generated by a set of units $\mathcal{X}$ of V, and that $1 \in \mathcal{X}$. We also assume $I \neq R$, hence $\mathcal{X}$ contains $\alpha \in U(V) \setminus R$. Then we may write $\alpha = a_0 + xz^na_1$, where $a_0, a_1 \in U(R)$ and $n \geq 0$. Assume that α has been chosen such that n is minimum. Note that $\langle 1, \alpha\rangle = \langle 1, xz^n\rangle$. Now pick any other $\beta \in \mathcal{X}$, with $\beta \notin R$. Then

$\beta = b_0 + xz^s b_1$, where $b_0, b_1 \in U(R)$ and $s \geq n$, by the minimality of n. It follows that $\beta = b_0 + b_1 z^{s-n}(xz^n) \in \langle 1, xz^n \rangle = \langle 1, \alpha \rangle$. Then $I = \langle 1, \alpha \rangle$, and the desired conclusion follows.

It remains to show that $V = R$, whenever $[V/P : R/\mathfrak{M}] = 1$ and $v(\mathfrak{M}) = 1$. Presently, we may assume $P = xV$, with $x \in R$. Pick any $r \in V$; then $r = x^n u$, where $u \in U(V)$ and $n \geq 0$. We have $u \equiv a$ modulo P, for a suitable $a \in U(R)$. Using $x \in R$ and $x^k V \subset R$, an argument similar to the preliminary observation in (i) shows that $u \in R$. Since $r \in V$ was arbitrary, we get $V \subseteq R$, as desired. □

As already observed in Example 1.2, the integral domain $R = K + X^2K[[X]]$ satisfies the hypothesis of Theorem 3.5 (ii).

The local stable domains constructed in [30,15,22,24] and discussed in Remark 4, provide examples of local almost perfect domains R dominated by a unique DVR V and not containing nonzero ideals of V. Actually, the integral closure of such an R is a DVR V, and $\mathcal{S}(R)$ is Boolean, since R is stable. Hence Theorem 3.5 shows that R cannot contain a nonzero ideal of V, since it admits ideals which are not two-generated.

It is important to remark that there is no hope to extend Theorem 3.5, just by asking if R is dominated by a unique DVR. In fact, in our final example we describe a local one-dimensional Noetherian domain R, dominated by a unique DVR $\hat{V}$, such that $[\hat{V}/P : R/\mathfrak{M}] = 1$, $v(\mathfrak{M}) = 1$, and $\mathcal{S}(R)$ is not Boolean. Thus R cannot contain a nonzero ideal of $\hat{V}$, by the preceding theorem. In the present circumstances, $\hat{V}$ coincides with the integral closure of R.

Example 3.6. Recall that Nagata, in his classical book [19], constructed an example of a field Q, of characteristic $p > 0$, endowed with a discrete real valuation v, such that its completion $\hat{Q}$, with respect to v, is a finite extension of Q (see [19], example (E3.1), p. 206). More precisely, $\hat{Q} = Q(u)$, where $u^p = a \in Q$. We may assume that $v(a) = 0$, i.e., a lies in the DVR V determined by v. We denote by $\hat{V} \subset \hat{Q}$ the completion of V, with respect to v. The extended valuation of $\hat{V}$ is still denoted by v. Let xV be the maximal ideal of V; then $x\hat{V}$ is the maximal ideal of $\hat{V}$. We suppose that $p \geq 3$.

Let $R = V[xu]$. Since $f^p \in V$ for every $f \in R$, we easily see that R is local with maximal ideal $\mathfrak{M} = \langle x, xu \rangle$. Since $\mathfrak{M}$ is finitely generated, R is Noetherian, hence almost perfect. Note that $\hat{Q}$ is the field of fractions of R and $R \subset \hat{V}$, hence $\hat{V}$ dominates R. From $\beta^p \in V$ for all $\beta \in \hat{V}$ it follows that $\hat{V} = \bar{R}$. Then R is one-dimensional, and $\hat{V}$ is the unique Archimedean valuation domain that dominates R. By construction, we have $v(\mathfrak{M}) = 1$. Moreover, $[\hat{V}/x\hat{V} : R/\mathfrak{M}] = 1$, since $\hat{V}$ is the completion of V, and so for any $\alpha \in U(\hat{V})$ and every $n > 0$, there exists $a_n \in U(V) \subset U(R)$ such that $\alpha - a_n \in x^n\hat{V}$.

To verify that $\mathcal{S}(R)$ is not a Clifford semigroup, we may use either Corollary 2.2, since $p \geq 3$ implies that u is not integral over R of degree two, or Theorem 3.5, since $\mathfrak{M}^2$ is not two-generated.

It is also interesting to note that we cannot find a result similar to Theorem 3.5, removing the hypothesis that V is a DVR and allowing $v(\mathfrak{M}) = 0$. This is shown in the next result. In the proof we will need the notion of maximal immediate extension of a valuation domain and of "Hahn formal power series", for which we refer the reader to [13], or to [31] for a more detailed exposition.

Proposition 3.7. *There exist local one-dimensional domains R satisfying the following properties:*

(i) *the integral closure of R is a nondiscrete Archimedean valuation domain V, and R contains a nonzero ideal of V;*

(ii) $[V/P : R/\mathfrak{M}] = 1$ *and* $v(\mathfrak{M}) = 0$, *where P is the maximal ideal of V and v its valuation;*

(iii) $\mathcal{S}(R)$ *is not a Clifford semigroup.*

Proof. Let K be a countable field, let

$$D = \left\{ f = \sum_{i=0}^{n} a_i x^{t_i} : a_i \in K,\ t_i \in \mathbb{Q}, 0 = t_0 < t_1 < \cdots < t_n \right\}$$

be the ring of the formal polynomials over K, with exponents in $\mathbb{Q}_{\geq 0}$, endowed with the convolution product. The degree of $f \in D$ is defined in the obvious way. For $f \in D$ we define $v(f) = \min\{t_i : a_i \neq 0\}$. This map gives rise to a valuation v on Q_0, the field of quotients of D (the usual "order" valuation). Let V_0 be the valuation domain of v. Now let V be the ring of the "Hahn formal power series" with coefficients in K and exponents in $\mathbb{Q}_{\geq 0}$. Then V is a maximal immediate extension of V_0. The valuation of V_0 extended to V is still denoted by v. Both V and V_0 are Archimedean nondiscrete valuation domains with value group $\mathbb{Q}$.

Let us fix a rational number $r > 0$, and let us define $R = V_0 + x^r V$. It is obvious that R and V have the same field of fractions Q. Let us verify that R is a local one-dimensional domain, whose maximal ideal $\mathfrak{M}$ consists of the elements $f \in R$ such that $v(f) > 0$. In fact, let us pick $f \in R$ with $v(f) = 0$. We may write $f = f_0 + x^r f_1$, where $f_0 \in V_0$ and $f_1 \in V$. Necessarily $v(f_0) = 0$, hence $f_0 \in U(V_0)$. Let $\eta = f_1 f_0^{-1}(1 + x^r f_1 f_0^{-1})^{-1} \in V$. Then we have $f^{-1} = f_0^{-1}(1 - x^r\eta) \in R$, as desired. It follows that V is unique among the Archimedean valuation domains dominating R, since R contains a nonzero ideal of V.

Let us show that every element of V is integral over R, hence V coincides with the integral closure and, in particular, R is one-dimensional. Take any $\eta \in V$. Since V is an immediate extension of V_0, there exists $f_0 \in V_0$ such that $v(\eta - f_0) > 0$. Then there exists an integer $k > 0$ such that $(\eta - f_0)^k \in x^r V \subset R$, which shows that η is integral over R.

By construction we have $v(\mathfrak{M}) = 0$. Moreover, since V is an immediate extension of V_0, for each $\alpha \in U(V)$ there exists $a \in U(V_0)$ such that $\alpha \equiv a$ modulo P. Since $V_0 \subset R$, we get $[V/P : R/\mathfrak{M}] = 1$. Moreover, an easy check shows that $V_0 = D + x^r V_0$, and hence $R = D + x^r V$.

It remains to show that $\mathcal{S}(R)$ is not a Clifford semigroup. Let us choose a rational number $s > 0$ such that $s < r/2$. Now we choose an increasing sequence $0 = s_0 < s_1 < \cdots < s_n < \cdots$ of rational numbers, whose sup is $s \in \mathbb{Q}$, and a formal power series $\alpha = \sum_{i=0}^{\infty} e_i x^{s_i} \in V$, where the $e_i \in K$ are such that α is transcendental over Q_0. This is certainly possible, since Q_0 is countable. In order to reach the desired conclusion, we will show that α is not integral of degree two over R. Assume for a contradiction that $\alpha^2 + f\alpha + g = 0$, where $f, g \in R$. Since $R = D + x^r V$, we may write $f = f_0 + x^r f_1$, $g = g_0 + x^r g_1$, where f_0, g_0 are formal polynomials of degree $< r$. Let us first observe that the monomials ax^t $(0 \neq a \in K)$, which appear in the expression of α^2, all satisfy the inequality $t < 2s < r$. Assume now that the formal polynomial f_0 contains some term, say bx^t $(0 \neq b \in K)$, where $t + s > r$. Since s is the sup of the s_i, we readily see that $bx^t\alpha \in D + x^r V$. Hence we may write $f_0\alpha = h_1\alpha + h_2 + x^r\beta$, where $h_1, h_2 \in D$, $\beta \in V$, the degree of h_2 is less than r, and the degree of h_1 is less than or equal to $r - s$. Thus we get the equality, valid in V,

$$\alpha^2 + h_1\alpha + h_2 + f_0 + g_0 = x^r\gamma,$$

where $\gamma \in V$ is a suitable formal power series. Now, by construction, the monomials cx^t $(0 \neq c \in K)$ which appear in the first member all satisfy $t < r$ (recall that the degree of h_1 is $\leq r - s$). This may happen only if $\alpha^2 + h_1\alpha + h_2 + f_0 + g_0 = 0$. But this is impossible, since α is transcendental over Q_0, the field of quotients of D.

The desired conclusion follows. $\square$

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