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ON THE MINIMALITY OF POWERS OF MINIMAL ω-BOUNDED ABELIAN GROUPS

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Abstract. We describe the structure of totally disconnected minimal ω-bounded abelian groups by reducing the description to the case of those of them which are subgroups of powers of the p-adic integers $\mathbb{Z}_p$. In this case the description is obtained by means of a functorial correspondence, based on Pontryagin duality, between topological and linearly topologized groups introduced by Tonolo. As an application we answer the question (posed in *Pseudocompact and countably compact abelian groups: Cartesian products and minimality*, Trans. Amer. Math. Soc. **335** (1993), 775–790) when arbitrary powers of minimal ω-bounded abelian groups are minimal. We prove that the positive answer to this question is equivalent to non-existence of measurable cardinals.

Introduction

All group topologies are assumed to be Hausdorff. We denote by $\hat{G}$ the two-sided (Raïkov) completion of a topological group G. A topological group G is *precompact* if $\hat{G}$ is compact, *pseudocompact* if every continuous real-valued function on G is bounded, *countably compact* if every open countable cover of G admits a finite subcover, *ω-bounded* if every countable subset of G has compact closure, *minimal* if every continuous group isomorphism $G \to H$ is open ([Ste]). Every compact group is minimal and ω-bounded, ω-bounded groups are countably compact, countably compact groups are pseudocompact. According to a deep theorem of Prodanov and Stoyanov minimal abelian groups are precompact ([PS]). In this paper we are interested mainly in ω-bounded minimal groups.

The first example of a ω-bounded minimal non-compact group was given by Comfort and Grant [CG]. The group they proposed was non-abelian, zero-dimensional, so in particular totally disconnected. An example of a totally disconnected minimal, ω-bounded non-compact abelian group was given in [DS3, Theorem 1.5]. A connected example was given in [DS2]; it was again non-abelian. In fact, item (b) of the following theorem from [D4] shows that the connected minimal countably compact abelian groups are "frequently" compact. (A cardinal α is *measurable* if there exists an ultrafilter on α which is closed under countable intersections.)

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Theorem A. *Let G be a countably compact abelian group with connected component $c(G)$.*

(a) *If $c(G)$ is compact, then G is minimal iff $G/c(G)$ is minimal.*

(b) *If G is minimal and $|c(G)|$ is not measurable, then $c(G)$ is compact.*

It should be stressed that the assumption on $|c(G)|$ in item (b) is very weak. In fact, the assumption that there exist no measurable cardinals is known to be consistent with ZFC (actually, [L=V] implies measurable cardinals do not exist [J]), while it is not known if their existence is consistent with ZFC. Examples of non-compact connected minimal ω-bounded abelian groups of every measurable cardinality are given in [D4].

It was proved in [DS3] that all finite powers of a countably compact minimal abelian group are minimal. The aim of this paper is to answer the following question ([DS3, Question 1.10]) in the case of ω-bounded groups.

Question B. Is G^ω minimal for a countably compact minimal abelian group G?

In the next theorem we answer positively Question B for minimal ω-bounded totally disconnected abelian groups (Corollary 2) and we show that the answer may be negative in the connected case if measurable cardinals exist (Corollary 3).

Main Theorem. *Let G be an ω-bounded minimal abelian group. Then G^ω is minimal iff $c(G)$ is compact.*

By this theorem G^ω is minimal for a connected ω-bounded minimal abelian group G iff G is compact.

The background of Question B is the important fact that if G^ω is minimal for a countably compact minimal abelian group, then *all* powers of G are minimal ([DS3, Corollary 1.9]). This permits one to define the *critical power of minimality* $\kappa(G)$ of a countably compact minimal abelian group G as 1 if all powers of G are minimal (i.e., G^ω is minimal), otherwise $\kappa(G)$ is the least cardinal λ such that G^λ is not minimal, i.e., $\kappa(G) = \omega$. The critical power of minimality $\kappa(\mathcal{P})$ of a class $\mathcal{P}$ of countably compact minimal abelian groups is defined as $\sup\{\kappa(G) : G \in \mathcal{P}\}$. This invariant was introduced and studied for larger classes of minimal abelian groups where it takes more than just two values, 1 and ω, as in the countably compact case (see [St], [D1], [DS1],[DS3] or [DPS, §6.3]). For example, it was shown in [DS3] that under the assumption of Martin's axiom (MA) the critical power of minimality of the class of pseudocompact minimal abelian groups is 2^ω. Recently ([D6]), a pseudocompact minimal abelian group G was found with $\kappa(G) = \omega_1$; moreover, under the assumption of MA, for every cardinal λ between ω and 2^ω a pseudocompact minimal abelian group was found with $\kappa(G) = \lambda$. This shows that κ provides a good measure to distinguish between pseudocompactness and ω-boundedness within the class of minimal abelian groups (see Corollary 2 and take into account that all pseudocompact minimal abelian groups mentioned above are also totally disconnected).

In view of Theorem A our Main Theorem entails:

Corollary 1. *Let G be an ω-bounded minimal abelian group such that $|c(G)|$ is not measurable. Then $\kappa(G) = 1$, i.e. all powers of G are minimal.*

By the Main Theorem G^ω is minimal for every totally disconnected ω-bounded minimal abelian group G. Hence we have:

Corollary 2. *The critical power of minimality of the class of totally disconnected ω-bounded minimal abelian groups is equal to 1.*

This corollary reduces the study of the critical power of minimality of the larger class of *all* ω-bounded minimal abelian groups to that of the connected ones. The proof of the next corollary is given in §2.

Corollary 3. *Let α denote the critical power of minimality of the class of all ω-bounded minimal abelian groups. Then:*

(a) *α is equal to the critical power of minimality of the class of ω-bounded minimal connected abelian groups,*
(b) *under the assumption that there exist no measurable cardinals, α is equal to 1,*
(c) *under the assumption that there exist measurable cardinals, α is equal to ω.*

In other words, Corollary 3 says that the assertion "$\alpha = 1$" is equivalent to non-existence of measurable cardinals.

The proof of the Main Theorem is obtained from Theorem A in two steps. The first one is a process of localization (described in §1) that permits one to describe locally the minimality of the powers (Fact 1.4) and reduces the study of totally disconnected ω-bounded minimal abelian groups to the particular case of minimal ω-bounded subgroups of powers of $\mathbb{Z}_p$ (Theorem 1.3 and Lemma 1.5). The second step is the following lemma that can be considered as the core of the proof of the Main Theorem.

Main Lemma. *Let α be a cardinal number and let p be a prime number. Then for every dense ω-bounded minimal subgroup G of $\mathbb{Z}_p^\alpha$ there exists $k \in \mathbb{N}$ such that $p^k\mathbb{Z}_p^\alpha \subseteq G$.*

The proof of the Main Lemma exploits a functorial correspondence based on Pontryagin duality (see §2). This correspondence was developed by the second author to solve a longstanding problem of Warner on the existence of a finest equivalent linear module topology ([T1]) and to study minimality and total minimality in topological modules covered by compact submodules ([T2]). Through this correspondence (see Lemma 2.1) the Main Lemma is translated in an equivalent question on linear topologies on direct sums of the Prüfer group $\mathbb{Z}(p^\infty)$.

The proof of the next corollary of the Main Lemma concerning the structure of totally disconnected ω-bounded minimal abelian groups is given in §2.

Corollary 4. *Let G be a totally disconnected ω-bounded minimal abelian group. Then there exists a compact subgroup N of G such that the quotient group G/N is a direct product (provided with the product topology) of ω-bounded torsion minimal groups G_p.*

We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of naturals and primes, respectively, by $\mathbb{Z}$ the integers, by $\mathbb{R}$ the reals, by $\mathbb{Z}_p$ the p-adic integers ($p \in \mathbb{P}$). For undefined symbols or notions see [E], [HR] or [J].

1. The localization – quasi-p-torsion elements

The following minimality criterion of Banaschewski-Prodanov-Stephenson is very important. A subgroup H of a topological group G is *essential* if for every non-trivial closed normal subgroup N of G the intersection $H \cap N$ is non-trivial.

1.1 Theorem ([B], [P], [Ste])**.** *Let G be a topological group and H be a dense subgroup of G. Then H is minimal iff G is minimal and H is essential in G.*

1.2 Definition ([St])**.** For $p \in \mathbb{P}$ and a topological abelian group G an element $x \in G$ is *quasi-p-torsion* if $\langle x \rangle$ is either a finite p-group or equipped with the induced topology is isomorphic to $(\mathbb{Z}, \tau_p)$, where τ_p is the p-adic topology of $\mathbb{Z}$.

The set $td_p(G)$ of all quasi-p-torsion elements of G is a subgroup of G ([DPS, Chap.4]).

Let us recall that ω-bounded groups are precompact and quasi-p-torsion precompact abelian groups are totally disconnected ([D2]).

We show next that in analogy with the compact case, every ω-bounded minimal totally disconnected abelian group G decomposes into a direct product of its quasi p-torsion subgroups $td_p(G)$. This settles the classification of the totally disconnected minimal ω-bounded abelian groups by reduction to the case of quasi-p-torsion minimal ω-bounded abelian groups. The counterpart of the following theorem for the case of countably compact groups is given in [D4]. Here we give a proof in the ω-bounded case which is different, entirely based on the corresponding property of the compact totally disconnected abelian groups.

1.3 Theorem. *Let G be a ω-bounded totally disconnected abelian group. Then for each $p \in \mathbb{P}$ the subgroup $td_p(G)$ is closed and*

$$G = \prod_{p\in\mathbb{P}} td_p(G) \tag{1}$$

with the product topology. Moreover, G is minimal iff each $td_p(G)$ is minimal.

Proof. It is known that for a compact totally disconnected abelian group K the subgroup $td_p(K)$ is closed for each $p \in \mathbb{P}$ and $K = \prod_{p\in\mathbb{P}} td_p(K)$ with the product topology (see [DPS, Example 4.1.3 (a)]). Applying this fact to the compact group $K = \hat{G}$ we get $\hat{G} = \prod_{p\in\mathbb{P}} td_p(\hat{G})$ with the product topology. Moreover, for each $p \in \mathbb{P}$ the subgroup $td_p(G)$ of G is closed as $td_p(G) = G \cap td_p(\hat{G})$. To prove (1) let us first observe that G, being ω-bounded, is covered by compact subgroups, and for every one of them, say H, we have $H = \prod_{p\in\mathbb{P}} td_p(H) \subseteq \prod_{p\in\mathbb{P}} td_p(G)$ since $td_p(H) \subseteq td_p(G)$. This proves the inclusion $\subseteq$ in (1). To prove the other inclusion observe that $\oplus td_p(G) \subseteq G$, hence an arbitrary element $x = (x_p) \in \prod_{p\in\mathbb{P}} td_p(G)$ can be presented as an element of the closure (taken in $\hat{G}$) of a countably generated subgroup of G. Since countably generated subgroups of G are contained in compact subgroups of G, this gives $x \in G$. So the inclusion $\supseteq$ in (1) is proved as well.

The last assertion follows from Corollary 6.1.3 of [DPS] (see also [D1]). □

For a topological abelian group G, a prime number p and $k \in \mathbb{N}$, G is *strongly p-dense (of degree k)* iff there exists $k \in \mathbb{N}$ with $p^k td_p(\hat{G}) \subseteq G$ (see [D1, Definition 1] or [DS1, p. 586]). For a quasi-p-torsion abelian group G this condition simplifies to $p^k\hat{G} \subseteq G$. Non-compact strongly p-dense quasi-p-torsion ω-bounded minimal abelian groups exist in profusion [DS3, Corollary 1.6].

The following fact explains the relation of strong p-density to minimality of powers.

1.4 Fact ([St], [D1, Corollaire 7]). Let G be a minimal abelian group. Then all powers of G are minimal iff G is strongly p-dense for every prime number p.

It follows from the minimality criterion, that the completion $\hat{G}$ of a torsion-free minimal abelian group G is torsion-free. Since torsion-free quasi-p-torsion compact abelian groups are of the form $\mathbb{Z}_p^\alpha$, the torsion-free quasi-p-torsion ω-bounded minimal abelian groups are subgroups of powers of $\mathbb{Z}_p$. Hence our Main Lemma says that *a torsion-free quasi-p-torsion ω-bounded minimal abelian group must be strongly p-dense.* Actually, by means of the localization developed in Theorem 1.3 and Lemma 1.5 we prove more: *every totally disconnected ω-bounded minimal abelian group is strongly p-dense for every prime p.*

Now we reduce the study of quasi-p-torsion ω-bounded minimal abelian groups to the study of those which are torsion-free, i.e., subgroups of powers of $\mathbb{Z}_p$.

1.5 Lemma. *Fix a prime p. Let G be a minimal quasi-p-torsion ω-bounded abelian group. Then, with $\alpha = w(G)$, there exist a dense minimal ω-bounded subgroup G_1 of $\mathbb{Z}_p^\alpha$ and a compact subgroup N of G_1 such that:*
(a) the quotient group G_1/N is isomorphic to G;
(b) G and G_1 have the same degree of strong p-density.

Proof. Since $\hat{G}$ is a compact $\mathbb{Z}_p$-module, there exists a continuous surjective homomorphism $f : K = \mathbb{Z}_p^\alpha \to \hat{G}$ (see [DPS]). Then $G_1 = f^{-1}(G)$ is a dense ω-bounded subgroup of K ([DS2, Lemma 2.1]). Minimality of G_1 follows from Theorem 1.1 and the fact that G_1 is essential in K.

Set $N = \ker f$; then $G \cong G_1/N$ by the Sulley-Grant lemma ([DPS, Lemma 4.3.2]). This proves (a).

To prove (b) note that $p^k\hat{G} \subseteq G$ is equivalent to $p^k\mathbb{Z}_p^\alpha \subseteq G_1$. □

2. Proof of the main results

We give briefly an outline of the functorial correspondence between precompact groups covered by their compact subgroups and linearly topologized groups developed in [T1] and [T2]. Let $\mathbb{T}$ be the circle group $\mathbb{R}/\mathbb{Z}$ endowed with the usual topology. The Pontryagin duality associates to each compact group K the abstract group $K^* = \mathrm{Chom}(K, \mathbb{T})$ of continuous homomorphisms of K in $\mathbb{T}$, to each abstract group X the group $X^* = \mathrm{Hom}(X, \mathbb{T})$ of homomorphisms of X in $\mathbb{T}$, endowed with the pointwise-convergence topology, and to each homomorphism, of compact or abstract groups, its transpose.

Let G be a precompact abelian group such that every element of G is contained in a compact subgroup of G. Then the family $\mathcal{M} = \{M_\lambda : \lambda \in \Lambda\}$ of compact separable subgroups of G (i.e., the subgroups which have a dense countable subset) is directed with respect to inclusion and $G = \bigcup_\lambda M_\lambda$. Let X be the Pontryagin dual of the compact completion $\hat{G}$ of G. For each $M \in \mathcal{M}$, consider the annihilator $Ann(M) = \{\xi \in X : \xi(M) = 0\}$. The family $\{Ann(M) : M \in \mathcal{M}\}$ is a filter base of subgroups of X; hence it defines a linear topology τ on X, i.e. a group topology which has a base of neighbourhoods of zero consisting of subgroups.

The idea is to express properties of G in terms of properties of (X, τ).

Lemma 2.1. *Let G, X and τ be as above.*

(i) *The group G is strongly p-dense with degree k if and only if for every subgroup L of X with $X/L \leq \mathbb{Z}(p^\infty)$ the subgroup $(L : p^k) = \{x \in X : p^k x \in L\}$ is τ-closed.*
(ii) *The group G is minimal if and only if (X, τ) has no dense proper subgroups.*

(iii) *The group G is ω-bounded if and only if the space (X, τ) is a P-space, i.e., countable intersections of τ-open sets are τ-open.*

Proof. (i) Consider the algebraic monomorphism $X/L \to \mathbb{Z}(p^\infty)$. The Pontryagin duality produces a surjective continuous homomorphism $\psi : \mathbb{Z}_p \to \operatorname{Ann} L$. The element $x = \psi(1)$ belongs to $td_p(\hat{G})$ and $(x) = \overline{\langle x \rangle} = \operatorname{Ann} L$. If G is strongly p-dense with degree k, then $p^k x \in G$, hence $(p^k x) = p^k(x) \leq G$. Now $p^k(x)$ is a separable compact subgroup of G, hence

$$\operatorname{Ann} p^k(x) = (\operatorname{Ann}(x) : p^k) = (L : p^k)$$

is τ-open, hence τ-closed. Conversely, given $x \in td_p(\hat{G})$, let us consider the surjective continuous epimorphism $\psi : \mathbb{Z}_p \to (x)$, defined by $\psi(1) = x$. The Pontryagin duality produces the monomorphism

$$(x)^* = X/\operatorname{Ann}(x) \to \mathbb{Z}(p^\infty).$$

By hypothesis $(\operatorname{Ann}(x) : p^k) = \operatorname{Ann} p^k(x)$ is τ-closed; since $X/[\operatorname{Ann} p^k(x)] \leq \mathbb{Z}(p^\infty)$ is Hausdorff and $\mathbb{Z}(p^\infty)$ is finitely cogenerated, $\operatorname{Ann} p^k(x)$ is also τ-open. Then $p^k(x)$ is contained in a (separable compact) subgroup of G; in particular $p^k x$ belongs to G.

(ii) Also the family of annihilators of all compact subgroups of G defines a linear topology σ on X, clearly finer than τ. By Proposition 2.2 (see the proof of point i)) of [T2], it is possible to verify that σ and τ are *equivalent*, i.e. they determine the same closed subgroups of X. In [T2, Theorem 3.1] the second author has proved that G is essential in $\hat{G}$ if and only if (X, σ) has no dense proper subgroups. Since τ and σ are equivalent, G is essential in $\hat{G}$ if and only if (X, τ) has no dense proper subgroups. Then we conclude by Theorem 1.1.

(iii) Clearly, the topology τ is a P-topology if and only if the family $\mathcal{M}$ of compact separable subgroups of G is ω-directed, i.e., for every countable family $\{L_i : i \in \mathbb{N}\}$ of $\mathcal{M}$ there exists $L \in \mathcal{M}$ containing all L_i. Let us prove that this happens if and only if the group G is ω-bounded. Let G be ω-bounded and let $\{L_i : i \in \mathbb{N}\}$ be a family in $\mathcal{M}$. For each $i \in \mathbb{N}$, denote by J_i a countable dense subset of L_i. Then the set $\bigcup_{i \in \mathbb{N}} J_i$ is countable and $L = \overline{\bigcup_{i \in \mathbb{N}} J_i} \in \mathcal{M}$ contains every L_i. Conversely, suppose the family $\mathcal{M}$ is ω-directed. Let J be a countable subset of G. Then $\overline{\langle j \rangle} \in \mathcal{M}$ for each $j \in J$. By our hypothesis there exists a compact subgroup $L \in \mathcal{M}$ that contains $\overline{\langle j \rangle}$ for every $j \in J$. Hence the closure of J in G is compact. □

Proof of the Main Lemma. Let G be a dense ω-bounded minimal subgroup of $\mathbb{Z}_p^\alpha$. Then by [DS3, Lemma 3.1] every element of G is contained in a compact subgroup of G. The Pontryagin dual of $\mathbb{Z}_p^\alpha$ is $\mathbb{Z}(p^\infty)^{(\alpha)}$, so that by Lemma 2.1, after dualizing, our Main Lemma takes the following form. Consider $\mathbb{Z}(p^\infty)^{(\alpha)}$ equipped with a linear Hausdorff topology τ satisfying the following conditions:

i) there exist no proper τ-dense subgroups of $(\mathbb{Z}(p^\infty)^{(\alpha)}, \tau)$;

ii) countable intersections of τ-open sets are τ-open.

For every subgroup L of $\mathbb{Z}(p^\infty)^{(\alpha)}$ with $[\mathbb{Z}(p^\infty)^{(\alpha)}]/L \cong \mathbb{Z}(p^\infty)$ there exists $n = n_L \in \mathbb{N}$ such that $(L : p^n)$ is τ-open. This is the number n determined by the τ-closure $\overline{L}$ of L: by (i) it cannot be the whole $\mathbb{Z}(p^\infty)^{(\alpha)}$, so that $\overline{L}/L$ must be isomorphic to a proper subgroup, say $\mathbb{Z}(p^n)$, of Prüfer's group. We have to prove that *the number n can be chosen independently of L.*

Let us denote by H_n the unique subgroup of $\mathbb{Z}(p^\infty)$ of order p^n. Suppose there exist subgroups $L_1, \dots, L_n, \dots$ of $\mathbb{Z}(p^\infty)^{(\alpha)}$ such that $[\mathbb{Z}(p^\infty)^{(\alpha)}]/L_i \cong \mathbb{Z}(p^\infty)$ and such that the orders $|\overline{L_i}/L_i| = p^{n_i}$ form a strictly increasing sequence of natural numbers. It is not restrictive to suppose $|\overline{L_i}/L_i| = p^i$. The subgroups L_i are kernels of morphisms

$$f_i : \mathbb{Z}(p^\infty)^{(\alpha)} \to \mathbb{Z}(p^\infty).$$

Since

$$Hom(\mathbb{Z}(p^\infty)^{(\alpha)}, \mathbb{Z}(p^\infty)) \cong [Hom(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty))]^\alpha$$

and

$$Hom(\mathbb{Z}(p^\infty), \mathbb{Z}(p^\infty)) \cong \mathbb{Z}_p,$$

for each f_i we have some $\mu_i = (\mu_{i,x})_{x\in\alpha}$ belonging to $\mathbb{Z}_p^\alpha$ such that

$$f_i((a_x)_{x\in\alpha}) = \sum_{x\in\alpha} \mu_{i,x} a_x.$$

Since the only Hausdorff linear topology on $\mathbb{Z}(p^\infty)$ is the discrete one, every $\overline{L_i}$ is open. Then, by the hypothesis ii) on τ, $W = \bigcap_{i\in\mathbb{N}} \overline{L_i}$ is an open subgroup of $\mathbb{Z}(p^\infty)^{(\alpha)}$. Clearly, for each $i \in \mathbb{N}$ and each open subgroup $U \leq W$, we have $L_i + U = \overline{L_i}$. Since $\overline{L_i}/L_i \cong H_i$, for each open subgroup $U \leq W$

$$f_i(\overline{L_i}) = f_i(L_i + U) = f_i(L_i) + f_i(U) = f_i(U) = H_i$$

hold. For $u \in \mathbb{Z}(p^\infty)^{(\alpha)}$ denote by $|u|$ the order of u. For each $i \in \mathbb{N}$, there exists a natural number $\nu(i) > 0$ such that for every open subgroup U there exists $u \in U$ with $|u| < \nu(i)$ such that $f_i(u)$ generates H_i. Otherwise, for each $n \in \mathbb{N}$ there would be an open subgroup U_n such that for no element $u \in U_n$ with $|u| < n$, does $f_i(u)$ generate H_i; then, $U_\infty = \bigcap_{n\in\mathbb{N}} U_n$ would be an open subgroup with $f_i(U_\infty)$ properly contained in H_i: a contradiction.

Now for $i \in \mathbb{N}$ define $\psi(i) = \max\{\nu(i^2), i+1\}$. Since

$$\psi(1) < \psi^2(1) < \cdots < \psi^j(1) < \dots, \tag{2}$$

the series

$$\mu_1 + \sum_{j\geq 1} p^{\psi^j(1)} \mu_{[\psi^j(1)]^2}$$

defines an element $\overline{\mu} \in \mathbb{Z}_p^\alpha$.

Let L be the kernel of the morphism f associated to $\overline{\mu}$, i.e.,

$$f : \mathbb{Z}(p^\infty)^{(\alpha)} \to \mathbb{Z}(p^\infty), \quad (a_x)_{x\in\alpha} \mapsto \sum_x \overline{\mu}_x a_x.$$

Let us prove that L is a proper dense subgroup of $\mathbb{Z}(p^\infty)^{(\alpha)}$ contradicting the hypotheses.

For every $U \leq W$ and every $j \in \mathbb{N}$ we have

$$p^{\psi^j(1)} f_{[\psi^j(1)]^2}(U) = p^{\psi^j(1)} H_{[\psi^j(1)]^2} = H_{[\psi^j(1)]^2 - \psi^j(1)}.$$

Moreover, there exists $u^{(j)} \in U$ with $|u^{(j)}| < \nu([\psi^j(1)]^2)$, such that $p^{\psi^j(1)} f_{[\psi^j(1)]^2}(u^{(j)})$ generates $H_{[\psi^j(1)]^2-\psi^j(1)}$. Now we prove that also $f(u^{(j)})$ generates $H_{[\psi^j(1)]^2-\psi^j(1)}$.

Indeed,

$$f(u^{(j)}) = \sum_x \overline{\mu}_x u_x^{(j)} = \sum_x (\mu_{1,x} + \sum_{i\geq 1} p^{\psi^i(1)} \mu_{[\psi^i(1)]^2,x}) u_x^{(j)}$$

$$= \left[\sum_x (\mu_{1,x} + \sum_{i=1}^{j-1} (p^{\psi^i(1)} \mu_{[\psi^i(1)]^2,x}) u_x^{(j)}) \right] + \sum_x (p^{\psi^j(1)} \mu_{[\psi^j(1)]^2,x}) u_x^{(j)}$$

$$+ \left[\sum_x (\sum_{i\geq j+1} (p^{\psi^i(1)} \mu_{[\psi^i(1)]^2,x}) u_x^{(j)}) \right].$$

The first summand is contained in $H_{[\psi^{j-1}(1)]^2-\psi^{j-1}(1)}$, the second one generates $H_{[\psi^j(1)]^2-\psi^j(1)}$ and the third one vanishes since for each $l \geq 1$ we have

$$\psi^{j+l}(1) = \psi^l(\psi^j(1)) > \psi(\psi^j(1)) \geq \nu([\psi^j(1)]^2) > |u^{(j)}|.$$

By (2) the sequence $[\psi^j(1)]^2 - \psi^j(1)$, $j \in \mathbb{N}$, is strictly increasing (note that the function $x \mapsto x^2 - x$ is strictly increasing for $x > 1$). Hence the family $\{f(u^{(j)}) : j \in \mathbb{N}\}$ generates $\mathbb{Z}(p^\infty)$.

We have proved in this way that for each open subgroup U contained in W, $f(U) = \mathbb{Z}(p^\infty)$. Therefore L is dense − a contradiction. □

Proof of Main Theorem. Since $C = c(G)$ is connected, its compact completion $\hat{C}$ is connected, hence divisible ([HR]). Now the minimality of G^ω yields that C^ω is minimal, as a closed subgroup of the minimal group G^ω. By [DS3, Corollary 1.9] this gives $\kappa(C) = 1$, hence, by Fact 1.4, C is strongly p-dense for every prime p. The divisibility of $\hat{C}$ yields that $td_p(\hat{C})$ is divisible for every prime p ([DPS, Proposition 4.1.2]). Hence $td_p(\hat{C}) \subseteq C$ for every prime p. Then C is totally minimal ([DPS, Theorem 4.3.7]). Since C is also countably compact, it follows from [DS2] that C is compact.

Suppose now that C is compact and let α be a cardinal. The quotient $H = G/C$ is totally disconnected and ω-bounded; moreover $c(G^\alpha) = C^\alpha$ and $H^\alpha \cong G^\alpha / c(G^\alpha)$. By Theorem A, $H = G/C$ is a minimal group. If we prove that H^α is minimal, Theorem A will imply that the group G^α is minimal as well. Hence it suffices to consider only the totally disconnected case.

From now on we assume G is an ω-bounded totally disconnected minimal abelian group. According to Fact 1.4 the minimality of all powers G^α for such group G follows from the strong p-density of G for every prime p. By Theorem 1.3 the subgroup $td_p(G)$ of quasi-p-torsion elements of G is still minimal and ω-bounded, moreover, $\overline{td_p(G)}$ coincides with $td_p(\hat{G})$. Therefore, strong p-density for G is equivalent with strong p-density for the quasi-p-torsion group $td_p(G)$. By Lemma 1.5 it suffices to consider quasi-p-torsion groups that are subgroups of powers of $\mathbb{Z}_p$. In such a case the strong p-density of G was established in our Main Lemma. □

Proof of Corollary 3. (a) If G^ω is not minimal for some ω-bounded abelian group G, then $c(G)$ is not compact by Main Theorem. Hence $c(G)$ is a connected ω-bounded minimal abelian group that is not compact, hence $c(G)^\omega$ is not minimal again by Main Theorem.

(b) Follows from Corollary 2 and Theorem A.

(c) For the connected ω-bounded minimal abelian group G of measurable weight constructed in [D4] the power G^ω is not minimal according to Main Theorem since G is not compact. □

Proof of Corollary 4. Let $f : K = \prod_{p\in\mathbb{P}} \mathbb{Z}_p^\sigma \to \hat{G}$ be a continuous surjective homomorphism (see [DPS]). Then $H = f^{-1}(G)$ is a dense ω-bounded minimal subgroup of K. Set $K_p = \mathbb{Z}_p^\sigma$; then for each p the subgroup $H_p = H \cap K_p$ of K_p is an essential ω-bounded subgroup of K_p. So by the Main Theorem there exists a natural k_p such that $p^{k_p}K_p \subseteq H_p$ for each p. Then for $N_1 = \prod p^{k_p+1}K_p$ and $N = f(N_1)$ obviously $G/N \cong K/N_1 \cong \prod_p K_p/(N_1 \cap K_p)$ and each quotient $K_p/(N_1 \cap K_p)$ is obviously a compact torsion abelian group. Moreover, every $f(H_p)$ is minimal as a dense subgroup of $L_p = K_p/p^{k_p+1}K_p \cong \mathbb{Z}(p^{k_p+1})^\sigma$ containing the socle $f(p^{k_p}K_p)$ of L_p. □

It follows from this corollary and Theorem A that the connected component C of an ω-bounded minimal abelian group G of non-measurable cardinality is compact and contained in a compact subgroup N of G such that the quotient G/N is a direct product (provided with the product topology) of ω-bounded torsion minimal groups.

We finish by showing a possible way of extension of our Main Lemma to the countably compact case. The following result on approximation by large ω-bounded subgroups was proved in [D4]:

2.2 Theorem. *Every minimal torsion-free connected countably compact abelian group G contains a minimal connected ω-bounded subgroup G_ω such that its closure $\overline{G_\omega}$ is a G_δ-subgroup of G.*

We do not know if one can replace "connected" by "totally disconnected" in this theorem. In case the answer is positive, then the Main Lemma implies that for every prime p, a torsion-free quasi-p-torsion countably compact minimal abelian group G must be strongly p-dense. In fact, $\hat{G} \cong \mathbb{Z}_p^\alpha$. Let O_n, $n \in \mathbb{N}$, be open subgroups of $\hat{G}$ such that for $V = \bigcap_n O_n$ we have $\overline{G_\omega} = G \cap V$. Since G is G_δ-dense in $\hat{G}$ (as G is pseudocompact, cf. [CR]), it follows that $\overline{G_\omega}$ is dense in V, i.e., V is the completion of G_ω. By the Main Lemma there exists $k \in \mathbb{N}$ such that $p^kV \subseteq G_\omega \subseteq G$. By the definition of the Tychonov topology of $\mathbb{Z}_p^\alpha = \hat{G}$, there exists a countable $D \subseteq \alpha$, such that $U = \{0\} \times \mathbb{Z}_p^{\alpha\setminus D} \subseteq V$. Since $U \subseteq V$, obviously $p^kU \subseteq G$. Now consider the closed subgroup $G_1 := G \cap \mathbb{Z}_p^D$ of G. By the minimality of G, it is essential in $\mathbb{Z}_p^D$. Since G_1 is compact (as a metrizable subgroup of a countably compact group), it is closed in $\mathbb{Z}_p^D$ as well. Now the essentiality means that the quotient group $\mathbb{Z}_p^D/G_1$ is torsion. Since it is compact abelian, it must have a finite exponent p^m. Thus $p^m\mathbb{Z}_p^D \subseteq G$. Then, with $n = \max\{k, m\}$, we get $p^n\hat{G} \subseteq G$.

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