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Quasi-Tilting Modules and Counter Equivalences

Riccardo Colpi*

Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Belzoni 7, 35131 Padova, Italy

Gabriella D'Este†

Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy

and

Alberto Tonolo‡

Istituto di Ingegneria Gestionale, Università di Padova, Viale X Guigno 22, 36100 Vicenza, Italy

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Given two rings R and S, we study the category equivalences $\mathcal{T} \rightleftarrows \mathcal{Y}$, where $\mathcal{T}$ is a torsion class of R-modules and $\mathcal{Y}$ is a torsion-free class of S-modules. These equivalences correspond to quasi-tilting triples (R, V, S), where ${}_RV_S$ is a bimodule which has, "locally," a tilting behavior. Comparing this setting with tilting bimodules and, more generally, with the torsion theory counter equivalences introduced by Colby and Fuller, we prove a local version of the Tilting Theorem for quasi-tilting triples. A whole section is devoted to examples in case of algebras over a field.

0. INTRODUCTION

Let R and S be rings, and denote by R-Mod and S-Mod the categories of left R-modules and S-modules. Given two torsion theories $(\mathcal{T}, \mathcal{F})$ and

* E-mail: colpi@math.unipd.it.
† E-mail: deste@vmimat.mat.unimi.it.
‡ E-mail: tonolo@math.unipd.it.

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$(\mathcal{X}, \mathcal{Y})$ in R-Mod and S-Mod, respectively, we say that the couple of functors $R\text{-Mod} \underset{G}{\overset{F}{\rightleftarrows}} S\text{-Mod}$ *induces* an equivalence between $\mathcal{T}$ and $\mathcal{Y}$ if the restrictions of F and G give a category equivalence between $\mathcal{T}$ and $\mathcal{Y}$ and the kernels of F and G coincide with $\mathcal{F}$ and $\mathcal{X}$, respectively. Moreover, we say that this equivalence is *represented* by the bimodule ${}_RV_S$ if it is induced by the functors $F = \mathrm{Hom}_R(V_S, -)$ and $G = {}_RV \otimes_S -$.

In a recent paper [CbF2], Colby and Fuller have investigated the existence of functors between R-Mod and S-Mod which induce pairs of equivalences, one between $\mathcal{T}$ and $\mathcal{Y}$, the other between $\mathcal{X}$ and $\mathcal{F}$. If such functors exist, the torsion theories $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are said to be *counter equivalent* and the pair of equivalences is said to be a *torsion theory counter equivalence*.

Since $(R\text{-Mod}, \{0\})$ and $(\{0\}, S\text{-Mod})$ are obviously torsion theories, Morita equivalences are examples of torsion theory counter equivalences. Nevertheless, the main examples follow from tilting theory: if ${}_RT_S$ is a tilting bimodule, the functors $F = \mathrm{Hom}_R(T, -)$, $G = T \otimes_S -$ and $F' = \mathrm{Ext}^1_R(T, -)$, $G' = \mathrm{Tor}^S_1(T, -)$ induce a counter equivalence between the torsion theories $(\mathrm{Ker}\, F', \mathrm{Ker}\, F)$ in R-Mod and $(\mathrm{Ker}\, G, \mathrm{Ker}\, G')$ in S-Mod.

In [CbF2] it is proved that a torsion theory counter equivalence is represented by a pair of bimodules. Moreover, necessary and sufficient conditions are given on a pair of bimodules to represent a torsion theory counter equivalence. Influenced by this paper, generalizing this point of view, we have studied the existence of category equivalences between a torsion class in R-Mod and a torsion-free class in S-Mod. These equivalences are shown to be represented by bimodules which have, "locally," a tilting behavior. Moreover, these bimodules are tilting exactly when every injective R-module is torsion and every projective S-module is torsion-free.

In Section 1 we recall the definition and the principal results about tilting modules, characterizing them by means of the equivalences that they represent (Theorem 1.5). We introduce a notion of cotilting module (Definition 1.6), and we prove that the cotiltings are the Ext-injective modules in the torsion-free classes cogenerated by them (Proposition 1.7).

In Section 2 we introduce quasi-tilting modules (Definition 2.2): they generalize tiltings as quasi-progenerators [F] generalize progenerators, i.e., the bimodules representing Morita equivalences. Indeed, a quasi-tilting module ${}_RV$ has, in the category $\overline{\mathrm{Gen}}({}_RV)$ of modules subgenerated by ${}_RV$, similar properties to tilting modules in R-Mod (Proposition 2.1). In [HRS] Happel, Reiten, and Smalø introduce the notion of tilting objects for an abelian category. Even if $\overline{\mathrm{Gen}}({}_RV)$ is an abelian full subcategory of R-Mod, nevertheless quasi-tiltings are not tilting objects in the sense of [HRS], since they may have high projective dimension (Example 5.4). If the ring is either finitely cogenerated or commutative, then a module is tilting if and only if it is faithful and quasi-tilting (Corollary 2.4). Similarly to the tilting case, if ${}_RQ$ is an injective cogenerator and ${}_RV$ is a quasi-tilting module, then $V^* = \mathrm{Hom}_R(V, Q)$ is a cotilting $\mathrm{End}({}_RV)$-module (Corollary 2.8). Next, generalizing the notion of a tilting triple, we introduce quasi-tilting triples. These triples (R, V, S) represent, by means of the bimodule ${}_RV_S$, any equivalence between a torsion class in R-Mod and a torsion-free class in S-Mod (Theorem 2.6).

In Section 3 we study the torsion theory counter equivalences which are given by a tilting module. In particular, we characterize them (Theorem 3.4) as those involving torsion theories (T, F) and (X, Y) such that one of the following conditions holds: (a) T cogenerates R-Mod and Y generates S-Mod; (b) T and Y are faithful, and T is closed under direct products; (c) the counter equivalence is induced by the covariant hom, tensor, ext, and tor functors associated to a single bimodule.

In Section 4 we prove that a local version of [CbF1, The Tilting Theorem] still holds true for quasi-tilting triples (R, V, S). More precisely, the equivalence $\mathsf{T} \rightleftarrows \mathsf{Y}$ represented by (R, V, S) can be completed, by means of the covariant ext and tor functors associated to ${}_RV_S$, to two couples of functors inducing a counter equivalence between torsion theories $(\mathsf{T}, \mathsf{F}')$ in $\overline{\mathrm{Gen}}({}_RV)$ and $(\mathsf{X}', \mathsf{Y})$ in $S/\mathrm{Ann}(V_S)$-Mod (Theorem 4.1). Next, comparing [CbF2] with our setting (Corollaries 4.3 and 4.4), we obtain a criterion (Theorem 4.6) on a pair of quasi-tilting triples to represent a torsion theory counter equivalence, which is close to [CbF2, Theorem 2.5].

In Section 5 we collect examples and counter-examples, confining ourselves to algebras over a fixed field. We stress the fact that these algebras are very special from several points of view. Indeed, with the exception of two cases (Examples 5.1 and 5.3), we deal only with finite-dimensional algebras of finite representation type. Moreover, with the exception of one case (Example 5.7), these representation-finite algebras are also directed [R]. As we shall see, it suffices to consider algebras with very few indecomposable modules to show that there are more than expected torsion theory counter equivalences. It actually turns out that the theory developed over arbitrary rings does not fade in this particular setting. On the contrary, we may often use finite-dimensional algebras to make sure that our results cannot be improved.

We recall now some definitions and notation used throughout the paper.

All rings have nonzero identity and all modules are unitary. If R is a ring, and $L \in R$-Mod, then:

$\mathrm{Gen}(L)$ (resp. $\mathrm{Cogen}(L)$) denotes the class of all left R-modules *generated* (resp. *cogenerated*) by L, that is, all $M \in R$-Mod such that there exist a cardinal λ and an epimorphism $L^{(\lambda)} \xrightarrow{\phi} M \to 0$ (resp. a monomorphism $0 \to M \xrightarrow{\phi} L^{\lambda}$).

Pres(L) (resp. Copres(L)) denotes the subclass of Gen(L) (resp. Cogen(L)) consisting of all left R-modules M which are *presented* (resp. *copresented*) by L; that is, there exists an exact sequence of the form $L^{(\mu)} \xrightarrow{\psi} L^{(\lambda)} \xrightarrow{\phi} M \to 0$ (resp. $0 \to M \xrightarrow{\phi} L^{\lambda} \xrightarrow{\psi} L^{\mu}$), so that M and $\operatorname{Ker}(\phi) = \operatorname{Im}(\psi)$ (resp. $\operatorname{Coker}(\phi) \cong \operatorname{Im}(\psi)$) are generated (resp. cogenerated) by L.

add(L) denotes the subclass of Pres(L) consisting of all summands of finite direct sums of copies of L.

$\operatorname{Tr}_L(M)$ (resp. $\operatorname{Rej}_L(M)$) denotes the *trace* $\sum\{\operatorname{Im}(f) \mid f \in \operatorname{Hom}_R(L, M)\}$ (resp. the *reject* $\bigcap\{\operatorname{Ker}(f) \mid f \in \operatorname{Hom}_R(M, L)\}$) of L in M, that is, the largest (resp. smallest) submodule M_0 of M such that $M_0 \in \operatorname{Gen}(L)$ (resp. $M/M_0 \in \operatorname{Cogen}(L)$).

$L^{\perp}$ (resp. ${}^{\perp}L$) denotes the class of all left R-modules M such that $\operatorname{Ext}^1_R(L, M) = 0$ (resp. $\operatorname{Ext}^1_R(M, L) = 0$).

$E(L)$ denotes the injective envelope of L.

A module $M \in R$-Mod is *self-small* if $\operatorname{Hom}_R(M, M^{(\lambda)}) \cong \operatorname{Hom}_R(M, M)^{(\lambda)}$ canonically for each cardinal λ. Of course, every finitely generated module is self-small, but the converse is not true in general [FuS, Lemma 24].

If $\mathcal{C}$ is a class of left R-modules, we denote by $\overline{\mathcal{C}}$ the smallest subclass of R-Mod containing $\mathcal{C}$ and closed under taking submodules and factor modules.

All the subcategories are full subcategories of modules, and all the functors are additive functors.

1. TILTINGS AND COTILTINGS

1.1. DEFINITION. A module ${}_RT$ is a *tilting module* if:

(i) ${}_RT$ is finitely presented and $\operatorname{proj\,dim}({}_RT) \leq 1$, i.e., there is an exact sequence $0 \to R' \to R'' \to T \to 0$ with $R', R'' \in \operatorname{add}(R)$;

(ii) $\operatorname{Ext}^1_R(T, T) = 0$;

(iii) there is an exact sequence $0 \to R \to T' \to T'' \to 0$ with $T', T'' \in \operatorname{add}(T)$.

It can be shown (see [C2, Theorem 3]) that, in Definition 1.1, condition (ii) can be replaced by

(ii′) $\operatorname{Ext}^1_R(T, T^{(k)}) = 0$ for each cardinal k

and condition (iii) can be replaced by

(iii′) for all $M \in R$-Mod, if $\operatorname{Hom}_R(T, M) = 0 = \operatorname{Ext}^1_R(T, M)$, then $M = 0$.

For instance, every progenerator (= finitely generated projective generator) is a tilting module.

In [CT, Proposition 1.3(iii)] the following result is proved:

1.2. PROPOSITION. *A module* ${}_RT$ *is tilting if and only if* ${}_RT$ *is self-small (in fact finitely generated) and* $\mathrm{Gen}({}_RT) = T^{\perp}$.

Hence, a tilting ${}_RT$ is a self-small module which is Ext-projective exactly in the class of all modules generated by ${}_RT$. In particular, $\mathrm{Gen}({}_RT)$ is a torsion class containing the injective modules, so that the corresponding torsion theory is hereditary if and only if $\mathrm{Gen}({}_RT) = R$-Mod. Moreover, every module $M \in \mathrm{Gen}({}_RT)$ has a T-resolution $\cdots \to T^{(\alpha_1)} \to T^{(\alpha_0)} \to M \to 0$, owing to the following:

1.3. PROPOSITION [CT, Lemma 1.2]. *If* $\mathrm{Gen}({}_RT) = T^{\perp}$, *then* $\mathrm{Gen}({}_RT) = \mathrm{Pres}({}_RT)$.

1.4. DEFINITION. A torsion class $\mathbf{T} \subseteq R$-Mod is called a *tilting torsion class* if $\mathbf{T}$ is generated by a tilting module.

Given a bimodule ${}_RT_S$, following [CbF1], we say that (R, T, S) is a *tilting triple* if ${}_RT$ is a tilting module and $S = \mathrm{End}({}_RT)$. As proved in [CbF1, Proposition 1.1], in this case ${}_RT_S$ is a faithfully balanced bimodule and T_S is a tilting module, too.

Tilting triples (R, T, S) characterize the equivalences between a torsion class of R-Mod containing the injectives and a torsion-free class of S-Mod containing the projectives:

1.5. THEOREM. *Let R and S be rings and let* ${}_RT_S$ *be a bimodule. Then the following conditions are equivalent*:

(i) (R, T, S) *is a tilting triple*;

(ii) *the functors* $\mathrm{Hom}_R(T, -)$ *and* $T \otimes_S -$ *give an equivalence between a torsion class* $\mathbf{T} \subseteq R$-Mod *containing the injective modules and a torsion-free class* $\mathbf{Y} \subseteq S$-Mod *containing the projective modules*.

In such a case $\mathbf{T} = \mathrm{Gen}({}_RT) = T^{\perp}$ *and* $\mathbf{Y} = \mathrm{Cogen}({}_ST^*) = {}^{\perp}T^*$, *where* ${}_ST^* = \mathrm{Hom}_R(T_S, Q)$ *for an arbitrary injective cogenerator* ${}_RQ$ *of* R-Mod.

Proof. It follows from [CbF1, Theorem 1.4] and [C2, Proposition 7], since ${}^{\perp}T^* = \mathrm{Ker}\,\mathrm{Tor}_1^S(T, -)$, by the canonical isomorphism $\mathrm{Ext}_S^1(-, T^*) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^S(T, -), Q)$ [CE, Proposition 5.1]. ∎

In analogy with [CbF1, Section 2], we give the following:

1.6. DEFINITION. A module ${}_RW$ is a *cotilting module* if:

(i) $\mathrm{inj\,dim}({}_RW) \leq 1$;

(ii) $\mathrm{Ext}_R^1(W^k, W) = 0$ for each cardinal k;

(iii) for all $M \in R$-Mod, if $\mathrm{Hom}_R(M, W) = 0 = \mathrm{Ext}^1_R(M, W)$, then $M = 0$.

For instance, every injective cogenerator is a cotilting module.

Note that in the above definition we cannot replace condition (iii) by the dual of Definition 1.1(iii) (see Example 5.3(c)). However, except for finiteness conditions, the notion of cotilting module is dual to that of tilting module (tiltings without finiteness conditions have been studied in [CT], and cotiltings are further investigated in [CTT]). Therefore, in light of Proposition 1.2, the following result is not surprising:

1.7. PROPOSITION. *${}_RW$ is a cotilting module if and only if* $\mathrm{Cogen}({}_RW) = {}^{\perp}W$.

Proof. Let ${}_RW$ be cotilting. Then ${}^{\perp}W$ is closed under submodules because of (i), and ${}_RW^k \in {}^{\perp}W$ for all k because of (ii). Therefore $\mathrm{Cogen}({}_RW) \subseteq {}^{\perp}W$. Now, let $M \in {}^{\perp}W$. Applying $\mathrm{Hom}_R(-, W)$ to the exact sequence

$$0 \to \mathrm{Rej}_W(M) \to M \to M/\mathrm{Rej}_W(M) \to 0,$$

we obtain

$$0 \to \mathrm{Hom}_R\big(M/\mathrm{Rej}_W(M), W\big) \overset{\cong}{\to} \mathrm{Hom}_R(M, W) \to \mathrm{Hom}_R\big(\mathrm{Rej}_W(M), W\big)$$
$$\to \mathrm{Ext}^1_R\big(M/\mathrm{Rej}_W(M), W\big) \to \mathrm{Ext}^1_R(M, W) \to \mathrm{Ext}^1_R\big(\mathrm{Rej}_W(M), W\big) \to 0,$$

where $\mathrm{Ext}^1_R(M/\mathrm{Rej}_W(M), W) = 0$ because $W/\mathrm{Rej}_W(M) \in \mathrm{Cogen}({}_RW) \subseteq {}^{\perp}W$ and $\mathrm{Ext}^1_R(M, W) = 0$ by assumption. Thus, $\mathrm{Hom}_R(\mathrm{Rej}_W(M), W) = 0 = \mathrm{Ext}^1_R(\mathrm{Rej}_W(M), W)$, therefore $\mathrm{Rej}_W(M) = 0$ by (iii). This proves that $M \in \mathrm{Cogen}({}_RW)$.

Conversely, if $\mathrm{Cogen}({}_RW) = {}^{\perp}W$, then conditions (ii) and (iii) are clearly satisfied. Moreover ${}^{\perp}W$ contains every projective module and it is closed under submodules. Therefore, for every module ${}_RM$ and exact sequence $0 \to K \to P \to M \to 0$ in R-Mod with P projective, we get the exact row

$$0 = \mathrm{Ext}^1_R(K, W) \to \mathrm{Ext}^2_R(M, W) \to \mathrm{Ext}^2_R(P, W) = 0$$

that produces $\mathrm{Ext}^2_R(M, W) = 0$. This proves (i). ∎

The last result shows that a cotilting ${}_RW$ is a module which is Ext-injective exactly in the class of all modules cogenerated by ${}_RW$. In particular, $\mathrm{Cogen}({}_RW)$ is a torsion-free class containing the projective modules, and the corresponding torsion theory can be hereditary and nontrivial (see

$(\mathcal{X}, \mathcal{Y})$ in Example 5.6). Moreover, every module $M \in \operatorname{Cogen}({}_RW)$ has a W-coresolution $0 \to M \to W^{\alpha_0} \to W^{\alpha_1} \to \cdots$, owing to the following:

1.8. PROPOSITION. *If* $\operatorname{Cogen}({}_RW) = {}^{\perp}W$, *then* $\operatorname{Cogen}({}_RW) = \operatorname{Copres}({}_RW)$.

Proof. Let $M \in \operatorname{Cogen}({}_RW)$ and $X = \operatorname{Hom}_R(M, W)$. Let η: $M \to W^X$ be the diagonal morphism $\eta(m) = (x(m))_{x \in X}$. Since M is cogenerated by ${}_RW$, η is injective. From the exact sequence

$$0 \to M \overset{\eta}{\to} W^X \to C = \operatorname{Coker}(\eta) \to 0,$$

we get the exact sequence

$$\operatorname{Hom}_R(W^X, W) \overset{\eta^*}{\to} \operatorname{Hom}_R(M, W) \to \operatorname{Ext}^1_R(C, W) \to \operatorname{Ext}^1_R(W^X, W) = 0.$$

The morphism η^* is surjective by construction. Thus we have $\operatorname{Ext}^1_R(C, W) = 0$, i.e., $C \in \operatorname{Cogen}({}_RW)$. ∎

1.9. DEFINITION. A torsion-free class $\mathcal{Y} \subseteq R$-Mod is called a *cotilting torsion-free class* if $\mathcal{Y}$ is cogenerated by a cotilting module.

1.10. *Remark.* Every tilting torsion class is equivalent, as a category, to a cotilting torsion-free class. In fact, if (R, T, S) is a tilting triple, then from Theorem 1.5 and Proposition 1.7 it follows that $\mathcal{T} = \operatorname{Gen}({}_RT)$ is equivalent to $\mathcal{Y} = \operatorname{Cogen}({}_ST^*)$, where ${}_RT$ is tilting and ${}_ST^*$ is cotilting. Even in this case, it can happen that ${}_ST^*$ is neither finitely generated nor finitely cogenerated: Example 5.1 shows that $\operatorname{Cogen}({}_ST^*) \neq \operatorname{Cogen}({}_SM)$ for any finitely generated or finitely cogenerated module ${}_SM$.

2. QUASI-TILTING MODULES

In [MO], Menini and Orsatti introduced a class of modules, later called $*$-modules, that generalizes both quasi-progenerators [F] and tiltings. A module ${}_RV$ is a $*$-module provided the functors $\operatorname{Hom}_R(V, -)$ and $V \otimes_S -$ give an equivalence between $\operatorname{Gen}({}_RV)$ and $\operatorname{Cogen}({}_SV^*)$, where $S = \operatorname{End}({}_RV)$. The following result introduces a subclass of $*$-modules that generalizes tilting modules similarly as the notion of quasi-progenerator extends that of progenerator.

2.1. PROPOSITION. *Let* ${}_RV$ *be a module. The following conditions are equivalent*:

(i) ${}_RV$ *is a* $*$*-module and* $\operatorname{Gen}({}_RV)$ *is a torsion class*;

(ii) ${}_RV$ *is a* $*$*-module and* $\operatorname{Gen}({}_RV) \subseteq V^{\perp}$;

(iii) ${}_RV$ *is self-small and* $\mathrm{Pres}({}_RV) = \mathrm{Gen}({}_RV) \subseteq V^{\perp}$;

(iv) ${}_RV$ *is finitely generated and* $\overline{\mathrm{Gen}}({}_RV) \cap V^{\perp} = \mathrm{Gen}({}_RV)$.

Proof. (i) ⇔ (ii) It is [C1, Proposition 4.4].

(ii) ⇔ (iii) It follows from [C1, Theorem 4.1, (1) ⇔ (3)].

(ii) ⇒ (iv) By [T], ${}_RV$ is finitely generated, and, by hypothesis, $\mathrm{Gen}({}_RV) \subseteq \overline{\mathrm{Gen}}({}_RV) \cap V^{\perp}$. Conversely, let $M \in \overline{\mathrm{Gen}}({}_RV) \cap V^{\perp}$. Then there is a short exact sequence $0 \to M \to M' \to M'' \to 0$, where $M' \in \mathrm{Gen}({}_RV)$, and $\mathrm{Ext}^1_R(V, M) = 0$. From [C1, Proposition 4.3], it follows that $M \in \mathrm{Gen}({}_RV)$.

(iv) ⇒ (ii) Clearly, $\mathrm{Gen}({}_RV) \subseteq V^{\perp}$. Let $M \leq {}_RV^{(X)}$. Since $M \in \overline{\mathrm{Gen}}({}_RV)$, we have that $M \in \mathrm{Gen}({}_RV)$ if and only if $M \in V^{\perp}$. This means that condition (5) of [C1, Theorem 4.1] is satisfied, so that ${}_RV$ is a $*$-module. ∎

2.2. DEFINITION. A module ${}_RV$ which satisfies the equivalent conditions of Proposition 2.1 is called a *quasi-tilting* module. A torsion class in R-Mod generated by a quasi-tilting module is called a *quasi-tilting torsion class*.

If ${}_RT$ is tilting, then $\overline{\mathrm{Gen}}({}_RT) = R$-Mod, as every injective module is generated by ${}_RT$. Therefore, comparing Proposition 1.2 to Proposition 2.1(iv), we obtain that each tilting module is quasi-tilting. Moreover, we can say—roughly speaking—that a module ${}_RV$ is quasi-tilting if and only if ${}_RV$ is "tilting in $\overline{\mathrm{Gen}}({}_RV)$." This situation is analogous to that of quasi-progenerators, which can be considered "progenerators in $\overline{\mathrm{Gen}}({}_RV)$."

The main aim of this section is to characterize the equivalences between a torsion class in R-Mod and a torsion-free class in S-Mod by means of the covariant hom and tensor functors associated to a quasi-tilting module. The ideas and techniques involved in this project have been suggested by studying recent results of Colby and Fuller in [CbF2].

We start with a comparison between tiltings and quasi-tiltings:

2.3. PROPOSITION. *Let ${}_RV$ be a quasi-tilting module. Then ${}_RV$ is a tilting module if and only if one of the following equivalent conditions hold*:

(i) $\overline{\mathrm{Gen}}({}_RV) = R$-Mod, *i.e.*, ${}_RR \in \overline{\mathrm{Gen}}({}_RV)$;

(ii) $\mathrm{Gen}({}_RV)$ *contains every injective left R-module*;

(iii) $E({}_RR) \in \mathrm{Gen}({}_RV)$;

(iv) ${}_RV$ *is faithful and* $\mathrm{Gen}({}_RV)$ *is closed under direct products*;

(v) ${}_RV$ *is faithful and V is finitely generated as an* $\mathrm{End}({}_RV)$*-module*;

(vi) *there is an exact sequence* $0 \to {}_RR \to {}_RV^n$ *for some* $n \in \mathbf{N}$;

(vii) $\mathrm{Gen}({}_RV)$ *is a tilting torsion class*.

Proof. It follows from Proposition 2.1, [CM, Proposition 1.5], [C2, Theorem 3], and [CT, Proposition 2.5]. ∎

2.4. COROLLARY. *Let R be a ring, ${}_RV$ a left R-module, and set $\overline{R} = R/\mathrm{Ann}({}_RV)$. If either*

(i) *$\overline{R}$ is finitely cogenerated, or*

(ii) *there exists a finite spanning set $\{v_1, \dots, v_n\}$ for V over the commutator of $\mathrm{Ann}_R(v_1, \dots, v_n)$,*

then $\overline{\mathrm{Gen}}({}_RV) = \overline{R}$-Mod.

If, moreover, ${}_RV$ is a quasi-tilting module, then ${}_{\overline{R}}V$ is a tilting module.

In particular, if R is either finitely cogenerated or commutative, then the class of tilting modules coincides with the class of faithful quasi-tilting modules.

Proof. In case (i), the position $r + \mathrm{Ann}({}_RV) \mapsto (rv)_{v \in V}$ defines a monomorphism $\overline{R} \to V^V$. As $\overline{R}$ is finitely cogenerated, by [AF, Proposition 10.2] there exists a monomorphism $\overline{R} \to V^n$, for some $n \in \mathbf{N}$. Similarly, in case (ii), the position $r + \mathrm{Ann}({}_RV) \mapsto (rv_1, \dots, rv_n)$ defines a monomorphism $\overline{R} \to V^n$. In both cases, $\overline{R} \in \overline{\mathrm{Gen}}({}_RV) \subseteq \overline{R}$-Mod, so that $\overline{\mathrm{Gen}}({}_RV) = \overline{R}$-Mod.

If, moreover, ${}_RV$ is a quasi-tilting module, then ${}_{\overline{R}}V$ is quasi-tilting, too, so that Proposition 2.3 applies to ${}_{\overline{R}}V$.

The last sentence is now clear. ∎

Comparing tiltings to quasi-tiltings, the question of measuring the gap between the three conditions in Definition 1.1 and the notion of quasi-tilting naturally arises. If ${}_RV$ is quasi-tilting, then condition (ii) clearly holds true. On the other hand, by Proposition 2.3(vi), condition (iii) is strong enough to imply that ${}_RV$ is tilting, and condition (i) is quite far, as Examples 5.3 and 5.4 show.

Let ${}_RV_S$ be a bimodule. Generalizing the notion of tilting triple, we say that (R, V, S) is a *quasi-tilting triple* if ${}_RV$ is a quasi-tilting module, $S/\mathrm{Ann}(V_S) \cong \mathrm{End}({}_RV)$, and $V \otimes_S \mathrm{Ann}(V_S) = 0$. Let us prove that quasi-tilting triples (R, V, S) characterize the equivalences between a torsion class in R-Mod and a torsion-free class in S-Mod, improving Theorem 1.5. First, we need the following:

2.5. LEMMA. *Let V_S be a right S-module and $I = \mathrm{Ann}(V_S)$. Let us consider the torsion class $\mathsf{X} = \{{}_SL \mid V \otimes_S L = 0\}$ in S-Mod, and let $\mathrm{t}(-)$ and Y be, respectively, the associated radical and torsion-free class. Then the following conditions are equivalent*:

(i) $I = \mathrm{t}(S)$;

(ii) $I \in \mathsf{X}$, *i.e.*, $V \otimes_S I = 0$;

(iii) $\operatorname{Tor}_1^S(V, S/I) = 0$;

(iv) $I \subseteq \operatorname{Ann}_S(\mathcal{Y})$, *i.e.*, $\mathcal{Y} \subseteq S/I$-Mod;

(v) $\overline{\mathcal{Y}} = S/I$-Mod.

Proof. (i) ⇒ (ii) It is trivial.

(ii) ⇔ (iii) From the exact sequence $0 \to I \to S \to S/I \to 0$, we get the exact row

$$0 \to \operatorname{Tor}_1^S(V, S/I) \overset{\cong}{\to} V \otimes_S I \overset{0}{\to} V \otimes_S S \overset{\cong}{\to} V \otimes_S S/I \to 0.$$

(ii) ⇒ (iv) Suppose that there is $Y \in \mathcal{Y}$ such that $IY \neq 0$, and let $Iy \neq 0$, with $y \in Y$. Then the right multiplication by y is a nonzero element of $\operatorname{Hom}_S(I, Y)$, a contradiction.

(iv) ⇒ (i) As $S/\mathrm{t}(S)$ is an element of $\mathcal{Y}$, by hypothesis we have that $I \subseteq \mathrm{t}(S)$. Conversely, $V \otimes_S \mathrm{t}(S) = 0$ implies $V\mathrm{t}(S) = 0$, i.e., $\mathrm{t}(S) \subseteq I$.

(i) & (iv) ⇒ (v) By (i) we have that $S/I \in \mathcal{Y}$, thence $S/I\text{-Mod} \subseteq \overline{\mathcal{Y}}$. The other inclusion follows from (iv).

(v) ⇒ (iv) It is trivial. ∎

2.6. THEOREM. (I) *Let R and S be rings, $\mathcal{T}$ a torsion class in R-*Mod, *$\mathcal{Y}$ a torsion-free class in S-*Mod, *and suppose that $\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ is a category equivalence. Let $I = \operatorname{Ann}_S(\mathcal{Y})$; then $S/I \in \mathcal{Y}$. Denote by Q an injective cogenerator of R-*Mod, *and let ${}_RV_S = T({}_SS/I_S)$ and ${}_SV^* = H(\mathrm{t}_{\mathcal{T}}({}_RQ))$. Then*:

(a) *(R, V, S) is a quasi-tilting triple and $I = \operatorname{Ann}(V_S)$*;

(b) *$H \cong \operatorname{Hom}_R(V, -)$ and $T \cong V \otimes_S -$*;

(c) *$\mathcal{T} = \operatorname{Gen}({}_RV)$ and $\mathcal{Y} = \operatorname{Cogen}({}_SV^*)$.*

(II) *Let (R, V, S) be a quasi-tilting triple. Let $I = \operatorname{Ann}(V_S)$, $H = \operatorname{Hom}_R(V, -)$, $T = V \otimes_S -$, and ${}_SV^* = H({}_RQ)$, where Q is a fixed injective cogenerator in R-*Mod. *Then*:

(a) *$\mathcal{T} = \operatorname{Gen}({}_RV)$ is a torsion class in R-*Mod *and $\mathcal{Y} = \operatorname{Cogen}({}_SV^*)$ is a torsion-free class in S-*Mod *with $I = \operatorname{Ann}_S(\mathcal{Y})$*;

(b) *$\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ is a category equivalence.*

(III) *Assume that* (I) *or* (II) *holds. Then $(\mathcal{T}, \operatorname{Ker}\operatorname{Hom}_R(V, -))$ and $(\operatorname{Ker} V \otimes_S -, \mathcal{Y})$ are torsion theories. Moreover, the following equalities hold*:

$$\mathcal{Y} = S/I\text{-Mod} \cap \operatorname{Ker}\operatorname{Tor}_1^S(V, -) = \operatorname{Ker}\operatorname{Tor}_1^{S/I}(V, -)$$

and $\overline{\mathcal{Y}} = S/I$-Mod.

Proof. (I) Under our hypotheses, [CbF2, Lemma 2.1]—which generalize [MO, Theorem 3.1]—works. Hence (b) and (c) are proved, and $S/I \cong \mathrm{End}({}_RV)$. Hence $\mathcal{Y} = \mathrm{Cogen}({}_SV^*)$ and ${}_SV^* \cong \mathrm{Hom}_R({}_RV_S, {}_RQ)$, so that $I = \mathrm{Ann}_S(\mathcal{Y}) = \mathrm{Ann}({}_SV^*) = \mathrm{Ann}(V_S)$. Moreover, ${}_RV$ is a $*$-module and $\mathrm{Gen}({}_RV)$ is a torsion class; thus, by Proposition 2.1(i), we have that ${}_RV$ is quasi-tilting. To complete the proof of (a), we show that $V \otimes_S I = 0$. Let $\mathcal{X}$ be the torsion class in S-Mod associated to $\mathcal{Y}$. Then ${}_SN \in \mathcal{X}$ if and only if $\mathrm{Hom}_S(N, V^*) = 0$. Since $\mathrm{Hom}_S(-, V^*) \cong \mathrm{Hom}_S(-, \mathrm{Hom}_R(V, Q)) \cong \mathrm{Hom}_R(V \otimes_S -, Q)$, we have that $\mathrm{Ker}\,\mathrm{Hom}_S(-, V^*) = \mathrm{Ker}\, V \otimes_S -$. Thus ${}_SN \in \mathcal{X}$ if and only if $V \otimes_S N = 0$. Hence we may apply Lemma 2.5, (iv) $\Rightarrow$ (ii), to obtain the thesis.

(II) By Proposition 2.1(i), $\mathrm{Gen}({}_RV)$ is a torsion class in R-Mod and ${}_RV$ is a $*$-module. Moreover, $S/I \cong \mathrm{End}({}_RV)$, so that

$$\mathrm{Gen}({}_RV) \underset{{}_RV\otimes_{S/I} -}{\overset{\mathrm{Hom}_R({}_RV_{S/I}, -)}{\rightleftarrows}} \mathrm{Cogen}({}_{S/I}V^*)$$

is an equivalence, and $\mathrm{Cogen}({}_{S/I}V^*)$ is a torsion-free class in S/I-Mod (see [CM, Proposition 1.2]). We can regard $\mathrm{Cogen}({}_{S/I}V^*) = \mathrm{Cogen}({}_SV^*)$ as a subcategory of S-Mod, obtaining (b). In order to prove that $\mathrm{Cogen}({}_SV^*)$ is a torsion-free class in S-Mod too, we have to check that it is closed under extensions. Let

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence in S-Mod, where $N', N'' \in \mathrm{Cogen}({}_SV^*)$. It remains to be proved that N belongs to S/I-Mod. Let us consider the torsion class $\mathcal{X} = \{{}_SL \mid V \otimes_S L = 0\}$ in S-Mod. As $\mathrm{Ker}\,\mathrm{Hom}_S(-, V^*) = \mathrm{Ker}\, V \otimes_S -$, we have that $\mathrm{Hom}_S(\mathcal{X}, V^*) = 0$. Then $\mathrm{Cogen}({}_SV^*)$ is $\mathcal{X}$-torsion-free. Therefore, N is $\mathcal{X}$-torsion-free, too. As $V \otimes_S I = 0$ by hypothesis, and Lemma 2.5, (ii) $\Rightarrow$ (iv), applies, we obtain $N \in S/I$-Mod.

(III) Since the torsion class $\mathcal{T}$ is generated by ${}_RV$, the corresponding torsion-free class coincides with the kernel of $\mathrm{Hom}_R(V, -)$. Moreover, the torsion-free class $\mathcal{Y}$ is cogenerated by ${}_SV^*$, and $\mathrm{Ker}\,\mathrm{Hom}_S(-, V^*) = \mathrm{Ker}\, V \otimes_S -$; hence the corresponding torsion class coincides with the kernel of $V \otimes_S -$. Thus, we may apply Lemma 2.5, (iv) $\Rightarrow$ (v), to obtain $\overline{\mathcal{Y}} = S/I$-Mod. Since ${}_RV$ is a $*$-module and $S/I \cong \mathrm{End}({}_RV)$, by [CM, Proposition 1.2] it follows that $\mathcal{Y} = \mathrm{Cogen}({}_{S/I}V^*) = \mathrm{Ker}\,\mathrm{Tor}_1^{S/I}(V, -)$. It remains to prove that a module $N \in S/I$-Mod belongs to $\mathcal{Y}$ if and only if $\mathrm{Tor}_1^S(V, N) = 0$. Let $0 \to K \xrightarrow{i} S/I^{(X)} \to N \to 0$ be exact in S/I-Mod. Since $S/I \in \mathcal{Y}$, we have that $S/I^{(X)}$ and K are in $\mathcal{Y}$. Applying $V \otimes_S -$

to the previous exact sequence, and using Lemma 2.5 to obtain $\mathrm{Tor}_1^S(V, S/I) = 0$, we get the exact row

$$0 \to \mathrm{Tor}_1^S(V, N) \to V \otimes_S K \xrightarrow{V \otimes_S i} V \otimes_S S/I^{(X)} \to V \otimes_S N \to 0.$$

Since $V \otimes_S - \cong V \otimes_{S/I} -$ in S/I-Mod, repeating an argument similar to [CM, Proposition 1.2], we can see that $N \in \mathcal{Y}$ if and only if $V \otimes_S i$ is a monomorphism, i.e., $\mathrm{Tor}_1^S(V, N) = 0$. ∎

2.7. *Remarks*. (a) By Theorem 2.6, if $\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ is the equivalence associated to the quasi-tilting triple (R, V, S), then it is represented (see Section 0) by the bimodule ${}_RV_S$. Moreover, using Proposition 2.1, the following identities hold:

$$\overline{\mathcal{T}} = \overline{\mathrm{Gen}}({}_RV), \qquad \overline{\mathcal{Y}} = \mathrm{End}({}_RV)\text{-Mod},$$

$$\mathcal{T} = \overline{\mathcal{T}} \cap \mathrm{Ker}\,\mathrm{Ext}_R^1(V, -), \qquad \mathcal{Y} = \overline{\mathcal{Y}} \cap \mathrm{Ker}\,\mathrm{Tor}_1^S(V, -).$$

This means that the two abelian categories really involved by (R, V, S) are $\overline{\mathrm{Gen}}({}_RV)$ and $\mathrm{End}({}_RV)$-Mod, rather than R-Mod and S-Mod. This will be definitely confirmed by Theorem 4.1.

(b) It is easy to see that if (R, V, S) is a quasi-tilting triple and I, J are ideals respectively of R and of S, such that $I \leq \mathrm{Ann}({}_RV)$ and $J \leq \mathrm{Ann}(V_S)$, then $(R/I, V, S/J)$ is a quasi-tilting triple too. In particular, if ${}_RV$ is a quasi-tilting module, we can put $\overline{R} = R/\mathrm{Ann}({}_RV)$ and $\overline{S} = \mathrm{End}({}_RV)$, obtaining a quasi-tilting triple $(\overline{R}, V, \overline{S})$, where ${}_{\overline{R}}V_{\overline{S}}$ is faithful on both sides.

(c) The notion of quasi-tilting triple is not left−right symmetric. In fact, [CM, Example 1.6] gives a faithfully balanced bimodule ${}_RV_S$ such that ${}_RV$ is a projective quasi-progenerator and V_S is not finitely generated. Then $\mathrm{Gen}({}_RV) = \overline{\mathrm{Gen}}({}_RV) \neq R$-Mod (see [F, Lemma 2.2] and [AF, Lemma 17.7]). By Proposition 2.1(iv), (R, V, S) is a quasi-tilting triple, but V_S is not finitely generated, hence not quasi-tilting. Moreover, ${}_RV_S$ represents an equivalence between $\mathrm{Gen}({}_RV)$ and S-Mod (see [F, Theorem 2.6]), which obviously cannot be extended to a counter equivalence between R-Mod and S-Mod. Therefore, the existence of an equivalence between a torsion class and a torsion-free class does not assure that the corresponding torsion-free and torsion classes are equivalent too (see also Remark 5.9(1) and Example 5.10).

By Theorem 2.6, we get the following generalization of Remark 1.10:

2.8. COROLLARY. *Let* (R, V, S) *be a quasi-tilting triple and let* $\mathcal{T} \underset{V \otimes_S -}{\overset{\mathrm{Hom}_R(V, -)}{\rightleftarrows}} \mathcal{Y}$ *be the represented equivalence. Then* $\mathcal{T}$ *is a quasi-tilting torsion class in* R-Mod *and* $\mathcal{Y}$ *is a cotilting torsion-free class in* $\mathrm{End}({}_RV)$-Mod.

Proof. By Theorem 2.6, $\mathcal{T} = \mathrm{Gen}({}_RV)$ is clearly a quasi-tilting torsion class. Since $\mathrm{End}({}_RV) \cong S/I$, with $I = \mathrm{Ann}_S(\mathcal{Y})$, by [CE, Proposition 5.1] we have, for every $N \in S/I$-Mod,

$$\mathrm{Ext}^1_{S/I}(N, V^*) = \mathrm{Ext}^1_{S/I}(N, \mathrm{Hom}_R(V, Q)) \cong \mathrm{Hom}_R\big(\mathrm{Tor}_1^{S/I}(V, N), Q\big).$$

Therefore, by Theorem 2.6,

$$\mathcal{Y} = \mathrm{Cogen}\big({}_{S/I}V^*\big) = \mathrm{Ker}\,\mathrm{Tor}_1^{S/I}(V, -) = \mathrm{Ker}\,\mathrm{Ext}^1_{S/I}(-, V^*) = {}^{\perp}_{S/I}V^*,$$

and ${}_{S/I}V^*$ is a cotilting module by Proposition 1.7. ∎

3. TORSION THEORY COUNTER EQUIVALENCES

One of the deepest results in tilting theory is a considerable generalization of Morita equivalence. A faithfully balanced progenerator ${}_RP_S$ represents an equivalence between R-Mod and S-Mod. Similarly, a tilting triple (R, T, S) gives a pair of category equivalences

$$\mathcal{T} \underset{T\otimes_S -}{\overset{\mathrm{Hom}_R(T,-)}{\rightleftarrows}} \mathcal{Y} \qquad \text{and} \qquad \mathcal{F} \underset{\mathrm{Tor}_1^S(T,-)}{\overset{\mathrm{Ext}^1_R(T,-)}{\rightleftarrows}} \mathcal{X},$$

where $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are torsion theories, respectively, in R-Mod and in S-Mod, canonically associated to (R, T, S) (see [CbF1, The Tilting Theorem]).

Recently Colby and Fuller in [CbF2], investigating a more general setting, have proved the following result: given two torsion theories $(\mathcal{T}, \mathcal{F})$ in R-Mod and $(\mathcal{X}, \mathcal{Y})$ in S-Mod, a pair of equivalences $\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ and $\mathcal{X} \underset{H'}{\overset{T'}{\rightleftarrows}} \mathcal{F}$ is represented by a pair of bimodules ${}_RV_S$ and ${}_SV'_R$, that is, $H \cong \mathrm{Hom}_R(V, -)$, $T \cong V \otimes_S -$, $T' \cong \mathrm{Hom}_S(V', -)$, $H' \cong V' \otimes_R -$ and $\mathcal{T} = \mathrm{Ker}\,H'$, $\mathcal{F} = \mathrm{Ker}\,H$, $\mathcal{X} = \mathrm{Ker}\,T$, $\mathcal{Y} = \mathrm{Ker}\,T'$. Following [CbF2], we call these pairs of functors a *torsion theory counter equivalence* between R-Mod and S-Mod, and we denote it by $(\mathcal{T}, \mathcal{F}) \underset{{}_SV'_R}{\overset{{}_RV_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$. A characterization of a pair of bimodules ${}_RV_S$ and ${}_SV'_R$ that represent a torsion theory counter equivalence is given in [CbF2, Theorem 2.5].

When (R, T, S) is a tilting triple, the corresponding pair of equivalences mentioned above produces a torsion theory counter equivalence $(\mathcal{T}, \mathcal{F}) \underset{{}_ST'_R}{\overset{{}_RT_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$, where ${}_ST'_R = \mathrm{Ext}^1_R(T, R)$. In this section we study when a torsion theory counter equivalence is of this kind.

3.1. LEMMA. *Let* $(\mathcal{T}, \mathcal{F}) \underset{{}_S V'_R}{\overset{{}_R V_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$ *be a torsion theory counter equivalence. Then*:

(i) $\mathrm{Ann}_R(\mathcal{T}) = \mathrm{Ann}({}_R V)$ *and* $\mathrm{Ann}_R(\mathcal{F}) = \mathrm{Ann}(V'_R) = \mathrm{Tr}_V(R)$;

(ii) $\mathrm{Ann}_S(\mathcal{X}) = \mathrm{Ann}({}_S V')$ *and* $\mathrm{Ann}_S(\mathcal{Y}) = \mathrm{Ann}(V_S) = \mathrm{Tr}_{V'}(S)$.

Proof. (i) Applying Theorem 2.6(I) to the equivalence $\mathcal{T} \rightleftarrows \mathcal{Y}$, we have $\mathcal{T} = \mathrm{Gen}({}_R V)$, so that $\mathrm{Ann}_R(\mathcal{T}) = \mathrm{Ann}({}_R V)$. Applying the same theorem to the equivalence $\mathcal{X} \rightleftarrows \mathcal{F}$, we have $\mathrm{Ann}_R(\mathcal{F}) = \mathrm{Ann}(V'_R)$. The equality $\mathrm{Ann}(V'_R) = \mathrm{Tr}_V(R)$ is contained in [CbF2, Lemma 3.2].

(ii) It can be proved in the same way. ∎

3.2. DEFINITION. A torsion theory counter equivalence of the form $(\mathcal{T}, \mathcal{F}) \underset{{}_S V'_R}{\overset{{}_R V_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$ is said to be *basic* if $\mathrm{Ann}_R(\mathcal{T}) \cap \mathrm{Ann}_R(\mathcal{F}) = 0$ and $\mathrm{Ann}_S(\mathcal{X}) \cap \mathrm{Ann}_S(\mathcal{Y}) = 0$.

This means that the rings R and S are minimal, in the following sense: there are no proper quotient rings R' of R or S' of S such that $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are still torsion theories in R'-Mod and S'-Mod, respectively.

There is a basic counter equivalence canonically associated to each torsion theory counter equivalence:

3.3. PROPOSITION. *Let* $(\mathcal{T}, \mathcal{F}) \underset{{}_S V'_R}{\overset{{}_R V_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$ *be a torsion theory counter equivalence. Put* $\hat{R} = R/\mathrm{Ann}_R(\mathcal{T}) \cap \mathrm{Ann}_R(\mathcal{F})$ *and* $\hat{S} = S/\mathrm{Ann}_S(\mathcal{X}) \cap \mathrm{Ann}_S(\mathcal{Y})$. *Then* $(\mathcal{T}, \mathcal{F}) \underset{{}_{\hat{S}} V'_{\hat{R}}}{\overset{{}_{\hat{R}} V_{\hat{S}}}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$ *is a basic counter equivalence between* $\hat{R}$-Mod *and* $\hat{S}$-Mod.

Proof. By the definition of $\hat{R}$ and $\hat{S}$, the pairs $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are torsion theories, respectively, in $\hat{R}$-Mod and $\hat{S}$-Mod. The thesis follows from [CbF2, Theorem 2.3] and Lemma 3.1. ∎

Every torsion theory counter equivalence given by a tilting triple (R, V, S) is basic: indeed, ${}_R V_S$ is faithfully balanced, so that $\mathrm{Ann}_R(\mathcal{T}) = 0 = \mathrm{Ann}_S(\mathcal{Y})$ by Lemma 3.1. Conversely, the following two results explain when a torsion theory counter equivalence, or the associated basic one, is given by a tilting bimodule ${}_R V_S$.

3.4. THEOREM. *Let* $(\mathcal{T}, \mathcal{F}) \underset{{}_S V'_R}{\overset{{}_R V_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$ *be a torsion theory counter equivalence. Then the following conditions are equivalent*:

(i) ${}_R V_S$ *is tilting bimodule*:

(ii) $\mathrm{Tor}_1^S(V, -)$ *and* $\mathrm{Ext}_R^1(V, -)$ *induce* (*see Section* 0) *an equivalence between* $\mathcal{X}$ *and* $\mathcal{F}$;

(iii) ${}_SV'_R$ *can be chosen so that there are natural isomorphisms of functors*: $\mathrm{Hom}_S(V', -) \cong \mathrm{Tor}_1^S(V, -)$ *in* S-Mod, $V' \otimes_R - \cong \mathrm{Ext}_R^1(V, -)$ *in* R-Mod;

(iv) $\mathcal{T}$ *contains every injective* R*-module and* $\mathcal{Y}$ *contains every projective* S*-module*;

(v) $\mathrm{E}({}_RR) \in \mathcal{T}$ *and* ${}_SS \in \mathcal{Y}$;

(vi) $\overline{\mathcal{T}} = R$-Mod *and* $\overline{\mathcal{Y}} = S$-Mod;

(vii) $\mathrm{Ann}_R(\mathcal{T}) = 0 = \mathrm{Ann}_S(\mathcal{Y})$ *and* $\mathcal{T}$ *is closed under direct products*.

Proof. (i) ⇒ (ii) It follows from [CbF1, Theorem 1.4].

(ii) ⇒ (iii) By [CbF2, Theorem 2.2], the bimodule ${}_SV'_R = \mathrm{Ext}_R^1(V, R)$ represents an equivalence between $\mathcal{X}$ and $\mathcal{F}$.

(iii) ⇒ (iv) By [CbF2, Theorem 2.2] we have the equalities $\mathcal{T} = \mathrm{Ker}\, V' \otimes_R -$ and $\mathcal{Y} = \mathrm{Ker}\,\mathrm{Hom}_S(V', -)$. Hence, by hypothesis, every injective R-module is in $\mathcal{T}$ and every projective S-module is in $\mathcal{Y}$.

(iv) ⇒ (v) It is obvious.

(v) ⇒ (vi) It is easy as, by hypothesis, $R \in \overline{\mathcal{T}}$.

(vi) ⇒ (vii) By hypothesis, ${}_RR$ is a submodule of a module in $\mathcal{T}$, so that $\mathrm{Ann}_R(\mathcal{T}) = 0$, and ${}_SS$ is a quotient of a module in $\mathcal{Y}$, so that $\mathrm{Ann}_S(\mathcal{Y}) = 0$. Moreover, $\mathcal{T} = \mathrm{Gen}({}_RV)$ by [CbF2, Lemma 2.1], and ${}_RV$ is a $*$-module by [CbF2, Theorem 2.5]. Since $\overline{\mathrm{Gen}}({}_RV) = R$-Mod, by [C1, Proposition 4.5] we have that $\mathcal{T} = \{{}_RM \mid \mathrm{Ext}_R^1(V, M) = 0\}$. This proves that $\mathcal{T}$ is closed under direct products.

(vii) ⇒ (i) By hypothesis, Lemma 3.1, and [CbF2, Theorem 2.5], we have that ${}_RV$ is a faithful $*$-module, $S \cong \mathrm{End}({}_RV)$ and $\mathrm{Gen}({}_RV)$ is closed under direct products. By [CM, Proposition 1.5], it follows that V_S is finitely generated. We can conclude by [C2, Theorem 3, (a) ⇔ (d)]. ∎

When the equivalent conditions of Theorem 3.4 hold true, we say that the torsion theory counter equivalence is a *tilting counter equivalence*, and we denote it simply by $(\mathcal{T}, \mathcal{F}) \overset{{}_RV_S}{\Longleftrightarrow} (\mathcal{X}, \mathcal{Y})$. The choice of the bimodule ${}_SV'_R$ is not necessarily unique, as Example 5.6 shows (see also [CbF2, "Remark Concerning Uniqueness"]).

From Proposition 3.3 and Theorem 3.4, (i) ⇔ (vii), we get immediately:

3.5. COROLLARY. *Let* $(\mathcal{T}, \mathcal{F}) \underset{{}_SV'_R}{\overset{{}_RV_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$ *be a torsion theory counter equivalence. Then the associated basic one is a tilting counter equivalence if and only if* $\mathrm{Ann}_R(\mathcal{T}) \subseteq \mathrm{Ann}_R(\mathcal{F})$, $\mathrm{Ann}_S(\mathcal{X}) \supseteq \mathrm{Ann}_S(\mathcal{Y})$, *and* $\mathcal{T}$ *is closed under direct products*.

3.6. *Remarks*. (1) In Theorem 3.4(iii), the natural isomorphisms $\mathrm{Hom}_S(V', -) \cong \mathrm{Tor}_1^S(V, -)$ in S-Mod and $V' \otimes_R - \cong \mathrm{Ext}_R^1(V, -)$ in R-Mod are both needed: Example 5.7 shows that there are non-tilting counter equivalences where, for instance, the second isomorphism holds in R-Mod and the first holds in $\overline{\mathcal{Y}}$, but not in S-Mod, even if $\overline{\mathcal{Y}}$ contains any simple module.

(2) Example 5.8 shows that there exist basic counter equivalences between the same torsion theory and torsion theories on nonisomorphic algebras of the same finite dimension.

4. QUASI-TILTING COUNTER EQUIVALENCES

The equivalence represented by a quasi-tilting triple (R, V, S), introduced in Theorem 2.6, can be completed to a "local form" of torsion theory counter equivalence. This is obtained, similarly to the tilting case, by means of the ext and tor functors associated to the bimodule ${}_RV_S$. The proof of the following result follows faithfully that of the Tilting Theorem given in [CbF1, Theorem 1.4], even if almost all the properties of quasi-tiltings basically work to guarantee the validity of the single steps of the proof.

4.1. THEOREM. *Let (R, V, S) be a quasi-tilting triple, $\overline{S} = S/\mathrm{Ann}(V_S)$, Q an injective cogenerator of R-Mod, and ${}_SV^* = \mathrm{Hom}_R(V, Q)$. Let*

$$H = \mathrm{Hom}_R(V, -), \qquad H' = \mathrm{Ext}_R^1(V, -),$$
$$T = V \bigotimes_S - , \qquad T' = \mathrm{Tor}_1^S(V, -)$$

to obtain pairs of functors

$$H, H' : \overline{\mathrm{Gen}}({}_RV) \to \overline{S}\text{-Mod} \qquad \textit{and} \qquad T, T' : \overline{S}\text{-Mod} \to \overline{\mathrm{Gen}}({}_RV)$$

and let

$$\mathcal{T} = \mathrm{Ker}\, H', \qquad \mathcal{F} = \mathrm{Ker}\, H, \qquad \mathcal{X} = \mathrm{Ker}\, T, \qquad \mathcal{Y} = \mathrm{Ker}\, T'.$$

Then:

(a) $TH' = 0_{\overline{\mathrm{Gen}}({}_RV)} = T'H$ *and* $HT' = 0_{\overline{S}\text{-Mod}} = H'T$;

(b) *there are natural transformations* θ *and* η *that, together with the canonical transformations* ρ *and* σ, *yield exact sequences*

$$0 \to TH(M) \overset{\rho_M}{\to} M \overset{\eta_M}{\to} T'H'(M) \to 0,$$

$$0 \to H'T'(N) \overset{\theta_N}{\to} N \overset{\sigma_N}{\to} HT(N) \to 0$$

for each $M \in \overline{\mathrm{Gen}}({}_RV)$ *and for each* $N \in \bar{S}$-Mod;

(c) $\mathcal{T} = \mathrm{Gen}({}_RV)$, $\mathcal{Y} = \mathrm{Cogen}({}_SV^*)$, *and, moreover*, $(\mathcal{T}, \mathcal{F})$ *and* $(\mathcal{X}, \mathcal{Y})$ *are torsion theories in* $\overline{\mathrm{Gen}}({}_RV)$ *and* $\bar{S}$-Mod, *respectively*;

(d) $\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ *and* $\mathcal{F} \underset{T'}{\overset{H'}{\rightleftarrows}} \mathcal{X}$ *are category equivalences*.

Proof. First of all, we check that H, T, H', and T' are well defined. This is clearly true for H and H'. Next, for every $N \in \bar{S}$-Mod there is an exact sequence of the form

$$0 \to K \to \bar{S}^{(X)} \to N \to 0, \tag{$*$}$$

from which we get the exact row

$$0 = T'(\bar{S}^{(X)}) \to T'(N) \to T(K) \to T(\bar{S}^{(X)}) \cong {}_RV^{(X)} \to T(N) \to 0,$$

where $T'(\bar{S}^{(X)}) = 0$ by Lemma 2.5, (ii) $\Rightarrow$ (iii), as $V \otimes_S \mathrm{Ann}(V_S) = 0$. This proves that $T(N) \in \mathrm{Gen}({}_RV) \subseteq \overline{\mathrm{Gen}}({}_RV)$ and, similarly, that $T(K) \in \mathrm{Gen}({}_RV)$. Therefore, $T'(N) \in \overline{\mathrm{Gen}}({}_RV)$.

Next, from Proposition 2.1 we have that $\overline{\mathrm{Gen}}({}_RV) \cap V^{\perp} = \mathrm{Gen}({}_RV)$ is a torsion class in R-Mod, so that $\mathcal{T} := \mathrm{Ker}\, H' = \mathrm{Gen}({}_RV)$ is a torsion class in $\overline{\mathrm{Gen}}({}_RV)$ too; the corresponding torsion-free class in $\overline{\mathrm{Gen}}({}_RV)$ is obviously $\mathcal{F} := \mathrm{Ker}\, H$. From Theorem 2.6, (II) and (III), we have that $\mathrm{Cogen}({}_SV^*)$ is a torsion-free class in S-Mod, so that $\mathcal{Y} := \mathrm{Ker}\, T' = \mathrm{Cogen}({}_SV^*)$ is a torsion-free class in $\bar{S}$-Mod too; the corresponding torsion class in $\bar{S}$-Mod is $\mathcal{X} := \mathrm{Ker}\, T$. Moreover, again by Theorem 2.6(II), $\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ is a category equivalence. This proves (c) and the first part of (d).

In order to prove (a), we start with a module $M \in \overline{\mathrm{Gen}}({}_RV)$. Then there is an exact sequence

$$0 \to M \to M' \to M'' \to 0, \tag{$**$}$$

where M' and M'' belong to $\mathrm{Gen}({}_RV)$. Applying $\mathrm{Hom}_R(V, -)$, we get the exact sequence $H(M') \to H(M'') \to H'(M) \to H'(M') = 0$, as $\mathrm{Gen}({}_RV) = \mathrm{Ker}\, H'$. Since T is right exact, $\mathcal{T} = \mathrm{Gen}({}_RV)$, and $\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ is an equivalence with counit ρ, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} M' & \longrightarrow & M'' & \longrightarrow & 0 & & \\ {\scriptstyle\rho_{M'}}\uparrow\cong & & {\scriptstyle\rho_{M''}}\uparrow\cong & & & & \\ TH(M') & \longrightarrow & TH(M'') & \longrightarrow & TH'(M) & \longrightarrow & 0 \end{array}$$

which shows that $TH'(M) = 0$. Moreover, there is an exact sequence $0 \to M \to Q^X$, from which we derive the exact row $0 \to H(M) \to H(Q^X) \cong ({}_SV^*)^X$. Thus $H(M) \in \mathrm{Cogen}({}_SV^*) = \mathrm{Ker}\, T'$. This proves that $T'H(M) = 0$.

Let $N \in \overline{S}$-Mod, and let us consider the exact sequence $(*)$. We have that K and $\overline{S}^{(X)} = H({}_RV^{(X)})$ belong to $\mathrm{Cogen}({}_SV^*) = \mathsf{Y}$. As $\mathsf{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathsf{Y}$ is an equivalence with unit σ, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & HT'(N) & \longrightarrow & HT(K) & \longrightarrow & HT(\overline{S}^{(X)}) \\ & & & & {\scriptstyle\sigma_K}\uparrow\cong & & {\scriptstyle\sigma_{\overline{S}^{(X)}}}\uparrow\cong \\ & & 0 & \longrightarrow & K & \longrightarrow & \overline{S}^{(X)} \end{array}$$

which shows that $HT'(N) = 0$. Since $T(N) \in \mathrm{Gen}({}_RV) = \mathrm{Ker}\, H'$, we have $H'T(N) = 0$, and so (a) is proved.

In order to prove the first part of (b), we consider $M \in \overline{\mathrm{Gen}}({}_RV)$. Since $\mathrm{Gen}({}_RV) \subseteq V^{\perp}$, the exact sequence $(**)$ induces the exact sequence

$$0 \to H(M) \to H(M') \overset{\psi}{\to} H(M'') \to H'(M) \to 0.$$

We obtain two short exact sequences

$$0 \to H(M) \to H(M') \to L \to 0,$$
$$0 \to L \to H(M'') \to H'(M) \to 0,$$

where $L = \mathrm{Im}(\psi) \in \mathrm{Cogen}({}_SV^*) = \mathrm{Ker}\, T'$. Then $T'(L) = TH'(M) = T'H(M'') = 0$, so that the same argument of [CbF1, proof of Theorem 1.4] gives a natural epimorphism $\eta_M\colon M \to T'H'(M)$ with kernel $\mathrm{Im}\, \rho_M$. The last part of (b) has a similar proof. Let us consider $N \in \overline{S}$-Mod and the exact sequence $(*)$. From the induced exact sequence

$$0 \to T'(N) \to T(K) \overset{\phi}{\to} T(\overline{S}^{(X)}) \to T(N) \to 0,$$

we obtain two short exact sequences

$$0 \to L \to T(\overline{S}^{(X)}) \to T(N) \to 0,$$
$$0 \to T'(N) \to T(K) \to L \to 0,$$

where $L = \mathrm{Im}(\phi) \in \mathrm{Gen}({}_RV) = \mathrm{Ker}\, H'$. Then $H'(L) = HT'(N) = H'T(K) = 0$, so that the same argument of [CbF1, proof of Theorem 1.4] gives a natural monomorphism θ_N: $H'T'(N) \to N$ with image $\mathrm{Ker}\, \sigma_N$.

From (a) we get $\mathrm{Im}(H') \subseteq \mathcal{X}$ and $\mathrm{Im}(T') \subseteq \mathcal{F}$. Moreover, from (b) we obtain that $\eta \restriction_{\mathcal{F}} : 1_{\mathcal{F}} \to T'H'$ and $\theta \restriction_{\mathcal{X}} : H'T' \to 1_{\mathcal{X}}$ are natural isomorphisms. This completes the proof of (d). ∎

Under the hypotheses and notation of Theorem 4.1, the pair of equivalences $\mathcal{T} \underset{V \otimes_S -}{\overset{\mathrm{Hom}_R(V,-)}{\rightleftarrows}} \mathcal{Y}$ and $\mathcal{F} \underset{\mathrm{Tor}_1^S(V,-)}{\overset{\mathrm{Ext}_R^1(V,-)}{\rightleftarrows}} \mathcal{X}$ given by a quasi-tilting triple (R, V, S) is called a *quasi-tilting counter equivalence*, and denoted by $(\mathcal{T}, \mathcal{F}) \overset{{}_RV_S}{\leftrightsquigarrow} (\mathcal{X}, \mathcal{Y})$. It must be noted that $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are torsion theories, respectively, in the subcategories $\overline{\mathrm{Gen}}({}_RV)$ of R-Mod and $\mathrm{End}({}_RV)$-Mod of S-Mod. $\overline{\mathrm{Gen}}({}_RV) = \overline{\mathcal{T}}$ and $\mathrm{End}({}_RV)$-Mod $= \overline{\mathcal{Y}}$ can be considered as localizations, respectively, of R-Mod and S-Mod, with respect to the bimodule ${}_RV_S$ (see Example 5.10).

In the artinian case, quasi-tilting and tilting counter equivalences are quite close:

4.2. COROLLARY. *Let R be a left artinian ring, (R, V, S) a quasi-tilting triple with associated quasi-tilting counter equivalence $(\mathcal{T}, \mathcal{F}) \overset{{}_RV_S}{\leftrightsquigarrow} (\mathcal{X}, \mathcal{Y})$. Let $\overline{R} = R/\mathrm{Ann}({}_RV)$ and $\overline{S} = S/\mathrm{Ann}(V_S)$. Then $(\overline{R}, V, \overline{S})$ is a tilting triple with associated counter equivalence $(\mathcal{T}, \mathcal{F}) \overset{{}_{\overline{R}}V_{\overline{S}}}{\Longleftrightarrow} (\mathcal{X}, \mathcal{Y})$.*

Proof. By Corollary 2.4, ${}_{\overline{R}}V$ is a tilting module, $\overline{\mathrm{Gen}}({}_RV) = \overline{R}$-Mod, and $\overline{S} = \mathrm{End}({}_RV) = \mathrm{End}({}_{\overline{R}}V)$. It follows that $(\overline{R}, V, \overline{S})$ is a tilting triple, and the quasi-tilting counter equivalence $(\mathcal{T}, \mathcal{F}) \overset{{}_RV_S}{\leftrightsquigarrow} (\mathcal{X}, \mathcal{Y})$ coincides with $(\mathcal{T}, \mathcal{F}) \overset{{}_{\overline{R}}V_{\overline{S}}}{\Longleftrightarrow} (\mathcal{X}, \mathcal{Y})$. ∎

The theory of Colby and Fuller on torsion theory counter equivalences is in fact a strong generalization of the tilting setting. As proved in Theorem 3.4, for a torsion theory counter equivalence $(\mathcal{T}, \mathcal{F}) \underset{{}_SV'_R}{\overset{{}_RV_S}{\Longleftrightarrow}} (\mathcal{X}, \mathcal{Y})$ the following properties can fail:

(a) $\mathcal{T}$ contains every injective module and $\mathcal{Y}$ contains every projective module;

(b) the functors $\mathrm{Tor}_1^S(V, -)$ and $\mathrm{Ext}_R^1(V, -)$ induce an equivalence between $\mathcal{X}$ and $\mathcal{F}$.

On the contrary, our generalization—from tilting to quasi-tilting triples—goes in the direction suggested by the previous conditions. Similarly to the tilting case, a quasi-tilting counter equivalence $(\mathcal{T}, \mathcal{F}) \overset{{}_RV_S}{\leftrightarrows} (\mathcal{X}, \mathcal{Y})$ satisfies conditions (a) and (b) restricted to the subcategories $\overline{\mathrm{Gen}}({}_RV)$ of R-Mod and $\mathrm{End}({}_RV)$-Mod of S-Mod, by means of Proposition 2.1(iv) and Theorems 2.6(III) and 4.1. Therefore, the quasi-tilting context seems closer than the Colby–Fuller one to tilting theory. Nevertheless, there is a natural connection between torsion theory counter equivalences and our setting:

4.3. COROLLARY. *Let* $(\mathcal{T}, \mathcal{F}) \underset{{}_SV'_R}{\overset{{}_RV_S}{\Leftarrow\!\!\Rightarrow}} (\mathcal{X}, \mathcal{Y})$ *be a torsion theory counter equivalence. Then* (R, V, S) *and* (S, V', R) *are quasi-tilting triples, giving, respectively, the quasi-tilting counter equivalences*

$$(\mathcal{T}, \mathcal{F} \cap \overline{\mathcal{T}}) \overset{{}_RV_S}{\leftrightarrows} (\mathcal{X} \cap \overline{\mathcal{Y}}, \mathcal{Y})$$

and

$$(\mathcal{X}, \mathcal{Y} \cap \overline{\mathcal{X}}) \overset{{}_SV'_R}{\leftrightarrows} (\mathcal{T} \cap \overline{\mathcal{F}}, \mathcal{F}).$$

Proof. From Lemma 3.1(ii) it follows that $V \otimes_S S/\mathrm{Ann}_S(\mathcal{Y}) \cong {}_RV_S$. Therefore, by Theorem 2.6(I) we get that (R, V, S) is a quasi-tilting triple and $\overline{\mathcal{T}} = \overline{\mathrm{Gen}}({}_RV)$. Moreover, by Theorem 2.6(III), we have $\overline{\mathcal{Y}} = S/\mathrm{Ann}_S(\mathcal{Y})$-Mod $= S/\mathrm{Ann}(V_S)$-Mod. Thus, applying Theorem 4.1 to (R, V, S), we obtain the quasi-tilting counter equivalence of the form $(\mathcal{T}, \mathcal{F} \cap \overline{\mathcal{T}}) \overset{{}_RV_S}{\leftrightarrows} (\mathcal{X} \cap \overline{\mathcal{Y}}, \mathcal{Y})$.

The same argument holds for ${}_SV'_R$. ∎

From Lemma 3.1 and Corollaries 2.8 and 4.3, we get immediately:

4.4. COROLLARY. *If* $(\mathcal{T}, \mathcal{F}) \underset{{}_SV'_R}{\overset{{}_RV_S}{\Leftarrow\!\!\Rightarrow}} (\mathcal{X}, \mathcal{Y})$ *is a torsion theory counter equivalence, then* $\mathcal{T}$ *and* $\mathcal{X}$ *are quasi-tilting torsion classes in* R-Mod *and* S-Mod, *respectively, and* $\mathcal{F}$ *and* $\mathcal{Y}$ *are cotilting torsion-free classes in* $R/\mathrm{Ann}_R(\mathcal{F})$-Mod *and* $S/\mathrm{Ann}_S(\mathcal{Y})$-Mod, *respectively*.

4.5. DEFINITION. Two quasi-tilting triples (R, V, S) and (S, V', R) are called *complementary* if the bimodules ${}_RV_S$ and ${}_SV'_R$ represent a torsion theory counter equivalence.

As already observed in Remark 2.7(c), a quasi-tilting triple does not necessarily admit a complement.

The following result, analogous to [CbF2, Theorem 2.5], complements Corollary 4.3:

4.6. THEOREM. *Let (R, V, S) and (S, V', R) be quasi-tilting triples, and consider the associated quasi-tilting counter equivalences $(\mathcal{T}, \mathcal{F}_0) \overset{{}_RV_S}{\leftrightarrows} (\mathcal{X}_0, \mathcal{Y})$ and $(\mathcal{X}, \mathcal{Y}_0) \overset{{}_SV'_R}{\leftrightarrows} (\mathcal{T}_0, \mathcal{F})$. Then the following conditions are equivalent:*

(i) *(R, V, S) and (S, V', R) are complementary;*

(ii) *$(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ are torsion theories in R-*Mod *and S-*Mod, *respectively;*

(iii) *$V' \otimes_R V = 0 = V \otimes_S V'$ and, for all $M \in R$-*Mod,

$$\operatorname{Hom}_R(V, M) = 0 = V' \otimes_S M \quad \textit{implies} \quad M = 0,$$

*and for all $N \in S$-*Mod,

$$\operatorname{Hom}_S(V', N) = 0 = V \otimes_S N \quad \textit{implies} \quad N = 0.$$

Proof. (i) ⇒ (iii) By hypothesis, $\operatorname{Gen}({}_RV) = \operatorname{Ker} V' \otimes_R -$. This gives immediately $V' \otimes_R V = 0$, and $M = 0$ whenever $\operatorname{Hom}_R(V, M) = 0 = V' \otimes_R M$. The other two conditions can be proved in the same way.

(iii) ⇒ (ii) As observed in Theorem 2.6(III), $\mathcal{Y}$ is a torsion-free class in S-Mod, associated to the torsion class $\operatorname{Ker} V \otimes_S -$. Since $\mathcal{X} = \operatorname{Gen}({}_SV')$, we have to prove that $\operatorname{Gen}({}_SV') = \operatorname{Ker} V \otimes_S -$. From $V \otimes_S V' = 0$ we get $\operatorname{Gen}({}_SV') \subseteq \operatorname{Ker} V \otimes_S -$. Conversely, let ${}_SL$ be such that $V \otimes_S L = 0$. Then $V \otimes_S L/\operatorname{Tr}_{V'}(L) = 0$ too. Since $\operatorname{Gen}({}_SV')$ is a torsion class, $\operatorname{Tr}_{V'}$ is a radical; hence $\operatorname{Hom}_S(V', L/\operatorname{Tr}_{V'}(L)) = 0$. Therefore, $L/\operatorname{Tr}_{V'}(L) = 0$ by assumption, i.e., $L \in \operatorname{Gen}({}_SV')$. This proves that $(\mathcal{X}, \mathcal{Y})$ is a torsion theory in S-Mod. A similar argument works for $(\mathcal{T}, \mathcal{F})$ in R-Mod.

(ii) ⇒ (i) It follows by hypotheses and Theorem 2.6(III). ∎

4.7. *Remarks*. (1) Remark 5.9(1) shows that there is a torsion theory counter equivalence $(\mathcal{T}, \mathcal{F}) \leftrightarrows (\mathcal{X}, \mathcal{Y})$ between R-Mod and S-Mod such that S is not isomorphic to the endomorphism ring of any R-module representing the equivalence $\mathcal{T} \rightleftarrows \mathcal{Y}$. Precisely, we construct a quasi-tilting triple (R, V, S) which has a complement, but, for every quasi-tilting module ${}_RU \in \operatorname{Gen}({}_RV)$, the triple $(R, U, \operatorname{End}({}_RU))$ has no complements.

(2) Looking at Theorem 4.6(iii), we observe in Remarks 5.9, (2) and (3), that the class of projective or injective modules and the class of semisimple modules are not large enough to test the complementarity of two quasi-tilting triples.

5. EXAMPLES

Throughout this section, K denotes an algebraically closed field, and all rings are K-algebras given by quivers according to [R]. If R is a finite-dimensional K-algebra given by a quiver Δ and i is a vertex of Δ, then we denote by $\mathrm{P}(i)$ (resp. $\mathrm{I}(i)$) the indecomposable projective (resp. injective) R-module associated with i, and we denote by $\mathrm{S}(i)$ the simple top of $\mathrm{P}(i)$.

In the following, we always identify indecomposable modules and their isomorphism classes. If the K-algebra R is of finite representation type, when we draw its Auslander–Reiten quiver Γ_R, we often replace indecomposable modules by some obvious pictures describing their composition series. In this way, it is easy to count the dimension of the K-vector space of all morphisms between two indecomposable modules belonging to the same torsion or torsion-free class. More generally, in order to discover more or less hidden torsion theory counter equivalences, it often suffices to compare some combinatorial data, for instance, the number of certain indecomposable modules and the dimension of certain vector spaces. We also note that some complicated objects involved in torsion theory counter equivalences are just the duals with respect to the field K of certain right modules. For instance, let R and S be K-algebras and let ${}_R V_S$ be a bimodule. Then it is well known that $D(R_R) = \mathrm{Hom}_K(R_R, K)$ is an injective cogenerator of R-Mod. Moreover, applying the adjoint isomorphism and proceeding as in tilting theory (see [R, page 171]) over finite-dimensional algebras, we see that $D(V_S) = \mathrm{Hom}_K(V_S, K)$ is isomorphic to $\mathrm{Hom}_R({}_R V_S, D(R_R))$.

As the next example shows, an S-module of the form $D(V_S)$ may be extremely large, and does not necessarily satisfy any finiteness condition.

5.1. EXAMPLE. There are K-algebras R and S and a tilting triple (R, V, S) such that if ${}_R Q$ is an injective cogenerator of R-Mod and ${}_S V^* = \mathrm{Hom}_R({}_R V_S, {}_R Q)$, then we have

$$\mathrm{Cogen}\big(D(V_S)\big) = \mathrm{Cogen}\big({}_S V^*\big) \neq \mathrm{Cogen}({}_S M)$$

for any finitely generated or finitely cogenerated module ${}_S M$.

First, the class of the right S-modules cogenerated by $\mathrm{Hom}_R({}_RV_S, {}_RQ)$ does not depend on the injective cogenerator ${}_RQ$ (see [C1, Lemma 3.2(a)]); moreover, the choice ${}_RQ = D(R_R)$ gives ${}_SV^* \cong D(V_S)$. Next, let R be the K-algebra given by the quiver

$$_{\alpha}\circlearrowleft a \xrightarrow{\beta} b,$$

let e_a and e_b denote the primitive idempotents of R corresponding to the vertices a and b, respectively, and let $\mathrm{P}(a) = Re_a$, $\mathrm{P}(b) = Re_b$.

Let ${}_RV$ denote the module

$${}_RV = \mathrm{P}(a) \oplus \mathrm{P}(a)/R\beta.$$

Then ${}_RV$ is a tilting module (see [D1, Proposition 5]), and it is easy to see that $\mathrm{End}({}_RV)$ is isomorphic to the K-algebra S given by the quiver

$$c \xrightarrow{\gamma} d \circlearrowright_{\delta}.$$

Next, let $(\mathcal{T}, \mathcal{F})$ be the torsion theory in R-Mod with $\mathcal{T} = \mathrm{Gen}({}_RV)$, and let $(\mathcal{X}, \mathcal{Y})$ be the torsion theory in S-Mod with $\mathcal{Y} = \mathrm{Cogen}(D(V_S))$. Then we clearly have $\mathcal{F} = \mathrm{Cogen}(\mathrm{P}(b))$; that is, $\mathcal{F}$ consists of all semisimple projective R-modules. This observation and the existence of an equivalence between $\mathcal{F}$ and $\mathcal{X}$ assure that $\mathcal{X}$ contains exactly one indecomposable module. Therefore, $\mathcal{X} = \mathrm{Gen}(\mathrm{I}(c))$, where $\mathrm{I}(c)$ is the unique simple injective module associated with the vertex c, while $\mathcal{Y}$ consists of all S-modules without simple injective summands. We claim that $\mathcal{Y} \neq \mathrm{Cogen}({}_SM)$ for any finitely generated or finitely cogenerated module ${}_SM$. To see this, let e_c and e_d denote the primitive idempotents of S corresponding to c and d, respectively. Next, let $\mathcal{Z} = \{M \in S\text{-Mod} \mid e_cM = 0\}$. Since S/Se_cS is isomorphic to $K[x]$ and $\mathcal{Z}$ is a subcategory of $\mathcal{Y}$ equivalent to $K[x]$-Mod, it follows that

(1) $\mathcal{Y}$ contains infinitely many nonisomorphic simple modules.

Moreover, if ${}_SZ \in \mathcal{Z}$ and ${}_SZ$ is finitely generated, then ${}_SZ$ is the direct sum of finitely many cyclic modules; hence $\bigcap_n \delta^n Z = 0$. Consequently,

(2) $\mathrm{Cogen}({}_SZ) \neq \mathcal{Z}$ for any finitely generated module ${}_SZ \in \mathcal{Z}$.

We also note that, if a module ${}_SM$ is generated by a subset L of the form $L' \cup L''$ with $L' \subseteq e_cM$, $L'' \subseteq e_dM$, then its submodule e_dM is generated by the subset $\gamma L' \cup L''$. This implies that

(3) e_dF is finitely generated for any finitely generated module ${}_SF$.

Assume now that ${}_S M$ is a module such that $\mathrm{Cogen}({}_S M) = \mathcal{Y}$. Since $\mathrm{Cogen}(e_d M) = \mathcal{Z}$, we deduce from (2) that $e_d M$ is not finitely generated. Hence the conclusion that ${}_S M$ is not finitely generated follows from (3). On the other hand, any simple module $Y \in \mathcal{Y}$ may be embedded in $\mathrm{Soc}\, {}_S M$. This remark and (1) assure that $\mathrm{Soc}\, {}_S M$ is not finitely generated, and so ${}_S M$ is not finitely cogenerated (see [AF, Proposition 10.7]). This completes the proof.

We state as a lemma the trick used in the sequel to give an example of a module, with a very easy structure, which is quasi-tilting but not finitely presented.

5.2. LEMMA. *Let ${}_R V$ be a module satisfying the following conditions*:

(a) *any module generated by ${}_R V$ is isomorphic to a direct sum of copies of ${}_R V$ and* $\mathrm{Ext}^1_R(V, V^{(\alpha)}) = 0$ *for any cardinal* α;

(b) *there is an exact sequence in* R-Mod *of the form*

$$0 \to L \to P \to {}_R V \to 0,$$

where P is a finitely generated projective module, while L is not finitely generated.

Then ${}_R V$ is a quasi-tilting module which is not finitely presented.

Proof. Since ${}_R V$ is finitely generated by (b), we deduce from (a) and Proposition 2.1(iii) that ${}_R V$ is a quasi-tilting module. Finally, the assertion that ${}_R V$ is not finitely presented follows from (b) and [K, Theorem 1, page 167]. ∎

The existence of quasi-tilting modules which are not finitely presented in an immediate consequence of Lemma 5.2. As we shall see, the simple module used to see this is a factor of a projective module with very special properties.

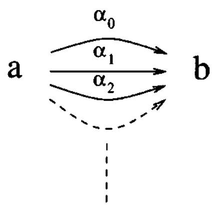
5.3. EXAMPLE. There is a K-algebra R such that the unique indecomposable faithful projective R-module P satisfies the following conditions:

(a) $P/\mathrm{Soc}\, P$ is a quasi-tilting module of projective dimension one, but $P/\mathrm{Soc}\, P$ is not finitely presented;

(b) $\mathrm{Gen}(P)$ is a torsion class containing any injective module, but it is not a tilting torsion class;

(c) P is a cotilting module, but there is no exact sequences of the form $0 \to P' \to P'' \to Q \to 0$, where Q is an injective cogenerator of R-Mod and P', P'' are direct summands of P^λ for some cardinal λ.

Indeed, let R denote the K-algebra given by the quiver



with infinitely many arrows, say α_n with $n \in \mathbf{N}$, from a to b, that is, let R be the direct limit of generalized Kronecker algebras [HU, p. 182]. Next, let e_a and e_b denote the primitive idempotents of R corresponding to the vertices a and b, respectively. Finally, let P denote the module Re_a. Then P is the unique indecomposable faithful projective R module, and P satisfies condition (b) [D2, Theorem 2]. On the other hand, $\text{Gen}({}_R P/\text{Soc } P)$ consists of all semisimple injective R-modules, while $\text{Soc } P$ is isomorphic to $Re_b^{(\aleph_0)}$. This observation and Lemma 5.2 prove that $P/\text{Soc } P$ is a quasi-tilting module, of projective dimension one, which is not finitely presented. Hence, P satisfies condition (a). We claim that P is a cotilting module. In fact, we clearly have

$$\text{inj dim}({}_R P) = 1. \tag{1}$$

Since the Jacobson radical J of R is the K-vector space generated by the arrows α_n, it follows that

$$R/J \text{ is semisimple and } J^2 = 0. \tag{2}$$

Let now H be a nonzero finitely generated right ideal of R. Assume first $H \subseteq e_a K + \sum_{n \in \mathbf{N}} \alpha_n K$. Then, for any $0 \neq h \in H$, we have $hR = hK$ and $\text{Ann}_R(h) = e_b R$. Consequently, there exists an exact sequence of the form $0 \to e_b R^d \to R^d \to H \to 0$, where $d = \dim_K H$. Now suppose $H \not\subseteq e_a K + \sum_{n \in \mathbf{N}} \alpha_n K$. Since H is a right ideal of R, we obtain $e_b R \subseteq H$. Consequently, we have either $H = R$ or $H = e_b R \cong R/e_a R$. This proves that R is right coherent, and so [AF, Theorem 19.20] implies that

$$\text{any direct product of flat left } R\text{-modules is flat}. \tag{3}$$

On the other hand, we deduce from (2) and [AF, Theorem 28.4] that

$$\text{any flat left } R\text{-module is projective}. \tag{4}$$

Putting (3) and (4) together, we get

$$\operatorname{Ext}^1_R(P^\lambda, P) = 0 \qquad \text{for any cardinal } \lambda. \tag{5}$$

Next, let M be a module such that $\operatorname{Hom}_R(M, P) = 0$ and $\operatorname{Ext}^1_R(M, P) = 0$. We claim that $M = 0$. Assume the contrary. Since $\operatorname{Hom}_R(Re_b, P) \neq 0$, our assumptions on M imply that M has a projective resolution of the form $0 \to X \cong Re_b^{(\lambda)} \xrightarrow{i} Y \cong P^{(\nu)} \to M \to 0$ for some cardinal $\lambda, \nu \neq 0$. Hence, the following sequence is exact:

$$0 = \operatorname{Hom}_R(M, P) \to \operatorname{Hom}_R(Y, P) \xrightarrow{i^*} \operatorname{Hom}_R(X, P) \to \operatorname{Ext}^1_R(M, P) = 0. \tag{6}$$

To find a contradiction, fix any $0 \neq x \in X$. Since $e_b P = \bigoplus_{n \in \mathbf{N}} \alpha_n P$, we have $e_b Y = \bigoplus_{n \in \mathbf{N}} \alpha_n Y$. Consequently, there is some m such that $i(x) \in \bigoplus_{n=0}^m \alpha_n Y$. This implies that

$$f(i(x)) \in \bigoplus_{n=0}^{m} \alpha_n P \qquad \text{for any } f \in \operatorname{Hom}_R(Y, P). \tag{7}$$

On the other hand, we have $\operatorname{Hom}_R(X, P) \cong \operatorname{Hom}_R(X, e_b P) \cong \operatorname{Hom}_K(X, e_b P)$. This means that

$$\{g(x) \mid g \in \operatorname{Hom}_R(X, P)\} = e_b P. \tag{8}$$

Since (8) is a contradiction to (6) and (7), we obtain $M = 0$, as claimed. Thus, by (1) and (5), M is a cotilting module. Finally, let Q be an injective cogenerator of R-Mod. Then Q has a projective resolution of the form $0 \to P_1 \to P_0 \to Q \to 0$, where $P_1 \notin \operatorname{Gen}(P)$, while P_0 is a direct sum of copies of P. Now, let L_0 and L_1 be direct summands of P^λ for some λ. Then we deduce from (3) and (4) that L_0 and L_1 are projective modules. This observation and our hypotheses on P_0 and P_1 imply that $P_0 \oplus L_1 \in \operatorname{Gen}(P)$, while $P_1 \oplus L_0 \notin \operatorname{Gen}(P)$. Hence, by [K, Theorem 1, page 167], there is no exact sequence of the form $0 \to L_1 \to L_0 \to Q \to 0$. This remark completes the proof of (c).

It suffices to deal with finite-dimensional algebras, of finite global dimension, to see that there is no restriction on the projective dimension of quasi-tilting modules, even of a very special kind (compare Proposition 2.3 with conditions (c), (d), and (e) in the next example).

5.4. EXAMPLE. For any $n \geq 2$, there are a K-algebra R and a quasi-tilting module ${}_RV$ satisfying the following conditions:

(a) $\operatorname{gl\,dim}(R) = n + 1$;

(b) $\operatorname{proj\,dim}({}_RV) = n$ and $\operatorname{Ext}^n_R(V, M) \neq 0$ for some $M \in \operatorname{Gen}({}_RV)$;

(c) $R\text{-Mod}\backslash\overline{\text{Gen}}({}_RV)$ contains exactly one indecomposable module;

(d) $R\text{-Mod}\backslash\text{Gen}({}_RV)$ contains exactly one indecomposable injective module;

(e) $\dim_K(\text{Ann}({}_RV)) = 1$.

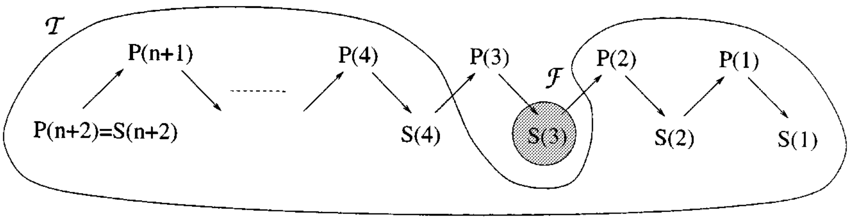
Fix some $n \geq 2$, and let R be the K-algebra given by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \longrightarrow \cdots \longrightarrow n \xrightarrow{\alpha_n} n+1 \xrightarrow{\alpha_{n+1}} n+2$$

with $\alpha_{i+1}\alpha_i = 0$ for any $i = 1, \dots, n$. Next, let ${}_RV$ denote the module

$${}_RV = \bigoplus_{i \neq 3} \text{P}(i) \oplus \text{S}(2).$$

Then $\text{Gen}({}_RV)$ is the torsion class $\mathcal{T}$ of a torsion theory $(\mathcal{T}, \mathcal{F})$, and Γ_R has the following shape:



Let $\overline{R}$ denote the algebra $R/\text{Ann}({}_RV)$. Then it is easy to check that ${}_{\overline{R}}V$ is a tilting module. Therefore, Theorem 2.6 implies that ${}_RV$ is a quasi-tilting module. Moreover, (a), (c), (d), and (e) clearly hold, and it is easy to check that the following K-vector spaces are isomorphic:

$$\text{Ext}^1_R(V, \text{S}(3)) \cong \text{Ext}^2_R(V, \text{S}(4)) \cong \cdots \cong \text{Ext}^n_R(V, \text{S}(n+2)).$$

Since $\text{proj}\dim({}_RV) = n$ and $\text{Ext}^1_R(V, \text{S}(3)) \neq 0$, this remark completes the proof of (b).

The next lemma shows that the degree of freedom in the choice of an equivalence between two subcategories of semisimple modules may be as large as possible.

5.5. LEMMA. *Let R (resp. S) be a K-algebra, let $\mathcal{C}$ (resp. $\mathcal{D}$) be a subcategory of R-*Mod *(resp. S-*Mod*) closed under direct sums and consisting of semisimple modules. Let* $\text{Ind}\,\mathcal{C}$ *(resp.* $\text{Ind}\,\mathcal{D}$*) be a representative system of the isomorphism classes of the indecomposable modules belonging to $\mathcal{C}$ (resp. $\mathcal{D}$), and assume that the following conditions hold*:

(a) $\text{End}({}_RC) \cong K$, $\text{End}({}_RD) \cong K$ *for any* $C \in \text{Ind}\,\mathcal{C}$, $D \in \text{Ind}\,\mathcal{D}$;

(b) $|\text{Ind}\,\mathcal{C}| = |\text{Ind}\,\mathcal{D}| = n$ *for some* $n \in \mathbf{N}$.

Then for any bijection F: Ind $\mathcal{C} \to$ Ind $\mathcal{D}$ *there is an equivalence* $\mathcal{C} \underset{G}{\overset{\overline{F}}{\rightleftarrows}} \mathcal{D}$, *where* $\overline{F}$ *extends* F.

Proof. Let Ind $\mathcal{C} = \{C_1, \dots, C_n\}$, Ind $\mathcal{D} = \{D_1, \dots, D_n\}$, and let $F(C_i) = D_i$ for any i. Fix some $C \in \mathcal{C}$. Then C is isomorphic to a direct sum of the form $\oplus_{i=1}^n C_i^{(\alpha_i)}$, where the cardinals $\alpha_1, \dots, \alpha_n$ are uniquely determined [P, Proposition 2.5]. This observation guarantees that F extends to a bijection between the objects of $\mathcal{C}$ and the objects of $\mathcal{D}$, sending $\oplus_{i=1}^n C_i^{(\alpha_i)}$ to $\oplus_{i=1}^n D_i^{(\alpha_i)}$. On the other hand, by (a) there is a unique choice to define the action on morphisms of a functor, say $\overline{F}$: $\mathcal{C} \to \mathcal{D}$, which extends F. Since $\overline{F}$ is faithful and full, the existence of a functor G, giving the desired equivalence, follows from [J, Proposition 1.3, page 27]. ∎

It is also easy to give an example where even all obvious functors, related to different pairs of candidate tilting bimodules, are not enough to obtain all the equivalences involved in a tilting counter equivalence.

5.6. EXAMPLE. There are K-algebras R and S and tilting counter equivalent torsion theories $(\mathcal{T}, \mathcal{F})$ in R-Mod and $(\mathcal{X}, \mathcal{Y})$ in S-Mod, with the following properties:

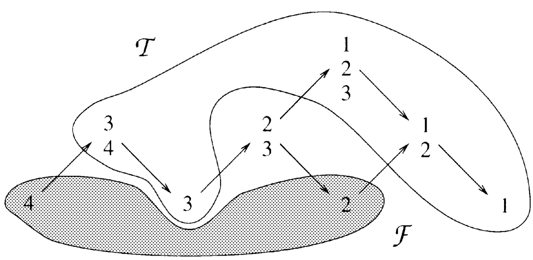
(a) there is exactly one (tilting) bimodule ${}_R V_S$ representing an equivalence between $\mathcal{T}$ and $\mathcal{Y}$;

(b) there are exactly two nonisomorphic S–R-bimodules representing an equivalence between $\mathcal{X}$ and $\mathcal{F}$.

In this case, let R denote the K-algebra given by the quiver

$$1 \overset{\alpha}{\to} 2 \overset{\beta}{\to} 3 \overset{\gamma}{\to} 4 \qquad \text{with } \gamma\beta = 0.$$

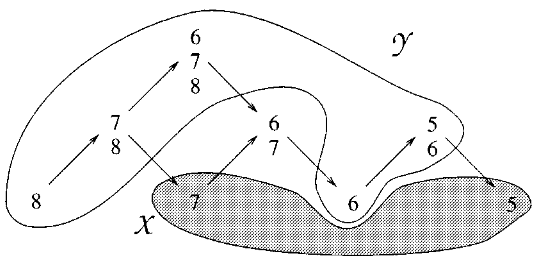
Then the injective R-modules generate the tilting torsion class $\mathcal{T}$ of the following torsion theory $(\mathcal{T}, \mathcal{F})$:



Next, let S denote the K-algebra given by the quiver

$$5 \xrightarrow{\delta} 6 \xrightarrow{\epsilon} 7 \xrightarrow{\eta} 8 \qquad \text{with } \epsilon\delta = 0.$$

Then the projective S-modules cogenerate the cotilting torsion-free class $\mathcal{Y}$ of the following torsion theory $(\mathcal{X}, \mathcal{Y})$:



Moreover, the tilting module ${}_RV = \mathrm{P}(3) \oplus \mathrm{S}(3) \oplus \mathrm{P}(1) \oplus \mathrm{S}(1)$, viewed as a right S-module in an obvious way, represents an equivalence between $\mathcal{T}$ and $\mathcal{Y}$. Using this remark, and comparing the dimension of the vector spaces of all morphisms between indecomposable modules in $\mathcal{T}$ and $\mathcal{Y}$, we immediately obtain (a). On the other hand, (b) follows from Lemma 5.5.

The next example shows that we cannot replace condition (iii) of Theorem 3.4 by a weaker one.

5.7. EXAMPLE. There are K-algebras R and S and counter equivalent torsion theories $(\mathcal{T}, \mathcal{F})$ in R-Mod and $(\mathcal{X}, \mathcal{Y})$ in S-Mod with the following properties:

(a) there is exactly one bimodule ${}_RV_S$ (resp. ${}_SV'_R$) representing an equivalence $\mathcal{T} \underset{T}{\overset{H}{\rightleftarrows}} \mathcal{Y}$ (resp. $\mathcal{X} \underset{H'}{\overset{T'}{\rightleftarrows}} \mathcal{F}$);

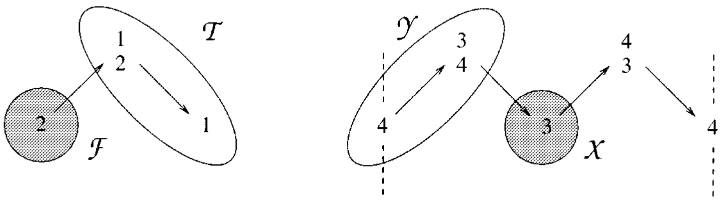
(b) ${}_RV_S$ is not a tilting bimodule;

(c) one of the following conditions holds:

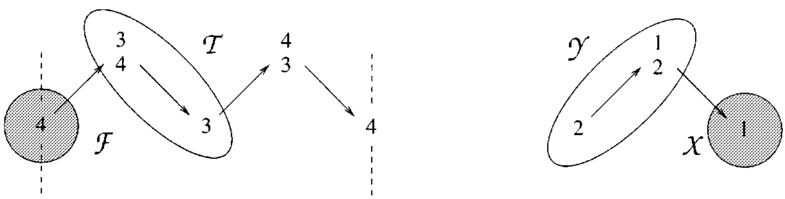
(1) $\mathrm{Ext}^1_R(V, -) \cong H'$ in R-Mod, $\mathrm{Tor}^S_1(V, -) \cong T'$ in $\overline{\mathcal{Y}}$, and $\overline{\mathcal{Y}}$ contains any simple module;

(2) $\mathrm{Tor}^S_1(V, -) \cong T'$ in S-Mod, $\mathrm{Ext}^1_R(V, -) \cong H'$ in $\overline{\mathcal{T}}$, and $\overline{\mathcal{T}}$ contains any simple module.

In the following, let A and B denote, respectively, the K-algebras given by the quivers $1 \to 2$ and $3 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 4$, with $\alpha\beta = 0$ and $\beta\alpha = 0$. Now, let $R = A$, $S = B$, and let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ be the torsion theories depicted in Γ_R and Γ_S (with identification along the vertical dashed lines):



Then (a) follows from the choice of $\mathcal{T}$, $\mathcal{Y}$, $\mathcal{X}$, and $\mathcal{F}$. On the other hand, it is easy to check that the bimodule ${}_RV_S$ defined in (a) satisfies (b) and condition (1) of (c). Finally, let $R = B$, $S = A$, and let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ be the following torsion theories:



Also in this case, (a) holds. Moreover, the bimodule ${}_RV_S$ defined in (a) satisfies (b) and condition (2) of (c).

We also note that, given a finite-dimensional K-algebra R and a torsion theory $(\mathcal{T}, \mathcal{F})$ in R-Mod, the K-algebras and torsion theories related to R and $(\mathcal{T}, \mathcal{F})$ by a basic counter equivalence may be more than expected. For instance, the next example shows that we cannot use one of the most obvious combinatorial data, that is the dimension, to distinguish noniso-morphic algebras with this property. Moreover, two of these algebras with the same dimension may be quite different from several points of view.

5.8. EXAMPLE. There are finite-dimensional K-algebras R and S and torsion theories $(\mathcal{T}, \mathcal{F})$, $(\mathcal{X}, \mathcal{Y})$ in R-Mod and $(\mathcal{X}', \mathcal{Y}')$ in S-Mod such that the following conditions hold:

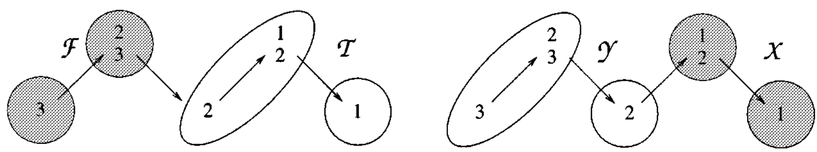
(a) there is a basic counter equivalence between $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ (resp. $(\mathcal{X}', \mathcal{Y}')$);

(b) R is not isomorphic to S, but $\dim_K R = \dim_K S$;

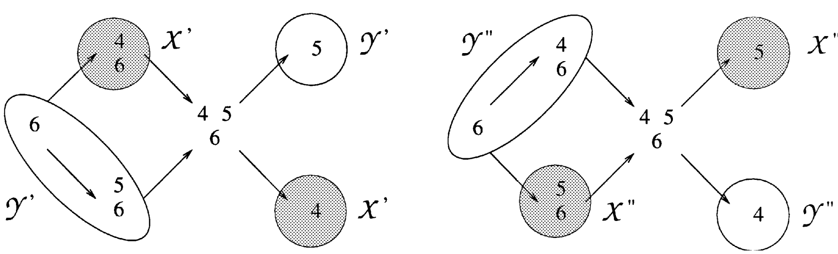
(c) $(\mathcal{X}, \mathcal{Y})$ is the unique torsion theory in R-Mod satisfying (a), but $(\mathcal{X}', \mathcal{Y}')$ is not the unique torsion theory in S-Mod satisfying (a);

(d) $R \cong \mathrm{End}({}_R M)$ for some finitely generated module ${}_R M$ such that $\mathcal{T} = \mathrm{Gen}({}_R M) \neq M^{\perp}$, but $S \not\cong \mathrm{End}({}_R N)$ for any finitely generated module ${}_R N \in \mathcal{T}$.

To see this, let R be the K-algebra given by the quiver $1 \overset{\alpha}{\to} 2 \overset{\beta}{\to} 3$ with $\beta\alpha = 0$, and let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ be the following torsion theories:



Next, let S be the K-algebra given by the quiver $4 \to 6 \leftarrow 5$, and let $(\mathcal{X}', \mathcal{Y}')$ and $(\mathcal{X}'', \mathcal{Y}'')$ be the following torsion theories:



Then (a), (b), and (c) obviously hold. On the other hand, let ${}_R M$ denote the module $S(2) \oplus P(1) \oplus S(1)$. Then we have $\mathrm{End}({}_R M) \cong R$, $\mathrm{Gen}({}_R M) = \mathcal{T}$, and $\mathrm{Ext}^1_R(M, M) \neq 0$. Finally, S admits two indecomposable projective modules with isomorphic socles. However, it is easy to see that $\mathrm{End}({}_R N)$

does not have this property for any finitely generated module ${}_RN \in \mathcal{T}$. Hence, also (d) holds.

In the next remarks we outline some properties of quasi-tilting triples with/without complements, and we show that the behavior of all but one indecomposable modules (either simple or projective–injective) does not characterize complementary quasi-tilting triples (compare with condition (iii) in Theorem 4.6).

5.9. *Remarks*. (1) Let (R, V, S) be a quasi-tilting triple representing an equivalence between the classes $\mathcal{T}$ and $\mathcal{Y}'$ defined in Example 5.8. Then we deduce from (a) that

$(*)$ (R, V, S) is a quasi-tilting triple admitting a complement.

Since $\mathrm{End}({}_RV)$ admits exactly three indecomposable modules, it follows that

$(**)$ $(R, V, \mathrm{End}({}_RV))$ is a quasi-tilting triple without complements.

More generally, it is easy to see that, for any nonzero quasi-tilting module ${}_RU \in \mathcal{T}$, the quasi-tilting triple $(R, U, \mathrm{End}({}_RU))$ does not have a complement.

(2) Let (R, W, R) be a quasi-tilting triple representing an equivalence between the classes $\mathcal{X}$ and $\mathcal{F}$ defined in Example 5.8. Then $\mathcal{X} \subsetneqq \mathcal{T}$, $\mathcal{F} \subsetneqq \mathcal{Y}$, and the following facts hold:

(i) (R, W, R) is not complementary to itself;

(ii) $W \otimes_R W = 0$;

(iii) $\mathrm{Hom}_R(W, M) = 0 = W \otimes_R M$ implies $M = 0$ for any R-module M of the form $M = P \oplus I$ with P projective and I injective (i.e., for any M of the form $M = X \oplus F$ with $X \in \mathcal{X}$ and $F \in \mathcal{F}$).

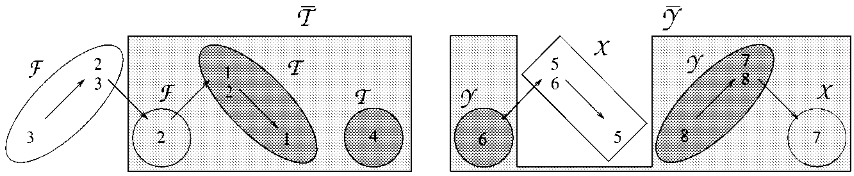
Since $\mathrm{Ker}\,\mathrm{Hom}_R(W, -) = \mathcal{Y}$ and $\mathrm{Ker}\, W \otimes_R - = \mathcal{T}$, there is exactly one indecomposable module M such that $\mathrm{Hom}_R(W, M) = 0$ and $W \otimes_R M = 0$, namely the simple module S(2).

(3) Using the K-algebra A given by the quiver $\cdot \to \cdot$ and the simple injective module I, we immediately obtain a quasi-tilting triple (A, I, A) satisfying the analogue of (i), (ii), and the following condition:

(iii′) $\mathrm{Hom}_A(I, M) = 0 = I \otimes_A M$ implies $M = 0$ for any semisimple module M (i.e., for any M without nonzero projective–injective summands).

Also dealing with K-algebras of finite representation type with the same number of indecomposable modules, it is easy to construct quasi-tilting counter equivalences which are not torsion theory counter equivalences.

5.10. Example. Let R (resp. S) be the K-algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \quad 4$ with $\beta\alpha = 0$ (resp. $5 \to 6 \quad 7 \to 8$), and let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{X}, \mathcal{Y})$ be the following splitting torsion theories:



Next, let (R, V, S) be a quasi-tilting triple representing an equivalence between $\mathcal{T}$ and $\mathcal{Y}$. Then the following facts hold:

(i) (R, V, S) does not admit a complement, because $\mathcal{F}$ and $\mathcal{X}$ are not equivalent;

(ii) the functors $\mathrm{Ext}^1_R(V, -)$ and $\mathrm{Tor}^S_1(V, -)$ (see Section 4) give the unique equivalence between $\overline{\mathcal{T}} \cap \mathcal{F}$ and $\overline{\mathcal{Y}} \cap \mathcal{X}$.

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