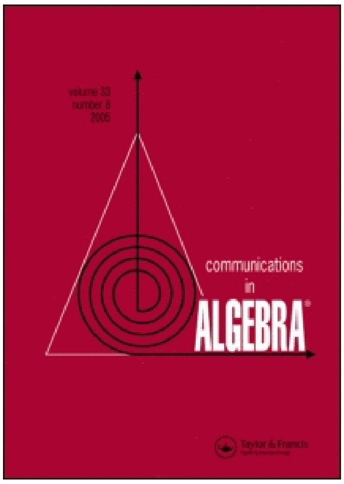

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Partial cotilting modules and the lattices induced by them

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PARTIAL COTILTING MODULES AND THE LATTICES INDUCED BY THEM

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ABSTRACT. We study a duality between (infinitely generated) cotilting and tilting modules over an arbitrary ring. Dualizing a result of Bongartz, we show that a module P is partial cotilting iff P is a direct summand of a cotilting module C such that the left Ext-orthogonal class ${}^{\perp}P$ coincides with ${}^{\perp}C$. As an application, we characterize all cotilting torsion-free classes. Each partial cotilting module P defines a lattice $\mathcal{L} = [\operatorname{Cogen} P, {}^{\perp}P]$ of torsion-free classes. Similarly, each partial tilting module P' defines a lattice $\mathcal{L}' = [[\operatorname{Gen} P', P'^{\perp}]]$ of torsion classes. Generalizing a result of Assem and Kerner, we show that the elements of $\mathcal{L}$ are determined by their Rej_P-torsion parts, and the elements of $\mathcal{L}'$ by their $\mathrm{Tr}_{P'}$-torsion-free parts.

§1 INTRODUCTION

Let R be an arbitrary associative ring with unit. Denote by Mod-R the category of all (unitary right R-) modules, and by mod-R the full subcategory of Mod-R consisting of all finitely generated modules. Similarly, R-Mod and R-mod are defined for left R-modules.

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Finitely generated cotilting modules first appeared in the context of finite dimensional algebras. There, they coincide with vector space duals of finitely generated tilting modules (see e.g. [H, IV.7.8]). Later, Colby studied them over noetherian rings [C1, §3]. He proved the following result which indicates their importance in a more general setting. In view of [HR, Theorem 2.1], [M, Theorem 1.16] and [CF, Theorem 1.4], the result may be called a Dual Tilting Theorem:

"Let R be right noetherian, S left noetherian and C be a cotilting module with $S = \operatorname{End} C_R$. Assume that C_R and ${}_SC$ are finitely generated. Then the contravariant functor $\operatorname{Hom}_R(-, C)$ defines a duality between the classes of all C-torsionless modules in mod-R and S-mod, and the contravariant functor $\operatorname{Ext}^1_R(-, C)$ defines a duality between the classes of all C-torsion modules in mod-R and S-mod," [C1, Theorems 2.4 and 3.3].

In [CF, §3], finitely generated cotilting modules were characterized over noetherian serial rings, while [C2] deals with further generalizations of the Dual Tilting Theorem.

Recently, (infinitely generated) cotilting modules and the corresponding torsion-free classes were investigated over arbitrary rings in [CDT, §1]. The latter work also reveals a duality with (infinitely generated) tilting modules, a notion coming from [CT].

In our paper, we study further aspects of this duality. We introduce the notion of a partial cotilting module, dualizing the notion of (an infinitely generated) partial tilting module, [CT]. Given a bimodule ${}_SV_R$, we describe when the dual $(V_R)^*$ is a partial cotilting right S-module. For example, if S is left noetherian and ${}_SV$ is finitely generated, this happens if and only if ${}_SV$ is partial tilting (see Proposition 2.8). In Theorem 2.11, we determine the relation between the cotilting and partial cotilting modules, using a generalization of a dual of the Bongartz Lemma, [T, Lemma 6.9].

As an application, we prove that a torsion-free class $\mathcal{F}$ is cogenerated by a cotilting module if and only if $\mathcal{F}$ is the left Ext-orthogonal class of an element of $\mathcal{F}$ (Corollary 2.12). In Proposition 2.15, we describe all partial cotilting abelian groups G such that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, G) = 0$, hence all cotilting torsion-free classes of abelian groups containing $\mathbb{Q}$. In Theorems 3.6 and 4.4, we give a characterization of the lattices of all torsion-free classes induced by a partial cotilting module, P, and of all torsion classes induced by a partial tilting module, P'. We prove that the classes are determined by their Rej_P-torsion parts, and their $\operatorname{Tr}_{P'}$-torsion-free parts, respectively. This may be viewed as a sequel to the works of Geigle and Lenzing, [GL], and Assem and Kerner, [AK].

Let $L \in$ Mod-R. Denote by $\operatorname{Gen} L$ ($\operatorname{Cogen} L$) the class of all modules *generated* (*cogenerated*) by L, that is of all $M \in$ Mod-R such that there exist a cardinal λ and an epimorphism $L^{(\lambda)} \xrightarrow{\phi} M \to 0$ (a monomorphism $0 \to M \xrightarrow{\phi} L^{\lambda}$).

For a module M, denote by $\operatorname{Tr}_L(M)$ ($\operatorname{Rej}_L(M)$) the *trace* $\sum\{\operatorname{Im}(f) \mid f \in \operatorname{Hom}_R(L, M)\}$ (the *reject* $\cap\{\operatorname{Ker}(f) \mid f \in \operatorname{Hom}_R(M, L)\}$) of L in M, that is the largest (least) submodule M_0 of M such that $M_0 \in \operatorname{Gen} L$ ($M/M_0 \in \operatorname{Cogen} L$). Note that $\operatorname{Tr}_L(-)$ is an idempotent preradical and $\operatorname{Rej}_L(-)$ is a radical for Mod-R.

For a left module V and a right module M we define the annihilator of V in M to be $\mathrm{Ann}_M(V) = \{m \in M \mid m \otimes v = 0 \text{ in } M \otimes V\ \forall v \in V\}$.

Further, $L^{\perp}$ (${}^{\perp}L$) denotes the class of all modules M such that $\mathrm{Ext}^1_R(L, M) = 0$ ($\mathrm{Ext}^1_R(M, L) = 0$).

For a module M, $\mathrm{proj\,dim}(M)$ ($\mathrm{inj\,dim}(M)$, $\mathrm{w\,dim}(M)$) denotes the projective (injective, weak) dimension of M.

As usual, $\mathbb{Z}$ and $\mathbb{Q}$ denote the group of all integers and all rational numbers, respectively. $\mathbb{P}$ denotes the set of all primes in $\mathbb{Z}$. For $p \in \mathbb{P}$, we denote by $\mathbb{J}_p$, $\mathbb{Z}_p$, and $\mathbb{Z}_{p^\infty}$, the group of all p-adic integers, the cyclic group of order p, and the Prüfer p-group, respectively.

For further notation, we refer to [F], [R] and [W].

§2 Partial cotilting modules

First, we recall the definition and the basic theorem on cotilting modules from [CDT, §1].

2.1. Definition. A module C is a *cotilting module* if:

i) $\mathrm{inj\,dim}\, C \leq 1$;
ii) $\mathrm{Ext}^1(C^\lambda, C) = 0$ for each cardinal λ;
iii) for any module M, if $\mathrm{Hom}(M, C) = 0 = \mathrm{Ext}^1(M, C)$ then $M = 0$.

Moreover, C is *strongly cotilting* provided that C satisfies i), ii), and

iii') there is an injective cogenerator Q for Mod-R and an exact sequence $0 \to C' \to C'' \to Q \to 0$ such that C' and C'' are direct summands of C^λ, for a cardinal λ.

2.2. Theorem. [CDT, Proposition 1.7] *A module C is cotilting if and only if* $\mathrm{Cogen}\, C = {}^{\perp}C$.

So each cotilting module cogenerates a torsion-free class in Mod-R, called the *cotilting torsion-free class.*

2.3. Examples. 1) An injective module is cotilting iff it is a cogenerator. If C is a cotilting module, then C^λ is cotilting for any cardinal λ.

2) Applying the functor $\mathrm{Hom}(M, -)$, it is easy to see that each strongly cotilting module is cotilting. The converse is not true in general by [CDT, Example 5.3]. This is surprising in view of the fact [CDT, Definition 1.1] that the corresponding dual notions coincide (with the notion of the tilting module). Nevertheless, by 2.13, every cotilting torsion-free class is cogenerated by a strongly cotilting module.

3) By [CM, Proposition 1.2 2)] and [R, Theorem 11.54], if P is a $*$-module and Q is an injective cogenerator for Mod-R, then the dual module $P^* = \mathrm{Hom}(P, Q)$ is a cotilting right S-module, where $S = \mathrm{End}\, P$. In particular, the dual of any (quasi-) tilting module is cotilting.

2.4. Proposition. *Let R be a ring and C be a cotilting module. Then the module* $I(C) \oplus I(C)/C$ *is an injective cogenerator for* Mod-R.

Proof. Since $\operatorname{inj}\dim C \leq 1$, the module $I(C)/C$ is injective. To prove the assertion it suffices to show that any simple module S embeds into $I(C) \oplus I(C)/C$. If S is torsion-free, then by 2.2, S is cogenerated by C, so that S, being simple, embeds into $I(C)$. On the other hand, if S is a torsion simple module, then by 2.2 it follows that $0 = \operatorname{Hom}_R(S, C) \cong \operatorname{Hom}_R(S, I(C))$ and $\operatorname{Ext}^1_R(S, C) \neq 0$. From the short exact sequence $0 \to C \to I(C) \to I(C)/C \to 0$ we get the exact sequence $0 = \operatorname{Hom}_R(S, I(C)) \to \operatorname{Hom}_R(S, I(C)/C) \to \operatorname{Ext}^1_R(S, C) \to \operatorname{Ext}^1_R(S, I(C)) = 0$ which shows that $\operatorname{Hom}_R(S, I(C)/C) \neq 0$. So S embeds into $I(C)/C$. □

Now, we dualize the notion of a partial tilting module introduced in [CT, Definition 1.4]:

2.5. Definition. A module P is a *partial cotilting module* if $\operatorname{Cogen} P \subseteq {}^{\perp}P$ and ${}^{\perp}P$ is a torsion-free class.

2.6. Lemma. *If P is a partial cotilting module then*

a) $\operatorname{Cogen} P$ *is a torsion-free class, with the associated radical* Rej_P.

b) P *satisfies conditions* i) *and* ii) *of* 2.1.

Proof. a): It is enough to prove that $\operatorname{Cogen} P$ is closed under extensions. Let $0 \to L \to M \to M/L \to 0$ be a short exact sequence with L and M/L in $\operatorname{Cogen} P$. As $\operatorname{Cogen} P \subseteq {}^{\perp}P$, every homomorphism from L to P extends to M. Therefore $\operatorname{Rej}_P(M) \cap L = \operatorname{Rej}_P(L) = 0$. On the other hand, $(\operatorname{Rej}_P(M) + L)/L \subseteq \operatorname{Rej}_P(M/L) = 0$. This proves that $\operatorname{Rej}_P(M) = 0$, i.e. $M \in \operatorname{Cogen} P$.

b): By the premise, ${}^{\perp}P$ contains every projective module and it is closed under submodules. For every module M and exact sequence $0 \to K \to F \to M \to 0$ with F projective, we get the exact row

$$0 = \operatorname{Ext}^1(K, P) \to \operatorname{Ext}^2(M, P) \to \operatorname{Ext}^2(F, P) = 0$$

and $\operatorname{Ext}^2(M, P) = 0$. This proves i). The condition ii) is clearly satisfied. □

2.7. Examples. 1) Any injective module is partial cotilting. If P is a partial cotilting module, then P^{λ} is partial cotilting for any cardinal λ.

2) $\operatorname{Cogen} P$ need not be a cotilting torsion free class even if P is a partial cotilting module (for example, if P is not faithful).

3) Clearly, conditions i) and ii) of 2.1 are equivalent to $\operatorname{Cogen} P \subseteq {}^{\perp}P$ and ${}^{\perp}P$ is closed under submodules (and extensions). Nevertheless, it is an open problem whether these conditions are sufficient for P to be a partial cotilting module.

By 2.3 3), a dual of any tilting module is cotilting. The next proposition determines when a dual of a module is (partial) cotilting.

2.8. Proposition. *Let ${}_SV_R$ be a bimodule, Q an injective cogenerator for* Mod-R *and $V^*_S = \operatorname{Hom}_R(V, Q)$. Then:*

a) *For all $M \in$ Mod-S, $\operatorname{Rej}_{V^*}(M) = \operatorname{Ann}_M(V)$;*

b) *V^*_S is cotilting if and only if for all $M \in$ Mod-S the condition $\operatorname{Tor}^S_1(M, V) = 0$ is equivalent to $\operatorname{Ann}_M(V) = 0$;*

c) *V_S^* is partial cotilting if and only if*
   1) $\operatorname{w\,dim}{}_SV \leq 1$;
   2) $\operatorname{Tor}_1^S(V^*, V) = 0$;
   3) $\operatorname{Ker}\operatorname{Tor}_1^S(-, V)$ *is closed under products;*

d) *Assume that S is left noetherian and ${}_SV$ is finitely generated. Then V_S^* is partial cotilting if and only if ${}_SV$ is a partial tilting module.*

Proof. a): Let $M \in \text{Mod-}S$. Then $\operatorname{Hom}_S(M, V^*) \cong \operatorname{Hom}_R(M \otimes_S V, Q)$ naturally. Also, $x \in \operatorname{Rej}_{V^*}(M)$ iff $f(x) = 0$ for all $f \in \operatorname{Hom}_S(M, V^*)$. The latter is equivalent to $x \otimes v = 0$ in $M \otimes_S V$ for all $v \in V$, i.e. to $x \in \operatorname{Ann}_M(V)$.

b): By part a), $\operatorname{Cogen} V^* = \operatorname{Ker}\operatorname{Rej}_{V^*}(-) = \operatorname{Ker}\operatorname{Ann}_-(V)$. Moreover, the natural isomorphism $\operatorname{Ext}_S^1(-, V^*) \cong \operatorname{Hom}_R(\operatorname{Tor}_1^S(-, V), Q)$ from [R, Theorem 11.54] yields ${}^\perp V^* = \operatorname{Ker}\operatorname{Tor}_1^S(-, V)$. Applying 2.2, we get the result.

c): As in b), we have ${}^\perp V^* = \operatorname{Ker}\operatorname{Tor}_1^S(-, V)$. Moreover, the latter class is closed under submodules iff $\operatorname{w\,dim}{}_SV \leq 1$.

d): First, the weak and the projective dimension of ${}_SV$ coincide by [CE, VI, Exercise 3 b)]. So condition 1) in c) is equivalent to $\operatorname{proj\,dim}{}_SV \leq 1$. Next, as a particular case of [R, Theorem 9.51], we obtain an isomorphism $\operatorname{Tor}_1^S(V^*, V) \cong \operatorname{Hom}_R(\operatorname{Ext}_S^1(V, V), Q)$. So condition 2) in c) is equivalent to $\operatorname{Ext}_S^1(V, V) = 0$. Finally, if ${}_SM$ is a finitely presented left S-module, then the functor $-\otimes_S M$ commutes with direct products (for instance, see [W]). Since there is an exact sequence $0 \to F' \to F \to V \to 0$ in S-mod such that F and F' are finitely presented and F is flat, it follows that the functor $\operatorname{Tor}_1^S(-, V)$ commutes with direct products. So condition 3) in c) is always satisfied in the given setting. By [CT, Section 1], a finitely generated left S-module V is partial tilting if and only if $\operatorname{proj\,dim}{}_SV \leq 1$ and $\operatorname{Ext}_S^1(V, V) = 0$, and c) applies. □

Note that in 2.8 b) and d), whether or not V^* is (partial) cotilting depends only on the properties of V as a left S-module.

2.9. Example. Let Λ be an artin algebra (over a commutative artinian ring k). Let J be the minimal injective cogenerator for Mod-k. Denote by $D = \operatorname{Hom}_k(-, J)$ the standard duality between Λ-mod and mod-Λ [ARS, II, §3]. Let $V \in \Lambda$-mod. Then V is a (Λ, k)-bimodule and 2.8 d) shows that ${}_\Lambda V$ is a partial tilting module iff $D(V)_\Lambda$ is partial cotilting. In particular, if k is a field, i.e. Λ is a finite dimensional k-algebra, then taking $J = k$ we see that for finitely generated modules, our notions of a partial tilting and a partial cotilting module are just k-duals of each other.

The following is a general version of a dual of Bongartz Lemma:

2.10. Lemma. [T, Lemma 6.9] *Let R and S be rings, let P be an S-R-bimodule and Q be an R-module. Let λ be the minimal number of generators of the left S-module $\operatorname{Ext}_R^1(Q, P)$. Assume $\operatorname{Ext}_R^1(P^\lambda, P) = 0$. Then there is a module M satisfying*
   1) $\operatorname{Ext}_R^1(M, P) = 0$, *and*
   2) *there is an exact sequence* $0 \to P^\lambda \to M \to Q \to 0$.

2.11. Theorem. *A module P is partial cotilting if and only if there is a (strongly) cotilting module C such that P is a direct summand of C and ${}^\perp C = {}^\perp P$.*

Proof. $\Rightarrow$: We use 2.10 for $S = \mathbb{Z}$ and Q an injective cogenerator for Mod-R. Take M as in 2.10. Then $M \in {}^{\perp}P$, and $P \in \operatorname{Cogen} M$.

Let $N \in {}^{\perp}P$. Applying the covariant functor $\operatorname{Hom}_R(N, -)$ to the exact sequence from 2.10 2), we obtain

$$0 \to \operatorname{Hom}(N, P^{\lambda}) \to \operatorname{Hom}(N, M) \xrightarrow{p} \operatorname{Hom}(N, Q) \to$$
$$\to \operatorname{Ext}^1(N, P^{\lambda}) = 0 \to \operatorname{Ext}^1(N, M) \to \operatorname{Ext}^1(N, Q) = 0.$$

Therefore $N \in {}^{\perp}M$. Since p is onto, we have $\operatorname{Rej}_M(N) \subseteq \operatorname{Rej}_Q(N) = 0$. This proves that ${}^{\perp}P \subseteq {}^{\perp}M \cap \operatorname{Cogen} M$.

Let $C = M \oplus P$. Then ${}^{\perp}C = {}^{\perp}M \cap {}^{\perp}P = {}^{\perp}P$. Moreover, $\operatorname{Cogen} C = \operatorname{Cogen} M \supseteq {}^{\perp}P$. Since $M \in {}^{\perp}P$ and ${}^{\perp}P$ is a torsion-free class, we get $\operatorname{Cogen} M \subseteq {}^{\perp}P$. Altogether, we have $\operatorname{Cogen} C = \operatorname{Cogen} M = {}^{\perp}P = {}^{\perp}C$, and C is cotilting by 2.2. Moreover, by 2.10 2), C is strongly cotilting.

$\Leftarrow$: Since ${}^{\perp}P = {}^{\perp}C = \operatorname{Cogen} C$, we infer that ${}^{\perp}P$ is a torsion-free class. Moreover, $\operatorname{Cogen} P \subseteq \operatorname{Cogen} C$, as P is a direct summand of C. □

2.12. Corollary. *Let $\mathcal{F}$ be a torsion-free class in* Mod-R. *Then $\mathcal{F}$ is cotilting if and only if $\mathcal{F} = {}^{\perp}F$ for some $F \in \mathcal{F}$.*

Proof. The necessity follows from 2.2. Conversely, if $F \in \mathcal{F} = {}^{\perp}F$, then F is a partial cotilting module. The conclusion follows by 2.11. □

2.13. Remarks. 1) Define a "strongly cotilting torsion-free class" as a class cogenerated by a strongly cotilting module. By 2.12, "strongly cotilting torsion-free classes" coincide with the cotilting ones.

2) Clearly, Mod-R is always a cotilting torsion-free class (the largest one). Denote by $\mathcal{P}$ the class of all projective modules. It is well-known that $\mathcal{P}$ is a torsion-free class if and only if R is a right hereditary left coherent semiprimary ring. In this case, by [T, Proposition 1.4], $\mathcal{P} = {}^{\perp}M$, where M is the direct sum of a representative set, A, of the class of all simple modules. Therefore, the free module $F = R^{(A)}$ satisfies $\operatorname{Cogen} F = {}^{\perp}F = \mathcal{P}$, and $\mathcal{P}$ is a cotilting torsion-free class (the smallest one).

Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory such that $\mathcal{F} = \operatorname{Cogen} C$ for a module C. A module N is said to have a *C-torsion-free resolution* provided that there exist modules $C', C'' \in \mathcal{F}$ such that C' is a direct summand of a direct product of copies of C and there is an exact sequence $0 \to C' \to C'' \to N \to 0$ in Mod-R.

As another application of 2.10, we dualize and generalize a construction of Tachikawa and Wakamatsu (cf. [TW]):

2.14 Corollary. *Let R be a ring and C be a cotilting module. Then each module has a C-torsion-free resolution.*

Proof. Take $P = C$, $Q = N$ and $S = \operatorname{End} C$ in 2.10. By 2.2, $C' = M$ and $C'' = C^{\lambda}$ have the required properties. □

We end this section by considering the particular case of (partial) cotilting abelian groups:

2.15. Proposition. *Let $R = \mathbb{Z}$ and $P \in$ Mod-$\mathbb{Z}$. Then the following two conditions are equivalent:*

(i) *P is a partial cotilting group such that $\mathbb{Q} \in {}^{\perp}P$,*

(ii) *P is a group of the form*

$$P = \bigoplus_{p\in A'} \mathbb{Z}_{p^\infty}^{(\alpha_p)} \oplus \mathbb{Q}^{(\beta)} \oplus \prod_{p\in A} C_p,$$

where A, A' are disjoint (possibly empty) subsets of $\mathbb{P}$, C_p is a p-adic completion of a non-zero direct sum of copies of the group $\mathbb{J}_p$, and $\alpha_p \geq 1$ for all $p \in A'$.

Moreover, if P is of the form (ii) *then P is cotilting iff $A' = \mathbb{P} \setminus A$, and $\beta > 0$ provided that $A' = \emptyset$. The corresponding cotilting torsion-free class consists of all groups G such that G contains no elements of order p for all $p \in A$.*

Proof. (i) $\Rightarrow$(ii): First, we show that the torsion part, T, of P is divisible, so $P = T \oplus P'$ for a torsion-free partial cotilting group P'. Indeed, for a prime $p \in \mathbb{P}$ denote by T_p the p-part of T. If $T_p \neq 0$, then $P/pP \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_p, P) = 0$. It follows that all the subgroups T_p, $p \in \mathbb{P}$, are divisible, and so is $T = \oplus_{p\in\mathbb{P}} T_p$.

We have $P' = D \oplus D'$, where $D \cong \mathbb{Q}^{(\beta)}$ is the divisible part of P' and D' is reduced. Clearly, ${}^{\perp}P = {}^{\perp}D'$.

Now, $\mathbb{Q} \in {}^{\perp}P$ implies that D' is a cotorsion group. (Actually, $\mathbb{Q} \in {}^{\perp}P$ always holds true whenever either $T \neq 0$ or $D \neq 0$, since then $\mathbb{Q} \leq P^\omega$). By [F, Corollary 54.5 and Proposition 40.1], $D' \cong \prod_{p\in\mathbb{P}} C_p$, where C_p is a p-adic completion of a direct sum of copies of the group $\mathbb{J}_p$. Let $A = \{p \in \mathbb{P} \mid C_p \neq 0\}$. Since $\mathrm{Ext}^1_{\mathbb{Z}}(H, F) \cong \mathrm{Hom}_{\mathbb{Z}}(H, I(F)/F)$ for each torsion group H and each torsion-free group F, we infer that ${}^{\perp}D'$ consists of all groups G without elements of order p, for all $p \in A$. Then $T \cong \oplus_{p\in A'} \mathbb{Z}_{p^\infty}^{(\alpha_p)}$, where $A' \subseteq \mathbb{P} \setminus A$, and $\alpha_p \geq 1$ for all $p \in A'$.

(ii) $\Rightarrow$(i): Since ${}^{\perp}P$ consists of all groups G such that G contains no elements of order p for all $p \in A$, P is partial cotilting and (i) holds.

The remaining assertions are now clear. □

The existence of partial cotilting (reduced torsion-free) groups P such that $\mathbb{Q} \notin {}^{\perp}P$ remains an open problem. This is closely related to another open problem, due to Schultz, asking for the structure of self-splitting abelian groups (a torsion-free group P is *self-splitting* if $\mathrm{Ext}^1_{\mathbb{Z}}(P, P) = 0$).

Clearly, each partial cotilting torsion-free group, P, is self-splitting. Moreover, P must contain a chain of subgroups of a specific form:

2.16. Proposition. *Let $P \neq 0$ be a torsion-free group such that $\mathrm{Ext}^1_{\mathbb{Z}}(P^\omega, P) = 0$. Then*

i) *P has no non-zero slender factor-groups.*

ii) *P contains an increasing continuous chain of subgroups, $(P_\alpha \mid \alpha \leq \kappa)$, such that $\kappa > 0$, $P_0 = 0$, $P_\kappa = P$, P_α is a pure subgroup of P and $P_{\alpha+1}/P_\alpha \in \{\mathbb{Z}^\omega, \mathbb{Q}\} \cup \{\mathbb{J}_p \mid p \in \mathbb{P}\}$ for all even ordinals $\alpha < \kappa$, and $P_{\alpha+1}/P_\alpha$ is a countable torsion group for all odd ordinals $\alpha < \kappa$. In particular,* card$(P) \geq$ 2^ω *provided that P is reduced.*

Proof. i): Assume $0 \neq P'$ is a slender factor-group of P. By the premise, we have $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^\omega, P') = 0$. By [EM, III, Corollary 1.3], $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\omega/\mathbb{Z}^{(\omega)}, P') = 0$. Starting with the short exact sequence $0 \to \mathbb{Z}^{(\omega)} \to \mathbb{Z}^\omega \to \mathbb{Z}^\omega/\mathbb{Z}^{(\omega)} \to 0$, we get

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\omega, P') \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(\omega)}, P') \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^\omega/\mathbb{Z}^{(\omega)}, P') \to 0.$$

Since the canonical isomorphism $\prod_{n<\omega} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, P') \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(\omega)}, P')$ restricts to the isomorphism $\oplus_{n<\omega} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, P') \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\omega, P')$ of [EM, III, Corollary 1.5], we infer that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^\omega/\mathbb{Z}^{(\omega)}, P') \cong (P')^\omega/(P')^{(\omega)}$. Since $\mathbb{Z}^\omega/\mathbb{Z}^{(\omega)}$ is torsion-free, the group $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^\omega/\mathbb{Z}^{(\omega)}, P')$ is divisible. It follows that P' is divisible, a contradiction.

ii): By a theorem of Nunke [EM, IX, Corollary 2.4], the torsion-free non-slender groups are exactly the groups containing a copy of $\mathbb{Z}^\omega$ or $\mathbb{Q}$ or $\mathbb{J}_p$, for a prime $p \in \mathbb{P}$. It follows that P contains a chain of the required form. $\square$

Of course, the existence of a chain of the form 2.16 ii) in P is a necessary, but not a sufficient, condition for the group P to satisfy $\mathrm{Ext}^1_{\mathbb{Z}}(P^\omega, P) = 0$.

§3 Lattices of torsion-free classes

In this section, P denotes a partial cotilting module. By 2.5 and 2.6, the classes $\mathrm{Cogen}\, P \subseteq {}^\perp P$ are torsion-free. By 2.2, the two classes coincide if and only if P is a cotilting module. Otherwise, we have a non-trivial interval $[\mathrm{Cogen}\, P, {}^\perp P]$ in the lattice of all torsion-free classes. In this section, we show that elements of this interval are characterized by their Rej_P-torsion parts.

3.1. Definition. a) ${}^\top P = \mathrm{Ker}\,\mathrm{Hom}(-, P)$ denotes the torsion class corresponding to the torsion free class $\mathrm{Cogen}\, P$ (see 2.6 a)).

b) ${}^+P = {}^\top P \cap {}^\perp P$.

3.2. Example. Let $R = \mathbb{Z}$. Define P by the relation as in 2.15 (ii), where A, A' are disjoint (possibly empty) subsets of $\mathbb{P}$, C_p is a p-adic completion of a non-zero direct sum of copies of the group $\mathbb{J}_p$, and $\alpha_p \geq 1$ for all $p \in A'$. Moreover, assume that $\beta > 0$ provided that $A' = \emptyset$. Then $\mathrm{Cogen}\, P$ is the class of all groups without elements of order p for all $p \in \mathbb{P} \setminus A'$, ${}^\perp P$ is the class of all groups without elements of order p for all $p \in A$, and ${}^+P$ is the class of all torsion groups without elements of order p for all $p \in A \cup A'$.

3.3. Lemma. *${}^+P$ is closed under taking direct summands, direct sums and extensions. Moreover*

a) *For all $M \in {}^+P$ and $L \leq M$ we get*

$$L \in {}^+P \Leftrightarrow L \in {}^\top P \Leftrightarrow M/L \in {}^\perp P \Leftrightarrow M/L \in {}^+P.$$

b) *$\forall M_\lambda \in {}^+P$, $\lambda \in \Lambda$, we get $\mathrm{Rej}_P(\prod_\lambda M_\lambda) \in {}^+P$.*

Proof. Both ${}^\top P$ and ${}^\perp P$ are closed under direct summands, direct sums and extensions.

a): Applying $\mathrm{Hom}(-, P)$ to the exact sequence $0 \to L \to M \to M/L \to 0$ we have

$$0 \to \mathrm{Hom}(M/L,P) \to \mathrm{Hom}(M,P) = 0 \to \mathrm{Hom}(L,P) \xrightarrow{\cong} \mathrm{Ext}^1(M/L,P) \to$$
$$\to \mathrm{Ext}^1(M,P) = 0 \to \mathrm{Ext}^1(L,P) \to \mathrm{Ext}^2(M/L,P) = 0,$$

and the assertion follows.

b): ${}^{\perp}P$ is a torsion-free class, so

$$\mathrm{Rej}_P(\prod_\lambda M_\lambda) \le \prod_\lambda M_\lambda \in {}^{\perp}P,$$

hence $\mathrm{Rej}_P(\prod_\lambda M_\lambda) \in {}^{\perp}P$. Since Rej_P is an idempotent radical (see 2.6 a)), $\mathrm{Rej}_P(\prod_\lambda M_\lambda) \in {}^{\top}P$. □

Let $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq$ Mod-R. Denote by $\langle \mathcal{C}_1, \mathcal{C}_2 \rangle$ the lattice of all $\mathcal{C} \subseteq$ Mod-R such that $\mathcal{C}_1 \subseteq \mathcal{C} \subseteq \mathcal{C}_2$. Moreover, $[\mathcal{C}_1, \mathcal{C}_2]$ denotes the lattice of all torsion-free classes $\mathcal{F}$ in Mod-R such that $\mathcal{C}_1 \subseteq \mathcal{F} \subseteq \mathcal{C}_2$.

3.4. Lemma. *Let*

$$\nu\colon \langle \mathrm{Cogen}\, P, {}^{\perp}P \rangle \to \langle \{0\}, {}^{+}P \rangle, \quad \mathcal{C} \mapsto \mathcal{C} \cap {}^{\top}P,$$
$$\mu\colon \langle \{0\}, {}^{+}P \rangle \to \langle \mathrm{Cogen}\, P, {}^{\perp}P \rangle, \quad \mathcal{C}' \mapsto \{M \in \text{Mod-}R \mid \mathrm{Rej}_P(M) \in \mathcal{C}'\},$$

then ν and μ are well defined lattice homomorphisms, and $\nu \circ \mu = \mathrm{id}$. Moreover, $\mu \circ \nu(\mathcal{C}) = \mathcal{C}$ iff $\forall M \in$ Mod-R : $M \in \mathcal{C} \Leftrightarrow \mathrm{Rej}_P(M) \in \mathcal{C}$. In particular, $\mu \circ \nu(\mathcal{C}) = \mathcal{C}$ provided that $\mathcal{C}$ is closed under submodules and extensions.

Proof. For the first part, it suffices to show that $\mu(\mathcal{C}') \in \langle \mathrm{Cogen}\, P, {}^{\perp}P \rangle$ provided $\{0\} \subseteq \mathcal{C}' \subseteq {}^{+}P$. Clearly, $\mathrm{Cogen}\, P \subseteq \mu(\mathcal{C}')$, and $\mu(\mathcal{C}') \subseteq {}^{\perp}P$ as P is partial cotilting. Moreover, μ and ν are lattice homomorphisms since they preserve arbitrary intersections. The fact that $\nu \circ \mu = \mathrm{id}$ is immediate. Finally, let $\mathcal{C} \in \langle \mathrm{Cogen}\, P, {}^{\perp}P \rangle$. Then $\mu \circ \nu(\mathcal{C}) = \{M \in \text{Mod-}R \mid \mathrm{Rej}_P(M) \in \nu(\mathcal{C})\} = \{M \in \text{Mod-}R \mid \mathrm{Rej}_P(M) \in \mathcal{C}\}$. □

3.5. Definition. a) A subclass $\mathcal{C} \subseteq {}^{+}P$ is called a *torsion-free class in* ${}^{+}P$ if

1) $\mathcal{C}$ is closed under extensions;
2) for all $M \in \mathcal{C}$, if $L \le M$ and $L \in {}^{+}P$ then $L \in \mathcal{C}$;
3) $\mathrm{Rej}_P(\prod_\lambda M_\lambda) \in \mathcal{C}$ for all families $\{M_\lambda\}_{\lambda \in \Lambda}$ of elements of $\mathcal{C}$.

Denote by $\mathbb{F}({}^{+}P)$ the lattice of all torsion-free classes in ${}^{+}P$.

b) Given $\mathcal{C}$, $\mathcal{D} \subseteq$ Mod-R, we define

$$\mathbb{E}(\mathcal{C}, \mathcal{D}) = \{M \in \text{Mod-}R \mid \exists C \in \mathcal{C}\ \exists D \in \mathcal{D} : 0 \to D \to M \to C \to 0 \text{ is exact}\}.$$

3.6. Theorem. *The mappings*

$$\bar{\nu}\colon [\mathrm{Cogen}\, P, {}^{\perp}P] \to \mathbb{F}({}^{+}P), \quad \mathcal{F} \mapsto \mathcal{F} \cap {}^{\top}P,$$
$$\bar{\mu}\colon \mathbb{F}({}^{+}P) \to [\mathrm{Cogen}\, P, {}^{\perp}P], \quad \mathcal{F}' \mapsto \mathbb{E}(\mathrm{Cogen}\, P, \mathcal{F}')$$

are restrictions of the mappings ν and μ, respectively, defined in 3.4. *Moreover, $\bar{\nu}$ and $\bar{\mu}$ are inverse lattice isomorphisms.*

Proof. First we prove that $\bar{\mu}(\mathcal{F}') = \mu(\mathcal{F}')$ for any $\mathcal{F}' \in \mathbb{F}({}^{+}P)$. The inclusion $\mu(\mathcal{F}') \subseteq \bar{\mu}(\mathcal{F}')$ is clear. Conversely, consider an exact sequence $0 \to L \to M \to$

$M/L \to 0$ where $L \in \mathcal{F}'$ and $M/L \in \operatorname{Cogen} P$. Then $\operatorname{Rej}_P(M) \leq L$ and $\operatorname{Rej}_P(M) \in {}^+P$, so that $\operatorname{Rej}_P(M) \in \mathcal{F}'$ by 3.5 a) 2).

Now we prove that the maps are well defined. For $\bar{\nu}$ it follows easily by 3.3 and 3.5 a). For the rest of the proof, let $\mathcal{F}' \in \mathbb{F}({}^+P)$.

Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of elements of $\bar{\mu}(\mathcal{F}')$. So $\operatorname{Rej}_P(M_\lambda) \in \mathcal{F}'$ for all $\lambda \in \Lambda$, and $\operatorname{Rej}_P(\prod_\lambda \operatorname{Rej}_P(M_\lambda)) \in \mathcal{F}'$. Moreover,

$$\operatorname{Rej}_P(\textstyle\prod_\lambda \operatorname{Rej}_P(M_\lambda)) \leq \operatorname{Rej}_P(\prod_\lambda M_\lambda) \leq \prod_\lambda \operatorname{Rej}_P(M_\lambda)$$

and Rej_P is idempotent, so that $\operatorname{Rej}_P(\prod_\lambda \operatorname{Rej}_P(M_\lambda)) = \operatorname{Rej}_P(\prod_\lambda M_\lambda) \in \mathcal{F}'$. This proves that $\prod_\lambda M_\lambda \in \bar{\mu}(\mathcal{F}')$.

Now, let $L \leq M \in \bar{\mu}(\mathcal{F}')$, so that $\operatorname{Rej}_P(M) \in \mathcal{F}'$. Then $\operatorname{Rej}_P(L) \in {}^+P$, and by 3.5 a) 2) $\operatorname{Rej}_P(L) \in \mathcal{F}'$, i.e. $L \in \bar{\mu}(\mathcal{F}')$.

Let $0 \to L \to M \to M/L \to 0$ be an exact sequence where L, $M/L \in \bar{\mu}(\mathcal{F}')$. In order to prove that M belongs to $\bar{\mu}(\mathcal{F}')$, we show that $\operatorname{Rej}_P(M) \in \mathcal{F}'$. First of all, the modules L, M and M/L belong to ${}^\perp P$; in particular, every homomorphism from L to P extends to M. Therefore $\operatorname{Rej}_P(L) = \operatorname{Rej}_P(M) \cap L$. Hence,

$$\operatorname{Rej}_P(M)/\operatorname{Rej}_P(L) \cong (\operatorname{Rej}_P(M) + L)/L \leq \operatorname{Rej}_P(M/L) \in \mathcal{F}'.$$

Therefore $\operatorname{Rej}_P(M)/\operatorname{Rej}_P(L) \in {}^\perp P$, and $\operatorname{Rej}_P(M) \in {}^+P$, as $\operatorname{Rej}_P(M) \leq M \in {}^\perp P$ and $\operatorname{Rej}_P(M)$ is a Rej_P-torsion module. Applying 3.3 a), we see that the module $\operatorname{Rej}_P(M)/\operatorname{Rej}_P(L)$ belongs to ${}^+P$. By 3.5 a) 2), $\operatorname{Rej}_P(M)/\operatorname{Rej}_P(L) \in \mathcal{F}'$. Since $\mathcal{F}'$ is closed under extensions, $\operatorname{Rej}_P(M) \in \mathcal{F}'$.

By 3.4, $\bar{\nu}$ and $\bar{\mu}$ are inverse lattice isomorphisms. □

Now, we characterize cotilting torsion-free classes in the interval $[\operatorname{Cogen} P, {}^\perp P]$.

3.7. Proposition. *Let P and C be modules. Assume P is partial cotilting. Then the following conditions are equivalent:*

(i) *C is a cotilting module and P is a direct summand of C^λ, for a cardinal λ;*
(ii) $\operatorname{Cogen} C = {}^\perp C \in [\operatorname{Cogen} P, {}^\perp P]$.

Proof. Dual of [CT, Lemma 2.9], using [CDT, Proposition 1.3]. □

3.8. Corollary. *Let C_1, C_2 be cotilting modules. Then the following conditions are equivalent:*

(i) *C_1 is a direct summand of C_2^λ for a cardinal λ;*
(ii) *C_2 is a direct summand of C_1^λ for a cardinal λ;*
(iii) *$C_1 \in {}^\perp C_2$ and $C_2 \in {}^\perp C_1$;*
(iv) $\operatorname{Cogen} C_1 = \operatorname{Cogen} C_2$.

§4 Lattices of torsion classes

This section contains a proof of the general version of the Assem-Kerner Theorem, [AK, Theorem 2.1]. The result has been announced in [CT, Proposition 2.8].

In the sequel, P denotes a partial tilting module over an arbitrary ring R, so $\operatorname{Gen} P \subseteq P^{\perp}$ are torsion classes (see [CT, Definition 1.4 and Lemma 2.4]).

Define $P^{\top} = \operatorname{Ker}\operatorname{Hom}_R(P,-)$ and $P^{+} = P^{\perp} \cap P^{\top}$.

4.1. Lemma. *The class P^+ is closed under taking direct summands, direct sums, direct products and extensions. Moreover, for all $M \in P^+$ and $L \leq M$, we have*

$$L \in P^+ \Leftrightarrow L \in P^{\perp} \Leftrightarrow M/L \in P^{\top} \Leftrightarrow M/L \in P^+.$$

Proof. The first part follows from basic properties of P. The second is dual to 3.3 a). □

4.2. Lemma. *Let*

$$\nu\colon \langle \operatorname{Gen} P, P^{\perp}\rangle \to \langle \{0\}, P^+\rangle, \quad \mathcal{C} \mapsto \mathcal{C} \cap P^{\top},$$
$$\mu\colon \langle \{0\}, P^+\rangle \to \langle \operatorname{Gen} P, P^{\perp}\rangle, \quad \mathcal{C}' \mapsto \{M \in \text{Mod-}R \mid M/\operatorname{Tr}_P(M) \in \mathcal{C}'\},$$

then ν and μ are well defined lattice homomorphisms, and $\nu \circ \mu = \mathrm{id}$. Moreover, $\mu \circ \nu(\mathcal{C}) = \mathcal{C}$ iff $\forall M \in$ Mod-R : $M \in \mathcal{C} \Leftrightarrow M/\operatorname{Tr}_P(M) \in \mathcal{C}$. In particular, $\mu \circ \nu(\mathcal{C}) = \mathcal{C}$ provided that $\mathcal{C}$ is closed under quotients and extensions.

Proof. Dual to the proof of 3.4. □

Assume $\mathcal{C}' \subseteq P^+$. Then $\mathcal{C}'$ is said to be *closed under submodules in P^+* if $L \leq M$ and $L \in P^+$ implies $L \in \mathcal{C}'$ for all $M \in \mathcal{C}'$.

$\mathcal{C}'$ is said to be *closed under quotients in P^+* if $L \leq M$ and $M/L \in P^+$ implies $M/L \in \mathcal{C}'$ for all $M \in \mathcal{C}'$.

$\mathcal{C}'$ is said to be *a torsion class in P^+* if $\mathcal{C}'$ is closed under quotients in P^+, and $\mathcal{C}'$ is closed under extensions and direct sums.

4.3. Lemma. *Let μ and ν be as in* 1.2. *Then*

i) *if $\mathcal{C} \in \langle \operatorname{Gen} P, P^{\perp}\rangle$ is closed under direct sums (direct products, extensions, direct summands, submodules, quotients) then $\nu(\mathcal{C})$ is closed under direct sums (direct products, extensions, direct summands, submodules in P^+, quotients in P^+).*

ii) *If $\mathcal{C}' \in \langle \{0\}, P^+\rangle$ is closed under direct sums (extensions and quotients in P^+, direct summands, submodules in P^+), then $\mu(\mathcal{C}')$ is closed under direct sums (extensions and quotients, direct summands, submodules).*

Proof. i): It follows by 4.1.

ii): The assertion concerning direct summands and direct sums is clear.

Suppose that $\mathcal{C}'$ is closed under quotients in P^+. Let $\pi\colon M \to N$ an epimorphism, with $M \in \mu(\mathcal{C}')$. Let us consider the commutative exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tr}_P(M) & \longrightarrow & M & \longrightarrow & M/\mathrm{Tr}_P(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Tr}_P(N) & \longrightarrow & N & \longrightarrow & N/\mathrm{Tr}_P(N) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By assumption, $M/\mathrm{Tr}_P(M) \in \mathcal{C}'$ and $N/\mathrm{Tr}_P(N) \in P^{\top}$. By 4.1, $N/\mathrm{Tr}_P(N)$ belongs to P^+,whence $N/\mathrm{Tr}_P(N) \in \mathcal{C}'$. So $N \in \mu(\mathcal{C}')$ and $\mu(\mathcal{C}')$ is closed under quotients.

Suppose that $\mathcal{C}'$ is closed under extensions and quotients in P^+. Let $0 \to L \to M \to M/L \to 0$ be an exact sequence, with $L, M/L \in \mu(\mathcal{C}')$. Since $L \in P^{\perp}$, we have $\mathrm{Tr}_P(M/L) = \dfrac{L + \mathrm{Tr}_P(M)}{L}$. Consider the epimorphism $\pi : M/\mathrm{Tr}_P(M) \to (M/L)/\mathrm{Tr}_P(M/L)$. We have $\mathrm{Ker}\,\pi = \dfrac{L + \mathrm{Tr}_P(M)}{\mathrm{Tr}_P(M)}$, whence $\mathrm{Ker}\,\pi$ is a quotient of $L/\mathrm{Tr}_P(L)$. So $\mathrm{Ker}\,\pi \in P^{\perp}$, and $\mathrm{Ker}\,\pi \in P^+$ as $M/\mathrm{Tr}_P(M) \in P^{\top}$. Since $\mathcal{C}'$ is closed under quotients in P^+, we infer that $\mathrm{Ker}\,\pi \in \mathcal{C}'$. Since $\mathcal{C}'$ is closed under extensions, we have $M/\mathrm{Tr}_P(M) \in \mathcal{C}'$, and $M \in \mu(\mathcal{C}')$. This proves that $\mu(\mathcal{C}')$ is closed under extensions.

Using a similar argument and the fact that P is partial tilting, it can be proved that if $\mathcal{C}'$ is closed under submodules in P^+, then $\mu(\mathcal{C}')$ is closed under submodules in $P^{\perp}$. □

Let $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq$ Mod-R. Denote by $[[\mathcal{C}_1, \mathcal{C}_2]]$ the lattice of all torsion classes $\mathcal{T}$ in Mod-R such that $\mathcal{C}_1 \subseteq \mathcal{T} \subseteq \mathcal{C}_2$.

4.4. Theorem. [CT, Proposition 2.8] *Let $\mathbb{T}(P^+)$ be the lattice of all torsion classes in P^+, and let*

$$\bar{\nu}\colon [[\mathrm{Gen}\,P, P^{\perp}]] \to \mathbb{T}(P^+), \quad \mathcal{T} \mapsto \mathcal{T} \cap P^{\top},$$

$$\bar{\mu}\colon \mathbb{T}(P^+) \to [[\mathrm{Gen}\,P, P^{\perp}]], \quad \mathcal{T}' \mapsto \mathbb{E}(\mathcal{T}', \mathrm{Gen}\,P) = \{M \mid M/\mathrm{Tr}_P(M) \in \mathcal{T}'\}$$

Then $\bar{\nu}$ and $\bar{\mu}$ are inverse lattice isomorphisms.

Proof. By 4.2 and 4.3. □

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