COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **5**, Number **4**, December **2006**

Website: http://AIMsciences.org
pp. **793–812**

MINIMAX AND VISCOSITY SOLUTIONS OF HAMILTON-JACOBI EQUATIONS IN THE CONVEX CASE

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(Communicated by Martino Bardi)

Abstract. The aim of this paper is twofold. We construct an extension to a non-integrable case of Hopf's formula, often used to produce viscosity solutions of Hamilton-Jacobi equations for p-convex integrable Hamiltonians. Furthermore, for a general class of p-convex Hamiltonians, we present a proof of the equivalence of the minimax solution with the viscosity solution.

1. **Introduction.** We review some aspects of the Cauchy Problem (CP) for Hamilton-Jacobi equations of evolutive type:

$$(CP)\left\{\begin{array}{l} \dfrac{\partial S}{\partial t}(t,q)+H\left(t,q,\dfrac{\partial S}{\partial q}(t,q)\right)=0, \\ S(0,q)=\sigma(q), \end{array}\right.$$

$t\in[0,T]$, $q\in N$, where N is a smooth connected manifold without boundary. For T small enough, the unique classical solution to (CP) is determined using the characteristics method. However, even though H and σ are smooth, in general there exists a critical time in which the classical solution breaks down: it becomes multivalued, i.e. the q-components of some characteristics cross each other. Hence, it arises the question of how to define, and then to determine, weak (e.g., continuous and almost everywhere differentiable) global solutions of (CP).

In the eighties, Crandall, Evans and Lions introduced the notion of viscosity solution for Hamilton-Jacobi equations, see [23] and [3] for a detailed review on the subject. Lions [23], Bardi and Evans [2], using Hopf's formulas, directly constructed viscosity solutions for convex Liouville-integrable Hamiltonians of the form $H = H(p)$.

Afterwards, in 1991 Chaperon and Sikorav proposed in a geometric framework a new type of weak solutions for (CP), called minimax solutions –sometimes also variational or Lagrangian solution (see [13], [33], [26]). Their definition is based on generating functions quadratic at infinity (G.F.Q.I.) of the Lagrangian submanifold L obtained by gluing together the characteristics of the Hamiltonian vector field $X_{\mathcal{H}}$ where $\mathcal{H}(t,q,\tau,p)=\tau+H(t,q,p)$. This global object L resumes geometrically the multi-valued features of the Hamilton-Jacobi problem, like a sort of Riemann surface (see e.g. [34]) occuring in complex analysis. A discussion on the construction of global generating functions of L related to viscosity solutions has been made in [8] in the special case of existence of a complete solution ("complete integral") of

2000 *Mathematics Subject Classification.* Primary: 35A15, 35A30; Secondary: 53D35.
Key words and phrases. Hamilton-Jacobi Theory, symplectic topology, viscosity solutions.

Hamilton-Jacobi equation. In this new topological framework, a lot of examples can be found and produced, even outside the classical mechanics: e.g. in control theory [6], or in multi-time theory of Hamilton-Jacobi equation [11].

Viscosity and minimax solutions have the same analytic properties, namely, theorems of existence and uniqueness hold, but in general they are different, see [26]. In [21] Joukovskaia indicated that viscosity and minimax solutions of (CP) coincide, provided that the Hamiltonian H is convex in the p variables. A task of the present paper is to give a detailed proof of this fact.

First, we construct an extension of (the above mentioned) Hopf's formula, for more general non-integrable Hamiltonians; this is performed on the torus $N = \mathbb{T}^n$. This result is caught by utilizing (i) a very fruitful, even though scarcely known, theorem of Hamilton (e.g. quoted by Gantmacher [19] as "Perturbation Theory"), (ii) a classical composition rule of generating functions in symplectic geometry [4], and (iii) the existence theorem by Chaperon-Laudenbach-Sikorav-Viterbo of global generating functions for Lagrangian submanifolds related to compactly supported Hamiltonians.

Furthermore, we present a proof of the coincidence of minimax and viscosity solutions, essentially based on an Amann-Conley-Zehnder reduction of the Action Functional of the Hamilton-Helmholtz variational principle. In our construction it is crucial the following representation of the candidate weak solution $S : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $(t,q) \mapsto S(t,q)$, where we require $(\tilde{q}(\cdot), \tilde{p}(\cdot)) \in H^1([0,T], T^*\mathbb{R}^n)$

$$S(t,q) := \inf_{\substack{\tilde{q}(\cdot): \\ \tilde{q}:[0,t]\to\mathbb{R}^n \\ \tilde{q}(t)=q}} \sup_{\substack{\tilde{p}(\cdot): \\ \tilde{p}:[0,t]\to\mathbb{R}^n, \\ \tilde{p}(0)=\frac{\partial\sigma}{\partial q}(\tilde{q}(0))}} \left\{ \sigma(\tilde{q}(0)) + \int_0^t (p\dot{q} - H)_{|(\tilde{q},\tilde{p})}\, ds \right\}, \quad (1)$$

$H(q,p) = \frac{1}{2}|p|^2 + V(q)$, V compactly supported. Indeed, under the convexity hypothesis on H, from one side, we see that (1) is the Hamiltonian version of the Lax-Oleinik formula hence defines the viscosity solution by Crandall-Evans-Lions, see [17], [18] and bibliography quoted therein; from the other one, we show that the minimax solution proposed by Chaperon-Sikorav-Viterbo, which is defined through the variational Hamilton-Helmholtz functional involved in (1), is *exactly* given by S. Incidentally, the paper [16] studies the same Hamiltonian from the perspective of idempotent analysis and arrives essentially to the above explicit formula. Moreover, in the interesting forthcoming paper [25] McCaffrey proposes technical conditions guaranteeing minimax solutions to be viscous ones and he points out that they do work also in some non convex cases.

The sequel is organized as follows.

In Sections 2-4 we recall some notions about symplectic topology, generating functions and minimax solutions of (CP). In Section 5, for $N = \mathbb{T}^n$, we explicitly write down a generating function with finite parameters for the Lagrangian submanifold geometric solution of (CP) for the Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + f(q,p)$, $q \in \mathbb{T}^n$, f compactly supported.

Section 6 is devoted to the proof of the equivalence of viscosity and minimax solutions of (CP), for Hamiltonians of mechanical type $H(q,p) = \frac{1}{2}|p|^2 + V(q)$. Here, we assume $(q,p) \in \mathbb{R}^{2n}$ and compactly supported energy potential $V(q)$.

In the literature the term minimax solution is often used to indicate a third approach to generalized solutions of (CP); this alternative approach –which applies properly in differential game theory and control theory, hence lies outside the tasks of the present paper– has been clarified to be equivalent to the concept of viscosity solution, see for example [29], [30] and [27].

One of the authors (F.C.) is friendly indebted to Claude Viterbo, because the need of investigating the relation between viscosity and minimax solutions arose during the redaction of the paper [11].

2. **Extension of exact Lagrangian isotopies. Conventions.** The considered functions are C^∞ and the involved spaces are assumed to be endowed with the C^∞ topology. Given a space of functions E, a *path* into E is a map $t \mapsto f_t$, denoted by (f_t), from $I := [0,1]$ into E, such that the application $(t,x) \mapsto f_t\,(x)$ is of class C^∞. For a generic manifold P, we denote by $E\,(P)$ the space of functions $f : P \to \mathbb{R}$.

Let now (P,ω) be a smooth, connected, symplectic manifold of dimension $2n$. An embedding $j : \Lambda \to P$ is called *Lagrangian* if Λ is of dimension n and $j^*\omega = 0$. In this case, $j\,(\Lambda)$ is called Lagrangian submanifold of (P,ω). The symplectic manifold (P,ω) is *exact* when ω admits a global primitive λ; given a primitive λ, an embedding $j : \Lambda \to P$ such that its pull-back $j^*\lambda$ is exact is called *exact Lagrangian embedding.*

The following results in symplectic geometry will be used in Section 4.

Theorem 1. *(Weinstein) For every Lagrangian embedding $j : \Lambda \to P$, there exists an open neighbourhood U of the zero section of $T^*\Lambda$ and an embedding $J : U \to P$ such that $J^*\omega = d\lambda_\Lambda|_U$, where λ_Λ denotes the Liouville 1-form on $T^*\Lambda$. Moreover, if $0_\Lambda : \Lambda \to T^*\Lambda$ is the zero 1-form on Λ, we have $j = J \circ 0_\Lambda$.*

We call J a *tubular neighbourhood of j for ω* when the open set $U \cap T_x^*\Lambda$ is star-shaped with respect to the origin for every $x \in \Lambda$.

We denote $Emb\,(\Lambda,\omega)$ the space of Lagrangian embeddings of Λ into P. A *Lagrangian isotopy* of a manifold Λ in (P,ω) is a path into $Emb\,(\Lambda,\omega)$.

Definition 1. (Exact Lagrangian isotopy) Let (j_t), $j_t : \Lambda \to P$, be a Lagrangian isotopy of a manifold Λ in (P,ω). Then (j_t) is called *exact* when, for every $t \in I$ and every local primitive λ of ω in a neighbourhood of $j_t\,(\Lambda)$, the 1-form $\frac{d}{dt} j_t^*\lambda$ is exact on Λ.

We note that a Lagrangian isotopy (j_t) such that j_t is exact for every $t \in I$, results an exact Lagrangian isotopy, in fact, in such a case:

$$\frac{d}{dt} j_t^*\lambda = d_x\left[\frac{\partial f}{\partial t}\,(t,x)\right].$$

We refer to [12] for a detailed proof of the following

Theorem 2. *(Extension of exact Lagrangian isotopies) For every isotopy (j_t) of a compact manifold Λ in (P,ω), the following two properties are equivalent:*
a) (j_t) is an exact Lagrangian isotopy of Λ in (P,ω).
b) j_0 is Lagrangian and there exists a Hamiltonian isotopy (ϕ_t), with compact support, such that $j_t = \phi_t \circ j_0$ for every $t \in I$.

3. **Generating functions.** Let N be a compact manifold and $L \subset T^*N$ a Lagrangian submanifold. A classical argument by Maslov and Hörmander shows that, at least locally, every Lagrangian submanifold is described by some generating function of the form

$$\begin{aligned} S : N \times \mathbb{R}^k &\longrightarrow \mathbb{R} \\ (q,\xi) &\longmapsto S\,(q,\xi) \end{aligned}$$

in the following way:

$$L := \left\{ \left(q, \frac{\partial S}{\partial q}(q,\xi) \right) : \ \frac{\partial S}{\partial \xi}(q,\xi) = 0 \right\},$$

where 0 is a regular value of the map

$$(q,\xi) \longmapsto \frac{\partial S}{\partial \xi}(q,\xi).$$

In order to apply the Calculus of Variations to generating functions, one needs a condition implying the existence of critical points. In particular, the following class of generating functions has been decisive in many issues:

Definition 2. A generating function $S : N \times \mathbb{R}^k \to \mathbb{R}$ is quadratic at infinity (G.F.Q.I.) if for $|\xi| > C$

$$S(q,\xi) = \xi^T Q \xi, \tag{2}$$

where $\xi^T Q \xi$ is a nondegenerate quadratic form.

There were known in literature (see e.g. [34], [22]) three main operations on generating functions which leave invariant the corresponding Lagrangian submanifolds:
• *Fibered diffeomorphism.* Let $S : N \times \mathbb{R}^k \to \mathbb{R}$ be a G.F.Q.I. and $N \times \mathbb{R}^k \ni (q,\xi) \mapsto (q, \phi(q,\xi)) \in N \times \mathbb{R}^k$ a map such that, $\forall q \in N$,

$$\mathbb{R}^k \ni \xi \longmapsto \phi(q,\xi) \in \mathbb{R}^k$$

is a diffeomorphism. Then

$$S_1(q,\xi) := S(q, \phi(q,\xi))$$

generates the same Lagrangian submanifold of S.
• *Stabilization.* Let $S : N \times \mathbb{R}^k \to \mathbb{R}$ be a G.F.Q.I. Then

$$S_1(q,\xi,\eta) := S(q,\xi) + \eta^T B \eta,$$

where $\eta \in \mathbb{R}^l$ and $\eta^T B \eta$ is a nondegenerate quadratic form, generates the same Lagrangian submanifold of S.
• *Addition of a constant.* Finally, as a third –although trivial– invariant operation, we observe that by adding to a generating function S any arbitrary constant $c \in \mathbb{R}$ the described Lagrangian submanifold is invariant.

Crucial problems in the global theory of Lagrangian submanifolds and their parameterizations are (1) the existence of a G.F.Q.I. for a Lagrangian submanifold $L \subset T^*N$, (2) the uniqueness of it (up to the operations described above).
The following theorem –see [28]– answers partially to the first question.

Theorem 3. *(Chaperon-Chekanov-Laudenbach-Sikorav) Let $\mathcal{O}_{T^*N}$ be the zero section of T^*N and (ϕ_t) a Hamiltonian flow. Then the Lagrangian submanifold $\phi_1(\mathcal{O}_{T^*N})$ admits a G.F.Q.I.*

The answer to the second problem is due to Viterbo:

Theorem 4. *(Viterbo) Let $\mathcal{O}_{T^*N}$ be the zero section of T^*N and (ϕ_t) a Hamiltonian flow. Then the Lagrangian submanifold $\phi_1(\mathcal{O}_{T^*N})$ admits a unique G.F.Q.I. up to the above operations.*

The theorems above –see also [31] and [32]– still hold in $T^*\mathbb{R}^n$, provided that (ϕ_t) is a flow of a compactly supported Hamiltonian vector field.

A generalization of Definition 2 –introduced by Viterbo and studied in detail by Theret [31], [32]– is the following:

Definition 3. A generating function $S : N \times \mathbb{R}^k \to \mathbb{R}$, $(q, \xi) \mapsto S(q, \xi)$, is asymptotically quadratic if for every fixed $q \in N$

$$||S\,(q, \cdot) - \mathcal{P}^{(2)}(q, \cdot)||_{C^1} < +\infty, \tag{3}$$

where $\mathcal{P}^{(2)}(q, \xi) = Q\,(q, \xi) + A(q)\xi + B(q)$ and $Q\,(q, \xi) = \xi^T Q(q)\xi$ is a nondegenerate quadratic form.

In particular, a Lagrangian manifold is generated by a generating function quadratic at infinity (in the sense of Definition 2) if and only if it is generated by an asymptotically quadratic function (in the sense of Definition 3).

4. **Geometric and minimax solutions.** Let N be a smooth, connected and closed (i.e. compact and without boundary) manifold. Let us consider the Cauchy problem (CP). We suppose that the Hamiltonian $H : \mathbb{R} \times T^*N \to \mathbb{R}$ is of class C^2 and the initial condition $\sigma : N \to \mathbb{R}$ is of class C^1.

Let $\mathbb{R} \times N$ be the "space-time", $T^*(\mathbb{R} \times N) = \{(t, q, \tau, p)\}$ its cotangent bundle (endowed with the standard symplectic form $dp \wedge dq + d\tau \wedge dt$) and $\mathcal{H}\,(t, q, \tau, p) = \tau + H\,(t, q, p)$.

In order to overcome the difficulties arising from the obstruction to existence of global solutions, we search for Lagrangian submanifolds $L \subset T^*(\mathbb{R} \times N)$ satisfying the following geometric version of Hamilton-Jacobi equation:

$$L \subset \mathcal{H}^{-1}(0)\,.$$

But how to obtain such an L? We explain now the procedure.

Let $\Phi^t : \mathbb{R} \times T^*(\mathbb{R} \times N) \to T^*(\mathbb{R} \times N)$ be the flow generated by the Hamiltonian $\mathcal{H} : T^*(\mathbb{R} \times N) \to \mathbb{R}$, $\mathcal{H}\,(t, q, \tau, p) = \tau + H\,(t, q, p)$:

$$\begin{cases} \dot{t} = 1 \\ \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{\tau} = -\dfrac{dH}{dt} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

and Γ_σ be the initial data submanifold:

$$\Gamma_\sigma := \{(0, q, -H\,(0, q, d\sigma\,(q)), d\sigma\,(q)) : \; q \in N\} \subset \mathcal{H}^{-1}(0) \subset T^*(\mathbb{R} \times N)\,.$$

We note that Γ_σ is the intersection of the Lagrangian submanifold

$$\Lambda_\sigma = \{(0, q, t, d\sigma\,(q)) : \; (t, q) \in \mathbb{R} \times N\}$$

with the hypersurface $\mathcal{H}^{-1}(0)$:

$$\Gamma_\sigma = \Lambda_\sigma \cap \mathcal{H}^{-1}(0)\,.$$

Definition 4. The geometric solution to (CP) is the submanifold

$$L := \bigcup_{0 \le t \le T} \Phi^t\,(\Gamma_\sigma) \subset T^*(\mathbb{R} \times N)\,.$$

Proposition 1. *The geometric solution L is an exact Lagrangian submanifold, contained into the hypersurface $\mathcal{H}^{-1}(0)$ and Hamiltonian isotopic to the zero section $\mathcal{O}_{T^*([0,T] \times N)} = \{(t, q, 0, 0) : \; 0 \le t \le T, \; q \in N\}$ of $T^*([0, T] \times N)$.*

Proof. A direct computation shows that every geometric solution is an exact Lagrangian submanifold. In order to prove that L is Hamiltonian isotopic to the zero section $\mathcal{O}_{T^*([0,T]\times N)} = \{(t,q,0,0):\ 0\le t\le T,\ q\in N\}$ of $T^*([0,T]\times N)$, we determine a continuous path of exact Lagrangian submanifolds in $T^*(\mathbb{R}\times N)$ connecting the zero section to L. Hence we conclude using Theorem 2.

Let us consider the following 1-parameter family of Cauchy problems related to Hamilton-Jacobi equations:

$$(CP)_\lambda \begin{cases} \frac{\partial S}{\partial t}(t,q) + \lambda H\left(t,q,\frac{\partial S}{\partial q}(t,q)\right) = 0 \\ S(0,q) = \lambda\sigma(q) \end{cases}$$

The initial data submanifold related to $(CP)_\lambda$ is:

$$\Gamma_{\lambda\sigma} = \{(0,q,-\lambda H(0,q,\lambda d\sigma(q)),\lambda d\sigma(q))\}$$

and the geometric solution to $(CP)_\lambda$ is

$$L_\lambda = \bigcup_{0\le t\le T} \Phi_\lambda^t(\Gamma_{\lambda\sigma}) = \{(t,\tilde{q}_\lambda(t),\tilde{\tau}_\lambda(t),\tilde{p}_\lambda(t))\}$$

with
1) Φ_λ^t the flow of $\mathcal{H}_\lambda = \tau + \lambda H$,
2) $(\tilde{q}_\lambda(t),\tilde{p}_\lambda(t))$ the characteristics of $X_{\lambda H}$ such that $\tilde{q}_\lambda(0) = q_0$ and $\tilde{p}_\lambda(0) = \lambda d\sigma(q_0)$,
3) $\tilde{\tau}_\lambda(t) = -\lambda H(t,\tilde{q}_\lambda(t),\tilde{p}_\lambda(t))$.
We point out that every L_λ, geometric solution to $(CP)_\lambda$, results an exact Lagrangian submanifold of $T^*(\mathbb{R}\times N)$ and that $L_1 = L$. On the other hand $L_0 = \mathcal{O}_{T^*([0,T]\times N)}$. Hence we have defined a continuous path $\lambda\mapsto L_\lambda$ connecting the zero section $\mathcal{O}_{T^*([0,T]\times N)}$ to the Lagrangian submanifold L. As a consequence of Theorem 2, this fact results equivalent to the existence of a Hamiltonian isotopy connecting the zero section $\mathcal{O}_{T^*([0,T]\times N)}$ to L. □

As a consequence of preceding Proposition 1 and of the compactness of N, Theorem 4 of Viterbo guarantees that the Lagrangian submanifold L admits essentially (that is, up to the three operations described above) a unique G.F.Q.I. $S:[0,T]\times N\times\mathbb{R}^k\to\mathbb{R}$, $(t,q;\xi)\mapsto S(t,q;\xi)$.

We can assume that the graph of $S(t,q;\xi)$ at $t=0$ coincides with Γ_σ:

$$\Gamma_\sigma = \left\{\left(0,q,\frac{\partial S}{\partial t}(0,q;\xi),\frac{\partial S}{\partial q}(0,q;\xi)\right):\ \frac{\partial S}{\partial \xi}(0,q;\xi) = 0\right\}.$$

The quadraticity at infinity property of $S(t,q;\xi)$ is crucial: minimax solutions arise from the application of the Lusternik-Schnirelman method to the G.F.Q.I. $S(t,q;\xi)$. In some more detail, let us consider the sublevel sets

$$S^c_{(t,q)} := \left\{\xi\in\mathbb{R}^k:\ S(t,q;\xi)\le c\right\},\ (t,q)\in[0,T]\times N \text{ fixed,}$$
$$Q^c := \left\{\xi\in\mathbb{R}^k:\ Q(\xi)\le c\right\}.$$

We observe that for $c>0$ large enough, $S^c_{(t,q)}$ and Q^c are invariant from a homotopical point of view:

$$S^{\pm c}_{(t,q)} = Q^{\pm c},$$

and $S^{\pm\bar{c}}_{(t,q)}$ retracts on $S^{\pm c}_{(t,q)}$ for every $\bar{c}>c$. Let $A := Q^{(c-\epsilon)}$, $\epsilon>0$ small. Then the isomorphisms below (the first one by excision and the second one by retraction) hold:

$$H^*\left(Q^c, Q^{-c}\right) \cong H^*\left(Q^c \setminus \mathring{A}, Q^{-c} \setminus \mathring{A}\right) \cong H^*\left(D^i, \partial D^i\right),$$

where i is the index of the quadratic form Q (that is, the number of negative eigenvalues of Q) and D^i denotes the disk (of radius $\sqrt{c}$) in $\mathbb{R}^i$. Hence $H^*\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ is 1-dimensional:

$$H^h\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right) \cong H^h\left(D^i, \partial D^i\right) = \begin{cases} 0 & \text{if } h \neq i \\ \alpha \cdot \mathbb{R} & \text{if } h = i \end{cases} \tag{4}$$

We remark that (4) holds also for generalized G.F.Q.I. of Definition 3, because the relative cohomology $H^h\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ is invariant.

Definition 5. (Minimax solution) Let $S(t,q;\xi)$ be any G.F.Q.I. for L, $S(t,q;\xi) = Q(\xi)$ out of a compact set in the parameters $\xi \in \mathbb{R}^k$. For $c > 0$ large enough and for every $(t,q) \in [0,T] \times N$, let $0 \neq \alpha \in H^i\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right)$ be the unique generator (up to a constant factor) as in (4) and

$$i_\lambda : S^\lambda_{(t,q)} \hookrightarrow S^c_{(t,q)}.$$

The function

$$(t,q) \mapsto u(t,q) := \inf\{\lambda \in [-c,+c] : \ i^*_\lambda \alpha \neq 0\} \tag{5}$$

is the minimax solution of (CP).

The following fundamental Theorem has been proved by Chaperon, see [13].

Theorem 5. *The minimax solution $u(t,q)$ is a weak solution to (CP), Lipschitz on finite times, which does not depend on the choice of the G.F.Q.I.*

The above independence is a remarkable consequence of Viterbo's Theorem, since u is invariant by stabilization and fibered diffeomorphisms.

We observe that the definition of minimax solutions arises naturally in the compact case, when the Uniqueness Theorem of Viterbo is satisfied. Moreover, for a fixed point on the manifold $[0,T] \times N$, the minimax critical value is unique and is determined by the Morse index of the quadratic form Q. We conclude with the following Proposition (see also Theorem 7.1 in [21]), which will be useful in the sequel.

Proposition 2. *Let $S(t,q;\xi)$ and $u(t,q)$ as in Definition 5. Let us suppose that the Morse index of the quadratic form Q is 0. Then*

$$u(t,q) = \min_{\xi \in \mathbb{R}^k} S(t,q;\xi).$$

Proof. Let us fix a point $(t,q) \in [0,T] \times N$. Since Q is positive definite, $S^{-c}_{(t,q)} = \emptyset$, and for $c > 0$ large enough, it results (see (4))

$$H^h\left(S^c_{(t,q)}, S^{-c}_{(t,q)}\right) = H^h\left(S^c_{(t,q)}\right) = \begin{cases} 0 & \text{if } h \neq 0 \\ 1 \cdot \mathbb{R} & \text{if } h = 0 \end{cases}$$

where 1 is the generator of $H^0\left(S^c_{(t,q)}\right)$. Consequently, the minimax solution (5)

$$u(t,q) = \inf\{\lambda \in [-c,+c] : \ i^*_\lambda 1 \neq 0\}$$

coincides with the minimum of the function $\xi \mapsto S(t,q;\xi)$, that is

$$u(t,q) = \min_{\xi \in \mathbb{R}^k} S(t,q;\xi).$$

□

5. **A global generating function for the geometric solution for** $H(q,p) = \frac{1}{2}|p|^2 + f(q,p)$ **on** $T^*\mathbb{T}^n$**.** Let us consider the Hamiltonian $H(q,p) \in C^2(T^*\mathbb{T}^n;\mathbb{R})$:

$$H(q,p) = \frac{1}{2}|p|^2 + f(q,p), \tag{6}$$

f compactly supported in the p variables, and the Cauchy Problem $(CP)_H$:

$$(CP)_H \begin{cases} \dfrac{\partial S}{\partial t}(t,q) + H\left(q, \dfrac{\partial S}{\partial q}(t,q)\right) = 0, \\ S(0,q) = \sigma(q), \end{cases}$$

where $t \in [0,T]$, $q \in \mathbb{T}^n$ and $\sigma \in C^1(\mathbb{T}^n;\mathbb{R})$.

In this section we investigate around the structure of the generating function for the geometric solution of $(CP)_H$, showing that its structure is naturally interpreted as an improvement of the Hopf's formula utilized by Bardi and Evans in order to build the viscosity solution for Liouville-integrable Hamiltonians.

It turns out useful to introduce the compactly supported Hamiltonian $K(t,q,p)$:

$$K(t,q,p) = \left(\left(\phi_0^t\right)^* f\right)(q,p) = H(q+tp,p) - \frac{1}{2}|p|^2, \tag{7}$$

where ϕ_0^t is the flow of $H_0(p) := \frac{1}{2}|p|^2$.

We recall now the following Proposition, which is, essentially, a result of Hamilton (see [20] and also [19], [5]).

Proposition 3. *Let $\phi_H^t, \phi_K^{t,0}$ and ϕ_0^t be the flows of H, K and H_0 respectively. We have:*

$$\phi_H^t(q,p) = \phi_0^t \circ \phi_K^{t,0}(q,p), \tag{8}$$

$\forall (q,p) \in T^*\mathbb{T}^n$ and $\forall t \in \mathbb{R}$.

Now let us consider the Cauchy Problem $(CP)_K$ related to K:

$$(CP)_K \begin{cases} \dfrac{\partial S}{\partial t}(t,q) + K\left(t, q, \dfrac{\partial S}{\partial q}(t,q)\right) = 0 \\ S(0,q) = \sigma(q) \end{cases}$$

and define $\mathcal{K}(t,q,\tau,p) := \tau + K(t,q,p)$, Φ_K^t its flow, and $(\Gamma_K)_\sigma$ the initial data submanifold

$$(\Gamma_K)_\sigma := \{(0,q,-K(0,q,d\sigma(q)), d\sigma(q)) : \ q \in \mathbb{T}^n\} \subset \mathcal{K}^{-1}(0).$$

Since the manifold $\mathbb{T}^n$ is compact, a consequence of Proposition 1 and Theorem 4 is the existence of a unique G.F.Q.I. $S_K(t,q;u)$ for the geometric solution L_K of $(CP)_K$:

$$L_K := \bigcup_{0 \le t \le T} \Phi_K^t\left((\Gamma_K)_\sigma\right).$$

Proposition 4. *Let $\mathcal{H}(q,\tau,p) := \tau + H(q,p)$, Φ_H^t its flow and $(\Gamma_H)_\sigma$ the initial data submanifold*

$$(\Gamma_H)_\sigma := \{(0,q,-H(q,d\sigma(q)), d\sigma(q)) : \ q \in \mathbb{T}^n\} \subset \mathcal{H}^{-1}(0).$$

Then the Lagrangian submanifold L_H

$$L_H := \bigcup_{0 \le t \le T} \Phi_H^t\left((\Gamma_H)_\sigma\right),$$

geometric solution of $(CP)_H$*, is generated by the function*

$$\widetilde{S}(t,q;\xi,u,v) := -\frac{1}{2}v^2 t + (q-\xi)\cdot v + S_K(t,\xi;u). \tag{9}$$

Proof. The generating function $S_K(t,\xi;u)$ generates the Lagrangian submanifold L_K, which can be written, more explicitly

$$L_K := \bigcup_{0\le t\le T} \Phi_K^t\left((\Gamma_K)_\sigma\right) = \left\{\left(t,\widetilde{q}(t),\widetilde{\tau}(t),\widetilde{p}(t)\right):\ 0\le t\le T\right\},$$

where $\widetilde{q}$ and $\widetilde{p}$ are the characteristics of X_K such that $\widetilde{q}(0)=q_0$ and $\widetilde{p}(0)=d\sigma(q_0)$, and $\widetilde{\tau}(t) = -K(t,\widetilde{q}(t),\widetilde{p}(t))$.

Now, by a direct computation, we prove that the Lagrangian submanifold generated by $\widetilde{S}(t,q;\xi,u,v)$ coincides with L_H.

$$L_{\widetilde{S}} = \left\{\left(t,q,\frac{\partial\widetilde{S}}{\partial t},\frac{\partial\widetilde{S}}{\partial q}\right):\ \frac{\partial\widetilde{S}}{\partial\xi}=0,\ \frac{\partial\widetilde{S}}{\partial u}=0,\ \frac{\partial\widetilde{S}}{\partial v}=0\right\}.$$

More precisely,

$$\frac{\partial\widetilde{S}}{\partial t}(t,q;\xi,u,v) = -\frac{1}{2}v^2 + \frac{\partial S_K}{\partial t}(t,\xi;u),$$

$$\frac{\partial\widetilde{S}}{\partial q}(t,q;\xi,u,v) = v,$$

$$\frac{\partial\widetilde{S}}{\partial\xi}(t,q;\xi,u,v) = 0 \text{ is and only if } -v + \frac{\partial S_K}{\partial\xi}(t,\xi;u) = 0$$

$$\text{if and only if } v = \frac{\partial S_K}{\partial\xi}(t,\xi;u),$$

$$\frac{\partial\widetilde{S}}{\partial u}(t,q;\xi,u,v) = 0 \text{ if and only if } \frac{\partial S_K}{\partial u}(t,\xi;u) = 0,$$

$$\frac{\partial\widetilde{S}}{\partial v}(t,q;\xi,u,v) = 0 \text{ if and only if } -vt+q-\xi = 0 \text{ if and only if } q = \xi + vt.$$

Hence $L_{\widetilde{S}}$ is equivalent to

$$\begin{aligned} L_{\widetilde{S}} \;=\; & \Big\{(t,q,-\tfrac{1}{2}v^2 + \frac{\partial S_K}{\partial t}(t,\xi;u),v): \\ & v = \frac{\partial S_K}{\partial\xi}(t,\xi;u),\ \frac{\partial S_K}{\partial u}(t,\xi;u)=0,\ q=\xi+vt\Big\}. \end{aligned}$$

Now we remind that $S_K(t,\xi;u)$ generates the Lagrangian submanifold L_K, hence

$$L_{\widetilde{S}} = \left\{\left(t,q,-\frac{1}{2}v^2 - K(t,\xi,v),v\right):\ (\xi,v)\in\phi_K^{t,0}(\mathrm{Im}(d\sigma)),\ q=\xi+vt\right\}.$$

But $K(t,\xi,v) = H(\xi+tv,v) - \frac{1}{2}v^2$, then $-\frac{1}{2}v^2 - K(t,\xi,v) = -H(\xi+tv,v)$. Hence

$$L_{\widetilde{S}} = \left\{(t,\xi+tv,-H(\xi+tv,v),v):\ (\xi,v)\in\phi_K^{t,0}(\mathrm{Im}(d\sigma))\right\}.$$

Now we also note that $(\xi+vt,v) = \phi_0^t(\xi,v)$, therefore, since $\phi_H^t = \phi_0^t\circ\phi_K^{t,0}$,

$$L_{\widetilde{S}} = \left\{(t,\bar{q}(t),-H(\bar{q}(t),\bar{p}(t)),\bar{p}(t)):\ 0\le t\le T\right\},$$

where $\bar{q}$ and $\bar{p}$ are the characteristics of X_H such that $\bar{q}(0)=q_0$ and $\bar{p}(0)=d\sigma(q_0)$. Equivalently

$$L_{\widetilde{S}} = L_H.$$

□

We remark the following interesting fact: the structure of the generating function (9) recalls the Hopf's formula used in 1984 by Bardi and Evans in order to construct viscosity solutions for Liouville-integrable and convex Hamiltonians $H(p)$.

In fact, their formula is

$$u_{visc}(t,q) = \inf_{\xi} \sup_{v} \left\{ -H(v)\, t + (q-\xi)\cdot v + \sigma(\xi) \right\}. \tag{10}$$

The above formula (10), in the case $H(p) = \frac{1}{2}|p|^2$, becomes

$$u_{visc}(t,q) = \inf_{\xi} \sup_{v} \left\{ -\frac{1}{2} v^2 t + (q-\xi)\cdot v + \sigma(\xi) \right\}. \tag{11}$$

Therefore, the generating function (9) can be considered the improvement of $-\frac{1}{2}v^2 t + (q-\xi)\cdot v + \sigma(\xi)$ in (11) when we take into account the perturbed non-integrable Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + f(q,p)$, f compactly supported. This correction is just provided by the term $S_K(t,\xi;u)$.

The relationship between the explicit generating function for the geometric solution and the Hopf's formula was already noticed e.g. in [8] and [15].

We finally note that the plan of construct viscosity solutions starting from generating functions has been rather fruitless; nevertheless, under suitable assumptions, we can find similar representation formulas for state-dependent Hamiltonians, see [7] and [24].

6. A relationship between minimax and viscosity solutions.

Here we prove in detail the coincidence of minimax and viscosity solutions for p-convex Hamiltonians of mechanical type. The equivalence is essentially established through an Amann-Conley-Zehnder reduction of an infinite parameters generating function arising from Hamilton-Helmholtz variational principle.

6.1. A global generating function for the geometric solution for $H(q,p) = \frac{1}{2}|p|^2 + V(q)$ on $T^*\mathbb{R}^n$.

We consider the Hamiltonian $H(q,p) = \frac{1}{2}|p|^2 + V(q) \in C^2(T^*\mathbb{R}^n;\mathbb{R})$, V compactly supported, and its related Cauchy problem $(CP)_H$:

$$(CP)_H \begin{cases} \dfrac{\partial S}{\partial t}(t,q) + \dfrac{1}{2}\left|\dfrac{\partial S}{\partial q}(t,q)\right|^2 + V(q) = 0, \\ S(0,q) = \sigma(q), \end{cases}$$

where $t \in [0,T]$, $q \in \mathbb{R}^n$, σ compactly supported. The starting point consists to take into account the below global generating function W for the geometric solution for H –see Theorem 6– arising from Hamilton-Helmholtz functional. The following deductions, here proposed in the mechanical case, still hold for more general Hamiltonians, see [26]; however, this special structure will be crucial in the next Sections.

Let us consider the set of curves:

$$\Gamma := \left\{ \gamma(\cdot) = (q(\cdot), p(\cdot)) \in H^1\left([0,T], \mathbb{R}^{2n}\right) : \ p(0) = d\sigma(q(0)) \right\}.$$

By Sobolev imbedding theorem,

$$H^1\left((0,T), \mathbb{R}^{2n}\right) \hookrightarrow C^0\left([0,T], \mathbb{R}^{2n}\right)$$

compactly, so in the above definition the elements of Γ are the natural continuous extensions of the curves of $H^1\left((0,T), \mathbb{R}^{2n}\right)$ (i.e. the continuous curves $t \mapsto \gamma(t)$, starting from the graph of $d\sigma$, such that $\dot{\gamma} = \frac{d\gamma}{dt} \in L^2 := L^2\left((0,T), \mathbb{R}^{2n}\right)$. Moreover, the set Γ has a natural structure of linear space, and then $T_\gamma \Gamma = \Gamma$, for all $\gamma \in \Gamma$.

An equivalent way to describe the curves of Γ is to assign the q-projection at time t, $q = q(t) \in \mathbb{R}^n$, and the velocity $\dot{\gamma}$ of the curve γ by means of a function $\Phi \in L^2$. This is summarized by the following bijection g:

$$
\begin{aligned}
& g : [0,T] \times \mathbb{R}^n \times L^2\left((0,T), \mathbb{R}^{2n}\right) \longrightarrow [0,T] \times \Gamma \\
& (t, q, \Phi) \longmapsto g(t,q,\Phi) = (t, \gamma(\cdot)), \\
& \gamma(s) = (\mathrm{pr}_\Gamma \circ g)(t,q,\Phi)(s) = (q(s), p(s)) \\
& \qquad = \left(q - \int_s^t \Phi_q(r)\, dr, d\sigma(q(0)) + \int_0^s \Phi_p(r)\, dr \right) \\
& \qquad = \left(q - \int_s^t \Phi_q(r)\, dr, d\sigma\left(q - \int_0^t \Phi_q(r)\, dr\right) + \int_0^s \Phi_p(r)\, dr \right). \quad (12)
\end{aligned}
$$

To be more clear, we remark that the second value of the map $g(t,q,\Phi)$ is the curve $\gamma(\cdot) = (q(\cdot), p(\cdot))$ which is
1) starting from $(q(0), d\sigma(q(0)))$, such that
2) $\dot{\gamma}(\cdot) = \Phi(\cdot)$, and
3) $q(t) = q$.

By composing the Hamilton-Helmholtz functional:

$$
A : [0,T] \times \Gamma \longrightarrow \mathbb{R}
$$

$$
(t, \gamma(\cdot)) \mapsto A[t, \gamma(\cdot)] := \sigma(q(0)) + \int_0^t [p(r) \cdot \dot{q}(r) - H(r, q(r), p(r))]\, dr.
$$

with the bijection g, we obtain the following global generating function $W = A \circ g$:

Theorem 6. *The infinite-parameters function:*

$$
W := A \circ g : [0,T] \times \mathbb{R}^n \times L^2 \longrightarrow \mathbb{R}, \quad (13)
$$

$$
(t, q, \Phi) \longmapsto W(t,q,\Phi) := A \circ g(t,q,\Phi),
$$

generates $L_H = \bigcup_{0 \le t \le T} \Phi_H^t\left((\Gamma_H)_\sigma\right)$, *the geometric solution for the Hamiltonian* $H(q,p) = \frac{1}{2}|p|^2 + V(q)$.

Proof. We first explicitly write down W:

$$
\begin{aligned}
W(t,q,\Phi) = \sigma(q(0)) + \int_0^t & \left[\left(d\sigma(q(0)) + \int_0^s \Phi_p(r)\, dr \right) \cdot \Phi_q(s) \right. \\
& \left. - H\left(s, q - \int_s^t \Phi_q(r)\, dr, d\sigma(q(0)) + \int_0^s \Phi_p(r)\, dr \right) \right] ds, \\
= \sigma\left(q - \int_0^t \Phi_q(r)\, dr \right) & + \int_0^t \left[\left(d\sigma\left(q - \int_0^t \Phi_q(r)\, dr \right) \right.\right. \\
& \left.\left. + \int_0^s \Phi_p(r)\, dr \right) \cdot \Phi_q(s) \right] ds \\
& - \int_0^t \left[H\left(s, q - \int_s^t \Phi_q(r)\, dr, d\sigma\left(q - \int_0^t \Phi_q(r)\, dr \right) + \int_0^s \Phi_p(r)\, dr \right) \right] ds.
\end{aligned}
$$

Then, for $\frac{DW}{D\Phi}=0$, we compute $\frac{\partial W}{\partial q}$ and $\frac{\partial W}{\partial t}$.

$$\begin{aligned}\frac{\partial W}{\partial q} &= d\sigma\left(q\left(0\right)\right)+\int_0^t d^2\sigma\left(q\left(0\right)\right)\cdot\Phi_q\left(s\right)ds-\int_0^t\frac{\partial H}{\partial q}ds-\int_0^t\frac{\partial H}{\partial p}\cdot d^2\sigma\left(q\left(0\right)\right)ds\\
&= d\sigma\left(q\left(0\right)\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\Phi_q\left(s\right)ds+\int_0^t\dot{p}\left(s\right)ds-d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\dot{q}\left(s\right)ds\\
&= d\sigma\left(q\left(0\right)\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\dot{q}\left(s\right)ds+\int_0^t\dot{p}\left(s\right)ds-d^2\sigma\left(q\left(0\right)\right)\cdot\int_0^t\dot{q}\left(s\right)ds\\
&= d\sigma\left(q\left(0\right)\right)+\int_0^t\Phi_p\left(s\right)ds=p\left(t\right).\end{aligned}$$

Finally, we compute $\frac{\partial W}{\partial t}$.

$$\begin{aligned}\frac{\partial W}{\partial t} &= d\sigma\left(q\left(0\right)\right)\cdot\frac{\partial q\left(0\right)}{\partial t}+\left(d\sigma\left(q\left(0\right)\right)+\int_0^t\Phi_p\left(r\right)dr\right)\cdot\Phi_q\left(t\right)\\
&\quad -H\left(t,q,d\sigma\left(q\left(0\right)\right)+\int_0^t\Phi_p\left(r\right)dr\right)+\int_0^t d^2\sigma\cdot\left(-\Phi_q\left(t\right)\right)\cdot\Phi_q\left(s\right)ds\\
&\quad +\int_0^t\frac{\partial H}{\partial q}\left(s,q-\int_0^t\Phi_q\left(r\right)dr,d\sigma\left(q\left(0\right)\right)+\int_0^s\Phi_p\left(r\right)dr\right)\cdot\Phi_q\left(t\right)ds\\
&\quad -\int_0^t\frac{\partial H}{\partial p}\left(s,q-\int_0^t\Phi_q(r)dr,d\sigma(q(0))+\int_0^s\Phi_p\left(r\right)dr\right)\cdot\frac{\partial^2\sigma}{\partial q^2}\left(q\left(0\right)\right)\cdot\left(-\Phi_q\left(t\right)\right)ds\\
&= d\sigma\left(q\left(0\right)\right)\cdot\left(-\Phi_q\left(t\right)\right)+p\left(t\right)\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),p\left(t\right)\right)\\
&\quad -d^2\sigma\left(q\left(0\right)\right)\cdot\Phi_q\left(t\right)\cdot\int_0^t\Phi_q\left(s\right)ds+\int_0^t\frac{\partial H}{\partial p}\left(s,q\left(s\right),p\left(s\right)\right)ds\cdot\Phi_q\left(t\right)\\
&\quad +d^2\sigma\left(q\left(0\right)\right)\cdot\Phi_q\left(t\right)\cdot\int_0^t\frac{\partial H}{\partial p}\left(s,q\left(s\right),p\left(s\right)\right)ds\\
&= -p\left(0\right)\cdot\dot{q}\left(t\right)+p\left(t\right)\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),p\left(t\right)\right)-d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\Phi_q\left(s\right)ds\\
&\quad +\int_0^t\frac{\partial H}{\partial q}\left(s,q\left(s\right),p\left(s\right)\right)ds\cdot\dot{q}\left(t\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\dot{q}\left(s\right)ds\\
&= -p\left(0\right)\cdot\dot{q}\left(t\right)+\frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right)-d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\dot{q}\left(s\right)ds\\
&\quad -\frac{\partial}{\partial q}\left(\int_0^t\left[p\dot{q}-H\right]d\tau\right)\cdot\dot{q}\left(t\right)+d^2\sigma\left(q\left(0\right)\right)\cdot\dot{q}\left(t\right)\cdot\int_0^t\dot{q}\left(s\right)ds\\
&= \frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right)-\frac{\partial}{\partial q}\left[\sigma\left(q\left(0\right)\right)+\int_0^t\left(p\dot{q}-H\right)d\tau\right]\cdot\dot{q}\left(t\right),\\
&= \frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right)-H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right)-\frac{\partial W}{\partial q}\cdot\dot{q}\left(t\right),\\
&= -H\left(t,q\left(t\right),\frac{\partial W}{\partial q}\right).\end{aligned}$$

□

6.2. **Fourier expansion and fixed point.** Results in this Section are classical, see also [14], [12] and [1].

Hamilton equations related to X_H are

$$\begin{cases} \dot{q} = p \\ \dot{p} = -V^{'}(q) \end{cases} \tag{14}$$

Using the p-components of the bijection g, (14) can be rewritten, almost everywhere, as

$$\begin{cases} \Phi_q(s) = d\sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) + \int_0^s \Phi_p(r)\,dr \\ \Phi_p(s) = -V^{'}\left(q - \int_s^t \Phi_q(r)\,dr\right) \end{cases} \tag{15}$$

Hence

$$\Phi_q(s) = d\sigma\left(q - \int_0^t \Phi_q(r)\,dr\right) - \int_0^s V^{'}\left(q - \int_r^t \Phi_q(\tau)\,d\tau\right)dr \tag{16}$$

Note that the reduction of (15) into (16) is equivalent to the displacement from the Hamiltonian formalism to the Lagrangian formalism through the Legendre transformation.

For every $\Phi_q \in L^2((0,T),\mathbb{R}^n)$, let us consider the Fourier expansion

$$\Phi_q(s) = \sum_{k\in\mathbb{Z}} (\Phi_q)_k\, e^{i(2\pi k/T)s}.$$

For each fixed $N \in \mathbb{N}$, let us consider the projection maps on the basis $\left\{e^{i(2\pi k/T)s}\right\}_{k\in\mathbb{Z}}$ of $L^2((0,T),\mathbb{R}^n)$,

$$\mathbb{P}_N\Phi_q(s) := \sum_{|k|\le N} (\Phi_q)_k\, e^{i(2\pi k/T)s},\ \ \mathbb{Q}_N\Phi_q(s) := \sum_{|k|>N} (\Phi_q)_k\, e^{i(2\pi k/T)s}.$$

Clearly,

$$\mathbb{P}_N L^2((0,T),\mathbb{R}^n) \oplus \mathbb{Q}_N L^2((0,T),\mathbb{R}^n) = L^2((0,T),\mathbb{R}^n),$$

and for $\Phi_q \in L^2((0,T),\mathbb{R}^n)$ we will write $u := \mathbb{P}_N\Phi_q$ and $v := \mathbb{Q}_N\Phi_q$.

We will try to solve (16) by a fixed point procedure.

Proposition 5. *(Lipschitz) Let* $\sup_{q\in\mathbb{R}^n} |V^{''}(q)| = C\ \ (< +\infty)$.
For fixed $(t,q) \in [0,T]\times\mathbb{R}^n$ *and* $u \in \mathbb{P}_N L^2((0,T),\mathbb{R}^n)$*, the map*

$$F : \mathbb{Q}_N L^2((0,T),\mathbb{R}^n) \longrightarrow \mathbb{Q}_N L^2((0,T),\mathbb{R}^n)$$

$$v \longmapsto \mathbb{Q}_N\left\{-\int_0^s V^{'}\left(q - \int_r^t (u+v)(\tau)\,d\tau\right)dr\right\}$$

is Lipschitz with constant

$$Lip(F) \le \frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right).$$

Before the proof of the Proposition 5, we premise some technical results.

Lemma 1. *Let* $f \in L^1((0,T),\mathbb{R}^n)\cap L^2((0,T),\mathbb{R}^n)$*. Then the function* $\int_0^t f(s)\,ds \in L^2((0,T),\mathbb{R}^n)$ *and*

$$||\int_0^t f(s)\,ds||_{L^2((0,T),\mathbb{R}^n)} \le T\cdot ||f||_{L^2((0,T),\mathbb{R}^n)} \tag{17}$$

Proof. Left to the reader.

Proof of Proposition 5 For each $v_1, v_2 \in \mathbb{Q}_N L^2\left((0,T), \mathbb{R}^n\right)$, let us consider the Fourier expansion

$$v := v_2 - v_1 = \sum_{|k|>N} v_k e^{i(2\pi k/T)\tau}.$$

We compute $F(v_2) - F(v_1)$:

$$\begin{aligned} F(v_2) - F(v_1) = \mathbb{Q}_N &\left\{ -\int_0^s \left[V'\left(q - \int_r^t (u+v_2)(\tau)\,d\tau \right) dr \right] \right. \\ &+ \left. \int_0^s \left[V'\left(q - \int_r^t (u+v_1)(\tau)\,d\tau \right) dr \right] \right\}, \\ = \mathbb{Q}_N &\left\{ -\int_0^s \left[V'\left(q - \int_r^t (u+v_2)(\tau)\,d\tau \right) - V'\left(q - \int_r^t (u+v_1)(\tau)\,d\tau \right) dr \right] \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} &||F(v_2) - F(v_1)||_{L^2((0,T),\mathbb{R}^n)} \\ &\leq T \cdot ||\mathbb{Q}_N \left\{ V'\left(q - \int_r^t (u+v_2)(\tau)\,d\tau \right) - V'\left(q - \int_r^t (u+v_1)(\tau)\,d\tau \right) \right\} ||_{L^2((0,T),\mathbb{R}^n)} \\ &\leq TC \cdot || - \int_r^t \sum_{|k|>N} v_k e^{i(2\pi k/T)\tau} d\tau ||_{L^2((0,T),\mathbb{R}^n)} \\ &\leq TC \cdot \left(|| \sum_{|k|>N} v_k e^{i(2\pi k/T)r} \cdot \frac{T}{i2\pi k} ||_{L^2((0,T),\mathbb{R}^n)} \right. \\ &\quad \left. + || \sum_{|k|>N} v_k e^{i(2\pi k/T)t} \cdot \frac{T}{i2\pi k} ||_{L^2((0,T),\mathbb{R}^n)} \right) \\ &\leq \frac{T^2 C}{2\pi N} \cdot ||v||_{L^2((0,T),\mathbb{R}^n)} + T^2 C \cdot || \sum_{|k|>N} \frac{|v_k|}{2\pi k} ||_{L^2((0,T),\mathbb{R}^n)}. \end{aligned}$$

We now use Cauchy-Schwartz inequality in $l^2 := L^2(\mathbb{Z}, \mathbb{C})$ as follows

$$\sum_{|k|>N} \frac{|v_k|}{k} = \left\langle (|v_k|)_{k\in\mathbb{Z}}, \left(\frac{1}{k}\right)_{|k|>N} \right\rangle_{l^2} \leq ||(|v_k|)_{k\in\mathbb{Z}}||_{l^2} \cdot ||\left(\frac{1}{k}\right)_{|k|>N}||_{l^2}.$$

Hence

$$\sum_{|k|>N} \frac{|v_k|}{2\pi k} \leq \frac{1}{2\pi} ||v||_{L^2((0,T),\mathbb{R}^n)} \sqrt{2 \sum_{|k|>N} \frac{1}{k^2}} \leq \frac{1}{2\pi} ||v||_{L^2((0,T),\mathbb{R}^n)} \sqrt{\frac{2}{N}},$$

obtaining

$$\begin{aligned} &\frac{T^2 C}{2\pi N} \cdot ||v||_{L^2((0,T),\mathbb{R}^n)} + T^2 C \cdot || \sum_{|k|>N} \frac{|v_k|}{2\pi k} ||_{L^2((0,T),\mathbb{R}^n)} \\ &\leq \frac{T^2 C}{2\pi N} \cdot ||v||_{L^2((0,T),\mathbb{R}^n)} + T^2 C \cdot \frac{1}{2\pi} \sqrt{\frac{2}{N}} ||v||_{L^2((0,T),\mathbb{R}^n)} \\ &= \frac{T^2 C}{2\pi N} \left(1 + \sqrt{2N} \right) \cdot ||v||_{L^2((0,T),\mathbb{R}^n)}, \end{aligned}$$

that is

$$\mathrm{Lip}\,(F) \leq \frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right).$$

□

Corollary 1. *(Contraction map) Let* $\sup_{q\in\mathbb{R}^n}|V''(q)| = C\ (<+\infty)$. *For fixed* $(t,q)\in[0,T]\times\mathbb{R}^n$, $u\in\mathbb{P}_N L^2((0,T),\mathbb{R}^n)$ *and* N *large enough:*

$$\frac{T^2C}{2\pi N}\left(1+\sqrt{2N}\right)<1,$$

the map $s\mapsto F(t,q,u)(s)$

$$F:\mathbb{Q}_N L^2((0,T),\mathbb{R}^n)\longrightarrow \mathbb{Q}_N L^2((0,T),\mathbb{R}^n)$$
$$v\longmapsto \mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}$$

is a contraction.

By Banach-Cacciopoli Theorem, for fixed $(t,q)\in[0,T]\times\mathbb{R}^n$ and $u\in\mathbb{P}_N L^2((0,T),\mathbb{R}^n)$, there exists one and only one fixed point $\mathcal{F}(t,q,u)(s)$, shortly $\mathcal{F}(u)$, for the above contraction:

$$\mathcal{F}(u)=\mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+\mathcal{F}(u))(\tau)\,d\tau\right)dr\right\}. \tag{18}$$

Beside (18), let us consider the finite-dimensional equation of unknown $u\in\mathbb{P}_N L^2((0,T),\mathbb{R}^n)$:

$$u=\mathbb{P}_N\left\{d\sigma\left(q-\int_0^t(u+\mathcal{F}(u))(r)\,dr\right)-\int_0^s V'\left(q-\int_r^t(u+\mathcal{F}(u))(\tau)\,d\tau\right)dr\right\}. \tag{19}$$

By adding (18) and (19), in correspondence to any solution u of (19), we gain

$$u+\mathcal{F}(u)=d\sigma\left(q-\int_0^t(u+\mathcal{F}(u))(r)\,dr\right)-\int_0^s V'\left(q-\int_r^t(u+\mathcal{F}(u))(\tau)\,d\tau\right)dr \tag{20}$$

in other words, the curve (see (12))

$$\gamma(s):=\mathrm{pr}_\Gamma\circ g\left(t,q,\left([u+\mathcal{F}(u)](s),-V'\left(q-\int_r^t(u+\mathcal{F}(u))(\tau)\,d\tau\right)\right)\right)$$

solves the Hamilton canonical equations starting from the graph of $d\sigma$ (so that $\gamma\in\Gamma$).

Furthermore, we point out that $\dim(\mathbb{P}_N L^2((0,T),\mathbb{R}^n)) = n(2N+1)$. As a consequence, substantially following the line of thought in [1], [9] and [10], we get that the geometric solution of Hamilton-Jacobi problem for H admits a finite-parameters generating function, denoted by $\overline{W}(t,q,u)$:

Theorem 7. *The finite-parameters function:*

$$\overline{W}:=A\circ g:[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n(2N+1)}\longrightarrow\mathbb{R},$$
$$(t,q,u)\longmapsto\overline{W}(t,q,u)=\left\{\sigma(q(0))+\int_0^t[p(s)\cdot\dot q(s)-H(s,q(s),p(s))]\,ds\right\}|_{(q(s),p(s))},$$
$$(q(s),p(s))=pr_\Gamma\circ g\left(t,q,\left([u+\mathcal{F}(u)](s),-V'\left(q-\int_r^t(u+\mathcal{F}(u))(\tau)\,d\tau\right)\right)\right),$$

generates $L_H=\bigcup_{0\le t\le T}\Phi_H^t((\Gamma_H)_\sigma)$, *the geometric solution for the Hamiltonian* $H(q,p)=\frac{1}{2}|p|^2+V(q)$.

6.3. **The quadraticity at infinity property.** We check the quadraticity at infinity property of $\overline{W}(t,q,u)$ with respect to u: this is a crucial step in order to catch the minimax critical point in the Lusternik-Schnirelman format. We premise the following technical Lemma (see also [12]).

Lemma 2. *For fixed $(t,q)\in[0,T]\times\mathbb{R}^n$, the function $u\mapsto\mathcal{F}(u)$ and its derivatives $u\mapsto\frac{\partial\mathcal{F}}{\partial u}(u)$ are uniformly bounded.*

Proof. We immediately get from (18) that $|\mathcal{F}(u)|\le TC$, where $C=\sup_{q\in\mathbb{R}^n}|V''(q)|<+\infty$. Moreover, by a direct computation, it can be proved that the derivatives $\frac{\partial\mathcal{F}}{\partial u}$ are uniformly bounded. In fact, the fixed point function $\mathcal{F}$ solves the equation of unknown v:

$$\mathcal{G}(t,q,u,v):=\mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}-v=0.$$

The implicit function theorem does work, since

$$\frac{\partial\mathcal{G}}{\partial v}(t,q,u,v)=\frac{\partial}{\partial v}\mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}-\mathbb{I},$$

and, by a classical argument, it can be proved that

$$\left[\frac{\partial\mathcal{G}}{\partial v}(t,q,u,v)\right]^{-1}=-\sum_{k=0}^{+\infty}\left[\frac{\partial}{\partial v}\mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}\right]^k.$$

Since a bound for the derivatives $\frac{\partial}{\partial v}\mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}$ is given by the Lipschitz constant $\alpha:=\frac{T^2C}{2\pi N}(1+\sqrt{2N})$,

$$\left|\frac{\partial}{\partial v}\mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}\right|\le\alpha<1,$$

we obtain that

$$\left|\left[\frac{\partial\mathcal{G}(t,q,u,v)}{\partial v}\right]^{-1}\right|\le\sum_{k=0}^{+\infty}\left|\frac{\partial}{\partial v}\mathbb{Q}_N\left\{-\int_0^s V'\left(q-\int_r^t(u+v)(\tau)\,d\tau\right)dr\right\}\right|^k=\frac{1}{1-\alpha}<+\infty.$$

$\mathcal{G}(t,q,u,\mathcal{F}(u))=0$ implies $\frac{\partial\mathcal{G}}{\partial u}+\frac{\partial\mathcal{G}}{\partial v}\frac{\partial\mathcal{F}}{\partial u}=0$, therefore the derivatives $\frac{\partial\mathcal{F}}{\partial u}=-\left(\frac{\partial\mathcal{G}}{\partial v}\right)^{-1}\frac{\partial\mathcal{G}}{\partial u}$ result uniformly bounded by the constant $\frac{\alpha}{1-\alpha}$:

$$\left|\frac{\partial\mathcal{F}}{\partial u}\right|\le\left|\left(\frac{\partial\mathcal{G}}{\partial v}\right)^{-1}\right|\cdot\left|\frac{\partial\mathcal{G}}{\partial u}\right|\le\frac{\alpha}{1-\alpha}<+\infty.$$

□

Theorem 8. *The finite-parameters function*

$$\begin{aligned}\overline{W}&:=A\circ g:[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n(2N+1)}\longrightarrow\mathbb{R},\\(t,q,u)&\longmapsto\bar{W}(t,q,u)\\&=\left\{\sigma(q(0))+\int_0^t[p(s)\cdot\dot{q}(s)-H(s,q(s),p(s))]\,ds\right\}|_{(q(s),p(s))},\end{aligned}$$

$(q(s),p(s))=pr_\Gamma\circ g\left(t,q,\left([u+\mathcal{F}(u)](s),-V'\left(q-\int_r^t(u+\mathcal{F}(u))(\tau)\,d\tau\right)\right)\right)$, *is asymptotically quadratic: there exists an u-polynomial $\mathcal{P}^{(2)}(t,q,u)$ such that for any fixed $(t,q)\in[0,T]\times\mathbb{R}^n$*

$$||\overline{W}(t,q,\cdot)-\mathcal{P}^{(2)}(t,q,\cdot)||_{C^1}<+\infty$$

and its leading term is positive defined (Morse index 0*).*

Proof.

$$\overline{W}\left(t,q,u\right)=\left\{\sigma\left(q\left(0\right)\right)+\int_0^t\left[p\left(s\right)\cdot\dot{q}\left(s\right)-H\left(s,q\left(s\right),p\left(s\right)\right)\right]ds\right\}|_{\left(q(s),p(s)\right)},$$

$$\left(q\left(s\right),p\left(s\right)\right)=\mathrm{pr}_{\Gamma}\circ g\left(t,q,\left(\left[u+\mathcal{F}\left(u\right)\right]\left(s\right),-V'\left(q-\int_r^t\left(u+\mathcal{F}\left(u\right)\right)\left(\tau\right)d\tau\right)\right)\right),$$

that is (through the Legendre transformation)

$$\overline{W}\left(t,q,u\right)=\left\{\sigma\left(q\left(0\right)\right)+\int_0^t\left[\frac{1}{2}|\dot{q}\left(s\right)|^2-V\left(q\left(s\right)\right)\right]ds\right\}|_{q(s)=q-\int_s^t\left[u(r)+\left(\mathcal{F}(u)\right)(r)\right]dr}$$

$$=\sigma\left(q-\int_0^t\left[u\left(r\right)+\left(\mathcal{F}\left(u\right)\right)\left(r\right)\right]dr\right)$$

$$+\int_0^t\left\{\frac{1}{2}|u\left(s\right)+\left(\mathcal{F}\left(u\right)\right)\left(s\right)|^2-V\left(q-\int_s^t\left[u\left(r\right)+\left(\mathcal{F}\left(u\right)\right)\left(r\right)\right]dr\right)\right\}ds.$$

As a consequence of the technical Lemma 2 above and the compactness of σ and V, for fixed $(t,q)\in[0,T]\times\mathbb{R}^n$ we obtain that

$$||\overline{W}\left(t,q,\cdot\right)-\mathcal{P}^{(2)}\left(t,q,\cdot\right)||_{C^1}<+\infty,$$

where $\mathcal{P}^{(2)}\left(t,q,u\right)$ is a function with positive defined leading term $\frac{1}{2}\int_0^t|u(s)|^2ds$ (hence with Morse index 0) and linear term with uniformly bounded coefficient, that is (see Definition 3) $\overline{W}\left(t,q,u\right)$ is an asymptotically quadratic generating function. □

6.4. **Minimax and viscosity solutions for** $H\left(q,p\right)=\frac{1}{2}|p|^2+V\left(q\right)$**.** We finally prove the main result: the equivalence of minimax and viscosity solutions for a large class of p-convex mechanical Hamiltonians.

Preliminarily, we point out the following quite natural technical fact:

Lemma 3. *Let* $H(t,q,p)$ *be a* C^2*-uniformely* p*-convex Hamiltonian function:*

$$\exists C\geq c>0:\quad c\,|\lambda|^2\leq\frac{\partial^2H}{\partial p_i\partial p_j}(t,q,p)\lambda_i\lambda_j\leq C\,|\lambda|^2,\tag{21}$$

$\forall\lambda\in\mathbb{R}^n,\ \forall t\in[0,T],\ \forall(q,p)\in\mathbb{R}^{2n}$*. Then, for every fixed* $q(\cdot)\in H^1([0,T],\mathbb{R}^n)$*,*

$$\sup_{\substack{p(\cdot)\in H^1([0,T],\mathbb{R}^n):\\ p(0)=\frac{\partial\sigma}{\partial q}(q(0))}}\int_0^T[p(t)\cdot\dot{q}(t)-H\left(t,q(t),p(t)\right)]\,dt=\int_0^T L\left(t,q(t),\dot{q}(t)\right)dt,\tag{22}$$

where

$$L(t,q,v)=\sup_{p\in\mathbb{R}^n}\left\{p\cdot v-H\left(t,q,p\right)\right\}\tag{23}$$

Proof. Convexity (21) guarantees us that global Legendre transformation holds and that, for any fixed $q(\cdot)$, the (unique, see below) critical curve of the functional:

$$\hat{A}:\ \left\{H^1([0,T],\mathbb{R}^n):\ p(0)=\frac{\partial\sigma}{\partial q}(q(0))\right\}\longrightarrow\mathbb{R}$$

$$p(\cdot)\longmapsto\int_0^T[p(t)\cdot\dot{q}(t)-H\left(t,q(t),p(t)\right)]\,dt$$

realizes the strong maximum (in the uniform convergence topology). In fact, $p(\cdot)$ is a critical curve iff, $\forall \triangle p \in H^1([0,T],\mathbb{R}^n)$ such that $\triangle p(0) = 0$,

$$d\hat{A}(p)\triangle p = d\Big\{ \int_0^T [p(t)\cdot \dot{q}(t) - H\,(t, q(t), p(t))]\,dt \Big\}\triangle p = 0.$$

$$\frac{d}{d\varepsilon}\Big\{ \int_0^T [(p(t)+\varepsilon\triangle p(t))\cdot \dot{q}(t) - H\,(t, q(t), p(t)+\varepsilon\triangle p(t))]\,dt \Big\}\Big|_{\varepsilon=0} = 0,$$

$$\int_0^T \left[\dot{q}(t) - \frac{\partial H}{\partial p}\,(t, q(t), p(t))\right]\triangle p(t)dt = 0,$$

that is,

$$\dot{q}(t) = \frac{\partial H}{\partial p}\,(t, q(t), p(t))\,, \tag{24}$$

and, by a standard argument laying on Legendre transformation, the unique solution $p(t)$ of (24), for *any* time t, is given by

$$p(t) = \frac{\partial L}{\partial v}(t, q(t), \dot{q}(t)).$$

Finally, $d^2\hat{A}(p)\big(\triangle p, \triangle p\big)$ is given by

$$d^2\Big\{ \int_0^T [p(t)\cdot \dot{q}(t) - H\,(t, q(t), p(t))]\,dt \Big\}\big(\triangle p, \triangle p\big) \le -cT \sup_{t\in[0,T]} |\triangle p(t)|^2.$$

From the identity

$$\begin{aligned} &\hat{A}(p+\triangle p) \\ &= \hat{A}(p) + \sum_{i=1}^n \frac{\partial \hat{A}}{\partial p_i}(p)\triangle p_i + \int_0^1 s \sum_{i,j=1}^n \frac{\partial^2 \hat{A}}{\partial p_i \partial p_J}\big((1-s)(p+\triangle p) + sp\big)\big(\triangle p_i, \triangle p_i\big)ds, \end{aligned}$$

we gain, at the critical p,

$$\hat{A}(p+\triangle p) - \hat{A}(p) \le -cT||\triangle p||^2_{C^0} \le 0$$

that is, p realizes the maximum of $\hat{A}$ in C^0, and then in $H^1(\hookrightarrow C^0)$. ☐

Theorem 9. *Let us consider* $H\,(q,p) = \frac{1}{2}|p|^2 + V\,(q)$, V *compactly supported and the related Cauchy Problem* $(CP)_H$*:*

$$(CP)_H \begin{cases} \dfrac{\partial S}{\partial t}\,(t,q) + \dfrac{1}{2}|\dfrac{\partial S}{\partial q}\,(t,q)\,|^2 + V\,(q) = 0, \\ S\,(0,q) = \sigma\,(q)\,, \end{cases}$$

where $t \in [0,T]\,, q \in \mathbb{R}^n$ *and* σ *compactly supported.*
The minimax and the viscosity solution of $(CP)_H$ *coincide with the function*

$$S\,(t,q) := \inf_{\substack{\tilde{q}\,(\cdot)\,: \\ \tilde{q}\,:\,[0,t]\to\mathbb{R}^n \\ \tilde{q}\,(t) = q}} \;\; \sup_{\substack{\tilde{p}\,(\cdot)\,: \\ \tilde{p}\,:\,[0,t]\to\mathbb{R}^n, \\ \tilde{p}(0) = \frac{\partial\sigma}{\partial q}(\tilde{q}(0))}} \left\{ \sigma\,(\tilde{q}\,(0)) + \int_0^t (p\dot{q} - H)_{|_{(\tilde{q},\tilde{p})}}\,ds \right\}. \tag{25}$$

Proof. In Subsection 6.1 we have proved that the Hamilton-Helmholtz functional involved in (25) can be interpreted as a global generating function W (with infinite parameters) for the geometric solution for the Hamiltonian H (Theorem 6). By Lemma 3, the sup-procedure on the curves $\tilde{p}$ in (25) represents exactly the Legendre transformation. Moreover, the fixed point technique described in Subsection 6.2 reduces the function W to a finite parameters G.F.Q.I., $\overline{W}$, with Morse index 0 (Theorems 7 and 8). As a consequence of Proposition 2, for such a function $\overline{W}$, the minimax critical value coincides with the minimum (which explains the inf-procedure on the curves $\tilde{q}$ in (25)). Hence the function $S\,(t, q)$ furnishes the minimax solution of $(CP)_H$.

On the other hand (see [17], [18] and bibliography quoted therein), the function $S\,(t, q)$ is the Hamiltonian version of the Lax-Oleinik formula producing the viscosity solution of $(CP)_H$.

Therefore (25) establishes the equivalence of the two solutions. □

Acknowledgements. We would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] B. Aebischer et al, Symplectic geometry, Progress in Mathematics, Basel, Vol XII, 124 pp. (1994).

[2] M. Bardi and L.C. Evans, *On Hopf's formulas for solutions of Hamilton-Jacobi equations,* Nonlinear Analysis, Theory, Methods & Applications, (11) 8 (1984), 1373–1381.

[3] M. Bardi and I. Capuzzo-Dolcetta, Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations, Systems and Control: Foundations and Applications. Boston, MA: Birkhauser. xvii, 570 pp. (1997).

[4] S. Benenti, *Symplectic relations in analytical mechanics.* Modern developments in analytical mechanics, Vol. I: Geometrical dynamics, Proc. IUTAM-ISIMM Symp., Torino, Italy 1982, 39-91, (1983).

[5] O. Bernardi and F. Cardin, *On Poincaré-Birkhoff periodic orbits for mechanical Hamiltonian systems on* $T^*\mathbb{T}^n$ Preprint, (2005).

[6] P. Bettiol and F. Cardin, *Landscapes of necessary conditions for the maxima in optimal control using Lagrangian submanifolds' global parameterizations and generalized solutions.* (Russian) Sovrem. Mat. Prilozh. No. 21, Geom. Zadachi Teor. Upr., 36-55, (2004).

[7] I. Capuzzo Dolcetta, *The Hopf solution of Hamilton-Jacobi equations,* Bemelmans, Josef (ed.) et al., Elliptic and parabolic problems. Proceedings of the 4th European conference, Rolduc, Netherlands, June 18-22, 2001 and Gaeta, Italy, September 24-28, 2001. Singapore: World Scientific., 343-351, (2002).

[8] F. Cardin, *On viscosity and geometrical solutions of Hamilton-Jacobi equations,* Nonlinear Anal., Theory Methods Appl., (6) 20 (1993), 713–719.

[9] F. Cardin, *The global finite structure of generic envelope loci for Hamilton-Jacobi equations,* J. Math. Phys., (1) 43 (2002), 417–430.

[10] F. Cardin and A. Marigonda, *Global world functions,* J. Geom. Symmetry Phys., 2 (2004), 1–17.

[11] F. Cardin and C. Viterbo, *Commuting Hamiltonians and multi-time Hamilton-Jacobi equations,* Preprint, arXiv:math.SG/0507418 v1 21 July (2005).

[12] M. Chaperon, *Quelques questions de géométrie symplectique,* Séminarie Bourbaki, 610 (1983), 231–249.

[13] M. Chaperon, *Lois de conservation et géométrie symplectique,* C. R. Acad. Sci. Paris 312 Séries 1, 345–348, (1991).

[14] C. Conley and E. Zehnder, *The Birkhoff-Lewis fixed point theorem and a conjecture of V.I. Arnol'd*, Invent. Math., 73 (1983), 33–49.

[15] M.V. Day, *On Lagrange manifolds and viscosity solutions,* J. Math. Systems Estim. Control, (3) 8 (1998).

[16] S.Yu. Dobrokhotov, V.N. Kolokoltsov and V.P. Maslov, *Quantization of the Bellman equation, exponential asymptotics and tunneling,* Adv. Soviet Math., 13, Amer. Math. Soc., Providence, RI, Idempotent analysis, 1–46, (1992).

[17] A. Fathi, *Thèoréme KAM faible et thèorie de Mather sur les systémes lagrangiens,* C. R. Acad. Sci. Paris, 324, Sèrie 1 (1997), 1043–1046.

[18] A. Fathi, Weak KAM Theorem in Lagrangian Dynamics, Preliminary version, (2005).

[19] F.R. Gantmacher, Lectures in analytical mechanics, Moscov, Mir, (1970).

[20] W.R. Hamilton, Second Essay on a General Method in Dynamics, Philosophical Transactions of the Royal Society, part I, 95–144, (1835).

[21] T. Joukovskaia, Singularités de minimax et solutions faibles d'équations aux dérivées partielles. PhD Thesis, Université Paris 7, (1993).

[22] P. Liebermann and C.M. Marle, Symplectic geometry and analytical mechanics. D. Reidel Publishing Co., Dordrecht, xvi+526 pp. (1987).

[23] P.L. Lions, *Generalized solutions of Hamilton-Jacobi equations,* Research Notes in Mathematics, 69. Boston - London - Melbourne: Pitman Advanced Publishing Program. 317 pp. (1982).

[24] J.J. Manfredi and B. Stroffolini, *A version of the Hopf-Lax formula in the Heisenberg group,* Commun. Partial Differ. Equations, (5–6) 27 (2002), 1139–1159.

[25] D. McCaffrey, *Graph Selectors and Viscosity Solutions on Lagrangian Manifolds,* To appear in ESAIM: Control, Optimisation and Calculus of Variations.

[26] A. Ottolenghi, C. Viterbo, *Solutions generalisees pour l'equation de Hamilton-Jacobi dans le cas d'evolution,* Unpublished.

[27] I.V.Rublev, *The relationship between two approaches to a generalized solution of the Hamilton-Jacobi equation,* Differ. Equ., (6) 38 (2002), 865–873.

[28] J.C.Sikorav, *Sur les immersions lagrangiennes dans un fibré cotangent admettant une phase génératrice globale.* C. R. Acad. Sci., Paris, 302, Sér. I, 3 (1986), 119–122.

[29] A.I.Subbotin, *Generalized solutions of first-order PDEs. The dynamical optimization perspective,* Basel: Birkhauser,. xi, 312 pp., (1994).

[30] A.I.Subbotin, *Minimax solutions to the Hamilton-Jacobi equations,* J. Math. Sci., New York 103, No.6, 772–777, (2001).

[31] D. Theret, *A complete proof of Viterbo's uniqueness theorem on generating functions,* Topology and its applications, 96 (1999), 249–266.

[32] D. Theret, Utilisation des fonctions génératrices en géométrie symplectique globale, PhD Thesis. Université Paris 7, (1996).

[33] C. Viterbo, *Solutions d'équations d'Hamilton-Jacobi et géométrie symplectique,* Sémin. Équ. Dériv. Partielles, Éc. Polytech., Cent. Math., Palaiseau Sémin. 1995-1996, Exp. No.22, 6 pp. (1996).

[34] A. Weinstein, Lectures on symplectic manifolds, Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics. Number 29. Providence, R. I.: AMS, (1977).

Received November 2005; revised June 2006.

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