Small parameter limit for ergodic, discrete-time, partially observed, risk-sensitive control problems

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Abstract

We show that discrete-time, partially observed, risk-sensitive control problems over an infinite time horizon converge, in the small noise limit, to deterministic dynamic games, in the sense of uniform convergence of the value function on compact subsets of its domain. We make use of new results concerning Large Deviations and existence of value functions.

Keywords: risk-sensitive control, dynamic games, large deviations.

1 Introduction

Risk-sensitive control is a branch of stochastic control dealing with performance indexes having the form of the expectation of the exponential of a cost function. More explicitly, in risk-sensitive control one aims at minimizing an index of the form

$$J_\mu(u) = \frac{1}{\mu}\log E\exp[\mu K(x^u(\cdot), u(\cdot))] \tag{1.1}$$

where K is a given real valued function acting on the paths of a controlled stochastic process x^u and on the applied control u. In the limit as $\mu \to 0$ one recovers the standard performance index in stochastic control

$$J_0(u) = E[K(x^u(\cdot), u(\cdot))]. \tag{1.2}$$

By increasing μ in (1.1), one penalizes large values of $K(x^u, u)$, modeling controller's aversion to risk; for this reason μ is commonly called the *risk parameter*.

When $x^u = (x^u_n)_{n\in\mathbb{N}}$ is a (discrete-time) controlled Markov process, and $K(x^u, u)$ is of the form

$$K(x^u, u) = \sum_{n=0}^{N} g_n(x^u_n, u_n) \quad \text{(finite time horizon cost)} \tag{1.3}$$

an optimal control for (1.1) can in principle be obtained by *Dynamic Programming.* To our knowledge, Dynamic Programming for risk-sensitive control has been first introduced in [15], and then subsequently developed by several authors [4, 18, 12, 13, 16, 10, 11, 5, 3]. The partially observed case, i.e. when admissible controls are nonanticipative functions of a noisy output signal rather than of the state of the system, has been first treated in [4], where the notion of *information state* is introduced. Extension to more general models can be found in [16] (see also [5] for results in continuous time).

A case of special interest is when the process x^u is driven by a "small" noise of magnitude, say, ϵ, and the risk parameter scales as ϵ^{-1}. In this situation, the big weight that (1.1) gives to paths that make $K(x^u, u)$ large may be balanced by their small probability. The behavior of the system in the limit as $\epsilon \to 0$ is usually referred to as the *small noise limit.* As shown in [12, 13, 16, 6, 10, 11, 1], finite time horizon, risk-sensitive control problems collapse to zero-sum deterministic differential games in the small noise limit, in the sense of convergence of the corresponding value functions.

In this paper we deal with risk-sensitive control in an infinite time horizon. Here, the performance index is taken to be of the form

$$J_\mu(u) = \limsup_{N\to\infty} \frac{1}{N\mu} \log E \exp\left[\mu \sum_{n=0}^{N-1} g(x_n, u_n)\right]. \tag{1.4}$$

Infinite time horizon risk-sensitive control is much less understood, both in terms of Dynamic Programming and of small noise limit. The existence of the value function has been proved under rather severe conditions. To our knowledge, the small noise limit has been treated rigorously only in the special case of finite state systems ([11]), and even in that case the convergence of the value function for partially observed problems has not been proved.

Here we consider partially observed models that are considerably more general than the ones in [11], and we prove convergence of the value function in the small noise limit. In comparison to [11], we introduce the following three ingredients, that play a fundamental role in the proof.

1. We use the Large Deviations framework developed in [1].

2. We take advantage of recent results ([8, 9]) on existence and concavity of the value function for partially observed systems.

3. We choose to define a value function that acts on a space of sequences (*information vectors*) rather than on a space of measures (*information states*). Although both spaces are infinite-dimensional, the first has a nicer topology and it is conceptually simpler.

The structure of this paper is as follows. In Section 2 we introduce the problem and state our main results. In Section 3 we recall recent results on ergodic control and Large Deviations, that are used in Section 4 to prove the results stated in Section 2. In Section 5 we discuss separately the case of complete observation.

The results that are proved in this paper have been announced in [2]

2 Model and Main Results

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. Moreover let $\mathcal{X}$, W be metric spaces, and U be a compact metric space. We will construct a family of controlled, partially observed stochastic systems evolving in discrete time ($n \in \mathbb{N}$), with state space $\mathcal{X}$, control space U, and observation space $\mathbb{R}^d$. For each $\epsilon > 0$, assume we are given two sequences of i.i.d. random variables $(W_n^\epsilon)_{n\geq 0}$, $(V_n^\epsilon)_{n\geq 0}$with values on W and $\mathbb{R}^d$ respectively. We denote by μ^ϵ the distribution of W_n^ϵ and by ν^ϵ the distribution of V_n^ϵ. Let $f : \mathcal{X} \times U \times W \to \mathcal{X}$ and $h : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}^d$ be two given measurable functions. We consider, for each $\epsilon > 0$, the controlled stochastic process defined recursively by:

$$\begin{cases} X_{n+1} &= f(X_n, u_n, W_n), \\ Y_n &= h(X_n, V_n), \end{cases} \tag{2.1}$$

together with the initial condition $X_0 = \xi \in \mathcal{X}$ deterministic and independent on ϵ. For simplicity of notation, we also assume the first observation to be deterministic and independent on ϵ, i.e. $Y_0 = \eta \in \mathbb{R}^d$. The assumption that the outputs are in $\mathbb{R}^d$ is for notational convenience; $\mathbb{R}^d$ could be replaced by any Riemannian manifold.

Definition 2.1 An infinite sequence $\mathbf{u} = (u_n)_{n\geq 0}$ of U-valued random variables is said to be an admissible control if $u_n = \phi_n(Y_1, \ldots, Y_n) = \phi_n(Y^n)$ for some measurable function $\phi_n : \mathbb{R}^{dn} \to U$. We will denote by $Ad(U)$ the set of admissible controls.

Let $c : \mathcal{X} \times U \to \mathbb{R}$ be a given measurable map called *cost function.* For each $\mathbf{u} \in Ad(U)$ we define the index:

$$J^\epsilon(\mathbf{u}) \;=\; \limsup_{N\to+\infty} \frac{\epsilon}{N} \log E \exp\left[\epsilon^{-1} \sum_{k=0}^{N-1} c(X_k, u_k)\right]. \tag{2.2}$$

The stochastic control problem we deal with consists in minimizing over $Ad(U)$ the index $J^\epsilon(\mathbf{u})$, i.e. finding $J_*^\epsilon = \inf\{J^\epsilon(\mathbf{u});\ \mathbf{u} \in Ad(U)\}$, and computing $\mathbf{u}_*^\epsilon \in Ad(U)$ such that $J^\epsilon(\mathbf{u}_*^\epsilon) = J_*^\epsilon$.

We now introduce the classical notions of Large Deviation Principle (LDP) and exponential tightness.

Definition 2.2 Let $\mathcal{X}$ be a metric space and $\{P^\epsilon : \epsilon > 0\}$ be a family of probability measures defined on its Borel σ-field. The family $\{P^\epsilon : \epsilon > 0\}$ is said to satisfy a *Large Deviation Principle* (LDP) with rate function $H_P : \mathcal{X} \to [0, +\infty]$ if
i) H_P is lower semicontinuous and $\{x : H_P(x) \leq l\}$ is compact for every $l \geq 0$.
ii) For every $A \subset \mathcal{X}$ measurable

$$- \inf_{x \in \overset{\circ}{A}} H_P(x) \leq \liminf_{\epsilon \to 0} \epsilon \log P^\epsilon(A) \leq \limsup_{\epsilon \to 0} \epsilon \log P^\epsilon(A) \leq - \inf_{x \in \bar{A}} H_P(x)$$

where $\overset{\circ}{A}, \bar{A}$ denote respectively the interior and the closure of A.

Definition 2.3 Let $\mathcal{X}$ be a metric space, and $\{P^\epsilon : \epsilon > 0\}$ be a family of probability measures defined on its Borel σ-field. The family $\{P^\epsilon : \epsilon > 0\}$ is called *exponentially tight* if, for every $L > 0$, there exists $C \subset \mathcal{X}$ compact such that

$$P^\epsilon(C^c) \leq e^{-\epsilon^{-1} L} \tag{2.3}$$

for all ϵ sufficiently small, where C^c is the complement of C.

Now we state the main assumptions on the model (equations (2.1) and (2.2)) which are needed in the rest of the paper.

H1) The function $f : \mathcal{X} \times U \times W \to \mathcal{X}$ is continuous.

H2) The family of distributions $(\mu^\epsilon)_{\epsilon > 0}$ is exponentially tight and satisfies a LDP with rate function $H(\cdot)$.

H3) For each $A \subset \mathcal{X}$ measurable, $x \in \mathcal{X}$, and $(\bar{x}, \bar{u}) \in \mathcal{X} \times U$ fixed, define:

$$\phi_x^{\epsilon, A}(u) = \epsilon \log \frac{\mu^\epsilon\left(f^{-1}(A; x, u)\right)}{\mu^\epsilon\left(f^{-1}(A; \bar{x}, \bar{u})\right)},$$

where $f^{-1}(A; x, u) = \{w \in W \mid f(x, u, w) \in A\}$ (we use the convention $0/0 = 1$).
The set of functions $\left\{\phi_x^{\epsilon, A} \mid \epsilon > 0, A \subset \mathcal{X}, x \in \mathcal{X}\right\}$ is uniformly bounded and equicontinuous.

H4) The function $h : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous, and for each fixed $x \in \mathcal{X}$ the function $v \to h(x, v)$ is a diffeomorphism of $\mathbb{R}^d$. The inverse map $h^{-1}(x, y)$ and its Jacobian $D_y h^{-1}(x, y)$ are continuous on $\mathcal{X} \times \mathbb{R}^d$; moreover the continuity in the y variable of the inverse map $h^{-1}(x, y)$ is uniform in x. For each $K \subset \mathbb{R}^d$ compact, the map $\log\left[\det \left(D_y h^{-1}(x, y)\right)\right]$ is bounded on $\mathcal{X} \times K$.

H5) The family of distributions $(\nu^\epsilon)_{\epsilon > 0}$ is exponentially tight and satisfy a LDP with rate function $K(\cdot)$. Moreover we assume that the map $K(\cdot)$ is finite valued and continuous.

H6) The measure ν^ϵ admits a strictly positive density ρ^ϵ w.r.t. the Lebesgue measure. The family of functions $\{\epsilon \log \rho^\epsilon\}$ is equicontinuous, uniformly bounded from below when restricted to any compact subset of $\mathbb{R}^d$ and uniformly bounded from above on all $\mathbb{R}^d$.

H7) the cost function $c : \mathcal{X} \times U \to \mathbb{R}$ which appears in $J^\epsilon(\mathbf{u})$ (equation (2.2)) is uniformly continuous and bounded (we will denote by $||c||_\infty$ its sup-norm).

Remark 2.4 1. Assumptions H1, H2, H4-H7 are similar but slightly stronger than Assumptions B in [1] (see Section 4.3). A possible example in which these assumptions hold, may be constructed as follows. Assume W is a Riemannian manifold, with dw denoting the Lebesgue measure on it. Let $f : \mathcal{X} \times U \times W \to \mathcal{X}$ be any continuous function and $c : \mathcal{X} \times U \to \mathbb{R}$ be any uniformly continuous and bounded function. Suppose we are given two continuous functions $\tilde{H} : W \to \mathbb{R}$, $\tilde{K} : \mathbb{R}^d \to \mathbb{R}$ such that

i) $\tilde{H}$ and $\tilde{K}$ have compact level sets;

ii) $e^{-\tilde{H}}$, $e^{-\tilde{K}}$ are integrable w.r.t. the Lebesgue measures on W and $\mathbb{R}^d$ respectively;

Then define

$$\mu^\epsilon(dw) = \frac{e^{-\epsilon^{-1}\tilde{H}(w)} dw}{\int e^{-\epsilon^{-1}\tilde{H}(w)} dw}, \quad \nu^\epsilon(dw) = \frac{e^{-\epsilon^{-1}\tilde{K}(v)} dv}{\int e^{-\epsilon^{-1}\tilde{K}(v)} dv}.$$

Then $\mu^\epsilon, \nu^\epsilon$ satisfy assumptions H2, H5 and H6, with corresponding rate functions $H(w) = \tilde{H}(w) - \inf \tilde{H}$, $K(v) = \tilde{K}(v) - \inf \tilde{K}$. To complete the description of the model we have to assign an output function h which satisfies assumption H4. Such examples are provided by functions of type

$$h(x, v) = \beta(x) + \gamma(x)v$$

where $\beta : \mathcal{X} \to \mathbb{R}^d$ and $\gamma : \mathcal{X} \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ are continuous functions and, for all $v \in \mathbb{R}^d$, the inequality $\|\gamma(x)v\|^2 \geq \delta\|v\|^2$ holds for a constant $\delta > 0$ independent of $x \in \mathcal{X}$. Note that no boundedness or growth assumptions on β are required.

2. Assumption H3 is the most restrictive one . If $\mathcal{X}$ and U are both finite (as in [11]), then H3 is a trivial requirement. It can be verified that the example given in 1. satisfies H3 if we also require that $\mathcal{X} = W$ is compact and that the map $w \to f(x, u, w)$ is a diffeomorphism for any pair (x, u).

3. All these assumptions can be easily adapted to the case treated in [11], where $\mathcal{X}$ and U are finite and $\mathbb{R}^d$ is replaced by a finite set.

Now we introduce the notion of information space which is very useful when studying the small parameter limit. Let $\emptyset$ be a special symbol. Define the set:

$$\mathcal{Z} \ = \ \cup_{n=0}^{+\infty} \left[\left(\mathbb{R}^d \times U \right)^n \times \{\emptyset\}^{\mathbb{N}} \right].$$

Let $z = (z^{(1)}, z^{(2)}, \ldots)$ be an element of $\mathcal{Z}$. Then either $z = (\emptyset, \emptyset, \emptyset, \ldots)$ or

$$z = (\underbrace{(y_n, u_{n-1})}_{z^{(1)}}, \underbrace{(y_{n-1}, u_{n-2})}_{z^{(2)}} \ldots, \underbrace{(y_1, u_0)}_{z^{(n)}}, \emptyset, \emptyset, \ldots),$$

and in this case z represents a finite sequence of outputs and controls.

Remark 2.5 The partially observed dynamics (2.1) induce the following completely observed dynamics on $\mathcal{Z}$:

$$\begin{cases} Z_{n+1} = (Y_{n+1}, u_n, Z_n), \\ Z_0 = (\emptyset, \emptyset, \ldots). \end{cases} \tag{2.4}$$

Notice that an admissible control u_n at time n can be thought as a function of Z_n. In the first equation of (2.4), we may interpret Z as the state variable, u as the control variable, and Y as the disturbance.

Definition 2.6 For each $z \in \mathcal{Z}$ let:

$$n(z) \ = \ \min\{\, n \mid z^{(k)} = \emptyset \ \forall\, k > n \,\}.$$

According to this definition we have:

$$\begin{array}{lll} n(z) = 0 & \Rightarrow & z = (\emptyset, \emptyset, \ldots), \\ n(z) > 0 & \Rightarrow & z = ((y, u), \Delta z), \quad \text{with} \quad \Delta z \in \mathcal{Z}, \quad \text{and} \quad n(\Delta z) = n(z) - 1. \end{array}$$

Now we will define a metric on $\mathcal{Z}$. Denote by $d_{\mathbb{R}^d}$, d_U metrics on $\mathbb{R}^d$ and U respectively. It is not restrictive to assume that these metrics are bounded above by 1. Let $z_i = (z_i^{(1)}, z_i^{(2)}, \ldots)$, $i = 1, 2$, be two elements of $\mathcal{Z}$. Then we let:

$$d(z_1, z_2) \ = \ \sum_{k \geq 1} \hat{d}(z_1^{(k)}, z_2^{(k)}), \tag{2.5}$$

where

$$\hat{d}(z_1^{(k)}, z_2^{(k)}) = \begin{cases} 0 & \text{if} \quad z_1^{(k)} = z_2^{(k)} = \emptyset, \\ 1 & \text{if} \quad z_1^{(k)} = \emptyset, z_2^{(k)} \in \mathbb{R}^d \times U, \\ 1 & \text{if} \quad z_2^{(k)} = \emptyset, z_1^{(k)} \in \mathbb{R}^d \times U, \\ \frac{1}{2}\left[d_{\mathbb{R}^d}(y_1^{(k)}, y_2^{(k)}) + d_U(u_1^{(k)}, u_2^{(k)}) \right] & \text{if} \quad z_1^{(k)}, z_2^{(k)} \in \mathbb{R}^d \times U. \end{cases}$$

Notice that the sum in (2.5) is always finite. It is easily checked that $d(\cdot, \cdot)$ is a metric on $\mathcal{Z}$.

Remark 2.7 Although the explicit form of the metric $d(\cdot,\cdot)$ will not be used, the following properties of $d(\cdot,\cdot)$ will play a key role in the proofs.

(a) Let $(z_i)_{i\geq 0}$ be a sequence of elements of $\mathcal{Z}$ and assume that this sequence converges to $z \in \mathcal{Z}$ with respect to the metric $d(\cdot,\cdot)$ defined above. Then there exists $k \in \mathbb{N}$ such that $n(z_i) = n(z)$ for all $i \geq k$.

(b) Let $C \subset \mathcal{Z}$ be a compact set; then there exists $\bar{n} \geq 0$ such that $n(z) \leq \bar{n}$ for all $z \in C$.

For each $x \in \mathcal{X}$ and $\epsilon > 0$, we define the following probability measure on $\mathbb{R}^d$

$$Q^\epsilon(A;x) = \nu^\epsilon\{v : h(x,v) \in A\}. \tag{2.6}$$

Assumption H4 and H6 imply, in particular, that this measure $Q^\epsilon(\cdot;x)$ has a density $q^\epsilon(y;x)$ w.r.t. the Lebesgue measure given by $q^\epsilon(y;x) = \rho^\epsilon(h^{-1}(x,y))|\ \mathrm{Det}\ \left(D_y h^{-1}(x,y)\right)|$. Now, for $(x,u) \in \mathcal{X}\times U$, we consider the probability measure $\Pi^\epsilon(d\xi;x,u)$ on $\mathcal{X}$ defined by

$$\Pi^\epsilon(A;x,u) = \mu^\epsilon(f^{-1}(A;x,u)) = \mu^\epsilon\{w \mid f(x,u,w) \in A\}.$$

Moreover, for $(y,u) \in \mathbb{R}^d \times U$, let $T^\epsilon(y,u)$ be the operator, acting on probability measures of $\mathcal{X}$, defined by:

$$\left(T^\epsilon(y,u)(\mu)\right)(A) = \frac{\int_{\mathcal{X}} e^{\epsilon^{-1}c(x',u)} \left[\int_A q^\epsilon(y;x)\Pi^\epsilon(dx;x',u)\right]\mu(dx')}{\int_{\mathcal{X}} e^{\epsilon^{-1}c(x',u)} \left[\int_{\mathcal{X}} q^\epsilon(y;x)\Pi^\epsilon(dx;x',u)\right]\mu(dx')} \tag{2.7}$$

For $z = ((y_n,u_{n-1}),\ldots,(y_1,u_0),\emptyset,\emptyset,\ldots) \in \mathcal{Z}$, we let $P^\epsilon(dx;z)$ be the probability measure on $\mathcal{X}$ defined by:

$$P^\epsilon(dx;z) = T^\epsilon(y_n,u_{n-1})\cdots T^\epsilon(y_1,u_0)\delta_\xi, \tag{2.8}$$

and

$$P^\epsilon(dx;(\emptyset,\emptyset,\ldots)) = \delta_\xi. \tag{2.9}$$

Finally, we define the following measure on $\mathbb{R}^d$

$$R^\epsilon(A;z,u) = \int_A \left[\int_{\mathcal{X}}\int_{\mathcal{X}} e^{\epsilon^{-1}c(x',u)} q^\epsilon(y;x)\Pi^\epsilon(dx;x',u)P^\epsilon(dx';z)\right] dy. \tag{2.10}$$

Next proposition proves the existence of a value function W^ϵ, and of an optimal control map u_*^ϵ in terms of the information state, for the problem described by equations (2.1), (2.2). This result is a consequence of Theorem 2 in [9], as we will show in Section 4. If $z = (z^{(1)}, z^{(2)},\ldots) \in \mathcal{Z}$ and $(y,u) \in \mathbb{R}^d \times U$, we denote by (y,u,z) the element of $\mathcal{Z}$ given by $((y,u), z^{(1)}, z^{(2)},\ldots)$.

Proposition 2.8 *Assume that for each $\epsilon > 0$, we are given a model as in (2.1) and (2.2) which satisfies assumptions H1-H7. Then there is a bounded and continuous function $W^\epsilon : \mathcal{Z} \to \mathbb{R}$, and a real number λ^ϵ that solve the following equation:*

$$W^\epsilon(z) + \lambda^\epsilon = \inf_{u\in U}\left[\epsilon \log \int_{\mathbb{R}^d} e^{\epsilon^{-1}W^\epsilon(y,u,z)} R^\epsilon(dy;z,u)\right] \tag{2.11}$$

and $W^\epsilon(\emptyset,\emptyset,\ldots) = 0$. We also have:

$$\lambda^\epsilon = \inf_{u\in Ad(U)} J^\epsilon(u). \tag{2.12}$$

Moreover there is a feedback $u_*^\epsilon = u_*^\epsilon(z)$ at which the infimum in (2.11) is attained. This feedback is an optimal control in the following sense:

for $z = (y_n,u_{n-1},\ldots,y_1,u_0,\emptyset,\emptyset,\ldots)$, the control $u_*^\epsilon = \{(u_*^\epsilon)_n\}_{n\geq 0}$ given by $(u_*^\epsilon)_n(y_n,u_{n-1},\ldots,y_1,u_0) = u_*^\epsilon(z)$ is optimal, i.e. $J^\epsilon(u_*^\epsilon) = \lambda^\epsilon$.

Before stating our main results we need some other definitions. Recall that $H(\cdot)$ and $K(\cdot)$ are the rate functions of the families $(\mu^\epsilon)_{\epsilon>0}$ and $(\nu^\epsilon)_{\epsilon>0}$ respectively, and $c(x,u)$ is the cost function. Let $H^S : \mathcal{X}\times\mathcal{X}\times U \to [0,+\infty]$ given by:

$$H^S(x;x',u) = \inf\{H(w) : f(x',u,w) = x\}, \tag{2.13}$$

and $H^O : Y \times \mathcal{X} \to [0,+\infty]$ given by:

$$H^O(y;x) = \inf\{K(v) : h(x,v) = y\}. \tag{2.14}$$

These two functions H^S and H^O represent the rate functions for the two parameterized families of measure $\Pi^\epsilon(d\xi;x,u)$ and $Q^\epsilon(dy;x)$ defined above (see Section 3 for the definition of rate function for a family of parameterized measures). Now let $\tilde{H}:\mathcal{X}\times\mathcal{Z}\to[0,+\infty]$ be the function defined by induction on $n=n(z)$ as follows:

$$\tilde{H}(x;(\emptyset,\ldots))=\begin{cases}0 & \text{if } x=\xi\\ +\infty & \text{otherwise}\end{cases} \tag{2.15}$$

$$\begin{aligned}\tilde{H}(x;(y,u,z)) &= H^O(y;x)+\inf_{x'\in\mathcal{X}}\left[\tilde{H}(x';z)-c(x',u)+H^S(x;x',u)\right]-\\ &- \inf_{x,x'\in\mathcal{X}}\left\{H^O(y;x)+\tilde{H}(x';z)-c(x',u)+H^S(x;x',u)\right\}.\end{aligned} \tag{2.16}$$

Finally, let:

$$\begin{cases}\bar{H}:Y\times Z\times U\to[0,+\infty],\\ \bar{H}(y;z,u)=\inf_{x,x'\in\mathcal{X}}\left\{H^O(y;x)-c(x',u)+H^S(x;x',u)+\tilde{H}(x';z)\right\}.\end{cases} \tag{2.17}$$

We are now ready to state our main result.

Theorem 1 *Assume that for each $\epsilon>0$, we are given a model as in (2.1) and (2.2) which satisfies assumptions H1-H7. For each $\epsilon>0$, let W^ϵ and λ^ϵ be the solutions of equation (2.11). Then the sets $\{\lambda^\epsilon:\epsilon>0\}$ and $\{W^\epsilon:\epsilon>0\}$ have at least a limit point λ and W respectively in $\mathbb{R}$ and $C(\mathcal{Z})$, the last space being provided with the topology of uniform convergence on compact sets. Each limit point is a solution of the equation:*

$$W(z)+\lambda=\inf_{u\in U}\sup_{y\in\mathbb{R}^d}\left\{W(y,u,z)-\bar{H}(y;z,u)\right\}. \tag{2.18}$$

Now we introduce a deterministic, partially observed dynamic game to give a proper interpretation to equation (2.18). Formally, the equations of dynamics are the same as in (2.1), where now $w_n\in W$ and $v_n\in\mathbb{R}^d$ are interpreted as deterministic disturbances. Thus we consider the deterministic, partially-observed, discrete-time system defined by:

$$\begin{cases}x_{n+1} &= f(x_n,u_n,w_n),\\ y_n &= h(x_n,v_n),\end{cases} \tag{2.19}$$

together with initial conditions $x_0=\xi\in\mathcal{X}$ and $y_0=\eta\in\mathbb{R}^d$. Admissible controls are those such that u_n is a function of the observations up to time n. Again, we denote by $Ad(U)$ the set of admissible controls, even if measurability issues are unimportant in this contest. The aim is to minimize over $Ad(U)$ the index:

$$J(u)=\limsup_{n\to+\infty}\frac{1}{n}\sup_{\substack{w_0,\ldots,w_{n-1}\\ v_1,\ldots,v_n}}\left[\sum_{k=0}^{n-1}\left(c(x_k,u_k)-H(w_k)-K(v_{k+1})\right)\right]. \tag{2.20}$$

Theorem 2 *Assume that we are given the model in equation (2.19). Moreover, assume that there exists a constant $\lambda\in\mathbb{R}$ and a bounded function $W\in C(\mathcal{Z})$ which are a solution of equation (2.18). Then*

$$\lambda=\inf_{u\in Ad(U)}J(u),$$

where $J(u)$ is the cost functional given by (2.20). Moreover there exists a feedback $u_*=u_*(z)$ which realizes the infimum in (2.18). This feedback provides and optimal control for the dynamic game given by equation (2.19), in the same sense as in Proposition 2.8.

3 Preliminaries

The results in Subsection 3.1 and 3.2 are taken from [1] and [9] respectively.

3.1 Large Deviations

In this section $\mathcal{X}$ and Θ are metric spaces. All measures on $\mathcal{X}$ are intended to be defined on its Borel σ-field. We first generalize Definitions 2.2 and 2.3 to the case of parameterized families of measures.

Definition 3.1 A family $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ of positive finite measures on $\mathcal{X}$ is called a *Weakly Uniform Large Deviation Family* (WULDF) with rate function $H_\mathcal{P} : \mathcal{X} \times \Theta \to (-\infty, +\infty]$ if
i) For every fixed $\theta \in \Theta$, $H_\mathcal{P}(\cdot, \theta)$ is lower semicontinuous and $\{x : H_\mathcal{P}(x,\theta) \leq l\}$ is compact for every $l \in \mathbb{R}$.
ii) The map $\theta \to \inf_{x\in\mathcal{X}} H_\mathcal{P}(x,\theta)$ is real valued, and is bounded on the compact subsets of Θ.
iii) For every $F : \mathcal{X} \to \mathbb{R}$ bounded and continuous

$$\lim_{\epsilon\to 0} \epsilon \log \int e^{\epsilon^{-1}F(x)} \mathcal{P}^\epsilon(dx;\theta) = \sup_{x\in\mathcal{X}} \Big[F(x) - H_\mathcal{P}(x,\theta)\Big] \tag{3.1}$$

uniformly for θ in the compact subsets of Θ.

Definition 3.2 A family $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ of positive finite measures on $\mathcal{X}$ is called *exponentially tight* if, for every $L > 0$ and every $K \subset \Theta$ compact, there exists $C \subset \mathcal{X}$ compact such that

$$\mathcal{P}^\epsilon(C^c;\theta) \leq e^{-\epsilon^{-1}L} \tag{3.2}$$

for all $\theta \in K$ and ϵ sufficiently small, where C^c is the complement of C.

Remarks 3.3

1. Note that we allow $\mathcal{P}^\epsilon$ in Definitions 3.1 and 3.2 to be a positive finite measure, not necessarily a probability measure.
2. When all the measures $\{\mathcal{P}^\epsilon\}_{\epsilon>0}$ in Definition 3.2 are probability measures and Θ is a singleton, Definition 3.2 reduces to Definition 2.3.
3. Condition ii) in Definition 3.1 roughly says that $\mathcal{P}^\epsilon(\mathcal{X};\theta)$ does not either go to zero or grow too fast as $\epsilon \to 0$. Indeed, by using ii) and letting $F \equiv 0$ in iii) the following statement is easy to prove: for each $K \subset \Theta$ compact, there exists $M(K) > 0$ such that, for ϵ sufficiently small,
$$e^{-\epsilon^{-1}M(K)} \leq \mathcal{P}^\epsilon(\mathcal{X};\theta) \leq e^{\epsilon^{-1}M(K)} \tag{3.3}$$
for all $\theta \in K$. If all $\mathcal{P}^\epsilon(dx;\theta)$ are probability measures, then condition ii) is automatically satisfied, since $\inf_{x\in\mathcal{X}} H_\mathcal{P}(x,\theta) \equiv 0$ (see [7]).
4. Suppose that $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ is a WULDF of probability measures on $\mathcal{X}$. Then, under the further assumption of exponential tightness, for every fixed $\theta \in \Theta$ the family $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0\}$ satisfies a LDP with rate function $H_\mathcal{P}(x;\theta)$ (Bryc Theorem, see [7]). This *pointwise* LDP does not imply uniform convergence in (3.1), but only pointwise convergence. On the other hand, (3.1) for all F continuous and bounded, does not imply any uniformity in θ of the Large Deviations bounds for $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0\}$ (see [1] for a counterexample).

Next proposition establishes the fact that a family of probability measures that satisfies a LDP transforms into a WULDF under a continuous, parameter dependent mapping.

Proposition 3.4 *Let $\mathcal{W}$ be a metric space, $f : \Theta \times \mathcal{W} \to \mathcal{X}$ a continuous map, and $\{\mu^\epsilon : \epsilon > 0\}$ a family of probability measures on $\mathcal{W}$ that satisfy a LDP with rate function $h(w)$. Define $\mathcal{P}^\epsilon(dx;\theta)$, a probability measure on $\mathcal{X}$, by*

$$\mathcal{P}^\epsilon(A;\theta) = \mu^\epsilon\{w : f(\theta,w) \in A\}. \tag{3.4}$$

Then $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ is a WULDF with rate function

$$H_\mathcal{P}(x;\theta) = \inf\{h(w) : f(\theta,w) = x\}. \tag{3.5}$$

Moreover, if $\{\mu^\epsilon : \epsilon > 0\}$ is exponentially tight, then so is $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$.

The notion of WULDF is normally used in the form given in next lemma, that is a slight modification of (3.1).

Lemma 3.5 *Suppose that $\{\mathcal{P}^\epsilon(dx;\theta) : \epsilon > 0, \theta \in \Theta\}$ is a WULDF with rate function $H_\mathcal{P}$, and it is exponentially tight. Let $F^\epsilon : \mathcal{X} \to \mathbb{R}$, $\epsilon \geq 0$ be such that $\sup_{\epsilon\geq 0} \|F^\epsilon\|_\infty < \infty$, $F^\epsilon \to F^0$ as $\epsilon \to 0$ uniformly on the compact subsets of $\mathcal{X}$ and F^0 is continuous. Then*

$$\lim_{\epsilon\to 0} \epsilon \log \int e^{\epsilon^{-1}F^\epsilon(x)} \mathcal{P}^\epsilon(dx;\theta) = \sup_{x\in\mathcal{X}} \Big[F^0(x) - H_\mathcal{P}(x,\theta)\Big] \tag{3.6}$$

uniformly on the compact subsets of Θ.

Next three propositions form the core of the theory of WULDF's. They state that, under suitable assumptions, the notion of WULDF is preserved by three operations on families of finite positive measures: composition, contraction, and conditioning. The reason for studying these operations is that they appear in the recursive equation giving the information state for a partially observed, risk-sensitive control problem.

In the rest of this section, $\mathcal{X}, Y$ and Θ are metric spaces.

Proposition 3.6 *(Composition). Let* $\{\mathcal{P}^\epsilon(dx; y, \theta) : \epsilon > 0, (y, \theta) \in \mathcal{Y} \times \Theta\}$ *and* $\{\mathcal{Q}^\epsilon(dy; \theta) : \epsilon > 0, \theta \in \Theta\}$ *be two exponentially tight WULDF's in* $\mathcal{X}$ *and* $\mathcal{Y}$ *respectively with rate functions* $H_\mathcal{P}(x; y, \theta)$ *and* $H_\mathcal{Q}(y; \theta)$. *Assume that the measures* $\mathcal{P}^\epsilon$ *are all probability measures. Moreover, assume that the rate function* $H_\mathcal{P}(x; y, \theta)$ *satisfies the following regularity conditions*

(i) Let $A = \{(x, y, \theta) \ : \ H_\mathcal{P}(x; y, \theta) < +\infty\}$. *Then for every* $(x, y, \theta) \in A$ *and every sequence* $(y_n, \theta_n) \to (y, \theta)$ *there exists a sequence* $x_n \to x$ *such that* $H_\mathcal{P}(x_n; y_n, \theta_n) \to H_\mathcal{P}(x; y, \theta)$,

(ii) $H_\mathcal{P}$ *is lower semicontinuous as a function of* (x, y, θ).

Then $\{\mathcal{R}^\epsilon(dx, dy; \theta) : \epsilon > 0, \theta \in \Theta\}$ *defined by:*

$$\int f(x, y)\mathcal{R}^\epsilon(dx, dy; \theta) = \int \left[\int f(x, y)\mathcal{P}^\epsilon(dx; y, \theta)\right] \mathcal{Q}^\epsilon(dy; \theta),$$

is an exponentially tight WULDF with rate function:

$$H_\mathcal{R}(x, y; \theta) = H_\mathcal{P}(x; y, \theta) + H_\mathcal{Q}(y; \theta).$$

Proposition 3.7 *(Contraction) Let* $\mathcal{R}^\epsilon(dx, dy; \theta)$ *be an exponentially tight WULDF on* $\mathcal{X} \times \mathcal{Y}$ *with rate function* $H_\mathcal{R}(x, y; \theta)$. *Then the family* $\mathcal{P}^\epsilon(dx; \theta)$, *defined by:*

$$\mathcal{P}^\epsilon(A; \theta) \ = \ \mathcal{R}^\epsilon(A \times \mathcal{Y}; \theta),$$

is, an exponentially tight WULDF on $\mathcal{X}$ *with rate function:*

$$H_\mathcal{P}(x; \theta) \ = \ \inf_{y \in \mathcal{Y}} H_\mathcal{R}(x, y; \theta).$$

Proposition 3.8 *(Conditioning). Let* $\{\mathcal{P}^\epsilon(dx; \theta) : \epsilon > 0, \theta \in \Theta\}$ *and* $\{\mathcal{Q}^\epsilon(dy; x) : \epsilon > 0, x \in \mathcal{X}\}$ *be two exponentially tight WULDF's on* $\mathcal{X}$ *and* $\mathcal{Y}$ *respectively, with rate functions* $H_\mathcal{P}(x; \theta)$ *and* $H_\mathcal{Q}(y; x)$. *Assume that the measures* $\mathcal{Q}^\epsilon(dy; x)$ *are all probability measures, and that both families of kernels are exponentially tight. Moreover assume that the rate function* $H_\mathcal{Q}(y; x)$ *is always finite and continuous, and that the following properties hold:*

1. *the measure* $\mathcal{Q}^\epsilon(dy; x)$ *is of the form:*

   $$\mathcal{Q}^\epsilon(dy; x) \ = \ q^\epsilon(y; x)\alpha(dy).$$

   where $q^\epsilon(y; x) > 0$ *and the measure* $\alpha(dy)$ *satisfies*

   $$\inf_{y \in K} \alpha\left(B(y, \gamma)\right) > 0;$$

   for every $K \subset \mathcal{Y}$ *compact and* $\gamma > 0$, *where* $B(y, \gamma)$ *is the ball centered at* y *with radius* γ.

2. *for any compact sets* $K \subseteq \mathcal{Y}$, $C \subseteq \mathcal{X}$, *and any* $\delta > 0$ *there exists* $\delta_1 > 0$ *and* $\epsilon(\delta)$ *such that:*

   $$|\epsilon \log q^\epsilon(y_1; x) \ - \ \epsilon \log q^\epsilon(y_2; x)| \ < \ \delta,$$

   for all y_1, $y_2 \in K$ *such that* $d(y_1, y_2) < \delta_1$, *for all* $\epsilon \leq \epsilon(\delta)$, *and for all* $x \in C$;

3. *for any compact sets* $K \subseteq \mathcal{Y}$, $C \subseteq \mathcal{X}$ *there exists* $n_{K,C} > 0$ *such that:*

   $$\epsilon \log q^\epsilon(y; x) \geq -n_{K,C};$$

   for all $y \in K$, $x \in C$, *and* $\epsilon > 0$

4. *for any compact set* $K \subseteq \mathcal{Y}$, *there exists* $N_K > 0$ *such that:*

   $$\epsilon \log q^\epsilon(y; x) \leq N_K;$$

   for all $y \in K$, *for all* $x \in \mathcal{X}$, *and for all* $\epsilon > 0$.

Then the measures on $\mathcal{X}$

$$\mathcal{S}^\epsilon(dx; y, \theta) \ = \ q^\epsilon(y; x)\mathcal{P}^\epsilon(dx; \theta)$$

form an exponentially tight WULDF with rate function

$$H_\mathcal{S}(x; y, \theta) \ = \ H_\mathcal{Q}(y; x) + H_\mathcal{P}(x; \theta).$$

3.2 Ergodic, risk-sensitive control under partial state observation

In this subsection we consider the risk-sensitive control problem given by the dynamics (2.1) and the cost function (2.2). The parameter $\epsilon > 0$ is assumed to be fixed; for later use, we keep it in all notations.

In Section 2 we introduced the probability kernels $\Pi^\epsilon(d\xi; x, u)$ and $Q^\epsilon(dy, x) = q^\epsilon(y; x)dy$ corresponding to the dynamics and the observation respectively. Denote by $\mathcal{M}_1(\mathcal{X})$ the space of probability measures on $\mathcal{X}$, provided with the weak topology. For $\mu \in \mathcal{M}_1(\mathcal{X})$ we introduce the probability measure $\rho^\epsilon_\mu(dy; u)$ on $\mathbb{R}^d$ given by

$$\rho^\epsilon_\mu(A; u) = \int_A \left[\int_{\mathcal{X}} \int_{\mathcal{X}} e^{\epsilon^{-1} c(x', u)} q^\epsilon(y; x) \Pi^\epsilon(dx; x', u) \mu(dx') \right] dy. \tag{3.7}$$

We also recall the notation

$$P^\epsilon(dx; z) = T^\epsilon(y_n, u_{n-1}) \cdots T^\epsilon(y_1, u_0) \delta_\xi$$

where $z = (y_n, u_{n-1}, \dots, y_1, u_0, \emptyset, \emptyset, \dots) \in \mathcal{Z}$.

Theorem 3 *Under assumptions H1, H3, H4, H6, H7, there exists a solution $(V^\epsilon, \lambda^\epsilon) \in \mathcal{C}(\mathcal{M}_1(\mathcal{X})) \times \mathbb{R}$ of the equation*

$$V^\epsilon(\mu) + \lambda^\epsilon = \inf_{u \in U} \left[\epsilon \log \int_{\mathbb{R}^d} e^{\epsilon^{-1} V^\epsilon(T^\epsilon(y, u)\mu)} \rho^\epsilon_\mu(dy; u) \right] \tag{3.8}$$

such that V^ϵ is bounded, continuous and $e^{\epsilon^{-1} V^\epsilon}$ is concave. We also have:

$$\lambda^\epsilon = \inf_{u \in Ad(U)} J^\epsilon(u). \tag{3.9}$$

Moreover, there exists a feedback $u^* = u^*(\mu)$ at which the infimum in (3.8) is attained. Finally, the control $u = (u_n)_{n \geq 0}$ given by

$$u_n = u_n(y_1, \dots, y_n, u_0, \dots, u_{n-1}) = u^*(P^\epsilon(dx; z))$$

with $z = (y_n, u_{n-1}, \dots, y_1, u_0, \emptyset, \emptyset, \dots)$ is an optimal control for the risk-sensitive control problem (2.1)-(2.2).

Theorem 3 is just a reformulation, in slightly different terms, of Theorem 2 in [9]. It is not hard to check that our assumptions imply those in [9]. Perhaps not totally obvious, it is to check that the following condition, required in [9], is actually implied by H1, H3, H4, H6, H7: if $\mu_n \to \mu$ weakly in $\mathcal{M}_1(\mathcal{X})$, then

$$\int \Pi^\epsilon(\cdot; x, u) \mu_n(dx) \to \int \Pi^\epsilon(\cdot; x, u) \mu(dx)$$

in total variation, uniformly in $u \in U$. This is equivalent to

$$\lim_{n \to \infty} \sup_{\substack{A \subset \mathcal{X} \\ u \in U}} \left| \int \Pi^\epsilon(A; x, u) \mu_n(dx) - \int \Pi^\epsilon(A; x, u) \mu(dx) \right| = 0. \tag{3.10}$$

If (3.10) fails to hold, then there exist $\delta > 0$, a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ and corresponding sequences $A_k \subset \mathcal{X}$, $u_k \in U$ such that

$$\left| \int \Pi^\epsilon(A_k; x, u_k) \mu_{n_k}(dx) - \int \Pi^\epsilon(A_k; x, u_k) \mu(dx) \right| \geq \delta \tag{3.11}$$

for all $k \geq 0$. Using assumption H3, it is easy to see that the functions $\{g_k\}$ from $\mathcal{X}$ to $[0, 1]$ given by

$$g_k(x) = \Pi^\epsilon(A_k; x, u_k)$$

form an equicontinuous family. By the Theorem of Ascoli-Arzela, there exists a limit point g of $\{g_k\}$ in the topology of uniform convergence on compact subsets of $\mathcal{X}$. Therefore, we can assume that the subsequence $\{\mu_{n_k}\}$ previously chosen, is such that the corresponding sequence $\{g_k\}$ converges to g. By using exponential tightness of $\{\mu_{n_k}\}$, it follows that

$$\int \Pi^\epsilon(A_k; x, u_k) \mu_{n_k}(dx) \to \int g(x) \mu(dx)$$

$$\int \Pi^\epsilon(A_k; x, u_k) \mu(dx) \to \int g(x) \mu(dx)$$

which contradicts (3.11).

4 Proofs

In this section we prove the results stated in Section 2. Throughout this section we assume that for each $\epsilon > 0$, a model as in (2.1) and (2.2), which satisfies assumptions H1-H7, is given.

Proof of Proposition 2.8
Let, for each $z \in \mathcal{Z}$:

$$W^\epsilon(z) = V^\epsilon\left(P^\epsilon(\cdot; z)\right), \tag{4.1}$$

where V^ϵ is the function given by Theorem 3 and $P^\epsilon(\cdot; z)$ is the probability measure defined by equation (2.8). Since V^ϵ is defined up to an additive constant, we may assume $V^\epsilon(\delta_\xi) = 0$ for all $\epsilon > 0$, i.e. $W^\epsilon(\emptyset, \emptyset, \ldots) = 0$ for all $\epsilon > 0$. Moreover let λ^ϵ be the constant given, again, by Theorem 3.

From the boundedness of the maps V^ϵ, we have that also the maps W^ϵ are bounded. Continuity of the maps W^ϵ follows from continuity of V^ϵ and weak continuity of the map $z \to P^\epsilon(\cdot; z)$. This last fact can be proved as follows. Suppose $z_k \to z$. If $n(z) = 0$, then $n(z_k) = 0$ for k large enough, so obviously $P^\epsilon(\cdot; z_k) \equiv P^\epsilon(\cdot; z)$. Otherwise we proceed by induction. Suppose z is of the form $z = (y, u, \Delta z)$ and assume $z_k \to z$. Then, for k sufficiently large, $z_k = (y_k, u_k, \Delta z_k)$, with $y_k \to y$, $u_k \to u$ and $\Delta z_k \to \Delta z$. From (2.7) and (2.8), it follows that all we have to show is that for all $f : X \to \mathbb{R}$ bounded and continuous

$$\lim_{k\to\infty} \int_X e^{\epsilon^{-1} c(x', u_k)} [\int_X f(x) q^\epsilon(y_k; x) \Pi^\epsilon(dx; x', u_k)] P^\epsilon(dx'; \Delta z_k) =$$

$$= \int_X e^{\epsilon^{-1} c(x', u)} [\int_X f(x) q^\epsilon(y; x) \Pi^\epsilon(dx; x', u)] P^\epsilon(dx'; \Delta z). \tag{4.2}$$

By inductive assumption and the fact that $c(x', u_k) \to c(x', u)$ uniformly, it is enough to show that

$$\int_X f(x) q^\epsilon(y_k; x) \Pi^\epsilon(dx; x', u_k) \to \int_X f(x) q^\epsilon(y; x) \Pi^\epsilon(dx; x', u) \tag{4.3}$$

uniformly on x' in any compact subset of $\mathcal{X}$, and that maps in the l.h.s. of (4.3) are uniformly bounded in $x' \in \mathcal{X}$. These last two facts follow readily from weak continuity of $\Pi^\epsilon(dx; x', u)$ and Assumption H4.

Now, notice that:

$$W^\epsilon(y, u, z) = V^\epsilon\left(P^\epsilon(\cdot; y, u, z)\right) = V^\epsilon\left(T^\epsilon(y, u) P^\epsilon(\cdot; z)\right),$$

$$R^\epsilon(A, z, u) = \rho^\epsilon_{P^\epsilon(\cdot; z)}(A, u)$$

(see (3.7) for the definition of ρ^ϵ_μ). From the previous equalities it is clear that equations (2.11), (2.12) and the existence on an optimal feedback law follow directly from equations (3.8), (3.9) and the existence of an optimal strategy given in Theorem 3. ∎

Now we will prove that the set of functions $\{W^\epsilon(\cdot)\}_{\epsilon>0}$ has limit points. To see this we will show that for any sequence ϵ_n, with $\epsilon_n \to 0$ as $n \to +\infty$, the maps $\{W^{\epsilon_n}(\cdot)\}$ are equibounded and equicontinuous. The conclusion then will follow by Ascoli-Arzela's Theorem.

For each $(y, u) \in Y \times U$ and each $\epsilon > 0$, let $\tilde{T}^\epsilon(y, u)$ be the operator acting on finite measures of $\mathcal{X}$ defined by:

$$\tilde{T}^\epsilon(y, u)(\mu)(A) = \int_{\mathcal{X}} e^{\epsilon^{-1} c(x', u)} \left[\int_A q^\epsilon(y; x) \Pi^\epsilon(dx; x', u)\right] \mu(dx'). \tag{4.4}$$

Notice that $\tilde{T}^\epsilon(y, u)$ is an unnormalized version of $T^\epsilon(y, u)$; in fact, if μ is a probability measure on $\mathcal{X}$ then:

$$T^\epsilon(y, u)(\mu)(A) = \frac{\tilde{T}^\epsilon(y, u)(\mu)(A)}{\tilde{T}^\epsilon(y, u)(\mu)(\mathcal{X})}.$$

Denote by $bB(M_1(\mathcal{X}))$ (resp. $bB(M(\mathcal{X}))$) the set of measurable, bounded and positive-real valued functions acting on probability (resp. finite) measure of $\mathcal{X}$. Define:

$$\begin{cases} S : bB(M_1(\mathcal{X})) \to bB(M(\mathcal{X})) \\ (Sf)(\mu) = \mu(X) f\left(\frac{\mu}{\mu(\mathcal{X})}\right) \end{cases}. \tag{4.5}$$

Lemma 2 of [9] establishes the following fact:

Fact 1 If $f \in bB(M_1(\mathcal{X}))$ is concave then also $(Sf) \in bB(M(\mathcal{X}))$ is concave.

Next proposition shows that the set of functions $\{W^\epsilon(\cdot)\}_{\epsilon>0}$ is equibounded.

Proposition 4.1 *Assume that for each $\epsilon > 0$, we are given a model as in (2.1) and (2.2) which satisfies assumptions H1-H7. Let $\{W^\epsilon(\cdot)\}_{\epsilon>0}$ be the set of functions that solve equation (2.11). Then there exists $\Lambda > 0$ such that:*

$$|W^\epsilon(z)| < \Lambda \qquad \forall \epsilon > 0,\ \forall z \in \mathcal{Z}. \tag{4.6}$$

Proof. Since the functions $\{W^\epsilon(\cdot)\}_{\epsilon>0}$ have a common zero ($W^\epsilon(\emptyset, \emptyset, \ldots) = 0,\ \forall \epsilon > 0$) to get equation (4.6) it suffices to prove that

$$|W^\epsilon(z) - W^\epsilon(z')| \le \Lambda \quad \forall \epsilon > 0,\ \forall z, z' \in \mathcal{Z}. \tag{4.7}$$

Let $M > 0$ be the uniform bound for the functions $\phi_x^{\epsilon,A}(u)$ given by Assumption H3. In particular it holds that:

$$\Pi^\epsilon(A; x, u) \ge e^{-\epsilon^{-1}M}\Pi^\epsilon(A; x', u'), \tag{4.8}$$

for all $x,\ x' \in \mathcal{X}$, $A \subset \mathcal{X}$ measurable, and $u,\ u' \in U$.

Given $x \in \mathcal{X}$ and $u \in U$, the following simple fact is easily established:

$$e^{\epsilon^{-1}c(x,u)} \ge \inf_{\xi\in\mathcal{X}} e^{\epsilon^{-1}c(\xi,u)} \ge e^{-2\epsilon^{-1}||c||_\infty} \sup_{\xi\in\mathcal{X}} e^{\epsilon^{-1}c(\xi,u)}. \tag{4.9}$$

Moreover by (4.8), we have, for every $A \subset \mathcal{X}$ measurable and $(y, u) \in Y \times U$,

$$\int_A q^\epsilon(y; x')\Pi^\epsilon(dx'; x, u) \ge \inf_{\xi\in\mathcal{X}} \int_A q^\epsilon(y; x')\Pi^\epsilon(dx'; \xi, u) \ge e^{-\epsilon^{-1}M} \sup_{\xi\in\mathcal{X}} \int_A q^\epsilon(y; x')\Pi^\epsilon(dx'; \xi, u). \tag{4.10}$$

From (4.9) and (4.10), for $z, z' \in \mathcal{Z}$, $A \subset \mathcal{X}$ measurable, and $(y, u) \in Y \times U$, we have:

$$\begin{gathered}
\tilde{T}^\epsilon(y, u)P^\epsilon(A; z) = \int_{\mathcal{X}} e^{\epsilon^{-1}c(x',u)} \left[\int_A q^\epsilon(y; x)\Pi^\epsilon(dx; x', u)\right] P^\epsilon(dx'; z) \ge \\
\left(\inf_{\xi\in\mathcal{X}} e^{\epsilon^{-1}c(\xi,u)}\right) \left(\inf_{\xi\in\mathcal{X}} \int_A q^\epsilon(y; x)\Pi^\epsilon(dx; \xi, u)\right) \ge \\
e^{-\epsilon^{-1}(2||c||_\infty+M)} \left(\sup_{\xi\in\mathcal{X}} e^{\epsilon^{-1}c(\xi,u)}\right) \left(\sup_{\xi\in\mathcal{X}} \int_A q^\epsilon(y; x)\Pi^\epsilon(dx; \xi, u)\right) \ge \\
e^{-\epsilon^{-1}(2||c||_\infty+M)} \int_{\mathcal{X}} e^{\epsilon^{-1}c(x',u)} \left[\int_A q^\epsilon(y; x)\Pi^\epsilon(dx; x', u)\right] P^\epsilon(dx'; z') = e^{-\epsilon^{-1}(2||c||_\infty+M)}\tilde{T}^\epsilon(y, u)P^\epsilon(A; z').
\end{gathered}$$

Let $\Lambda = 2||c||_\infty + M$, and consider the measure m^ϵ on $\mathcal{X}$ defined by:

$$m^\epsilon(A; z, z', y, u) = \tilde{T}^\epsilon(y, u)P^\epsilon(A; z) - e^{-\epsilon^{-1}\Lambda}\tilde{T}^\epsilon(y, u)P^\epsilon(A; z').$$

From the previous inequalities, we have that m^ϵ is positive for all $z, z' \in \mathcal{Z}$ and all $(y, u) \in Y \times U$. Therefore $\tilde{T}^\epsilon(y, u)P^\epsilon(\cdot; z)$ may be written as convex combination of positive measures as follows:

$$\tilde{T}^\epsilon(y, u)P^\epsilon(\cdot; z) = e^{-\epsilon^{-1}\Lambda}\tilde{T}^\epsilon(y, u)P^\epsilon(\cdot; z') + \left(1 - e^{-\epsilon^{-1}\Lambda}\right) \frac{m^\epsilon(\cdot; z, z', y, u)}{1 - e^{\epsilon^{-1}\Lambda}}.$$

If S denote the operator defined by (4.5), then since $e^{\epsilon^{-1}V^\epsilon}$ is concave (see Theorem 3), Fact 1 implies that also the maps $S\left(e^{\epsilon^{-1}V^\epsilon}\right)$ are concave. Thus:

$$\begin{aligned}
&S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y, u)P^\epsilon(\cdot; z)\right) \ge \\
&e^{-\epsilon^{-1}\Lambda}S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y, u)P^\epsilon(\cdot; z')\right) + \left(1 - e^{-\epsilon^{-1}\Lambda}\right) S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tfrac{m^\epsilon(\cdot;z,z',y,u)}{1-e^{\epsilon^{-1}\Lambda}}\right) \ge \\
&e^{-\epsilon^{-1}\Lambda}S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y, u)P^\epsilon(\cdot; z')\right).
\end{aligned} \tag{4.11}$$

Now we are ready to prove (4.7). Using equation (2.11), we may write:

$$|W^\epsilon(z)-W^\epsilon(z')| = |W^\epsilon(z)+\lambda^\epsilon-(W^\epsilon(z')+\lambda^\epsilon)| \le \left|\sup_{u\in U} \epsilon\log\left(\frac{\int_{\mathbb{R}^d} e^{\epsilon^{-1}W^\epsilon(y,u,z)}R^\epsilon(dy;z,u)}{\int_{\mathbb{R}^d} e^{\epsilon^{-1}W^\epsilon(y,u,z')}R^\epsilon(dy;z',u)}\right)\right|.$$

Noticing that:

$$e^{\epsilon^{-1}W^\epsilon(y,u,z)}R^\epsilon(dy;z,u) = e^{\epsilon^{-1}V^\epsilon\left(\frac{\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)}{\tilde{T}^\epsilon(y,u)P^\epsilon(X;z)}\right)}\left[\tilde{T}^\epsilon(y,u)P^\epsilon(X;z)\right]dy = S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)\right)dy,$$

we may write:

$$|W^\epsilon(z)-W^\epsilon(z')| \le \sup_{u\in U}\left|\epsilon\log\frac{\int_{\mathbb{R}^d} S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)\right)dy}{\int_{\mathbb{R}^d} S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z')\right)dy}\right|.$$

Now, using equation (4.11), we have:

$$|W^\epsilon(z)-W^\epsilon(z')| \le \sup_{u\in U}\left|\epsilon\log\frac{\int_{\mathbb{R}^d} S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)\right)dy}{e^{-\epsilon^{-1}\Lambda}\int_{\mathbb{R}^d} S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)\right)dy}\right| = \Lambda.$$

So inequality (4.7) holds, as desired. ∎

Next lemma proves a continuity property of the probability measures $P^\epsilon(\cdot;z)$ which will be crucial for getting equicontinuity of the functions $\{W^\epsilon(\cdot)\}_{\epsilon>0}$.

Lemma 4.2 *Assume that for each $\epsilon>0$, we are given a model as in (2.1) and (2.2) which satisfies assumptions H1-H7. Then there exists a map $K:\mathcal{Z}\times\mathcal{Z}\to[0,+\infty]$ such that*

(a) for each $C\subset\mathcal{Z}$ compact, there is a map $O^\epsilon_C: C\times C\to\mathbb{R}$ such that $O^\epsilon_C\to 0$ uniformly as $\epsilon\to 0$, and

$$P^\epsilon(\cdot;z_1)-e^{-\epsilon^{-1}[K(z_1,z_2)+O^\epsilon_C(z_1,z_2)]}P^\epsilon(\cdot;z_2)\ge 0 \tag{4.12}$$

for every $z_1,\ z_2\in C$;

(b) $\lim_{z\to\bar{z}}K(z,\bar{z})=0$, and $K(z,\bar{z})<+\infty$ if $n(z)=n(\bar{z})$.

Proof. Before proving (a) and (b) we state some useful inequalities which are implied by our Assumptions H1-H7. Assumption H7 implies, in particular, that there exists a map $K_c: U\times U\to\mathbb{R}$ such that $\lim_{u\to u'}K_c(u,u')=0$, and

$$c(x,u)-c(x,u')\ge -K_c(u,u'), \tag{4.13}$$

for all $x\in\mathcal{X}$ and all $u,u'\in U$.

Since the function $\phi^{\epsilon,A}_x(u)$ are equicontinuous (by Assumption H3) we have that there exists a function $K_\Pi: U\times U\to\mathbb{R}$ such that $\lim_{u\to u'}K_\Pi(u,u')=0$ and

$$\Pi^\epsilon(\cdot;x,u)\ge e^{-\epsilon^{-1}K_\Pi(u,u')}\Pi^\epsilon(\cdot;x,u'), \tag{4.14}$$

for all $x\in\mathcal{X}$ and all $u,u'\in U$.

From the equicontinuity of the maps $\epsilon\log\rho^\epsilon$ (Assumption H6) and from the continuity in y uniformly in x of the map $h^{-1}(x,y)$ (Assumption H4), we get that there exists a function $K_q:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, such that:

$$\rho^\epsilon\left(h^{-1}(x,y)\right)\ge e^{-\epsilon^{-1}K_q(y,y')}\rho^\epsilon\left(h^{-1}(x,y')\right),$$

with $\lim_{y\to y'}K_q(y,y')=0$.

Assumption H4 says, in particular, that for each $I\subset\mathbb{R}^d$ compact, the map $\log|\text{ Det } D_y h^{-1}(x,y)|$ is bounded on $\mathcal{X}\times I$. From this fact and the previous inequality we also have that for each compact set $I\subset\mathbb{R}^d$ there exists $M_I>0$ such that:

$$q^\epsilon(y;x)\ge e^{-\epsilon^{-1}\left(K_q(y,y')+\epsilon M_I\right)}q^\epsilon(y';x), \tag{4.15}$$

for all $x\in X$ and $y,y'\in I$.

We are now ready to construct the maps K and O^ϵ_C. Note that we are allowed to define $K(z_1,z_2)=+\infty$ and $O^\epsilon_C(z_1,z_2)=0$ for $n(z_1)\neq n(z_2)$, and any compact C containing $z_1,\ z_2$. For $n=n(z_1)=n(z_2)=0$, let $K(z_1,z_2)=0$. Inductively, we define, for $n=n(z_1)=n(z_2)>0$,

$$K(z_1,z_2)=2\left[K(\Delta z_1,\Delta z_2)+K_\Pi(u_1,u_2)+K_c(u_1,u_2)+K_q(y_1,y_2)\right]. \tag{4.16}$$

It is easily shown by induction that statement (b) holds for this $K(\cdot,\cdot)$. Now let $C\subset\mathcal{Z}$ compact. By the properties of the metric on $\mathcal{Z}$, it easily follows that there exists a compact set $I_C\subset\mathbb{R}$ such that for every $z=(z^{(1)},z^{(2)},\cdots)\in C$ and any $n\geq 1$, either $z^{(n)}=\emptyset$ or $z^{(n)}=(y,u)$ with $y\in I_C$. Let M_{I_C} be the corresponding constant in equation (4.15). We inductively define $O^\epsilon_C(z_1,z_2)$ for $z_1,\ z_2\in C$, $n=n(z_1)=n(z_2)$ by:

$$O^\epsilon_C(z_1,z_2)=0,$$

for $n=0$, and, for $n>0$:

$$O^\epsilon_C(z_1,z_2)=2O^\epsilon_C(\Delta z_1,\Delta z_2)+2\epsilon M_{I_C}. \tag{4.17}$$

Since there exists $\bar{n}\geq 0$ such that if $z\in C$ then $n(z)\leq\bar{n}$ (see Remark 2.7 (b)), it is easy to prove that $O^\epsilon_C\to 0$ uniformly as $\epsilon\to 0$.

We now show that (4.12) holds. For $n(z_1)\neq n(z_2)$, (4.12) is obvious. So we may prove (4.12) by induction on $n=n(z_1)=n(z_2)$. For $n=0$ there is nothing to prove. For $n>0$ we have:

$$P^\epsilon(A;z_1)=T^\epsilon(y_1,u_1)P^\epsilon(A;\Delta z_1)=\frac{\int_{\mathcal{X}}e^{\epsilon^{-1}c(x',u_1)}\left[\int_A q^\epsilon(y_1;x)\Pi^\epsilon(dx;x',u_1)\right]P^\epsilon(dx';\Delta z_1)}{\int_{\mathcal{X}}e^{\epsilon^{-1}c(x',u_1)}\left[\int_X q^\epsilon(y_1;x)\Pi^\epsilon(dx;x',u_1)\right]P^\epsilon(dx';\Delta z_1)}. \tag{4.18}$$

By inductive assumption the above expression is greater or equal than:

$$e^{-2e^{-1}[K(\Delta z_1,\Delta z_2)+O^\epsilon_C(\Delta z_1,\Delta z_2)]}\frac{\int_{\mathcal{X}}e^{\epsilon^{-1}c(x',u_1)}\left[\int_A q^\epsilon(y_1;x)\Pi^\epsilon(dx;x',u_1)\right]P^\epsilon(dx';\Delta z_2)}{\int_{\mathcal{X}}e^{\epsilon^{-1}c(x',u_1)}\left[\int_X q^\epsilon(y_1;x)\Pi^\epsilon(dx;x',u_1)\right]P^\epsilon(dx';\Delta z_2)}. \tag{4.19}$$

Then by (4.13), (4.14), and (4.15),

$$P^\epsilon(A;z_1)\geq e^{-2\epsilon^{-1}\left[K(\Delta z_1,\Delta z_2)+K_\Pi(u_1,u_2)+K_C(u_1,u_2)+K_q(y_1,y_2)+O^\epsilon_C(\Delta z_1,\Delta z_2)+\epsilon M_{I_C}\right]}P^\epsilon(A;z_2),$$

that, by (4.16) and (4.17), completes the proof. ∎

Proposition 4.3 *Assume that for each $\epsilon>0$, we are given a model as in (2.1) and (2.2) which satisfies assumptions H1-H7. Let $\{W^\epsilon(\cdot)\}_{\epsilon>0}$ be the set of functions that solve equation (2.11). Then, for any sequence $\epsilon_n\to 0$, as $n\to+\infty$, the set of function $\{W^{\epsilon_n}(\cdot)\}_{n>0}$ is equicontinuous.*

Proof. The idea of the proof is very similar to the one used to get equiboundedness of same set of functions (see Proposition 4.1). Fix any $C\subset\mathcal{Z}$ compact and any $z,\ z'\in\mathcal{Z}$. Then, by using Lemma 4.2, we have:

$$\tilde{T}^\epsilon(y,u)P^\epsilon(A;z)=\int_{\mathcal{X}}e^{\epsilon^{-1}c(x',u)}\left[\int_A q^\epsilon(y;x)\Pi^\epsilon(dx;x',u)\right]P^\epsilon(dx';z)\geq$$

$$e^{-\epsilon^{-1}\left(K(z,z')+O^\epsilon_C(z,z')\right)}\int_{\mathcal{X}}e^{\epsilon^{-1}c(x',u)}\left[\int_A q^\epsilon(y;x)\Pi^\epsilon(dx;x',u)\right]P^\epsilon(dx';z')=$$

$$=e^{-\epsilon^{-1}\left(K(z,z')+O^\epsilon_C(z,z')\right)}\tilde{T}^\epsilon(y,u)P^\epsilon(A;z')$$

From this, and using the same arguments used in Proposition 4.1, we have, for $z,\ z'\in C$:

$$|W^\epsilon(z)-W^\epsilon(z')|\leq\sup_{u\in U}\left|\epsilon\log\frac{\int_{\mathbb{R}^d}S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)\right)dy}{\int_{\mathbb{R}^d}S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z')\right)dy}\right|$$

$$\leq\sup_{u\in U}\left|\epsilon\log\frac{\int_{\mathbb{R}^d}S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)\right)dy}{e^{-\epsilon^{-1}\left(K(z,z')+O^\epsilon_C(z,z')\right)}\int_{\mathbb{R}^d}S\left(e^{\epsilon^{-1}V^\epsilon}\right)\left(\tilde{T}^\epsilon(y,u)P^\epsilon(\cdot;z)\right)dy}\right|=K(z,z')+O^\epsilon_C(z,z').$$

Thus we have

$$|W^\epsilon(z) - W^\epsilon(z')| \le K(z, z') + O^\epsilon_C(z, z'), \quad \forall\ z,\ z' \in C. \tag{4.20}$$

Next we show that continuity of the maps $\{W^\epsilon(\cdot)\}_{\epsilon>0}$ together with equation (4.20) implies equicontinuity of any sequence of functions $\{W^{\epsilon_n}(\cdot)\}_{n>0}$. Fix a sequence $\epsilon_n \to 0$. Let $\bar{z} \in \mathcal{Z}$, and $\delta > 0$ be arbitrary. It is possible to choose a compact neighborhood $V_{\bar{z}} \subset \mathcal{Z}$ of $\bar{z}$ such that:

$$\text{if } \ z \in V_{\bar{z}} \quad \Rightarrow \quad K(z, \bar{z}) < \frac{\delta}{2}.$$

Moreover, since $O^\epsilon_{V_{\bar{z}}}(\bar{z}, z)$ goes to zero uniformly on $V_{\bar{z}}$, there exists $\bar{n} > 0$ such that for all $z \in V_{\bar{z}}$, and for all ϵ_n with $n \ge \bar{n}$ we have:

$$|W^{\epsilon_n}(z) - W^{\epsilon_n}(\bar{z})| \le \delta. \tag{4.21}$$

Equicontinuity of the set $\{W^{\epsilon_n}(\cdot)\}_{n>0}$ follows from (4.21) by elementary arguments. ∎

Proposition 4.4 *Assume that for each $\epsilon > 0$, we are given a model as in (2.1) and (2.2) which satisfies assumptions H1-H7. Then the family $\{P^\epsilon(\cdot; z)\}$ of probability measures on $\mathcal{X}$, given by equation (2.8), is an exponentially tight WULDF with rate function $\tilde{H}(x; z)$ (see equation (2.16)).*

Proof. Since the maps $f : \mathcal{X} \times U \times W \to \mathcal{X}$ and $h : \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous (assumptions H1 and H4), by Proposition 3.4, we get that the families of probability measures $\Pi^\epsilon(dx; x', u)$ and $Q^\epsilon(dy; x)$ are exponentially tight WULDF with rate functions $H^S(x; x', u)$ (see (2.13)) and $H^O(y; x)$ (see (2.14)) respectively.

Remark 2.7 (b) implies, in particular, that to get our thesis, it suffices to prove that $P^\epsilon(\cdot; z)$ is an exponentially tight WULDF with rate function $\tilde{H}(x; z)$ when restricted to any $\mathcal{Z}_n = \{z \in \mathcal{Z} \,|\, n(z) \le n\}$. We will prove this by induction on n. For $n = 0$ the statement is true by definition (see (2.9) and (2.15)). Assume that it holds for $n \ge 0$, and let $z \in \mathcal{Z}_{n+1}$. Then $z = ((y, u), \Delta z)$ with $n(\Delta z) = n$, and

$$P^\epsilon(dx; z) = T(y, u) P^\epsilon(dx; \Delta z).$$

Since, by inductive assumption, the family $P^\epsilon(\cdot; \Delta z)$, with $\Delta z \in \mathcal{Z}_n$, is an exponentially tight WULDF with rate function $\tilde{H}(x; \Delta z)$, it is easy to see that the family of positive measures:

$$\alpha^\epsilon(dx; \Delta z, u) \ = \ e^{\epsilon^{-1} c(x,u)} P^\epsilon(dx; \Delta z), \qquad (\Delta z, u) \in \mathcal{Z}_n \times U,$$

is again an exponentially tight WULDF with rate function $H_\alpha(x; \Delta z, u) = \tilde{H}(x; \Delta z) - c(x, u)$.

Now we apply Proposition 3.6 (Composition) to the two families of measures $\Pi^\epsilon(dx; x', u)$ and $\alpha^\epsilon(dx; \Delta z, u)$. Notice that the Π^ϵ are families of probability measures. For the proof that their rate function $H^S(x; x', u)$ satisfy both requirements (i) and (ii) of Proposition 3.6, we refer to Proposition 4.3 of [1]. Thus we have that the family $\beta^\epsilon(dx', dx; \Delta z, u)$ defined by:

$$\int f(x', x) \beta^\epsilon(dx', dx; \Delta z, u) \ = \ \int \left[\int f(x', x) \Pi^\epsilon(dx; x', u) \right] \alpha^\epsilon(dx; \Delta z, u), \tag{4.22}$$

is an exponentially tight WULDF with rate function:

$$H_\beta(x', x; \Delta z, u) = H_\alpha(x; \Delta z, u) + H^S(x; x', u) = \tilde{H}(x; \Delta z) - c(x, u) + H^S(x; x', u).$$

Now we apply Proposition 3.7 (Contraction) to the family β^ϵ, to get that also the family:

$$\delta^\epsilon(A; \Delta z, u) \ = \ \int_{\mathcal{X} \times A} \beta^\epsilon(dx', dx; \Delta z, u) \ = \ \int_{\mathcal{X}} e^{\epsilon^{-1} c(x', u)} \left[\int_A \Pi^\epsilon(dx; x', u) \right] P^\epsilon(\cdot; \Delta z), \quad (\Delta z, u) \in \mathcal{Z}_n \times U,$$

is an exponentially tight WULDF with rate function:

$$H_\delta(x; \Delta z, u) \ = \ \inf_{x' \in \mathcal{X}} H_\beta(x', x; \Delta z, u) \ = \ \inf_{x' \in \mathcal{X}} \left(\tilde{H}(x; \Delta z) - c(x, u) + H^S(x; x', u) \right).$$

Next step is to apply Proposition 3.8 (Conditioning) to the two families $\delta^\epsilon(dx; \Delta z, u)$ (defined above) and $Q^\epsilon(dy; x)$ (defined by (2.6)). Notice that $Q^\epsilon(dy; x)$ are all probability measures. Moreover it is easy to show that their rate function $H^O(y; x)$ is always finite and continuous by assumption H5. For the proof that assumptions

H4, H5, and H6 imply that also requirement 1,2,3 and 4 of Proposition 3.8 hold, we refer to Proposition 4.3 of [1]. Thus, we conclude that the family of measures:

$$\tilde{T}(y,u)P^{\epsilon}(A;\Delta z) \;=\; q^{\epsilon}(y;x)\delta^{\epsilon}(A;\Delta z,u) \;=\; \int_{\mathcal{X}} e^{\epsilon^{-1}c(x',u)}\left[\int_A q^{\epsilon}(y;x)\Pi^{\epsilon}(dx;x',u)\right]P^{\epsilon}(dx';\Delta z,u), \tag{4.23}$$

is an exponentially tight WULDF (for $(y,u,\Delta z)\in \mathbb{R}^d\times U\times \mathcal{Z}_n$) with rate function:

$$H^O(y;x)+\inf_{x'\in\mathcal{X}}\left(\tilde{H}(x;\Delta z)-c(x,u)+H^S(x;x',u)\right).$$

Now, since, for $z=((y,u),\Delta z)\in\mathcal{Z}_{n+1}$, we have:

$$P^{\epsilon}(dx;z) \;=\; T^{\epsilon}(y,u)P^{\epsilon}(dx;\Delta z) \;=\; \frac{\tilde{T}^{\epsilon}(y,u)P^{\epsilon}(dx;\Delta z)}{\tilde{T}^{\epsilon}(y,u)P^{\epsilon}(\mathcal{X};\Delta z)},$$

it is easy to see that also the family $P^{\epsilon}(dx;z)$ for $z\in\mathcal{Z}_{n+1}$ is an exponentially tight WULDF with rate function:

$$H^O(y;x)+\inf_{x'\in\mathcal{X}}\left(\tilde{H}(x;\Delta z)-c(x,u)+H^S(x;x',u)\right)-\inf_{x,x'\in\mathcal{X}}\left(H^O(y;x)+\tilde{H}(x;\Delta z)-c(x,u)+H^S(x;x',u)\right),$$

which coincides with the one in (2.16), as desired. ∎

Proposition 4.5 *Assume that for each $\epsilon>0$, we are given a model as in (2.1) and (2.2) which satisfies assumptions H1-H7. Then the family of measures on $\mathbb{R}^d$, $R^{\epsilon}(\cdot;z,u)$ given by equation (2.10), is an exponentially tight WULDF with rate function $\bar{H}(y;z,u)$ defined in (2.17).*

Proof. Let $\beta^{\epsilon}(dx',dx;z,u)$ be the parameterized family of measures given by (4.22). We have seen in the proof of Proposition 4.4 that $\beta^{\epsilon}(dx',dx;z,u)$ is an exponentially tight WULDF with rate function:

$$H_{\beta}(x',x;z,u) \;=\; \tilde{H}(x;z)-c(x,u)+H^S(x;x',u).$$

Now we apply Proposition 3.6 (Composition) to the two families of measures $\beta^{\epsilon}(dx',dx;z,u)$ and $Q^{\epsilon}(dy;x)$ (defined in (2.6)). Notice that $Q^{\epsilon}(dy;x)$ is a probability measure for all ϵ, x. Moreover, by assumption H5, the rate function $H^O(y;x)$ is always finite and continuous. Thus both requirements (i) and (ii) of Proposition 3.6 hold, and we obtain that the family of measures $\delta^{\epsilon}(dx',dx,dy;z,u)$ defined by:

$$\int f(x',x,y)\delta^{\epsilon}(dx',dx,dy;z,u) \;=\; \int\left[\int f(x',x,y)Q^{\epsilon}(dy,x)\right]\beta^{\epsilon}(dx',dx;z,u),$$

is an exponentially tight WULDF with rate function:

$$H_{\delta}(x',x,y;z,u) \;=\; H^O(y;x)+\tilde{H}(x;z)-c(x,u)+H^S(x;x',u).$$

To conclude our proof, it remains to apply Proposition 3.7 (Contraction) to this family. In fact:

$$R^{\epsilon}(A;z,u)=\int_A\left[\int_X\int_X \delta^{\epsilon}(dx',dx,dy;z,u)\right].$$

So, the family $R^{\epsilon}(dy;z,u)$ is an exponentially tight WULDF with rate function:

$$\inf_{x',x\in\mathcal{X}} H_{\delta}(x',x,y;z,u) \;=\; \inf_{x',x\in\mathcal{X}}\left[H^O(y;x)+\tilde{H}(x;z)-c(x,u)+H^S(x;x',u)\right].$$

Since this last function is exactly $\bar{H}(y;z,u)$, we are done. ∎

Proof of Theorem 1
Let, for each $\epsilon>0$, $W^{\epsilon},\lambda^{\epsilon}$ be, respectively, the map and the constant that satisfy equation (2.11), i.e.:

$$W^{\epsilon}(z)+\lambda^{\epsilon} \;=\; \inf_{u\in U}\left[\epsilon\log\int_{\mathbb{R}^d} e^{\epsilon^{-1}W^{\epsilon}(y,u,z)}R^{\epsilon}(dy;z,u)\right].$$

First notice that $\lambda^\epsilon = \inf_{u\in Ad(U)} J^\epsilon(u) \le ||c||_\infty$. Thus $\{\lambda^\epsilon : \epsilon > 0\}$ admits limit points. Now, applying Propositions 4.1 and 4.3, we have that for each sequence $\epsilon_n \to 0$ as $n \to +\infty$, the set W^{ϵ_n} is equibounded and equicontinuous, and therefore by Ascoli-Arzela's Theorem it admits limit points.

Denote by $(W(\cdot), \lambda)$ any limit point of $(W^\epsilon, \lambda^\epsilon)$. Since the family of measures $R^\epsilon(\cdot; z, u)$ is an exponentially tight WULDF with rate function $\bar{H}(y; z, u)$, using Lemma 3.5, we have:

$$\inf_{u\in U} \lim_{\epsilon\to 0} \left[\epsilon \log \int_{\mathbb{R}^d} e^{\epsilon^{-1} W^\epsilon(y,u,z)} R^\epsilon(dy; z, u)\right] = \inf_{u\in U} \sup_{y\in\mathbb{R}^d} [W(y, u, z) - \bar{H}(y; z, u)].$$

To conclude the proof, we need to interchange, in the left hand side of the previous equation, the two operations of inf and lim. To see that this is possible we refer to Lemma 4.6 of [1]. So, we have:

$$W(z) + \lambda = \lim_{\epsilon\to 0} \inf_{u\in U} \left[\epsilon \log \int_{\mathbb{R}^d} e^{\epsilon^{-1} W^\epsilon(y,u,z)} R^\epsilon(dy; z, u)\right] = \inf_{u\in U} \sup_{y\in\mathbb{R}^d} [W(y, u, z) - \bar{H}(y; z, u)],$$

as desired. ∎

Before proving Theorem 2, we need some preliminary lemmas, and a general result on totally observed dynamic games.

Lemma 4.6 *Let $\tilde{H}(x; z)$ be the function defined in (2.16). For all $n \ge 1$, if*

$$z_n = ((y_n, u_{n-1}), z_{n-1}) = ((y_n, u_{n-1}), (y_{n-1}, u_{n-2}), \ldots, (y_1, u_0), \emptyset, \emptyset, \ldots),$$

then we have:

$$\begin{aligned} \tilde{H}(x_n; y_n, u_{n-1}, z_{n-1}) &= \inf_{x_1,\ldots,x_{n-1}} \textstyle\sum_{k=0}^{n-1} \left[H^O(y_{k+1}; x_{k+1}) - c(x_k, u_k) + H^S(x_{k+1}; x_k, u_k)\right] \\ &- \inf_{x_1,\ldots,x_n} \textstyle\sum_{k=0}^{n-1} \left[H^O(y_{k+1}; x_{k+1}) - c(x_k, u_k) + H^S(x_{k+1}; x_k, u_k)\right], \end{aligned} \tag{4.24}$$

with $x_0 = \xi$.

Proof. We will prove equation (4.24) by induction on n. For $n = 1$, we have:

$$\begin{aligned} \tilde{H}(x_1; y_1, u_0, \emptyset, \emptyset, \ldots) &= H^O(y_1; x_1) + \inf_{x_0} \left[H^S(x_1; x_0, u_0) - c(x_0, u_0) + \tilde{H}(x_0; \emptyset, \emptyset, \ldots)\right] \\ &- \inf_{x_0,x_1} \left[H^O(y_1; x_1) + H^S(x_1; x_0, u_0) - c(x_0, u_0) + \tilde{H}(x_0; \emptyset, \emptyset, \ldots)\right] \end{aligned}$$

From this equality (4.24) easily follows, since, by definition of $\tilde{H}(x; \emptyset, \emptyset, \ldots)$ the infimum is obtained when $x_0 = \xi$ and $\tilde{H}(\xi; \emptyset, \emptyset, \ldots) = 0$. Assume that equation (4.24) holds for n. We have:

$$\begin{aligned} \tilde{H}(x_{n+1}; y_{n+1}, u_n, z_n) &= \inf_{x_n} \left[H^O(y_{n+1}; x_{n+1}) - c(x_n, u_n) + H^S(x_{n+1}; x_n, u_n) + \tilde{H}(x_n; z_n)\right] \\ &- \inf_{x_n,x_{n+1}} \left[H^O(y_{n+1}; x_{n+1}) - c(x_n, u_n) + H^S(x_{n+1}; x_n, u_n) + \tilde{H}(x_n; z_n)\right]. \end{aligned}$$

To simplify notation we let:

$$l(k) = \left[H^O(y_{k+1}; x_{k+1}) - c(x_k, u_k) + H^S(x_{k+1}; x_k, u_k)\right]$$

Using the inductive assumption, we get:

$$\begin{aligned} \tilde{H}(x_{n+1}; y_{n+1}, u_n, z_n) &= \inf_{x_n} \left\{ l(n) + \inf_{x_1,\ldots,x_{n-1}} \sum_{k=0}^{n-1} l(k) - \inf_{x_1,\ldots,x_n} \sum_{k=0}^{n-1} l(k) \right\} \\ &- \inf_{x_n,x_{n+1}} \left\{ l(n) + \inf_{x_1,\ldots,x_{n-1}} \sum_{k=0}^{n-1} l(k) - \inf_{x_1,\ldots,x_n} \sum_{k=0}^{n-1} l(k) \right\} = \\ &= \inf_{x_n} \left\{ l(n) + \inf_{x_1,\ldots,x_{n-1}} \sum_{k=0}^{n-1} l(k) \right\} - \inf_{x_n,x_{n+1}} \left\{ l(n) + \inf_{x_1,\ldots,x_{n-1}} \sum_{k=0}^{n-1} l(k) \right\}. \end{aligned}$$

Thus (4.24) holds also for $n + 1$, as desired. ∎

Lemma 4.7 *Let $\bar{H}(y;z,u)$ be the function defined in (2.17). If, for all $n \geq 1$,*

$$z_n = ((y_n,u_{n-1}),z_{n-1}) = ((y_n,u_{n-1}),(y_{n-1},u_{n-2}),\ldots,(y_1,u_0),\emptyset,\emptyset,\ldots),$$

then we have:

$$\sum_{k=0}^{n-1} \bar{H}(y_{k+1};z_k,u_k) \;=\; \inf_{x_1,\ldots,x_n} \sum_{k=0}^{n-1} \left[H^O(y_{k+1};x_{k+1}) - c(x_k,u_k) + H^S(x_{k+1};x_k,u_k)\right], \tag{4.25}$$

with $x_0 = \xi$.

Proof. We will prove (4.25) by induction on n. The case $n=1$ is trivial. For $n>1$, we have:

$$\sum_{k=0}^{n} \bar{H}(y_{k+1};z_k,u_k) \;=\; \bar{H}(y_{n+1};z_n,u_n) + \sum_{k=0}^{n-1} \bar{H}(y_{k+1};z_k,u_k) =$$

(using the inductive assumption and the definition of $\bar{H}$)

$$= \inf_{x_{n+1},x_n} \left[H^O(y_{n+1};x_{n+1}) - c(x_n,u_n) + H^S(x_{n+1};x_n,u_n) + \tilde{H}(x_n;z_n)\right] +$$

$$\inf_{x_1,\ldots,x_n} \sum_{k=0}^{n-1} \left[H^O(y_{k+1};x_{k+1}) - c(x_k,u_k) + H^S(x_{k+1};x_k,u_k)\right].$$

Now by (4.24),

$$\begin{aligned}
\textstyle\sum_{k=0}^{n} \bar{H}(y_{k+1};z_k,u_k) \;&=\; \inf_{x_{n+1},x_n} \Big\{ H^O(y_{n+1};x_{n+1}) - c(x_n,u_n) + H^S(x_{n+1};x_n,u_n) \\
&+\; \inf_{x_1,\ldots,x_{n-1}} \textstyle\sum_{k=0}^{n-1} \left[H^O(y_{k+1};x_{k+1}) - c(x_k,u_k) + H^S(x_{k+1};x_k,u_k)\right] \\
&-\; \inf_{x_1,\ldots,x_n} \textstyle\sum_{k=0}^{n-1} \left[H^O(y_{k+1};x_{k+1}) - c(x_k,u_k) + H^S(x_{k+1};x_k,u_k)\right] \Big\} \\
&+\; \inf_{x_1,\ldots,x_n} \textstyle\sum_{k=0}^{n-1} \left[H^O(y_{k+1};x_{k+1}) - c(x_k,u_k) + H^S(x_{k+1};x_k,u_k)\right] =
\end{aligned}$$

$$= \inf_{x_1,\ldots,x_{n+1}} \sum_{k=0}^{n} \left[H^O(y_{k+1};x_{k+1}) - c(x_k,u_k) + H^S(x_{k+1};x_k,u_k)\right]$$

∎

Now we present a general result on totally observed deterministic dynamic games. Assume we are given $\mathcal{X}$, W metric spaces, U a compact metric space, and a discrete-time model, whose dynamics are described by:

$$\begin{cases} x_{n+1} \;=\; l(x_n,u_n,w_n), \\ x_0 \;=\; \xi \in \mathcal{X}, \end{cases} \tag{4.26}$$

where $x_n \in \mathcal{X}$, $u_n \in U$, and $w_n \in W$ and l is a fixed continuous function. In this case, admissible controls are those such that u_n is a function of the states up to time n. Again, we denote by $Ad(U)$ the set of admissible controls; our aim is to minimize over $Ad(U)$ the following performance index:

$$J(u) \;=\; \limsup_{n\to+\infty} \frac{1}{n} \sup_{w_0,\ldots,w_{n-1}} \sum_{k=0}^{n-1} h(x_k,u_k,w_k), \tag{4.27}$$

where h is a given continuous map.

Theorem 4 *Assume we are given a model as in equation (4.26) with performance index as in (4.27). Moreover, assume that there exists a constant $\lambda \in \mathbb{R}$, and a continuous bounded function $W : \mathcal{X} \to \mathbb{R}$ which satisfy the equation:*

$$W(x) + \lambda \;=\; \inf_{u\in U} \sup_{w\in W} \left\{W(l(x,u,w)) + h(x,u,w)\right\}, \tag{4.28}$$

for all $x \in \mathcal{X}$. Then for all $\bar{u} \in Ad(U)$, we have

$$\lambda \leq J(\bar{u}).$$

Moreover there exists a feedback $u_* = u_*(x)$ which realizes the infimum in (4.28). This feedback provides and optimal control for the given dynamic game, in the same sense as in Proposition 2.8.

Proof. Fix a control $\bar{u} = (u_0, u_1, \ldots) \in Ad(U)$. We first prove, by induction on n, that

$$\lambda \leq \frac{1}{n} \sup_{w_0,\ldots,w_{n-1}} \left\{ \sum_{k=0}^{n-1} h(x_k, u_k, w_k) + W(x_n) \right\}. \tag{4.29}$$

Using (4.28) and the fact that it is not restrictive to assume $W(x_0) = W(\xi) = 0$, we have:

$$\lambda = \inf_u \sup_{w_0} \{W(l(x_0, u, w_0)) + h(x_0, u, w_0)\} \leq \sup_{w_0} \{W(x_1) + h(x_0, u_0, w_0).\}$$

So (4.29) holds for $n = 1$. Now observe that, by (4.28), for $n > 0$, we have:

$$W(x_n) \leq \left[\sup_{w_n} \{W(x_{n+1}) + h(x_n, u_n, w_n)\} - \lambda \right],$$

that, together with the inductive assumptions, yields

$$\lambda \leq \frac{1}{n} \sup_{w_0,\ldots,w_{n-1}} \left\{ \sum_{k=0}^{n-1} h(x_k, u_k, w_k) + W(x_n) \right\}$$

$$\leq \frac{1}{n} \sup_{w_0,\ldots,w_{n-1}} \left\{ \sum_{k=0}^{n-1} h(x_k, u_k, w_k) + \left[\sup_{w_n} \{W(x_{n+1}) + h(x_n, u_n, w_n)\} - \lambda \right] \right\}$$

$$= \frac{1}{n} \left[\sup_{w_0,\ldots,w_n} \left\{ \sum_{k=0}^{n} h(x_k, u_k, w_k) + W(x_{n+1}) \right\} - \lambda \right].$$

Thus we have obtained (4.29) for $n + 1$.

Since W is bounded, there exists a constant $M > 0$ such that $W(x) \leq M$ for all $x \in \mathcal{X}$. Thus, by (4.29), we get:

$$\lambda \leq \frac{1}{n} \sup_{w_0,\ldots,w_{n-1}} \left\{ \sum_{k=0}^{n-1} h(x_k, u_k, w_k) \right\} + \frac{M}{n}, \qquad \forall n \geq 1,$$

and therefore

$$\lambda \leq \limsup_{n\to+\infty} \left[\frac{1}{n} \sup_{w_0,\ldots,w_{n-1}} \left\{ \sum_{k=0}^{n-1} h(x_k, u_k, w_k) \right\} + \frac{M}{n} \right] = J(\bar{u}).$$

The second part of this theorem (i.e. the existence of an optimal feedback) is proved similarly. First one uses the compactness assumption on U to find the value $u_* = u_*(x)$ which realizes the infimum in (4.28). Then, it is possible to repeat all the previous steps changing all the inequalities into equalities, and thus obtaining optimality of the feedback. ∎

Proof of Theorem 2

The idea of this proof is to find a totally-observed dynamic game which is equivalent to the one given by equations (2.19) and (2.20), and then to use Theorem 4. Assume given the dynamic game

$$\begin{cases} x_{n+1} &= f(x_n, u_n, w_n), \\ y_n &= h(x_n, v_n), \end{cases}$$

together with initial conditions $x_0 = \xi \in \mathcal{X}$ and $y_0 = \eta \in \mathbb{R}^d$, and:

$$J(u) = \limsup_{n\to+\infty} \frac{1}{n} \sup_{\substack{w_0,\ldots,w_{n-1} \\ v_1,\ldots,v_n}} \left[\sum_{k=0}^{n-1} (c(x_k, u_k) - H(w_k) - K(v_{k+1})) \right].$$

To this partially observed dynamic game, we associate a totally observed one with state space $\mathcal{Z}$, control space U, disturbance space $\mathbb{R}^d$, with dynamics

$$\begin{cases} z_{n+1} = (y_{n+1}, u_n, z_n), \\ z_0 = (\emptyset, \emptyset, \ldots),\ y_1 = \eta, \end{cases} \tag{4.30}$$

and performance index:

$$\bar{J}(u) = \limsup_{n\to+\infty} \frac{1}{n} \sup_{y_1,\ldots,y_n} \left[\sum_{k=0}^{n-1} -\bar{H}(y_{k+1}; z_k, u_k)\right], \tag{4.31}$$

where $\bar{H}$ is the function defined in (2.17). If we prove that:

$$J(u) = \bar{J}(u), \tag{4.32}$$

for all sequences of controls $u \in Ad(U)$, then Theorem 2 follow. In fact, if $\lambda \in \mathbb{R}$ and $W \in C(\mathcal{Z})$ solve

$$W(z) + \lambda = \inf_{u\in U} \sup_{y\in\mathbb{R}^d} \Big\{W(y, u, z) - \bar{H}(y; z, u)\Big\},$$

then, by Theorem 4, we get $\lambda = \inf_u \bar{J}(u) = \inf_u J(u)$. Moreover there exists a feedback $u_* = u_*(z)$ which realizes the infimum in the previous equation and that is admissible for both models. This feedback provides and optimal control for both dynamic games defined above.

Thus, it remains to prove (4.32). By using (4.25) we have:

$$\sup_{y_1,\ldots,y_n} \left[\sum_{k=0}^{n-1} -\bar{H}(y_{k+1}; z_k, u_k)\right] =$$

$$= \sup_{y_1,\ldots,y_n} - \left\{ \inf_{x_1,\ldots,x_n} \sum_{k=0}^{n-1} \left[H^O(y_{k+1}; x_{k+1}) - c(x_k, u_k) + H^S(x_{k+1}; x_k, u_k)\right] \right\} =$$

$$= \sup_{y_1,\ldots,y_n} \sup_{x_1,\ldots,x_n} \sum_{k=0}^{n-1} \left[-H^O(y_{k+1}; x_{k+1}) + c(x_k, u_k) - H^S(x_{k+1}; x_k, u_k)\right]$$

Now, by using the definition of H^O and H^S we have:

$$\sup_{y_1,\ldots,y_n} \left[\sum_{k=0}^{n-1} -\bar{H}(y_{k+1}; z_k, u_k)\right] = \sup_{y_1,\ldots,y_n} \sup_{x_1,\ldots,x_n}$$

$$\sum_{k=0}^{n-1} \left\{ \sup_{v_{k+1}, w_k} \left[(c(x_k, u_k) - H(w_k) - K(v_{k+1})) \Big| f(x_k, u_k, w_k) = x_{k+1},\ h(x_{k+1}, v_{k+1}) = y_{k+1}\right] \right\}$$

From the previous equality, it is easy to get:

$$\sup_{y_1,\ldots,y_n} \left[\sum_{k=0}^{n-1} -\bar{H}(y_{k+1}; z_k, u_k)\right] = \sup_{\substack{w_0,\ldots,w_{n-1} \\ v_1,\ldots,v_n}} \left[\sum_{k=0}^{n-1} (c(x_k, u_k) - H(w_k) - K(v_{k+1}))\right],$$

from which (4.32) follows. ∎

5 The completely observed case

Assumptions H4, H5 and H6, that we have assumed throughout this paper, clearly do not cover the completely observed case $h(x, v) = x$. However by using the results in [8] on the existence of the value function for totally observed models, and by making straightforward modifications to the proofs above (for details see [17]), the following theorem can be established.

Theorem 5 *Assume that for each $\epsilon > 0$ we are given a model as in (2.1) and (2.2), with the output function $h(x, v) = x$. Suppose also that assumptions H1, H2, H3, H7 are satisfied. Then for each $\epsilon > 0$*

i) there exist a bounded continuous function $V^\epsilon : \mathcal{X} \to \mathbb{R}$ and a real number l^ϵ such that the equation

$$V^\epsilon(x) + l^\epsilon = \inf_{u\in U} \left[c(x, u) + \epsilon \log \int e^{\epsilon^{-1} V^\epsilon(f(x,u,w))} \mu^\epsilon(dw)\right]. \tag{5.1}$$

The infimum in (5.1) is attained and gives rise to a feedback optimal control, l^ϵ representing the corresponding optimal cost.

ii) The family $\{V^\epsilon, l^\epsilon : \epsilon > 0\}$ admits limit points (V, l), each one being a solution of the equation

$$V(x) + l = \inf_{\mathbf{u} \in U} \sup_{w \in W} [c(x, u) + V(f(x, u, w)) - H(w)]. \tag{5.2}$$

The infimum in (5.2) is attained and gives rise to a feedback optimal control for the deterministic dynamic game

$$x_{n+1} = f(x_n, u_n, w_n)$$

$$J(u) = \limsup_{n \to +\infty} \frac{1}{n} \sup_{w_0, \ldots, w_{n-1}} \left[\sum_{k=0}^{n-1} (c(x_k, u_k) - H(w_k)) \right],$$

for which l is the optimal cost.

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