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LINEAR ALGEBRA AND ITS APPLICATIONS

The algebraic Riccati inequality: parametrization of solutions, tightest local frames and generalized feedback matrices

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Abstract

Extending on previous work by Faurre, Scherer and Pavon, a parametrization of the symmetric solutions of the algebraic Riccati inequality is established. This is then applied to derive new results on tightest local frames, and on generalized feedback matrices that arise in stochastic realization theory. © 1999 Published by Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $Z(s) = H(sI - F)^{-1}G + J$ be a minimal realization of the $m \times m$ positive real matrix function Z. Assume that $F \in \mathbb{R}^{n\times n}$ is a stability matrix ($\sigma(F) \subset \mathbb{C}_-$) so that the $m \times m$, real-rational spectral density

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$$\Phi(s) := Z(s) + Z(-s)^{\mathrm{T}} \tag{1.1}$$

has McMillan degree equal to $2n$. Moreover, assume that $\Phi(s)$ is *coercive*, i.e., $\Phi(\mathrm{i}\omega) > cI, c > 0, \ \forall \omega \in \mathbb{R}$. It follows, in particular, that $R := \Phi(\infty) = J + J^{\mathrm{T}}$ is positive definite. The coercivity assumption is made to avoid obscuring results by technicalities. In Section 6, we show that the parametrization result of Section 2 may be suitably extended to spectra satisfying the weaker assumption that R be positive definite.

Consider the *Algebraic Riccati Inequality* (ARI)

$$\Lambda(P) := FP + PF^{\mathrm{T}} + (G - PH^{\mathrm{T}})R^{-1}(G^{\mathrm{T}} - HP) \leqslant 0. \tag{1.2}$$

The set $\mathscr{P}$ of symmetric solutions of the ARI has been intensively studied since the work of Anderson [1,2] and Faurre [6,7], see [18,25,15,13] and references therein. In particular, the parametrization problem was first addressed and partially solved by Faurre in [6]. This result is recalled at the beginning of Section 2. Other papers on the parametrization problem are [25] and [22]. Extending on the latter references, we provide in Section 2 what we believe to be the first *complete* and *bijective* description of the set $\mathscr{P}$. The latter allows us to derive in Section 3 new results on the *tightest local frames* [18] of different solutions. We then proceed in Section 4 to establish further results concerning *zero* and *generalized feedback matrices*. Section 5 is devoted to the motivation, in a stochastic realization framework, of the introduction of the above mentioned generalized feedback matrices. In Section 6, we deal with the noncoercive case where $\Phi(s)$ may have zeros on the imaginary axis. Finally, in Section 7, we briefly indicate a dual form of the results.

It is well known [2] that $\mathscr{P}$ is precisely the set of the symmetric solutions P of the *Linear Matrix Inequality* (LMI)

$$\begin{bmatrix} FP + PF^{\mathrm{T}} & PH^{\mathrm{T}} - G \\ HP - G^{\mathrm{T}} & -R \end{bmatrix} \leqslant 0 \tag{1.3}$$

or, equivalently, the set of the symmetric matrices P for which there exist the matrices B and D such that (P, B, D) solves the *Positive Real Lemma* equations

$$FP + PF^{\mathrm{T}} + BB^{\mathrm{T}} = 0, \tag{1.4a}$$

$$G = PH^{\mathrm{T}} + BD^{\mathrm{T}}, \tag{1.4b}$$

$$DD^{\mathrm{T}} = R. \tag{1.4c}$$

It is also well known [7] that the set $\mathscr{P}$ is a compact, convex subset of the cone of n-dimensional positive definite matrices. It admits a minimum element P_{-} and a maximum element P_{+} with respect to the natural partial order. These belong in fact to $\mathscr{P}_0 := \{P = P^{\mathrm{T}} : \Lambda(P) = 0\}$, the subset of $\mathscr{P}$ consisting of the symmetric solutions of the corresponding ARE. For $P \in \mathscr{P}$, let B_2 be a full column rank matrix such that $B_2 B_2^{\mathrm{T}} = -\Lambda(P)$ and $B_1 := (G - PH^{\mathrm{T}})R^{-1/2}$, where $R^{1/2}$ denotes the square root of R. A central result of system and control theory [2] is that

$$W(s) = H(sI - F)^{-1}[B_1 \mid B_2] + [R^{1/2} \mid 0] \tag{1.5}$$

is a minimal stable spectral factor of $\Phi(s)$, i.e., a (matrix valued) function, analytic on the right half plane and of least possible McMillan degree, such that

$$W(s)W(-s)^{\mathrm{T}} = \Phi(s). \tag{1.6}$$

Moreover, this is indeed a parametrization of minimal stable spectral factors of Φ, provided we identify spectral factors differing by multiplication on the right by a constant orthogonal matrix.

A large amount of literature has been produced on the parametrization of the set $\mathscr{P}_0$ of the solutions of ARE and on the connection of this problem with spectral factorization, system and control theory, optimal filtering and estimation, see [27,16,18,12] and references therein. We recall that for any $P \in \mathscr{P}_0$ the corresponding spectral factor

$$W_P(s) = H(sI - F)^{-1}B_P + R^{1/2} \tag{1.7}$$

with $B_P := (G - PH^{\mathrm{T}})R^{-1/2}$ is *square* (i.e., it is a square transfer matrix). Notice that a minimal realization of the inverse $W_P^{-1}(s)$ is given by

$$W_P^{-1}(s) = -R^{-1/2}H(sI - F + B_P R^{-1}H)^{-1}B_P R^{-1/2} + R^{-1/2}, \tag{1.8}$$

so that the zero dynamics of the spectral factor $W_P(s)$ (which plays a crucial role in estimation theory [18]), is governed by the Jordan structure of $\Gamma_P := F - B_P R^{-1}H$. For this reason, the matrix Γ_P is often referred to as the *zero matrix* corresponding to P. Observe also that Γ_P is obtained by state feedback from the pair (F, B_P), so that it is also named *feedback matrix* corresponding to P.

Particularly important in many applications such as LQ optimal control, Kalman filtering, network theory, and stochastic realization is the *minimum phase* spectral factor

$$W_-(s) = H(sI - F)^{-1}B_- + R^{1/2} \tag{1.9}$$

with $B_- := (G - P_- H^{\mathrm{T}})R^{-1/2}$, corresponding to the minimum element P_- of $\mathscr{P}$. Let $\Gamma_- := F - B_- R^{-1}H$ be the corresponding feedback matrix. Under the coercivity assumption, Γ_- is a stability matrix [7]. Moreover, it follows essentially from Jan Willems fundamental work [27] that the left-invariant subspaces of Γ_- (i.e., the invariant subspaces of Γ_-^{T}) parametrize the set of minimal stable square spectral factors. More precisely, to any Γ_-^{T}-invariant subspace M, there corresponds a minimal square stable spectral factor with zero matrix Γ_M such that

$$\Gamma_-^{\mathrm{T}}\big|_M = \Gamma_M^{\mathrm{T}}\big|_M. \tag{1.10}$$

Moreover, the map induced by Γ_M^{T} on the quotient space $\mathbb{R}^n/M$ is similar to the map induced by $-\Gamma_-^{\mathrm{T}}$ on the same quotient space, see [27,9].

At this point, it is important to observe that the parametrization of the solutions of ARE, or equivalently of the set of stable square spectral factors, may also be given in a coordinate-free setting by means of *inner divisors* of a certain "maximum" inner function. [2] In this way the problem becomes a particular case of the classical factorization problem which has attracted a great interest both in the mathematics and in the electrical engineering literature see, e.g., [26,24,4,11,10] and references therein. The latter approach extends naturally beyond the finite dimensional (rational) case [17] and has been recently employed by Fuhrmann and Gombani [13] to study the class of possibly nonsquare spectral factors. In this case a *rigid* function [3] can be naturally associated to any minimal spectral factor. A fundamental issue is then the minimal inner extension of such function. In Section 4, we describe some properties of this class of rigid functions, and we address the inner extension problem.

2. Parametrization of the set $\mathscr{P}$

As mentioned in Section 1, we recall that in [6] Faurre gave a parametrization of the subset

$$\bar{\mathscr{P}} := \{P \in \mathscr{P} : P - P_- > 0\} \subset \mathscr{P} \tag{2.1}$$

containing the elements of $\mathscr{P}$ which have positive distance from P_-. The nonparametrized subset, however, contains interesting solutions laying in the boundary of $\mathscr{P}$, such as all the solutions of the corresponding ARE $\Lambda(P) = 0$ except for the maximum one P_+. The same result may also be found in [7, p. 91].

Next, we seek to obtain a parametrization of the whole set $\mathscr{P}$, and consequently of the set of all minimal stable spectral factors (even nonsquare).

For $v = 0, 1, \ldots, n$, let $\mathscr{I}^v(\Gamma_-^{\mathrm{T}})$ denote the set of v-dimensional Γ_-^{T}-invariant subspaces of $\mathbb{R}^n$. For each subspace $M \subseteq \mathbb{R}^n$, we denote by Π_M the matrix

[2] A square matrix valued function is called inner if it is analytic on the right half plane and unitary on the imaginary axis, see, e.g., [11].

[3] A matrix valued function is called rigid if it is analytic on the right half plane and $Q(\mathrm{i}\omega)Q^*(\mathrm{i}\omega) = I$ for all $\omega \in \mathbb{R}$, see, e.g., [11].

representation of the orthogonal projection onto M with respect to the canonical basis of $\mathbb{R}^n$. Moreover, let V_M be the $n \times v$ matrix whose columns are obtained by *Gram–Schmidt orthonormalization* [19, pp. 54–55] of the columns of Π_M and U_M be the $n \times (n-v)$ matrix whose columns are obtained by Gram–Schmidt orthonormalization of the columns of $\Pi_{M^\perp}$. Hence,

$$T_M = [U_M \mid V_M] \tag{2.2}$$

is an orthogonal matrix uniquely associated to M. Let M be an invariant subspace of Γ_-^{T}. A change of basis induced by T_M carries the matrix Γ_- to the block triangular form:

$$\Gamma_-^M := T_M^{\mathrm{T}} \Gamma_- T_M = \begin{bmatrix} \Gamma_{-11}^M & \Gamma_{-12}^M \\ 0 & \Gamma_{-22}^M \end{bmatrix}. \tag{2.3}$$

Now set $H_{1M} := HU_M$, $H_{2M} := HV_M$ and

$$L_{ij}^M := H_{iM}^{\mathrm{T}} R^{-1} H_{jM}, \quad i,j = 1,2, \tag{2.4}$$

and consider the $(n-v)$-dimensional homogeneous ARE

$$\Gamma_{-11}^M Q + Q(\Gamma_{-11}^M)^{\mathrm{T}} + Q(D + L_{11}^M)Q = 0, \tag{2.5}$$

where $D = D^{\mathrm{T}} > 0$. The observability of the pair (F, H) implies that of (Γ_-, H). Consequently, the pair (Γ_{-11}^M, H_{1M}) is also observable. This, in turn, implies that $(\Gamma_{-11}^M, (D + L_{11}^M)^{1/2})$ is observable. Hence, Eq. (2.5) admits a unique positive definite solution.

The following result parametrizes the set $\mathscr{P}$ in terms of the parameter set

$$\begin{aligned}\mathscr{C} := \{(M, D) : M \in \mathscr{I}^v(\Gamma_-^{\mathrm{T}}),\ D \in \mathbb{R}^{(n-v)\times(n-v)},\ D = D^{\mathrm{T}} > 0, \\ v = 0, 1, \ldots, n\}.\end{aligned} \tag{2.6}$$

Theorem 2.1. *There exists a one-to-one correspondence between the set $\mathscr{P}$ and the set $\mathscr{C}$. Indeed, the map $\varphi : \mathscr{C} \to \mathscr{P}$ assigning to each $(M, D) \in \mathscr{C}$ the matrix*

$$\varphi[(M, D)] := P_- + T_M \begin{bmatrix} Q_D & 0 \\ 0 & 0 \end{bmatrix} T_M^{\mathrm{T}} \tag{2.7}$$

with T_M defined by (2.2) *and Q_D the unique positive definite solution of* (2.5) *is bijective.*

In order to prove this theorem, we first need to set the notation and establish some preliminary facts. For any $P \in \mathscr{P}$ define

$$Q_+(P) := P - P_-. \tag{2.8}$$

Moreover, denote by $\mathscr{Q}$ the set

$$\mathscr{Q} := \{Q_+(P) : P \in \mathscr{P}\} \tag{2.9}$$

obtained from $\mathscr{P}$ by translation (the minimum element of $\mathscr{Q}$ is clearly $Q_+(P_-) = 0$). Finally, denote by $\mathscr{Q}_0$ the subset of $\mathscr{Q}$ defined by

$$\mathscr{Q}_0 := \{Q_+(P) : P \in \mathscr{P}_0\}. \tag{2.10}$$

Let $P \in \mathscr{P}$. Then a standard calculation shows that $Q_+(P)$ is a solution of the matrix equation

$$\Gamma_- Q + Q\Gamma_-^{\mathrm{T}} + QH^{\mathrm{T}}R^{-1}HQ = \Lambda(P). \tag{2.11}$$

Let $\Delta^\sharp$ denote the Moore–Penrose pseudo-inverse of Δ. Extending on [25], it was observed in [22] that

$$\begin{aligned}\Lambda(P) &= Q_+(P)Q_+(P)^\sharp\Lambda(P) = \Lambda(P)Q_+(P)^\sharp Q_+(P)\\ &= Q_+(P)Q_+(P)^\sharp\Lambda(P)Q_+(P)^\sharp Q_+(P).\end{aligned} \tag{2.12}$$

The latter follows from the obvious inclusion

$$\ker Q_+(P) \subseteq \ker \Lambda(P). \tag{2.13}$$

Hence, $Q_+(P)$ may be viewed as a solution of the homogeneous ARE

$$\Gamma_- Q + Q\Gamma_-^{\mathrm{T}} + Q(H^{\mathrm{T}}R^{-1}H - Q^\sharp\Lambda(P)Q^\sharp)Q = 0. \tag{2.14}$$

We also have the following lemma, that may be proved by straightforward calculations.

Lemma 2.1. *Let $M = M^{\mathrm{T}} \geqslant 0$ and $Q = Q^{\mathrm{T}}$ be a symmetric solution of the homogeneous ARE*

$$\Gamma_- Q + Q\Gamma_-^{\mathrm{T}} + Q(H^{\mathrm{T}}R^{-1}H + M)Q = 0. \tag{2.15}$$

Then $Q \in \mathscr{Q}$, i.e., $P := Q + P_- \in \mathscr{P}$. Moreover, if $Q_0 = Q_0^{\mathrm{T}}$ is a symmetric solution of the homogeneous ARE

$$\Gamma_- Q + Q\Gamma_-^{\mathrm{T}} + QH^{\mathrm{T}}R^{-1}HQ = 0, \tag{2.16}$$

then $Q_0 \in \mathscr{Q}_0$, i.e., $P_0 := Q_0 + P_- \in \mathscr{P}_0$.

Proof of Theorem 2.1. Let $(M, D) \in \mathscr{C}$, T_M be defined by (2.2) and Q_D be the unique positive definite solution of (2.5). It is clear that

$$Q_D^e := \begin{bmatrix} Q_D & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

solves the equation

$$\Gamma_-^M Q + Q(\Gamma_-^M)^{\mathrm{T}} + Q\left(\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L_{11}^M & L_{12}^M \\ L_{21}^M & L_{22}^M \end{bmatrix}\right)Q = 0. \tag{2.17}$$

Equivalently, $T_M Q_D^e T_M^{\mathrm{T}}$ solves (2.15), where

$$M = T_M \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} T_M^{\mathrm{T}}.$$

Then, in view of Lemma 2.1, $P = P_- + T_M Q_D^e T_M^{\mathrm{T}}$ belongs to $\mathscr{P}$. We now prove that φ is injective. Let $(M_i, D_i) \in \mathscr{C}$, $i = 1,2$ and P_i, $i = 1,2$, be the corresponding elements of $\mathscr{P}$. It is easy to check that $\ker(P_i - P_-) = M_i$. Therefore, if $P_1 = P_2$, or equivalently $P_1 - P_- = P_2 - P_-$, then $M_1 = \ker(P_1 - P_-) = \ker(P_2 - P_-) = M_2$ and hence $T_{M_1} = T_{M_2}$. Therefore, if $P_1 = P_2$, the two equations

$$\Gamma_{-11}^{M_1} Q + Q(\Gamma_{-11}^{M_1})^{\mathrm{T}} + Q(D_1 + L_{11}^{M_1})Q = 0, \tag{2.18}$$

and

$$\Gamma_{-11}^{M_2} Q + Q(\Gamma_{-11}^{M_2})^{\mathrm{T}} + Q(D_2 + L_{11}^{M_2})Q = 0, \tag{2.19}$$

obtained specializing (2.5) to $M = M_1$ and $M = M_2$, respectively, have the same positive definite solution Q_D which is equal to the upper left block of the matrix $T_{M_1}^{\mathrm{T}}[P_1 - P_-]T_{M_1} = T_{M_2}^{\mathrm{T}}[P_2 - P_-]T_{M_2}$. Moreover, from $T_{M_1} = T_{M_2}$ it also follows $\Gamma_{-11}^{M_1} = \Gamma_{-11}^{M_2}$ and $L_{-11}^{M_1} = L_{-11}^{M_2}$. Then, by subtraction, we easily obtain $Q_D(D_1 - D_2)Q_D = 0$ and hence $D_1 = D_2$.

We now prove that φ is surjective. Let $\bar{P} \in \mathscr{P}$, $\bar{Q} := \bar{P} - P_-$, $\bar{M} = \ker \bar{Q}$, $T_{\bar{M}}$ be the corresponding matrix defined by (2.2) and $\bar{v} = \dim \bar{M}$. Since $\ker \bar{Q} = \ker \bar{Q}^\sharp$ the matrix $D^e := T_{\bar{M}}^{\mathrm{T}} \bar{Q}^\sharp \Lambda(\bar{P}) \bar{Q} T_{\bar{M}}$ has a block-diagonal structure

$$D^e = \begin{bmatrix} \bar{D} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.20}$$

where the matrix $\bar{D} \in \mathbb{R}^{(n-\bar{v})\times(n-\bar{v})}$ is symmetric and positive semidefinite. Taking into account identity (2.12), it is now easy to check that $\varphi[(\bar{M}, \bar{D})] = \bar{P}$. $\square$

3. Connections with the theory of tightest local frames

It is proved in [18] that given any solution $\bar{P} \in \mathscr{P}$ of the ARI, the set $\{P_0 \in \mathscr{P}_0 : P_0 \leqslant \bar{P}\}$ has a maximal element P_{0-} and the set $\{P_0 \in \mathscr{P}_0 : P_0 \geqslant \bar{P}\}$ has a minimal element P_{0+}. The *tightest local frame* of $\bar{P}$, denoted by $[P_{0-}, P_{0+}]$, is the subset of $\mathscr{P}$ defined by

$$[P_{0-}, P_{0+}] := \{P \in \mathscr{P} : P_{0-} \leqslant P \leqslant P_{0+}\}. \tag{3.1}$$

Corresponding to P_{0-} and P_{0+}, we define the feedback matrices

$$\Gamma_{0-} := F - (G - P_{0-}H^{\mathrm{T}})R^{-1}H, \tag{3.2a}$$

$$\Gamma_{0+} := F - (G - P_{0+}H^{\mathrm{T}})R^{-1}H. \tag{3.2b}$$

For more details on such construction and for the applications of the tightest local frame and of the corresponding feedback matrices to the stochastic realization theory and in particular to the smoothing problem we refer to [18, Section 9]. See also [15,13,8].

The next result shows that the solutions $P \in \mathscr{P}$ having the same upper bound in the tightest local frame are exactly those corresponding to the s ame Γ_-^{T}-invariant subspace. In this way the set $\mathscr{P}$ is partitioned into classes each one corresponding to a different P_{0+}.

Theorem 3.1. *For $i = 1, 2$, let $(M_i, D_i) \in \mathscr{C}$, $P_i = \varphi[(M_i, D_i)]$ and $[P_{i0-}, P_{i0+}]$ be the corresponding tightest local frames. Then $P_{10+} = P_{20+}$ if and only if $M_1 = M_2$.*

Proof. Consider $P \in \mathscr{P}$ with its tightest local frame $[P_{0-}, P_{0+}]$. It is clear that $Q_+(P_{0+}) = \min\{Q_0 \in \mathscr{Q}_0 : Q_0 \geqslant Q_+(P)\}$. In view of Lemma 2.1 and identity (2.12), the elements of $\mathscr{Q}_0$ are the symmetric solutions of Eq. (2.16). The map assigning to each $Q \in \mathscr{Q}_0$ its null space $\ker Q$ establishes a one to one correspondence between the set $\mathscr{Q}_0$ and the set of invariant subspaces of Γ_-^{T} [27]. Moreover, given $Q_1, Q_2 \in \mathscr{Q}_0$, $Q_1 \geqslant Q_2$ if and only if $\ker Q_1 \subseteq \ker Q_2$ [28]. Since $Q_+(P)$ solves Eq. (2.14), $\ker Q_+(P)$ is a Γ_-^{T}-invariant subspace. Then there exists a unique solution $\bar{Q}$ of (2.16) such that

$$\ker \bar{Q} = \ker Q_+(P). \tag{3.3}$$

We now prove that $Q_+(P_{0+}) = \bar{Q}$. Comparing Eqs. (2.14) and (2.16) and taking into account (3.3) it easily follows that

$$\bar{Q} \geqslant Q_+(P). \tag{3.4}$$

On the other hand if $\tilde{Q} \in \mathscr{Q}_0$ is such that $\tilde{Q} \leqslant \bar{Q}$ and $\tilde{Q} \neq \bar{Q}$, then $\ker \tilde{Q} \supsetneqq \ker \bar{Q} = \ker Q_+(P)$. The latter implies that $\nexists\ \tilde{Q} \in \mathscr{Q}_0$ such that $\tilde{Q} \leqslant \bar{Q}$, $\tilde{Q} \neq \bar{Q}$ and $\tilde{Q} \geqslant Q_+(P)$. Taking into account that, in view of identity (2.12), $Q_+(P_{0+})$ solves Eq. (2.11) with $\Lambda(P) = 0$, or equivalently Eq. (2.16), it immediately follows that $Q_+(P_{0+}) = \bar{Q}$. This proves that $Q_+(P_{0+})$ (and hence P_{0+} itself) can be uniquely associated to $\ker Q_+(P)$. But it is clear that if $P_i = \varphi[(M, D_i)]$, $i = 1, 2$, then $\ker Q_+(P_1) = \ker Q_+(P_2) = M$, and hence $P_{10+} = P_{20+}$. Conversely, if $P_{10+} = P_{20+}$ then $M_1 = \ker Q_+(P_{10+}) = \ker Q_+(P_{20+}) = M_2$. $\square$

Remark 3.1. In [13, Proposition 20] a characterization of the local frames in terms of inner factorizations of certain inner functions is presented. Such characterization is strictly related to the above theorem. In fact, the connection relays on the well known correspondence between invariant subspaces and inner divisors, see, e.g., [11].

4. Minimal inner extension of a rigid function

Let $P \in \mathscr{P}$, let B_2 be such that $B_2 B_2^{\mathrm{T}} = -\Lambda(P)$ and let $W(s)$ be given by (1.5). It is clear that $Q_r(s) := W_-^{-1}(s)W(s)$ is a rigid function, i.e., it is an analytic function such that $Q_r(\mathrm{i}\omega)Q_r^*(\mathrm{i}\omega) = I$, where $*$ denotes transposition plus conjugation. Moreover, it is easy to check that $Q_r(s)$ has the realization

$$Q_r(s) = R^{-1/2}H(sI - \Gamma_-)^{-1}[B_1 - B_- \mid B_2] + [I \mid 0]. \tag{4.1}$$

In a stochastic realization framework, the question arises whether it is possible to augment $Q_r(s)$ and obtain an *inner* function with the same McMillan degree, see [18, Section 5.3] and the next section. More precisely, the problem amounts to finding a $\tilde{Q}(s)$ such that

$$Q_i(s) := \begin{bmatrix} Q_r(s) \\ \tilde{Q}(s) \end{bmatrix}$$

is a square stable transfer function with $\deg Q_i(s) = \deg Q_r(s)$ and $Q_i(\mathrm{i}\omega)Q_i^*(\mathrm{i}\omega) = I$. This is a classical problem that may be viewed as a special case of Darlington synthesis [5]. It has been addressed in [21] and studied more extensively in [13, pp. 295 and ff.].

Lemma 4.1. *The transfer function*

$$Q_i(s) = \begin{bmatrix} R^{-1/2}H \\ -B_2^{\mathrm{T}}(P - P_-)^{\sharp} \end{bmatrix}(sI - \Gamma_-)^{-1}[B_1 - B_- \mid B_2] + I \tag{4.2}$$

is an inner extension of $Q_r(s)$.

Proof. It is obvious that $Q_i(s)$ is a stable square transfer function which is obtained by augmenting $Q_r(s)$. Moreover, since the pair (H, F) is observable, such is also the pair $(R^{-1/2}H, \Gamma_-)$, so that the McMillan degree of $Q_i(s)$ equals that of $Q_r(s)$. Finally, taking into account Eq. (2.11) and identity $B_1 - B_- = (P - P_-)H^{\mathrm{T}}R^{-1/2}$, a cumbersome but straightforward calculation shows that $Q_i(\mathrm{i}\omega)Q_i^*(\mathrm{i}\omega) = I$. $\square$

Remark 4.1. Observe that the realization (4.1) as well as (4.2) is minimal if and only if $P - P_-$ is nonsingular. Conversely, if $P - P_-$ has a nontrivial kernel and U is a matrix whose columns are an orthonormal basis for $[\ker\,(P - P_-)]^{\perp}$, it is easy to check that

$$Q_r(s) = R^{-1/2}HU(sI - U^{\mathrm{T}}\Gamma_- U)^{-1}[U^{\mathrm{T}}(B_1 - B_-) \mid U^{\mathrm{T}}B_2] + [I \mid 0] \tag{4.3}$$

and

$$Q_i(s) = \begin{bmatrix} R^{-1/2}HU \\ -B_2^{\mathrm{T}}(P - P_-)^{\sharp}U \end{bmatrix}(sI - U^{\mathrm{T}}\Gamma_- U)^{-1}[U^{\mathrm{T}}(B_1 - B_-) \mid U^{\mathrm{T}}B_2] + I \tag{4.4}$$

are minimal realizations of $Q_r(s)$ and $Q_i(s)$, respectively.

We denote by $\Gamma_+(P)$ the matrix of the inverse dynamics of the realization (4.2) of $Q_i(s)$:

$$\begin{aligned} \Gamma_+(P) :=& \Gamma_- - [B_1 - B_- \mid B_2]\begin{bmatrix} R^{-1/2}H \\ -B_2^{\mathrm{T}}(P - P_-)^{\sharp} \end{bmatrix} \\ =& \Gamma_- + Q_+(P)H^{\mathrm{T}}R^{-1}H - \Lambda(P)Q_+(P)^{\sharp} \\ =& F - (G - PH^{\mathrm{T}})R^{-1}H - \Lambda(P)(P - P_-)^{\sharp}. \end{aligned} \tag{4.5}$$

Notice that the *generalized feedback matrix* $\Gamma_+(P)$ depends indeed only on the solution P of the ARI and not on the choice of B_2. Their significance in a stochastic realization framework is discussed in Section 5. The relevance of feedback matrices in the parametrization problem for square spectral factors (corresponding to elements of $\mathscr{P}_0$) has been thoroughly investigated in [9]. Our results on generalized feedback matrices may then be viewed as an extension of the corresponding results in [9]. The following theorem connects the matrix $\Gamma_+(P)$ with the theory of tightest local frames.

Theorem 4.1. *Let $P \in \mathscr{P}$, $\Gamma_+(P)$ be the corresponding matrix given by* (4.5) *and Γ_{0+} be the zero matrix corresponding to the upper bound P_{0+} in the tightest local frame of P. Then, $\Gamma_+(P)$ and Γ_{0+} are similar.*

Proof. Let $P \in \mathscr{P}$, $M = \ker [Q_+(P)]$ and T_M be defined by (2.2). As a first step, observe that from (3.2b) we readily get

$$\Gamma_{0+} = \Gamma_- + Q_+(P_{0+})H^{\mathrm{T}}R^{-1}H. \tag{4.6}$$

Moreover, as a straightforward consequence of Theorem 3.1, we have

$$\ker [Q_+(P_{0+})] = \ker [Q_+(P)]. \tag{4.7}$$

Hence, the matrices $T_M^{\mathrm{T}}Q_+(P)T_M$ and $T_M^{\mathrm{T}}Q_+(P_{0+})T_M$ have both the block diagonal structure

$$T_M^{\mathrm{T}}Q_+(P)T_M = \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad T_M^{\mathrm{T}}Q_+(P_{0+})T_M = \begin{bmatrix} Q_{+11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.8}$$

where Q_{11} and Q_{+11} are positive definite matrices of dimension equal to $(n - \dim M)$. It then clearly follows from (4.6) that

$$\begin{aligned} T_M^{\mathrm{T}} \Gamma_{0+} T_M &= \begin{bmatrix} \Gamma_{-11}^M & \Gamma_{-12}^M \\ 0 & \Gamma_{-22}^M \end{bmatrix} + \begin{bmatrix} Q_{+11} L_{11}^M & Q_{+11} L_{12}^M \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{-11}^M + Q_{+11} L_{11}^M & \Gamma_{-12}^M + Q_{+11} L_{12}^M \\ 0 & \Gamma_{-22}^M \end{bmatrix}, \end{aligned} \tag{4.9}$$

where Γ_{-ij}^M and L_{ij}^M, $i,j = 1,2$ are defined in (2.3) and (2.4), respectively. Taking into account Lemma 2.1, we have the identity

$$\Gamma_- Q_+(P_{0+}) + Q_+(P_{0+}) \Gamma_-^{\mathrm{T}} + Q_+(P_{0+}) H^{\mathrm{T}} R^{-1} H Q_+(P_{0+}) = 0. \tag{4.10}$$

The latter, by pre- and post-multiplication by T^{T} and T, respectively, yields

$$\Gamma_{-11}^M Q_{+11} + Q_{+11} (\Gamma_{-11}^M)^{\mathrm{T}} + Q_{+11} L_{11}^M Q_{+11} = 0, \tag{4.11}$$

so that (4.9) may be rewritten as

$$T_M^{\mathrm{T}} \Gamma_{0+} T_M = \begin{bmatrix} -Q_{+11} (\Gamma_{-11}^M)^{\mathrm{T}} Q_{+11}^{-1} & \Gamma_{-12}^M + Q_{+11} L_{12}^M \\ 0 & \Gamma_{-22}^M \end{bmatrix}. \tag{4.12}$$

Now recall (identity (2.13)) that $\ker [Q_+(P)] \subseteq \ker [\Lambda(P)]$ so that

$$T_M^{\mathrm{T}} \Lambda(P) T_M = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.13}$$

From the definition (4.5) of $\Gamma_+(P)$, it also easily follows that

$$\begin{aligned} T_M^{\mathrm{T}} \Gamma_+(P) T_M &= \begin{bmatrix} \Gamma_{-11}^M & \Gamma_{-12}^M \\ 0 & \Gamma_{-22}^M \end{bmatrix} + \begin{bmatrix} Q_{11} L_{11}^M & Q_{11} L_{12}^M \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \Lambda_{11} Q_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{-11}^M + Q_{11} L_{11}^M + \Lambda_{11} Q_{11}^{-1} & \Gamma_{-12}^M + Q_{11} L_{12}^M \\ 0 & \Gamma_{-22}^M \end{bmatrix}. \end{aligned} \tag{4.14}$$

Pre- and post-multiplying by T^{T} and T, respectively, the identity obtained by plugging the solution $Q_+(P)$ in (2.11) we get

$$\Gamma_{-11}^M Q_{11} + Q_{11} (\Gamma_{-11}^M)^{\mathrm{T}} + Q_{+11} L_{11}^M Q_{+11} + \Lambda_{11} = 0, \tag{4.15}$$

so that (4.14) may be rewritten as

$$T_M^{\mathrm{T}} \Gamma_+(P) T_M = \begin{bmatrix} -Q_{11} (\Gamma_{-11}^M)^{\mathrm{T}} Q_{11}^{-1} & \Gamma_{-12}^M + Q_{11} L_{12}^M \\ 0 & \Gamma_{-22}^M \end{bmatrix}. \tag{4.16}$$

The conclusion now follows by comparing (4.12) and (4.16). $\square$

The following interesting result connecting different generalized feedback matrices is a straightforward consequence of the above theorem and of Theorem3.1.

Corollary 4.1. *For $i = 1, 2$, let $(M_i, D_i) \in \mathscr{C}$ and $P_i = \varphi[(M_i, D_i)]$. If $M_1 = M_2$, then the matrices $\Gamma_+(P_1)$ and $\Gamma_+(P_2)$ are similar.*

5. Stochastic realizations

Let $\{y(t); t \in \mathbb{R}\}$ be a zero-mean, $\mathbb{R}^m$-valued, purely nondeterministic stochastic process defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Suppose that y has stationary Gaussian increments, and a rational, coercive incremental spectral density $\Phi(\cdot)$, namely

$$y(t) - y(s) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t} - e^{i\omega s}}{i\omega}\, d\hat{y}(\omega), \quad E[d\hat{y}(\omega) d\hat{y}(\omega)^*] = \frac{\Phi(i\omega)}{2\pi}\, d\omega, \tag{5.1}$$

where $d\hat{y}$ is an orthogonal stochastic measure. Let, as in Section 1, $\Phi(s) = Z(s) + Z(-s)^{\mathrm{T}}$, with Z positive real, let $R := \Phi(\infty)$, and let $Z(s) = H(sI - F)^{-1}G + \frac{1}{2}R$ be a minimal realization of Z of dimension n. Consider the following version of the *strong stochastic realization problem*: Characterize all linear stochastic models

$$dx = Ax\, dt + B\, dw, \tag{5.2a}$$

$$dy = Cx\, dt + D\, dw, \tag{5.2b}$$

such that A is a stability matrix of minimal dimension and w is a standard p-dimensional Wiener process defined on $(\Omega, \mathscr{F}, \mathbb{P})$. The quintuplet $[A, B, C, D, w]$ is called a *minimal, stable stochastic realization* of y. Notice that the transfer function of (5.2) $W(s) = C(sI - A)^{-1}B + D$ is a stable, minimal spectral factor of Φ. It is apparent that if $[A, B, C, D, w]$ is a stochastic realization, so is $[A, BV, C, DV, V^{\mathrm{T}}w]$, where V is any $p \times p$ orthogonal matrix. Moreover, we can change the (deterministic) realization of W. It is then easy to show [16] that it suffices to consider minimal stochastic realizations of the form

$$dx = Fx\, dt + B_1\, du + B_2\, dv, \tag{5.3a}$$

$$dy = Hx\, dt + R^{1/2}\, du, \tag{5.3b}$$

where B_1 is $n \times m$ and B_2 is $n \times (p - m)$. The state covariance $P := E[x(t)x^{\mathrm{T}}(t)]$ is a symmetric solution of the ARI. It solves, together with $B = [B_1 | B_2]$, the positive real lemma equations. The model with state covariance P_- is a steady-state *Kalman–Bucy filter*

$$\mathrm{d}x = Fx_- \, \mathrm{d}t + B_- \, \mathrm{d}u_-, \tag{5.4a}$$

$$\mathrm{d}y = Hx_- \, \mathrm{d}t + R^{1/2} \, F \, \mathrm{d}u_-, \tag{5.4b}$$

with driving noise the normalized *innovations* $\mathrm{d}u_-$ of y. Similarly, the realization corresponding to P_+

$$\mathrm{d}x = Fx_+ \, \mathrm{d}t + B_+ \, \mathrm{d}u_+,$$

$$\mathrm{d}y = Hx_+ \, \mathrm{d}t + R^{1/2} \, \mathrm{d}u_+,$$

corresponds to a backward, steady-state Kalman–Bucy filter.

Consider now a realization (5.3) with state covariance P. Define the estimation error process as $\tilde{x}(t) = x(t) - x_-(t)$. It follows from (5.3)–(5.4) that $\tilde{x}$ is the (Markovian) state process of the system

$$\mathrm{d}\tilde{x} = \Gamma_- \tilde{x} \, \mathrm{d}t + [B_1 - B_-]\mathrm{d}u + B_2 \, \mathrm{d}v, \tag{5.5a}$$

$$\mathrm{d}u_- = R^{-1/2} H \tilde{x} \, \mathrm{d}t + \mathrm{d}u, \tag{5.5b}$$

where, recall, $\Gamma_- = F - B_- R^{-1/2} H$. Under the present assumptions on Φ, Γ_- is a stability matrix. The model (5.5) may be viewed as a (nonminimal) realization of the innovations process u_-. Since the latter is a standard Wiener process, the transfer function $Q_r(s) = R^{-1/2} H (sI - \Gamma_-)^{-1} [B_1 - B_- | B_2] + [I|0]$ is a rigid function. Next, we wish to derive from (5.5a) a reverse-time representation for the process $\tilde{x}$. In order to do that, we need a slight generalization of the results in [16, Section 3] and [3, Theorem 4.3], since the covariance of $\tilde{x}$ may be singular.

Lemma 5.1. *Suppose that the n-dimensional process ξ satisfies on the whole real line the stochastic differential equation*

$$\mathrm{d}\xi = M\xi \, \mathrm{d}t + N \, \mathrm{d}v, \tag{5.6}$$

where M is asymptotically stable and v is a standard Wiener process. Then ξ is Gaussian and stationary. It also satisfies the reverse-time stochastic differential equation

$$\mathrm{d}\xi = [M + NN^{\mathrm{T}} \Pi^{\sharp}]\xi \, \mathrm{d}t + N \, \mathrm{d}\bar{v} \tag{5.7}$$

where

$$\mathrm{d}\bar{v} := \mathrm{d}v - N^{\mathrm{T}} \Pi^{\sharp} \xi \, \mathrm{d}t \tag{5.8}$$

is a Wiener process such that, if $\sigma < \tau \leqslant t$, the components of $\bar{v}(\tau) - \bar{v}(\sigma)$ are independent of $\xi(r)$, $r \geqslant t$.

Proof. The first assertion follows immediately from the representation

$$\xi(t) = \int_{-\infty}^{t} \mathrm{e}^{M(t-\tau)} N \, \mathrm{d}v(\tau). \tag{5.9}$$

To prove the second part, let $\mathscr{F}_t^+(\xi)$ be the σ-algebra induced by the components of $\xi(r)$, $r \geqslant t$. The same argument as in [16, Lemma 3.1] gives that, for $s < t$, $E[\xi(s)|\mathscr{F}_t^+(\xi)] = \Pi \mathrm{e}^{M^{\mathrm{T}}(t-s)} \Pi^\sharp \xi(t)$. From the Lyapunov equation $M\Pi + \Pi M^{\mathrm{T}} + NN^{\mathrm{T}} = 0$, we get

$$-\Pi M^{\mathrm{T}} \Pi^\sharp = M \Pi \Pi^\sharp + NN^{\mathrm{T}} \Pi^\sharp. \tag{5.10}$$

Moreover, by computing the covariance of $(I - \Pi\Pi^\sharp)\xi(t)$, it is easy to see that $\xi(t) = \Pi\Pi^\sharp \xi(t)$. Hence, we have

$$(D_-\xi)(t) := \lim_{h \searrow 0} E\left[\frac{\xi(t) - \xi(t-h)}{h} \middle| \mathscr{F}_t^+(\xi) \right] = [M + NN^{\mathrm{T}}\Pi^\sharp]\xi(t), \tag{5.11}$$

where the limit is taken in $L_n^2(\Omega, \mathscr{F}, \mathbb{P})$. By [20, Theorem 11.3], the process

$$\theta(s,t) := \xi(t) - \xi(s) - \int_s^t [M + NN^{\mathrm{T}}\Pi^\sharp]\xi(\tau) \, \mathrm{d}\tau \tag{5.12}$$

is a reverse-time *difference martingale* with respect to the decreasing family $\{\mathscr{F}_t^+(\xi)\}$. By the forward representation of ξ, we also have

$$\xi(t) - \xi(s) = \int_s^t M\xi(\tau) \, \mathrm{d}\tau + N[v(t) - v(s)]. \tag{5.13}$$

We conclude that

$$\theta(s,t) := N[v(t) - v(s)] - \int_s^t NN^{\mathrm{T}}\Pi^\sharp \xi(\tau) \, \mathrm{d}\tau = N[\bar{v}(t) - \bar{v}(s)]. \tag{5.14}$$

Finally observe that, letting $\mathrm{d}_-\bar{v}(t) := \bar{v}(t) - \bar{v}(t - \mathrm{d}t)$, $(\mathrm{d}t > 0)$, we have, up to $\mathrm{o}(\mathrm{d}t)$,

$$E[\mathrm{d}_-\bar{v}(t)\mathrm{d}_-\bar{v}^{\mathrm{T}}(t)|\xi(t)] = I \, \mathrm{d}t. \tag{5.15}$$

By a version of Levy's theorem [20, Theorem 11.8], $\bar{v}$ is a standard Wiener process. $\square$

Applying this lemma to (5.5a), and using the positive real lemma equations, we get the reverse time representation

$$\mathrm{d}\tilde{x} = [F - B_1 R^{-1/2} H + B_2 B_2^{\mathrm{T}} (P - P_-)^\sharp]\tilde{x} \, \mathrm{d}t + [B_1 - B_-]\mathrm{d}u_- + B_2 \, \mathrm{d}\zeta, \tag{5.16}$$

where

$$\mathrm{d}\zeta := \mathrm{d}v - B_2^{\mathrm{T}}(P - P_-)^{\sharp}\tilde{x}\,\mathrm{d}t. \tag{5.17}$$

Notice that $\mathrm{d}\zeta$ is $p - m$ dimensional. Since

$$\begin{bmatrix} u_-(t) \\ \zeta(t) \end{bmatrix}$$

is a standard Wiener process, $\mathrm{d}\zeta$ is independent of the innovations $\mathrm{d}u_-$ or, equivalently, independent of y. Thus, by a purely probabilistic argument, we have identified the"exogenous" portion of the input in (5.3) and (5.5). Adjoining Eq. (5.17) to the model (5.5) solves the minimal extension problem for the rigid function $Q_r(s)$ discussed in the previous section. Indeed, the transfer function of

$$\mathrm{d}\tilde{x} = \Gamma_-\tilde{x}\,\mathrm{d}t + [B_1 - B_-]\mathrm{d}u + B_2\,\mathrm{d}v, \tag{5.18a}$$

$$\mathrm{d}u_- = R^{-1/2}H\tilde{x}\,\mathrm{d}t + \mathrm{d}u, \tag{5.18b}$$

$$\mathrm{d}\zeta = -B_2^{\mathrm{T}}(P - P_-)^{\sharp}\tilde{x}\,\mathrm{d}t + \mathrm{d}v \tag{5.18c}$$

is precisely the inner function $Q_i(s)$. Inverting this system, we get

$$\mathrm{d}\tilde{x} = [F - B_1R^{-1/2}H + B_2B_2^{\mathrm{T}}(P - P_-)^{\sharp}]\tilde{x}\,\mathrm{d}t + [B_1 - B_-]\mathrm{d}u_- + B_2\,\mathrm{d}\zeta, \tag{5.19a}$$

$$\mathrm{d}u = -R^{-1/2}H\tilde{x}\,\mathrm{d}t + \mathrm{d}u_-, \tag{5.19b}$$

$$\mathrm{d}v = B_2^{\mathrm{T}}(P - P_-)^{\sharp}\tilde{x}\,\mathrm{d}t + \mathrm{d}\zeta. \tag{5.19c}$$

Hence, $\Gamma_+(P) = F - B_1R^{-1/2}H + B_2B_2^{\mathrm{T}}(P - P_-)^{\sharp}$ is the state matrix of the inverse system (5.19).

6. The noncoercive case

The results presented in Section 2 may be extended beyond the coercive case provided that $R = \Phi(\infty)$ remains positive definite. This is equivalent to say that $\Phi(s)$ may have zeros on the imaginary axis but not at infinity. Notice that this precisely the case of the spectral density of a purely nondeterministic Gaussian process of full rank [23].

In this case, the ARI still admits a minimum solution P_- [14, Theorem 2.10], but the eigenvalues of the corresponding feedback matrix Γ_- lay in the *closed* left half plane.

Lemma 2.1 continues to hold in the noncoercive case, however Eqs. (2.15) and (2.16) do not, in general, admit a positive definite solution any more.

The next proposition allows to extend Theorem 2.1 to the noncoercive case since it essentially states that, along the subspace corresponding to the eigenvalues of Γ_- with zero real part, the solutions of (2.15) and (2.16) are zero.

Proposition 6.1. *Let $\Gamma_-, L \in \mathbb{R}^{n\times n}$ be such that*

$$\sigma(\Gamma_-) \subset \mathbb{C}_{0-}, \tag{6.1a}$$

$$L = L^{\mathrm{T}} \geqslant 0, \tag{6.1b}$$

$$(L, \Gamma_-) \text{ is observable.} \tag{6.1c}$$

Then the homogeneous ARE

$$\Gamma_- Q + Q\Gamma_-^{\mathrm{T}} + QLQ = 0 \tag{6.2}$$

admits a maximum solution $Q_M \geqslant 0$. Moreover, let

$$E_< = \bigoplus_{\Lambda\in\sigma(\Gamma_-)\cap\mathbb{C}_-} [\ker\,(\Gamma_- - \lambda I)^n] \tag{6.3}$$

and

$$T = [T_< \mid T_0], \tag{6.4}$$

where the, say, v columns of $T_<$ are an orthonormal basis for $E_<$ and the $n - v$ columns of T_0 are an orthonormal basis for $E_<^{\perp}$. Then:

1. *$E_<$ is a Γ_-^{T}-invariant subspace.*
2. *A change of basis induced by T carries Γ_- to the form*

$$\bar{\Gamma}_- = T^{\mathrm{T}}\Gamma_- T = \begin{bmatrix} \Gamma_s & \Gamma_{so} \\ 0 & \Gamma_o \end{bmatrix} \tag{6.5}$$

   with $\Gamma_s \in \mathbb{R}^{v\times v}$ being a stability matrix.

3. *The matrix $\bar{Q}_M := T^{\mathrm{T}} Q_M T$ has the structure*

$$\bar{Q}_M = \begin{bmatrix} Q_+ & 0 \\ 0 & 0 \end{bmatrix} \tag{6.6}$$

where Q_+ is the unique positive definite solution of the reduced order homogeneous ARE

$$\Gamma_s Q + Q\Gamma_s^{\mathrm{T}} + QL_s Q = 0 \tag{6.7}$$

with L_s being the upper left block of the matrix $\bar{L} := T^{\mathrm{T}}LT$.

Proof. Employing again [14, Theorem 2.10], it is clear that (6.2) admits a maximum solution Q_M and a minimum solution Q_m. The latter is clearly $Q_m = 0$ while Q_M is obviously positive semidefinite. The fact that $E_<$ is Γ_-^{T}-invariant, or equivalently the structure (6.5) of $\bar{\Gamma}_-$) is a direct consequence of the definition of $E_<$. Finally,

$$\begin{bmatrix} Q_+ & 0 \\ 0 & 0 \end{bmatrix}$$

is the maximum solution of the homogeneous ARE

$$\bar{\Gamma}_- Q + Q\bar{\Gamma}_-^{\mathrm{T}} + Q\bar{L}Q = 0. \tag{6.8}$$

In fact, it is immediate to check that it is *a* solution. To prove that it is the maximum one assume, by contradiction, that there exists a solution $\tilde{Q}$ of (6.8) such that:

$$\begin{cases} \tilde{Q} \geqslant \begin{bmatrix} Q_+ & 0 \\ 0 & 0 \end{bmatrix}, \\ \tilde{Q} \neq \begin{bmatrix} Q_+ & 0 \\ 0 & 0 \end{bmatrix}. \end{cases} \tag{6.9}$$

Since Q_+ is the unique positive definite solution of (6.7), this implies that $\ker \tilde{Q}$ is strictly contained in

$$\ker \begin{bmatrix} Q_+ & 0 \\ 0 & 0 \end{bmatrix}$$

so that it has dimension $n - \tilde{v}$, with $\tilde{v} > v$. Therefore, employing a standard argument, we may conclude that there exists a $\bar{\Gamma}_-^{\mathrm{T}}$-invariant subspace $\tilde{M}$ of dimension $\tilde{v} > v$ such that $\bar{\Gamma}_-^{\mathrm{T}}\big|_M$ is stable which is against the assumptions.

The conclusion now follows by observing that $\bar{Q}_M$ is the maximum solution of (6.8) if and only if $\bar{Q}_M := T^{\mathrm{T}}Q_M T$ with Q_M being the maximum solution of (6.2). $\square$

Corollary 6.1. *Under the assumptions of Proposition* 6.1*, Q is solution of* (6.2) *if and only if it has the form*

$$Q = T \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} T^{\mathrm{T}}, \tag{6.10}$$

with Q_1 *being a solution of* (6.7).

At this point we are in the position of extending the parametrization of the set $\mathscr{Q}$ (or equivalently of the set $\mathscr{P}$). In fact, by performing the preliminary change of basis induced by the matrix T defined as in (6.4), we may restrict the ARI to the subspace $E_{<}$ where Γ_{-} is a stability matrix. We can then employ Theorem 2.1.

We finally remark that the results of Sections 3 and 4 may also be extended to the noncoercive case along the same lines.

7. Dual results

The results obtained in the previous sections have a dual counterpart. To avoid overburdening the paper, we only give some indication in this direction. By defining $Q_{-}(P) := P - P_{+}$, a dual parametrization of the set $\mathscr{P}$ may be established in terms of the parameter set

$$\begin{aligned} \bar{\mathscr{C}} := \{ & (M, D) : M \in \mathscr{I}^{v}(\Gamma_{+}^{\mathrm{T}}),\ D \in \mathbb{R}^{(n-v)\times(n-v)},\ D = D^{\mathrm{T}} > 0, \\ & v = 0, 1, \ldots, n\}, \end{aligned} \tag{7.1}$$

where $\Gamma_{+} := F - (G - P_{+}H^{\mathrm{T}})R^{-1}H$ and $\mathscr{I}^{v}(\Gamma_{+}^{\mathrm{T}})$ is the set of v-dimensional Γ_{+}^{T}-invariant subspaces of $\mathbb{R}^{n}$. Thus, the set $\mathscr{P}$ may be partitioned into classes each one corresponding to a Γ_{+}^{T}-invariant subspace and it is possible to prove a theorem (dual with respect to Theorem 3.1) stating that elements of the same class have the same lower bound P_{0-} in the tightest local frame and that elements of different classes have different P_{0-}. Moreover, an alternative extension of the rigid function $Q_{r}(s)$ may be obtained which leads to a model (of the type of (5.18)) where the state matrix of the inverse dynamics has the form

$$\begin{aligned} \Gamma_{-}(P) &:= F - (G - PH^{\mathrm{T}})R^{-1}H - \Lambda(P)(P - P_{+})^{\sharp} \\ &= \Gamma_{+} + Q_{-}(P)H^{\mathrm{T}}R^{-1}H - \Lambda(P)Q_{-}(P)^{\sharp}. \end{aligned} \tag{7.2}$$

It may also be shown that $\Gamma_{-}(P)$ and Γ_{0-} are similar.

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