Numerical Functional Analysis and Optimization
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lnfa20

Approximating fixed-points of decreasing operators in spaces of continuous functions*

Alvise Sommariva [a] & Marco Vianello [a]

[a] Dipartimento di hla,tematica Pura e Applicata, Universiti di Padova., Via Belzoni 7, Padova, 35131, Italy E-mail:

Available online: 26 Jun 2007

NUMER. FUNCT. ANAL. AND OPTIMIZ., 19(5&6), 635-646 (1998)

Approximating fixed-points of decreasing operators in spaces of continuous functions*

Alvise Sommariva and **Marco Vianello**

Dipartimento di Matematica Pura e Applicata
Università di Padova
Via Belzoni 7, 35131 Padova, Italy
e-mail:{*alvise,marcov*}@*brouwer.math.unipd.it*

Abstract

We prove constructively existence and uniqueness of a *continuous* fixed-point for a class of *decreasing* and pointwise *monotonically demi-continuous* (nonlinear) operators in *spaces* of (vector-valued) *functions*, *without* resorting to *complete continuity*. As an application, we rediscuss and improve recent results for a class of perturbed nonlinear integral equations.

Keywords and phrases: fixed-point equations, spaces of continuous functions, decreasing operators, noncompact operators, monotone demicontinuity, monotone approximations, Chandrasekhar H- equation, (perturbed) Hammerstein integral equations.

1991 AMS subject classifications: 45GXX, 47H07, 47H10, 65J15.

*Work supported, in part, by the MURST, and by the GNIM-CNR.

1 The main result.

Consider a classical nonlinear integral equation associated with a *decreasing* operator, i.e. the "*Chandrasekhar*-type" equation

$$u(x) = \int_D \frac{\phi(x,t)}{1+u(t)}\, dt \ , \quad x \in D \ , \tag{1}$$

where $\phi(x,t)$ is measurable and nonnegative in D^2, D being a closed subset of $\mathbb{R}^N$, $0 < \sup_{x\in D} \int_D \phi(x,t)dt < +\infty$, $\lim_{x\to x_0} \int_D |\phi(x,t) - \phi(x_0,t)|dt = 0$ for every $x_0 \in D$.

Equations of the form (1) have been extensively and deeply investigated in the literature, in view of their important meaning as physical models (especially in heat transfer problems and nuclear physics), and also of their interesting mathematical features (cf., e.g., [3, 9, 12]). Indeed, the well-known *Chandrasekhar H*- equation, which plays a key role in the theory of radiative heat transfer in semi-infinite atmosphere, can be easily rewritten in the form (1), with $D = [0,1]$, cf. [9, 12, 14].

Starting from the late '40s, various instances, as well as generalizations, of (1) have been studied, both analitically and numerically, by means of successive approximations (cf. the pioneering work of Chandrasekhar [2]), resting essentially (if not explicitly) on the fact that they are fixed-point equations of the form

$$u = \psi(u) \ , \tag{2}$$

where $\psi : C^+(D) \to C^+(D)$, is a *decreasing* (nonlinear) integral operator. What is usually shown is that, given the sequence

$$u_{n+1} = \psi(u_n) \ , \quad u_0 = 0 \ , \tag{3}$$

then $u_{2n} \leq u_{2n+1}$, $u_{2n} \uparrow u^*$, $u_{2n+1} \downarrow v^*$, where $u^* = \psi(v^*)$ and $v^* = \psi(u^*)$, and more that $u^* = v^*$ by exploiting the special structure of the operator; cf., e.g., [5, 8, 9, 12].

It should be noticed that *complete continuity* of ψ (with respect to the sup-norm on $C(D)$ for D compact) is typically a key ingredient of the proofs, in that it ensures uniform convergence of the subsequences $\{u_{2n}\}$ and $\{u_{2n+1}\}$. In particular, equation (1) and some generalizations have been solved by applying a general result on completely continuous decreasing operators in the framework of abstract cones, cf. [7, 9].

We recall, in fact, that the integral operator in (1) is certainly completely continuous when D is bounded and the limit as $x \to x_0$ is uniform, as proved for a generalization of (1) in [5, 9] (this being indeed the case of the classical

Chandrasekhar H-equation). On the contrary, in absence of uniformity for the limit above, compactness (and hence complete continuity) may fail (cf. [8]).

The lackness of compactness has been cleverly circumvented for (a generalization of) (1) with $D = \mathbb{R}^m$ in [8], where uniform (monotone) convergence of u_{2n} and u_{2n+1} to the unique fixed-point is obtained by an extremely sophisticated use of the structure of the nonlinearity. Such a proof rests on a quite general fixed-point result for a class of noncompact (and even discontinuous) decreasing operators in ordered Banach spaces. We recall also another important fixed-point theorem for $(-\alpha)$-convex decreasing operators [9, Thm.2.2.6], which does not require complete continuity at all: it is not applicable, however, to the integral equations above.

We started our analysis from the following question: is compactness really needed in the analysis of the iterative process (3), to obtain constructively existence and uniqueness of a nonnegative *continuous* fixed-point? The answer is negative by a quite simple argument, since we can conveniently exploit the properties that the corresponding integral operator ***preserves*** pointwise ***monotone*** convergence, and ***regularizes*** nonnegative measurable functions. These entail that $u^* = \psi(v^*)$ and $v^* = \psi(u^*)$ (see (3) and below) are continuous, hence we are done as soon as we prove that $u^* = v^*$ (by resorting in the usual way to the special structure of the operator). Moreover, in view of a famous theorem of Dini, convergence of u_{2n} to u^* and of u_{2n+1} to v^*, being pointwise monotone, becomes *uniform* on compact subsets of the support D. In such a way we recover all the features of the iterative process (3) usually obtained via complete continuity; observe that neither continuity with respect to the uniform convergence topology is used.

The approach just sketched can be extended to a general class of decreasing operators in spaces of (vector-valued) functions, as it will be shown in Theorem 1.1 below. For basic notations, definitions, and results concerning monotone operators in ordered Banach spaces we refer the reader. e.g, to [9, 10, 11]. Let X be a topological space, and Y a ***regularly*** partially ***ordered*** Banach space; we are concerned with nonlinear operators

$$\psi : (P \subset V) \to P , \tag{4}$$

where P is the cone of positive functions of V, V being a linear subspace of Y-valued functions defined on X. We assume that V is ***closed*** with respect to ***pointwise convergence***, i.e., it is characterized by some additional properties which are mantained by pointwise limits (for example measurability when X is a measure space). Making a little abuse, we shall denote by $\succeq$ the order relation in Y as well as that canonically induced in spaces of Y-valued functions defined on X. Moreover we shall denote by θ the zero element of Y as well as the zero function on X. Following [9], the "order interval" $\{z : u \preceq z \preceq v\}$ will be written as $[u, v]$. We recall that an operator $\psi : (P \subset V) \to P$ is termed *decreasing* when

$$\psi(u) \succeq \psi(v) \text{ for every } u, v \in P \ , \quad u \preceq v \ . \tag{5}$$

We are now ready to state and prove the following

Theorem 1.1 *Consider the fixed-point equation* (2), *where*

$$\psi = \psi_1 + \psi_2 \ , \tag{6}$$

$\psi_1, \psi_2 : P \to P$ *being two decreasing operators which are pointwise monotonically demicontinuous, i.e., if* $v_n \in P$ *converges pointwise increasing [decreasing] to* v, *then* $\psi_i(v_n)$ *converges pointwise weakly (decreasing [increasing]) to* $\psi_i(v)$, $i = 1, 2$ *(observe that necessarily* $v \in P$, *since* V *is closed under pointwise convergence). Moreover assume that*

(*i*) $\psi_1(\theta) \succ \theta$ *or* $\psi_2(\theta) \succ \theta$.

(*ii*) $\psi_1 : P \to P \cap C(X; Y)$, $\psi_2(P \cap C(X; Y)) \subseteq P \cap C(X; Y)$.

(*iii*) *The fixed-point equation*

$$u = \psi_2(u) + \beta \ , \tag{7}$$

has a unique solution in P, *say* u_β *with* $u_\beta \in C(X; Y)$, *for every* $\beta \in [\theta, \psi(\theta)] \cap C(X; Y)$.

(*iv*) *There exists* $\varepsilon_0 > 0$ *such that* $\psi^2(\theta) \succeq \varepsilon_0 \psi(\theta)$; *for every* $u \in [\psi^2(\theta), \psi(\theta)]$, *for every* $\tau \in (0, 1)$, *there exists* $\eta_i = \eta_i(\tau, u) > 0$ *such that*

$$\psi_i(\tau u) \preceq (\tau(1 + \eta_i))^{-1} \psi_i(u) \ , \quad i = 1, 2 \ . \tag{8}$$

Then, the sequence $\{u_n\}$ *defined recursively by*

$$u_{n+1} = \psi(u_n) \ , \quad u_0 = \theta \ , \quad n = 0, 1, 2, \ldots \tag{9}$$

converges (pointwise) to the unique fixed-point of ψ *in* P, *say* $u^* \in C(X; Y)$, *and moreover,*

$$\theta \prec \varepsilon_0 \psi(\theta) \preceq \psi^2(\theta) \preceq \ldots \preceq u_{2n} \preceq \ldots \preceq u^* \preceq \ldots \preceq u_{2n-1} \preceq \ldots \preceq \psi(\theta) \ , \tag{10}$$

$n = 2, 3, \ldots$. *Finally, the iterative process (9) converges uniformly to the fixed-point on compact subsets of* X, *starting from any initial choice* $u_0 \in P$.

Before proving the theorem, some observations are in order. First, we stress once again that the result is *constructive*: from the computational point of view, the inequality $u_{2n} \preceq u^* \preceq u_{2n-1}$, when applied on a suitable discretization of X, can provide a simple and reliable *stopping criterion* in the implementation of the successive approximations algorithm. Moreover

Remark 1.2 Observe that by (ii) we require that the operator ψ_1 has some "regularizing" effect. This is a typical feature of *integral* operators, such as for example that in (1) (where we can take $\psi_2 \equiv 0$), as well as the feature of preserving pointwise monotone convergence (essentially in view of Beppo-Levi's monotone convergence theorem). On the other hand, by (ii) ψ_2 must preserve continuity; this feature can be exhibited even by local operators, like the superposition (or Nemytskii [1]) operator generated by a continuous function (see the example below).

Remark 1.3 If we take as topological space X the singleton $\{1\}$, then both V and $C(X;Y)$ can be trivially identified with the space Y. This entails that Theorem 1.1 provides also a constructive fixed-point result in regularly partially ordered Banach spaces, where the operators ψ_1 and ψ_2 are assumed to be only monotonically demicontinuous. This is reminiscent of a well-known fixed-point (existence) theorem for increasing demicontinuous operators, cf., e.g., [9, Thm.2.1.1]. Observe also that when $Y = \mathbb{R}^N$ (whatever be X), or more generally when $Y = Z^N$, with Z regularly partially ordered Banach space, then Theorem 1.1 becomes a tool for studying *systems* of operator equations.

Proof of Theorem 1.1. The sum $\psi = \psi_1 + \psi_2$ is clearly positive and decreasing, as such are its summands. First we observe that the order interval $[\theta, \psi(\theta)]$ is ψ-invariant, and that $u_n \in [\theta, \psi(\theta)] \cap C(X;Y)$ for every n, by (ii). Now, we prove by induction on n that every order interval $[u_{2n}, u_{2n+1}]$ is ψ-invariant. This is true for $n = 0$ (see above); assuming that $[u_{2n}, u_{2n+1}]$ is ψ-invariant, we obtain easily that if $u \in [u_{2n+2}, u_{2n+1}]$ where $u_{2n+2} = \psi(u_{2n+1}) \succeq u_{2n}$, then $u_{2n+1} \succeq \psi(u) \succeq \psi(u_{2n+1}) = u_{2n+2}$, i.e. that $[u_{2n+2}, u_{2n+1}]$ is ψ-invariant. In a similar way, from the ψ-invariance of $[u_{2n+2}, u_{2n+1}]$ we recover that of $[u_{2n+2}, u_{2n+3}]$, which terminates the inductive argument. The such proved invariance implies that $u_{2n} \preceq u_{2n+2} \preceq u_{2n+3} \preceq u_{2n+1}$ for every $n \geq 0$, i.e.,

$$\theta \prec \varepsilon_0 \psi(\theta) \preceq \psi^2(\theta) \preceq \dots \preceq u_{2n} \preceq \dots \preceq u_{2n-1} \preceq \dots \preceq \psi(\theta)\,.$$

Now, the sequences $\{u_{2n}\}$ and $\{u_{2n+1}\}$ are monotone (increasing and decreasing respectively), and pointwise bounded in order: hence, they are *pointwise convergent* since the cone of Y is regular, i.e., there exist $u^*, v^* \in P \cap [\psi^2(\theta), \psi(\theta)]$, such that $u_{2n}(x) \uparrow u^*(x)$, $u_{2n+1}(x) \downarrow v^*(x)$ in Y for every $x \in X$ (recall that V is closed with respect to pointwise convergence).

The operators ψ_1 and ψ_2, and thus ψ, are pointwise monotonically demi-continuous, so that, taking the pointwise weak limits as $n \to \infty$, we obtain immediately from (9)

$$v^* = \psi(u^*) \; , \quad u^* = \psi(v^*) \; . \tag{11}$$

It remains to prove that $u^* = v^* \in C(X;Y)$, and that this is the unique fixed point of ψ in P. As for the former, if (iv) is verified, we can proceed as in [9, Thm.2.1.5]. Set

$$\tau_0 = \sup \{\tau \in (0,1] : \tau v^* \preceq u^*\} \; , \tag{12}$$

which is well defined since $u^* \succeq \psi^2(\theta) \succeq \varepsilon_0 \psi(\theta) \succeq \varepsilon_0 v^*$. If $\tau_0 = 1$ we are done; if $\tau_0 < 1$, taking $\eta = \min\{\eta_1, \eta_2\}$, we get

$$\begin{aligned} v^* = \psi(u^*) \preceq \psi(\tau_0 v^*) &\preceq (\tau_0(1+\eta_1))^{-1}\psi_1(v^*) + (\tau_0(1+\eta_2))^{-1}\psi_2(v^*) \\ &\preceq (\tau_0(1+\eta))^{-1}(\psi_1(v^*) + \psi_2(v^*)) = (\tau_0(1+\eta))^{-1}u^* \; , \end{aligned}$$

which contradicts the definition of τ_0. Moreover, any fixed-point of ψ in P, say $z = \psi_1(z) + \psi_2(z)$, is *continuous*. In fact, $\psi_1(z) \in [\theta, \psi(\theta)] \cap C(X;Y)$ by (ii), so taking $\beta = \psi_1(z)$ in (iii) we get $u_\beta = z \in C(X;Y)$.

As for the uniqueness of the fixed-point, if $z = \psi(z)$, $\theta \preceq z \in P$, then $\psi(\theta) \succeq z \succeq \theta$, and so by induction on n, $u_{2n} \preceq z \preceq u_{2n+1}$ for every $n = 0, 1, 2, \ldots$. Taking the pointwise limit of both sides in the inequality above, and using the fact that the cone of Y is regular, and hence normal, we get $z = u^* = \lim u_{2n} = \lim u_{2n+1}$, in view of the so-called "two militia-men" theorem (cf. [10, Thm.4.1.6]).

At this point, a straightforward generalization to functions valued in ordered Banach spaces of a well-known theorem of Dini [4, Thm.7.2.2] (we omit details for brevity), gives uniform convergence of the sequences $\{u_{2n}\}$ and $\{u_{2n+1}\}$ on compact subsets of X, since they are both *monotonically pointwise* convergent to the *continuous function* u^*.

Finally, for any initial choice $u_0 \in P$, the inequalities $u_{2n} \preceq \psi^{2n}(u_0) \preceq u_{2n-1}$, $u_{2n} \preceq \psi^{2n+1}(u_0) \preceq u_{2n+1}$, can be easily proved by induction on $n \geq 1$. Observe that, in view of the just written inequalities, we have $\theta \preceq \psi^n(u_0) \preceq \psi(\theta)$ for every $n \geq 1$, wherefrom $\psi^n(u_0) \in B(K;Y) = \{u : K \to Y, \sup_{x\in K} \|u(x)\|_Y < +\infty\}$ for every $n \geq 1$ (recall that $\psi(\theta)$ is continuous on the compact K by (ii)). This shows that $\psi^n(u_0)$ converges uniformly to u^* on every fixed compact subset $K \subseteq D$. by applying the "two militia-men" theorem in the normally ordered Banach space $B(K; Y)$ (endowed with the norm $\|u\| = \sup_{x \in K} \|u(x)\|_Y$). □

Remark 1.4 The assumption at the beginning of (iv) is essential in proving that $u^* = v^*$, since it ensures that the set in (12) is nonempty. Existence of such ε_0 is guaranteed, for example, when $Y = \mathbb{R}$, X is compact, and $\psi^2(\theta)(x) > 0$ for every $x \in X$. In fact, $\psi(\theta)$ and $\psi^2(\theta)$ are continuous functions of x by (ii), so they are bounded by positive numbers from above and from below, respectively.

Observe also that, in this case, (8) holds with $i = 1$ whenever $\tau\psi_1(\tau u)(x) < \psi_1(u)(x)$, and $\psi_1(u)(x) > 0$ for every $x \in X$, since $\tau\psi_1(\tau u)/\psi_1(u)$ is continuous on the compact X.

2 Applications.

In two recent papers [5, 8], certain (possibly perturbed [5]) Hammerstein integral equations, associated with decreasing (possibly *noncompact* [8]) operators, have been studied. We show that Theorem 1.1 above can be used to rediscuss, generalize, and improve in part such results, as in the following

Example 2.1 (*Perturbed Hammerstein equations.*) Consider the perturbed Hammerstein integral equation

$$u(x) = \lambda \int_D \phi(x,t)\, f(t, u(t))\, dt + g(u(x)), \quad \lambda > 0, \tag{13}$$

where

(I) the kernel $\phi(x,t)$ is measurable and nonnegative in D^2, D being a *closed* subset of $\mathbb{R}^N$ (not necessarily bounded), $0 < M := \sup_{x \in D} \int_D \phi(x,t)dt < +\infty$, $\lim_{x \to x_0} \int_D |\phi(x,t) - \phi(x_0,t)|dt = 0$ for every fixed $x_0 \in D$;

(II) $f : D \times [0, +\infty) \to [0, +\infty)$ is measurable, $f(t, \cdot)$ is continuous and nonincreasing for *a.e.* $t \in D$, $f(\cdot, 0) \in L^\infty(D)$, and there exists $0 < \xi = \xi(\tau, u) < 1$, with $\xi(\tau, \cdot) \in C[0, +\infty)$, such that

$$\tau f(t, \tau u) \leq \xi(\tau, u) f(t, u), \tag{14}$$

for every $\tau \in (0, 1)$, $u > 0$, and for a.e. $t \in D$; moreover, if $g \equiv 0$, we assume that there exists a constant $c_\mu > 0$ such that $f(t, \mu) \geq c_\mu$ for *a.e.* $t \in D$, where $\mu = \lambda M \|f(\cdot, 0)\|_\infty$;

(III) either $g \equiv 0$, or $g : [0, +\infty) \to (0, +\infty)$ is continuous and nonincreasing, and

$$\tau g(\tau u) < g(u) , \tag{15}$$

for every $\tau \in (0, 1)$, $u > 0$.

Let u be a nonnegative measurable function; we can define an operator ψ as $\psi(u) = \psi_1(u) + \psi_2(u)$, where

$$\psi_1(u)(x) = \lambda \int_D \phi(x, t)\, f(t, u(t))\, dt , \tag{16}$$

and

$$\psi_2(u)(x) = g(u(x)) . \tag{17}$$

It is easily seen that $\psi_1(u)$ is continuous on D by (I), (II), while $\psi_2(u)$ is continuous whenever such is u, in view of (III). It is also immediate that any fixed point equation like $u = \psi_2(u) + \beta$ has a unique nonnegative and bounded measurable solution, say u_β such that

$$u_\beta(x) = (id - g)^{-1}(\beta(x)) , \quad x \in D , \tag{18}$$

for every nonnegative and bounded measurable β. Clearly, $u_\beta(x)$ is continuous on D if such is β.

Observe that, for every $x \in D$,

$$0 \le \psi(\theta)(x) = \lambda \int_D \phi(x, t)\, f(t, 0)\, dt + g(0)$$

$$\le \lambda \|f(\cdot, 0)\|_\infty \int_D \phi(x, t)\, dt + g(0) \le \lambda M \|f(\cdot, 0)\|_\infty + g(0) =: \mu , \tag{19}$$

and that

$$\psi^2(\theta)(x) = \lambda \int_D \phi(x, t)\, f(t, \psi(\theta)(t))\, dt + g(\psi(\theta)(x))$$

$$\ge \lambda \operatorname{ess} \inf_{t \in D} f(t, \mu) \int_D \phi(x, t)\, dt + g(\mu) . \tag{20}$$

Moreover, if $y \in [0, \mu]$, then by (14), (15), for every fixed τ and *a.e.* fixed t,

$$\tau f(t, \tau y) \le \sigma_1 f(t, y) , \quad \sigma_1 := \max_{y \in [0, \mu]} \xi(\tau, y) < 1 , \tag{21}$$

and, if g is not identically zero,

$$\tau g(\tau y) \le \sigma_2 g(y) , \quad \sigma_2 := \max_{y \in [0, \mu]} \frac{\tau g(\tau y)}{g(y)} < 1 . \tag{22}$$

In fact, $g(y) \geq g(\mu) > 0$ for $y \in [0, \mu]$, so that both $\xi(\tau, y)$ and $\tau g(\tau y)/g(y)$ are continuous functions on $[0, \mu]$, strictly bounded by 1. If $g \equiv 0$, we can take, e.g., $\sigma_2 = \sigma_1$ in (22).

By (21), (22) we obtain that, if u is a measurable function such that $0 \leq u(x) \leq \psi(\theta)(x)$ for (almost) every $x \in D$, then

$$\tau\psi_i(\tau u)(x) \leq \sigma_i\psi_i(u)(x) \ , \quad i = 1, 2 \ , \tag{23}$$

for (almost) every $x \in D$, where σ_1, σ_2 depend only on τ and μ.

With these considerations at hand, equation (13) can be studied in two different functional frameworks, via Theorem 1.1.

First approach. Taking into account Remark 1.3, when D is *bounded* (hence compact), we could consider ψ as a decreasing operator from $L^p(D)$ into $L^p(D)$, $1 \leq p < \infty$: observe that if $u \in L^p(D)$ is nonnegative, then both $\psi_1(u)$ and $\psi_2(u)$ are measurable, nonnegative, and *bounded*, and hence their p-th power is integrable. As known, $L^p(D)$ can be partially ordered in the obvious way, and its cone is fully *regular* but *non solid* [9, Ch.1].

The properties discussed above, see (16)-(23), entail that all assumptions of Theorem 1.1 are satisfied. In particular, in (*iv*) we can take $\varepsilon_0 = c_\mu/\|f(\cdot, 0)\|_\infty$ if $g \equiv 0$, otherwise $\varepsilon_0 = g(\mu)/\mu$, and $\eta_i = \sigma_i^{-1} - 1$, $i = 1, 2$, cf. (19)-(23). The only remaining point concerns monotone demicontinuity of ψ_1 and ψ_2: this is an almost immediate consequence of the continuity of $f(t, \cdot)$ and g, and of Beppo-Levi's monotone convergence theorem, in view of the Riesz representation theorem of linear continuous functionals on $L^p(D)$ (cf. [6, Thm.6.15]).

The final result is that equation (13) has a unique nonnegative and bounded fixed-point $u^* \in L^p(D)$, and that the successive approximations converge to u^* in the norm $\| \cdot \|_{L^p}$, for any initial nonnegative function $u_0 \in L^p(D)$. The fixed-point turns out to be strictly positive whenever g is not identically zero, or $\int_D \phi(x, t)\, dt > 0$ for every $x \in D$. Moreover, in view of the considerations above (cf. (16)-(18)), the fixed-point u^* turns out to be *continuous* on D, as well as the successive approximations starting from any *continuous* nonnegative function u_0 (or even from any nonnegative L^p function, when g is constant).

Second approach. If we make the natural choice $X = D$, $Y = \mathbb{R}$ (with D in general unbounded), and we take as V the space of measurable functions defined on D, again (16)-(23) show that all assumptions of Theorem 1.1 are fulfilled. As for monotone demicontinuity, it can be easily derived by the continuity of $f(t, \cdot)$ and g, and by Beppo-Levi's monotone convergence theorem.

The final result is now that equation (13) has a unique nonnegative and bounded fixed-point $u^* \in C(D)$, and the sequences u_{2n} and u_{2n+1} defined in (9) converge, increasing and decreasing respectively, to u^*, *uniformly* on compact subsets of D. Convergence of the successive approximations is also uniform on compact subsets of D, starting from any nonnegative measurable function u_0. The observations made above, concerning strict positivity of u^* and continuity of the successive approximations, still apply.

Various instances of the nonlinear integral equation (13) have been investigated in [5, 8], under the common assumption that

$$f(t,u) = \left[\sum_{i=0}^{m} a_i(t)u^{\alpha_i}\right]^{-1}, \tag{24}$$

where $0 = \alpha_0 < \alpha_1 < \dots < \alpha_m = 1$, the a_i's are nonnegative and bounded on D, and $\text{ess inf}_{t\in D}\, a_0(t) > 0$.

Note that the function f in (24) satisfies our assumption (II), in particular $\|f(\cdot,0)\|_\infty = (\text{ess inf}_{t\in D}\, a_0(t))^{-1}$, $c_\mu = (\sum_{i=0}^{m} \sup_{t\in D} a_i(t)\, \mu^{\alpha_i})^{-1}$, and by simple calculations

$$0 < \frac{\tau f(t,\tau u)}{f(t,u)} = \frac{1 + \sum_{i=1}^{m} a_i(t)u^{\alpha_i}/a_0(t)}{1/\tau + \sum_{i=1}^{m} a_i(t)u^{\alpha_i}/a_0(t)}$$

$$\leq \frac{\text{ess inf}_{t\in D}\, a_0(t) + \sum_{i=1}^{m} \sup_{t\in D} a_i(t)u^{\alpha_i}}{\frac{1}{\tau}\text{ess inf}_{t\in D}\, a_0(t) + \sum_{i=1}^{m} \sup_{t\in D} a_i(t)u^{\alpha_i}} =: \xi(\tau,u) < 1\ , \tag{25}$$

where we used the fact that $(1+w)/(1/\tau + w)$ is an increasing function of $w \in \mathbb{R}^+$ (clearly $\xi(\tau,u)$ in (25) is a continuous function of u). Note that when g is not identically zero, boundedness of $a_0(\cdot)$ is not required, since it is explicitly needed only in the evaluation of the lower bound c_μ.

We recall that in [5] the support D is bounded (hence compact) and the limit in (I) is uniform with respect to x_0, while in [8] $D = \mathbb{R}^N$, the limit is not assumed to be uniform, but only the unperturbed case ($g \equiv 0$) is considered. Indeed, in absence of uniformity for the limit above, compactness may fail, cf. [8]. Here we are able to treat the case of a general (not necessarily bounded) integration domain D, and of a continuous and nonincreasing perturbation g, either identically zero, or satisfying *only* assumption (15). In [5] the analysis of the "perturbed" case is regarded as a difficult problem, which requires special tools and strong additional hypotheses on the perturbation.

The application of Theorem 1.1 to equation (13), via the second approach described above, represents a generalization of both [5] and [8], and an improvement with respect to [5], since there, *in addition*, g is assumed to be a *contraction* (to allow the use of Darbo's fixed-point theorem); a trivial example of a function g which satisfies (III) but is *not* a contraction, is given by $g(u) = 1/(1+u)$.

On the other hand, when D is unbounded (e.g. $D = \mathbb{R}^N$, cf. [8]), we obtain uniform convergence of the successive approximations only restricted to compact subsets of D. In a forthcoming paper [13], we'll show how Guo's theorem [8] can be used as an alternative approach to the constructive analysis of the nonlinear integral equation (13), under the general assumptions $(I) - (III)$.

References

[1] R.K. Bose and M.C. Joshi, "Some topics in nonlinear functional analysis", Wiley Eastern Limited, New Dehli, 1985.

[2] S.Chandrasekhar and F.H. Breen, *On the radiative equilibrium of a stellar atmosphere. XVI*, Astrophys. J. **105** (1947), 435-440.

[3] S. Chandrasekhar, "Radiative Transfer", Dover, New York, 1960.

[4] J. Dieudonné, "Foundations of Modern Analysis", Academic Press, New York, 1969.

[5] L. Erbe, D. Guo and X. Liu, *Positive solutions of a class of nonlinear integral equations and applications*, J. Integral Equations Appl. **4** (1992), 179-195.

[6] G.B. Folland, "Real Analysis", J. Wiley, New York, 1984.

[7] D. Guo, *Some fixed point theorems and applications*, Nonlinear Anal. **10** (1986), 77-84.

[8] D. Guo, *Positive fixed points and eigenvectors of noncompact decreasing operators with applications to nonlinear integral equations*, Chin. Ann. of Math. (Ser. B) **4** (1993), 419-426.

[9] D. Guo and V. Lakshmikantham, "Nonlinear Problems in Abstract Cones", Academic Press Inc., New York, 1988.

[10] M.A. Krasnoselskii, "Positive solutions of operator equations", P. Noordhoff, Groningen, 1964.

[11] M.A. Krasnoselskii, G. Vainikko, P. Zabreiko, Ya. Rutitskii, V. Stetsenko, "Approximate solution of operator equations" (translated from the Russian by D. Louvish), Wolters-Noordhoff Publishing, Groningen, 1972.

[12] R.W. Leggett, *A new approach to the H-equation of Chandrasekhar*, SIAM J. Math. Anal. **7** (1976), 542-550.

[13] A. Sommariva and M. Vianello, *Constructive approximation for a class of perturbed Hammerstein integral equations*, December 1997, submitted.

[14] C.A. Stuart, *Positive solutions of a nonlinear integral equation*, Math. Ann. **192** (1971), 119-124.